(1.) A particle is moving on a path with position vector $\vec{R}(t) = m\cos t\,\vec{i} + \sin t\,\vec{j}$ where $m>1$.

a) Its path is a known curve. Identify and sketch it.

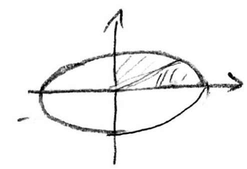
b) Find the velocity vector.

c) For which t is the speed maximal? minimal?

d) Find the acceleration vector. Find the tangential and normal components of the acceleration.

Solution:

a) $\left.\begin{array}{l} x = m\cos t \\ y = \sin t \end{array}\right\}$ eliminate t: $\frac{x^2}{m^2} + y^2 = 1$ Ellipse!



b) $\vec{v} = \frac{d\vec{R}}{dt} = -m\sin t\,\vec{i} + \cos t\,\vec{j}$

c) Speed $= |\vec{v}| = \sqrt{m^2\sin^2 t + \cos^2 t}$ Need max/min

$= \sqrt{m^2\sin^2 t + 1 - \sin^2 t)} = \sqrt{\underbrace{(m^2-1)}_{>0}\sin^2 t + 1}$

so max when $\sin^2 t = \max = 1$ $\max = m$ for $t = \frac{\pi}{2}, \frac{3\pi}{2}$

min when $\sin^2 t = 0$, $\min = 1$ for $t = 0, \pi$

d) $\vec{a} = \frac{d\vec{v}}{dt} = -m\cos t\,\vec{i} - \sin t\,\vec{j}$

Tangential component $a_T = \vec{a}\cdot\vec{T}$ where $\vec{T}$ = unit tan vector

$= \frac{\vec{v}}{|\vec{v}|} = \frac{-m\sin t\,\vec{i} + \cos t\,\vec{j}}{\sqrt{(m^2-1)\sin^2 t + 1}}$

So
$$a_T = \frac{m^2\sin t\cos t + \sin t\cos t}{\sqrt{(m^2-1)\sin^2 t + 1}} = \frac{(m^2-1)\sin t\cos t}{\sqrt{(m^2-1)\sin^2 t + 1}}$$

Normal component
$$a_N = \sqrt{|\vec{a}|^2 - a_T^2} = \sqrt{m^2\cos^2 + \sin^2 - \frac{(m^2-1)^2\sin^2\cos^2}{(m^2-1)}}$$

② Find the value of the parameter λ so that the planes $\lambda x + y - z = 2$ and $x + \lambda y - z = 0$ are parallel.

<u>Solution</u> Normal vectors: $\vec{N}_1 = \lambda \vec{i} + \vec{j} - \vec{k}$

$\vec{N}_2 = \vec{i} + \lambda \vec{j} - \vec{k}$

parallel iff $\vec{N}_1 = k \vec{N}_2$ for some k:

$(\lambda \vec{i} + \vec{j} - \vec{k}) = k(\vec{i} + \lambda \vec{j} - \vec{k})$

$$\begin{cases} \lambda = k \\ 1 = \lambda k \\ -1 = -k \end{cases} \qquad k = 1 \Rightarrow \lambda = 1.$$

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③ Consider the planes $2x + y - z = 2$ and $x + 2y - z = 0$

Why do they intersect? Solution: $\vec{N}_1 \nparallel \vec{N}_2$

Find the symmetric eq of line of $\cap$. Find parametric eq. too.

L

<u>Solution</u>

Since $\vec{N}_1 \perp L$ and $\vec{N}_2 \perp L$ then $L \perp \text{plane}(\vec{N}_1, \vec{N}_2)$

so $L \parallel \vec{N}_1 \times \vec{N}_2$. $\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & -1 \end{vmatrix} = (-1+2)\vec{i} - (-2+1)\vec{j} + (4-1)\vec{k}$

$= \vec{i} + \vec{j} + 3\vec{k}$

Point on L: solve $\begin{array}{l} 2x - z = 2 - y \\ x - z = -2y \end{array}$ for $y = 0$

$$\begin{cases} 2x - z = 2 \\ x - z = 0 \end{cases} \qquad x = z = 2 \qquad \text{Point } (2,0,2) \; (= (x_0, y_0, z_0))$$

Recall: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ if $L \parallel a\vec{i} + b\vec{j} + c\vec{k}$ so $\dfrac{x-2}{1} = \dfrac{y}{1} = \dfrac{z-2}{3}$

Parametric: set $= t$ so $x - 2 = t$

$y = t$

$z - 2 = 3t$

$$\begin{cases} x = t + 2 \\ y = t \\ z = 3t + 2 \end{cases}$$

Solution II Suppose we did not have the idea to use $N_1 \times N_2$

Line of $\cap$ = the set of points (x, y, z) which satisfy both eq

$$\begin{aligned} 2x + y - z &= 2 \\ \text{and} \quad x + 2y - z &= 0 \end{aligned}$$

2 Equations, 3 unknowns.

Set one of the unknowns $= t$, solve for the other two.

Say $y = t$: $2x - z = 2 - t$ (1) Solve for x and z

$x - z = -2t$ (2)

(1) - (2) : $x = 2 - t + 2t$ $x = 2 + t$

(1) - 2×(2) : $-z + 2z = 2 - t + 4t$ $z = 2 + 3t$

$$\begin{cases} x = 2 + t \\ y = t \\ z = 2 + 3t \end{cases}$$ parametric eq.

Line has direction?

Recall $\vec{R}(t) = \vec{r_0} + t\vec{v}$ direction

$\vec{v} = \vec{i} + \vec{j} + 3\vec{k}$

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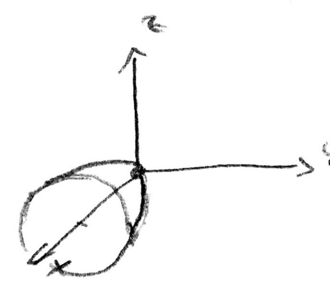
(4) a) Identify and sketch the surface $-2x + y^2 + z^2 = 0$

b) Choose appropriate cylindrical coordinates in which the eq has a simple form and write this equation

c) Show that the point $(1, -1, -1)$ is on this surface and find the eq of the tan plane at this point.

Solution



a) one square missing, $+x^2, +z^2$ paraboloid

$x = 0$ only 0

$x = \text{const} > 0 \rightsquigarrow$ circles circular paraboloid

$y = 0$: $2x = z^2$

b) choose polar coordinates in the yz plane

$y = r\cos\theta$

$z = r\sin\theta$ $-2x + r^2 = 0$

$x = x$

c) Plug in point $-2\cdot 1 + (-1)^2 + (-1)^2 = 0$

Recall $\vec{N} = \frac{dz}{dx}\Big|_{P_0}\vec{i} + \frac{dz}{dy}\Big|_{P_0}\vec{j} - \vec{k}$ And

Implicit differentiation:

$\frac{\partial}{\partial x}$: $-2 + 2z\frac{\partial z}{\partial x} = 0$ so $\frac{\partial z}{\partial x} = \frac{1}{z}$ $\quad \frac{\partial z}{\partial x}\Big|_{(1,-1,-1)} = -1$

$\frac{\partial}{\partial y}$: $-2y + 2z\frac{\partial z}{\partial y} = 0$ so $\frac{\partial z}{\partial y} = \frac{y}{z}$ $\quad \frac{\partial z}{\partial y}\Big|_{(1,-1,-1)} = 1$

$\boxed{\text{Recall}}$ eq of plane through (x_0, y_0, z_0), $\perp$ to $\vec{N} = a\vec{i} + b\vec{j} + c\vec{k}$

is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

So $-(x-\cancel{1}) + (y+1) - (z+\cancel{1}) = 0 \qquad -x + y - z + 1 = 0$

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⑤ Find the symmetric eq of the line of $\cap$ of the plane $-x+y-z+1=0$ with the xy plane.

Solution

xy plane has eq $z = 0$ so $\begin{cases} -x+y-z+1=0 \\ z=0 \end{cases}$

$\begin{cases} -x+y+1=0 \\ z=0 \end{cases}$ $\quad$ Set $x = t$ $\quad \begin{cases} x = t \\ y = -1+t \\ z = 0 \end{cases}$

Symmetric : $t = x = y+1 = \frac{z}{0}$

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(6) Wave equation: $c^2 \frac{\partial^2 w}{dx^2} = \frac{\partial^2 w}{\partial t^2}$

Show that if f, g are twice differentiable functions (of one variable) then $w = f(x+ct) + g(x-ct)$ is a solution

Solution $\frac{\partial w}{\partial x} = f'(x+ct) + g'(x-ct)$ $\qquad \frac{\partial w}{\partial t} = c f'(x+ct) - c g'(x-ct)$

$\frac{\partial^2 w}{\partial x^2} = f''(x+ct) + g''(x-ct)$ $\qquad \frac{\partial^2 w}{dt^2} = c^2 f''(x+ct) + c^2 g''(x-ct)$

$= c^2 \frac{\partial^2 w}{dx^2}$ indeed!

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(7) Show that any function of the form $z = f(\frac{y}{x})$ satisfies the PDE: $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$

Sol. $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x \cdot f'(\frac{y}{x}) \frac{-y}{x^2} + y f'(\frac{y}{x}) \frac{1}{x} = 0$ indeed

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(8) a) Find the domain of the function $f(x,y) = 1 + \sqrt{x^2+y^2}$. Is it continuous?

b) Write its linear approximation at (a, b)

c) At which points there is no linear approximation?

d) plot the graph

Solution a) defined everywhere. Continuous

$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ $\qquad L(x,y) = 1 + \sqrt{a^2+b^2} + \frac{a}{\sqrt{a^2+b^2}}(x-a) + \frac{b}{\sqrt{a^2+b^2}}(y-b)$

No lin approx at $(0,0)$ Not differentiable

Plot: $z = 1 + \sqrt{x^2+y^2}$ so $(z-1)^2 = x^2+y^2$

$\frac{1}{2}$ cone!