NONLINEAR PERTURBATIONS OF FUCHSIAN SYSTEMS: 
CORRECTIONS AND LINEARIZATION, NORMAL FORMS

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ABSTRACT. Nonlinear perturbation of Fuchsian systems are studied in a region 
including two or more singularities. It is proved that such systems are generally 
not analytically equivalent to their linear part (they are not linearizable) and the 
obstructions are found constructively, as a countable set of numbers. It is also 
shown that there exists a unique “correction” of the nonlinear part so that the 
"corrected" system is formally linearizable.

Normal forms of these systems are found, providing also their classification.

1. INTRODUCTION

1.1. Setting. The paper studies linearization criteria and the formal classification 
of nonlinear perturbations of Fuchsian systems:

\[ \frac{du}{dx} = M(x)u + g(x, u) \]

for \( u \in \mathbb{C}^d \) small, and \( x \) in a connected, simply connected domain \( D \subset \mathbb{C} \) which 
includes two, or more singular points of the matrix \( M(x) \).

The linear part of (1)

\[ \frac{dw}{dx} = M(x)w \quad w \in \mathbb{C}^d, \quad x \in \mathbb{C} \]

is a Fuchsian system. Then that all its singularities in the complex plane, including 
the point \( x = \infty \), are regular (see the Appendix, §4.1 for details).

The function \( g(x,u) \), which gathers the nonlinear terms, can have simple poles at 
the singularities of \( M(x) \):

\[ g(x,u) = \frac{1}{Q(x)}f(x,u) \]

where: \( Q(x) = \prod (x - x_i) \), with \( x_i \in D \) singular for \( M \), \( f(x,.) \) has a zero of order 
two at \( u = 0 \) and \( f \) is holomorphic for \( x \in D \) and small \( u \).

The case when the domain \( D \) contains just one singular point of \( M(x) \) was studied 
in [16] and the main results are summarized in §1.3.1.
The present paper considers the case when $M(x)$ has three singularities in the extended complex plane. Placing one at $\infty$, the other two are conventionally placed at $x = 1$ and $x = -1$ (but their location can be arbitrarily placed using a linear fractional change of the variable $x$). In this case the matrix $M$ has the form

$$M(x) = \frac{1}{x - 1} A + \frac{1}{x + 1} B \quad A, B \in \mathcal{M}_d(\mathbb{C})$$

and

$$Q(x) = x^2 - 1$$

The systems studied have therefore the form

$$\frac{du}{dx} = \left( \frac{1}{x - 1} A + \frac{1}{x + 1} B \right) u + \frac{1}{x^2 - 1} f(x, u)$$

where $f$ is analytic for $x$ in the connected, simply connected domain $D \ni \{\pm 1\}$ and for $u \in \mathbb{C}^d$, $|u| < r$, $f(x, \cdot)$ has a zero of order two at $u = 0$.

Generalizations of the results to Fuchsian systems with a larger number of singularities are discussed in §3.3.

1.2. Motivation. The question of linearization and, more generally, of equivalence, is a fundamental problem in the theory of differential equations.

Besides the clear theoretical interest, linearization and equivalence are used as instrumental methods in control theory, and in devising algorithms for numerical and symbolic calculations (see, to cite just a few authors, [1]-[5]). Often the methods used are developed from the method of equivalence introduced by Cartan [6] to decide whether two differential structures can be mapped one to another by a transformation taken in a given pseudogroup [7], [8]. In the case of differential equations, the method was used for regular systems and extended for a neighborhood of one singular manifold [9], [10].

Integrability is global, and it is natural to look for linearizability in wide regions. Existence of an analytic linearization of differential systems is closely related to integrability (see, e.g., [1],[3]).

The present paper uses direct and constructive methods in the study of linearizability and normal forms for equations in the class (3) in domains which include several singular manifolds.

The study of vector fields with an eigenvalue equal to 1 at a singular point can be reduced to the study of a Fuchsian system near one singularity (7) by eliminating time in a time-independent system. Similarly, the study of vector fields in regions containing two, or more singular points is reducible to that of equations of type (1).
More generally, the study of Hamiltonian systems with polynomial potentials near particular, periodic, or doubly-periodic solutions can be reduced to the study of nonlinear perturbations of Fuchsian systems [11], [12].

The result of Theorem 3 has a very interesting similarity with needed corrections found in other problems: Écalle and Vallet showed that resonant systems are linearizable after appropriate correction [13]; also Gallavotti showed that there exists appropriate corrections of Hamiltonian systems so that the new system is integrable [14], convergence being proved later by Eliasson [15]. This suggests the possible existence of an underlying general structure.

1.3. Linearization in a neighborhood of one regular singular point and connection to integrability.

1.3.1. Region containing one regular singularity. Consider a nonlinear perturbation of a system with one regular singular point in the complex plane:

\[
\frac{du}{dx} = \frac{1}{x} M(x)u + \frac{1}{x} \tilde{f}(x,u), \quad u \in \mathbb{C}^d, \quad x \in \mathbb{C}
\]

where \(M(x)\) is a matrix analytic at \(x = 0\).

There is no loss of generality in assuming that the matrix \(M(x)\) is constant (see §4.2 for details):

\[
\frac{du}{dx} = \frac{1}{x} Mu + \frac{1}{x} f(x,u), \quad u \in \mathbb{C}^d, \quad x \in \mathbb{C}
\]

\(f(x,u)\) is assumed analytic for small \(u\): \(|u| < r\) and for \(x\) in a domain \(D\) which is either a disk centered at the origin: \(|x| < r'\), or an annulus: \(r'' < |x| < r'\). Such systems are analytically linearizable, generically, [16]:

**Theorem 1.** Assume that the eigenvalues \(\mu_1, \ldots, \mu_d\) of the matrix \(M\) satisfy the Diophantine condition:

\[
|n \cdot \mu + l - \mu_s| > C (|n| + |l|)^{-\nu}
\]

for all \(l \in \mathbb{N}\) if \(D\) is a disk (respectively all \(l \in \mathbb{Z}\) if \(D\) is an annulus), and for all \(s \in \{1, \ldots, d\}\), and \(n \in \mathbb{N}^d\) with \(|n| \geq 2\).

Then the system (8) is analytically equivalent to its linear part \(w' = \frac{1}{x} Mw\) for \(x \in \tilde{D} \subset D\) and \(|u| < \tilde{r} < r\).

**Remark 1.**

(i) The condition (9) is satisfied by generic matrices \(M\).

(ii) The domain \(\tilde{D}\) can be made arbitrarily close to \(D\) for \(\tilde{r}\) small enough.

\(^1\)Allowing the nonlinear part to be singular at \(x = 0\) accommodates systems corresponding to higher order equations.
(iii) The analytic equivalence map is unique if no eigenvalue $\mu_j$ is integer.

1.3.2. Region containing several regular singularities. Consider equation (6). By Theorem 1, there (generically) exists an equivalence map of this equation to its linear part (2), for short, a linearization map, analytic for $x$ in a neighborhood of 1. This map is unique under the requirement of analyticity at $x = 1$ (see Remark). Similarly, there exists a unique linearization map which is analytic at $x = -1$. Generally these two maps do not coincide: the map which is analytic at $x = \pm 1$ is (usually) ramified at $x = \mp 1$. Indeed, the first term of the right-hand-side of (25) below displays the ramification.

Therefore equations (1) are not necessarily linearizable in a region containing more than one singularity.

Absence of linearizability has important consequences for the system, since it implies non-integrability (at least generically) [17]:

**Theorem 2.** Consider the nonlinear equation in one dimension:

\[
\frac{du}{dx} = \left( \frac{a}{x - 1} + \frac{b}{x + 1} \right) u + \frac{1}{x^2 - 1} f(x,u)
\]

for $x$ in a connected domain $D$ containing $x = \pm 1$ and small $u$.

If equation (10) is not analytically linearizable then for generic $a, b$ (precise conditions are given in [17]) solutions have dense branching and therefore no single-valued integrals exist. Among integrable cases, first integrals are not meromorphic (generically).

It is therefore important to find what are the linearization criteria for equations (1), and furthermore to describe the equivalence classes - to find normal forms for these equations.

2. Main Results

2.1. Assumptions. Consider the system (6). The following conditions (a)-(c), satisfied by generic matrices, are assumed.

(a) The eigenvalues of $A$, respectively $B$, satisfy the Diophantine condition (9), and (b) none of these eigenvalues is an integer.

(c) The eigenvalues $\lambda_1, \ldots, \lambda_d$ of the matrix $A + B$ satisfy the following nonresonance condition:

\[
k + n \cdot \lambda - \lambda_j \not\in \mathbb{Z}_- \text{ for all } n \in \mathbb{N}^d, k \in \mathbb{N}, j = 1, \ldots, d
\]

**Notation:** for $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$ and $m \in \mathbb{N}^d$ denote

\[w^m = w_1^{m_1} \cdots w_d^{m_d}, \quad |m| = m_1 + \cdots + m_d\]
2.2. Obstructions to linearization and existence of corrections.

Theorem 3. Consider the system (6) under the assumptions of §2.1. Assume that the nonlinear part is of polynomial type in $x$, in the sense that

\begin{equation}
\mathbf{f}(x, \mathbf{u}) = \sum_{|m| \geq 2} \mathbf{f}_m(x) \mathbf{u}^m \quad \text{with } \mathbf{f}_m(x) \text{ polynomials}
\end{equation}

Then there exists a unique "correction" $\phi(\mathbf{u})$ of $\mathbf{f}(x, \mathbf{u})$ as a formal series

\begin{equation}
\phi(\mathbf{u}) = \sum_{|m| \geq 2} \phi_m \mathbf{u}^m
\end{equation}

so that the "corrected" equation

\begin{equation}
\frac{d\mathbf{u}}{dx} = \left( \frac{1}{x-1} \mathbf{A} + \frac{1}{x+1} \mathbf{B} \right) \mathbf{u} + \frac{1}{x^2-1} [\mathbf{f}(x, \mathbf{u}) - \phi(\mathbf{u})]
\end{equation}

is linearizable by a formal series

\begin{equation}
\mathbf{u} = \mathbf{h}(x, \mathbf{w}) = \mathbf{w} + \sum_{|m| \geq 2} \mathbf{h}_m(x) \mathbf{w}^m
\end{equation}

where $\mathbf{h}_m(x)$ are functions analytic on $D$. Moreover, $\mathbf{h}_m(x)$ are polynomials.

Remarks

1. The coefficients $\phi_m \in \mathbb{C}^d$ of the correction $\phi$ (see (13)) represent the obstructions to linearization, in the sense that the system (6) is linearizable by a formal power series if and only if all these vectors are zero.

2. $\phi_m$ are obtained constructively, see (20), (21), Lemmas 5 and 6.

3. The nonresonance condition (11) is essential for the result of Theorem 3 to hold. However, the condition that $\mathbf{f}(\cdot, \mathbf{u})$ be of polynomial type is not. In a forthcoming paper [18] it is shown that if the matrices $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonalizable, then Theorem 3 holds more generally, for $\mathbf{f}(x, \mathbf{u})$ analytic in $x$. Furthermore, it is also shown in [18] that if in addition, the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{B}$ have positive real part, then all the series converge and the corrected system is analytically linearizable.

2.3. Formal normal forms. Since equations (6) are not necessarily linearizable, then two such equations are not necessarily equivalent. The following theorem finds normal forms for equations (6).
Theorem 4. Consider the system (6) under the assumptions of §2.1.
Furthermore, assume that the nonlinear part is of polynomial type in \( x \), in the sense of (12).

Then there exists a unique formal series \( \psi(w) = \sum_{|m| \geq 2} \psi_m w^m \) so that (6) is equivalent to

\[
(16) \quad w' = \left( \frac{1}{x-1} A + \frac{1}{x+1} B \right) w + \frac{1}{x^2 - 1} \psi(w)
\]

through a formal series

\[
(17) \quad u = h(x, w) = w + \sum_m h_m(x) w^m
\]

with \( h_m(x) \) analytic functions on \( D \).
Moreover, \( h_m(x) \) are polynomials.

3. Proofs

3.1. Proof of Theorem 3

3.1.1. The recursive system. The map (15) is a linearization map of (6) iff \( h \) satisfies the nonlinear PDE

\[
(18) \quad \partial_x h + d_w h \ M w = M h + \frac{1}{x^2 - 1} \left[ f(x, w + h) - \phi(w + h) \right]
\]

where \( M \) is given by (4). Expanding in power series in \( w \) and denoting by \( h_n \) the homogeneous part of degree \( n \) in (15):

\[
(19) \quad h_n(x, w) = \sum_{|m|=n} h_m(x) w^m
\]

we obtain \( h_n \) recursively in \( n \). The functions \( \{h_m\}_{|m|=n} \) satisfy the linear differential system

\[
(20) \quad \partial_x h_n + d_w h_n \ M w - M h_n = \frac{1}{x^2 - 1} R_n(x, w)
\]

where

\[
(21) \quad R_n = f_n - \phi_n + \tilde{R}_n
\]

with \( \tilde{R}_n \) a polynomial in \( \phi_m, h_m, f_m \) with \( |m| < n \).
3.1.2. *Numerical obstructions to linearizability.* Denote by $Y(x)$ a fundamental matrix of the linear system (2): $Y' = MY$. A particular solution of (20) is given by

$$h_n(x, w) = Y(x) \int_{-1}^{x} Q(t)^{-1}Y(t)^{-1}R_n(t, Y(t)Y(x)^{-1}w) \, dt$$

where it was assumed that $Q^{-1}Y^{-1}$ is Lebesgue integrable at $x = -1$ (otherwise the integral is understood as the the Hadamard principal part, or, a finite number of terms must first be subtracted out).

Let $G$ be the monodromy matrix of (2) at $x = -1$: after analytic continuation along a closed loop around $x = -1$ the matrix $Y(x)$ becomes $AC(-1)Y(x) = Y(x)G$.

The analytic continuation of (22) on a closed loop around $x = -1$ yields

$$AC(-1)h_n(x, w) = Y(x)G \int_{-1}^{x} Q(t)^{-1}G^{-1}Y(t)^{-1}R_n(t, Y(t)GG^{-1}Y(x)^{-1}w) \, dt$$

$$= h_n(x, w)$$

which means that (22) is the unique linearization map of (6) which is analytic at $x = -1$.

Rewriting (22) as

$$h_n(x, w) = Y(x) \int_{-1}^{1} Q^{-1}Y^{-1}R_n \, dt + Y(x) \int_{1}^{x} Q^{-1}Y^{-1}R_n \, dt$$

a similar argument shows that the last term in (25) is the unique linearization map which is analytic at $x = 1$. Then $u = h(x, w)$ is analytic at both $x = 1$ and $x = -1$ if and only if

$$\int_{-1}^{1} Q(t)^{-1}Y(t)^{-1}R_n(t, Y(t)Y(x)^{-1}w) \, dt = 0 \quad \text{for all } w \in \mathbb{C}^d, \, n \geq 2$$

The recursive formulas (26) contain the obstructions to linearization: there is one numerical condition (vector-valued in $\mathbb{C}^d$) for every $m \in \mathbb{N}^d, |m| \geq 2$.

3.1.3. *Existence of the correction $\phi$.* The existence of $\phi$ (as a formal series) is established directly on the differential equation (20) with (21), rather than solving (26). This result, which completes the proof of Theorem 3, follows directly from the following lemma:

**Lemma 5.** Denote by $P_n$ the space of vector-valued polynomials in $w$, homogeneous degree $n$ ($P_n \subset \mathbb{C}^d[w_1, \ldots, w_d]$).

Assume (11) holds.
Then for any $F = F(x, w) \in \mathcal{P}_n[x]$, there exists a unique $\phi \in \mathcal{P}_n$ so that the differential equation

$$\partial_x P(x, w) + d_w P(x, w) M(x) w - M(x) P(x, w) = \frac{1}{x^2 - 1} [F(x, w) - \phi(w)]$$

has a solution $P \in \mathcal{P}_n[x]$.

The proof of Lemma 6 is given in §3.1.4. Note that if $F$ has degree $k + 1$ in $x$, then the solution $P$ of (27) has degree $k$.

**Remark 2.** The only solutions of (27) which are analytic at both $x = 1$ and $x = -1$ are polynomials, and they are given by the following generalized Rodrigues formulas. Consider the polynomials given by

$$P_{k+1}(x, w; q) = Q(x)Y(x)\left[\frac{d^{k+1}}{dt^{k+1}} Q(t)Y(t)^{-1}q(Y(t)Y(x)^{-1}w)\right], \quad q \in \mathcal{P}_n, \; k \in \mathbb{N}$$

(these, together with $\mathcal{P}_n$, generate the vector space $\mathcal{P}_n[x]$). Then

$$P_k(x, w; q) = Y(x)\left[\frac{d^k}{dt^k} Q(t)Y(t)^{-1}q(Y(t)Y(x)^{-1}w)\right]$$

is the polynomial solution of (27) for $F = F_{k+1}$ and $\phi = 0$.

The functions (28) are indeed polynomials in $x$, of degree $k$. For example

$$P_0(x, w; q) = q(w), \quad P_1(x, w; q) = (Q' - Q M) q(w) + Q d q(w) M w$$

$$P_2(x, w; q) = [(Q^2)'' + Q^2(M^2 - M') - 2(Q^2)M] q(w) + 2 [(Q^2)' + Q^2 M] d q(w) M w + Q^2 d q(w) (M' + M^2) w + Q^2 d^2 q(w) (M w, M w)$$

3.1.4. Proof of Lemma 6. The proof uses the following solvability Lemma:

**Lemma 6.** Let $\Lambda$ be a $d \times d$ matrix with eigenvalues satisfying the nonresonance condition (17). Let $J_\Lambda$ be the following linear operator on $\mathcal{P}_n$:

$$J_\Lambda = (d p) \Lambda w - \Lambda p$$

Let $l \in \mathbb{N}$.

(i) If $\Lambda$ is diagonal, then the operator $l + J_\Lambda$ is diagonal, with eigenvectors $w^m e_j$, and corresponding eigenvalues $l + m \cdot \lambda_j - \lambda_j$, for all $j = 1, \ldots, d$ and $m$, $|m| = n$.

(ii) If $\Lambda$ is in Jordan normal form, then $l + J_\Lambda$ is in Jordan normal form and the eigenvalues are as at (i).

(iii) As a consequence, the operator $l + J_\Lambda$ is one-to-one and onto $\mathcal{P}_n$.

Lemma 6 follows by calculation (see also [20] Ch.5, §22).

To prove Lemma 5 denote by $L$ and $N$ the following linear operators on $\mathcal{P}_n$: $L = J_{(A+B)}$ and $N = J_{(A-B)}$ (see (29)).
Let $F = F(x, w) \in P_n[x]$ be a polynomial in $x$ degree $k + 1$: $F = \sum_{j=0}^{k+1} x^j f_j$ with $f_j \in P_n$. Identifying the coefficients of $x$, a solution $P$ of (27) has degree $k$ and, if $P = \sum_{j=0}^{k} x^j p_j$ then $p_j$ must satisfy the recursive system

\begin{align*}
(k + L)p_k &= f_{k+1} \\
(k - 1 + L)p_{k-1} &= Np_k + f_k \\
(l - 1 + L)p_{l-1} &= Np_l + (l + 1)p_{l+1} + f_l \quad \text{for } 1 \leq l \leq k - 1 \\
0 &= Np_0 + p_1 + f_0 - \phi
\end{align*} \tag{30}

By Lemma [6] the system (30) can be solved recursively for $p_k$, then $p_{k-1}$, and all the way to $p_0$, for any $f_{k+1}, f_k, \ldots, f_1$. The last relation in (30) determines uniquely $\phi$. \hfill \Box

3.2. **Proof of Theorem [4]**. The map (17) is an equivalence map of (6) and (16) iff $h$ satisfies the nonlinear PDE

\[ \partial_x h + d_w h M w - M h = \frac{1}{x^2 - 1} [f(x, w + h) - \psi(w) - (d_w h) \psi] \]

where $M$ is given by (4). Expanding in power series in $w$ and denoting by $h_n$ the homogeneous part of degree $n$ in (15):

\[ h_n(x, w) = \sum_{|m|=n} h_m(x) w^m \]

we obtain that $h_n$ are obtained recursively by $n$, as solutions of the equations

\[ \partial_x h_n + d_w h_n M w - M h_n = \frac{1}{x^2 - 1} R_n(x, w) \quad (n \geq 2) \]

where

\[ R_n = f_n - \psi_n + \tilde{R}_n \]

with $\tilde{R}_n$ a polynomial in $\psi_m, h_m, f_m$ with $|m| < n$.

As in the proof of Theorem [3] the terms $\psi_n$ are recursively determined from (33), (34) using Lemma [5] for $F = f_n + \tilde{R}_n$.

3.3. **Generalizations to more than two singularities**. The obstructions to linearizability of equations (6) appear as one numerical condition for all $w^m e_j$ ($m \in \mathbb{N}^d, |m| \geq 2, j = 1, \ldots, d$) - see [3.1.2] The correction $\phi(u)$ chosen in Theorem [3] does not depend on $x$. But this correction can be chosen to depend on $x$, say as $x \phi(u)$, or $x^2 \phi(u)$, or $P(x) \phi(u)$ for almost any polynomial $P(x)$.

If a system (1) is studied for $x$ in a connected, simply connected domain containing more than two singularities of $M$, say $x_1, x_2, \ldots, x_s$, then the correction should
be searched in the form $\phi_0(\mathbf{u}) + x\phi_1(\mathbf{u}) + \ldots + x^{s-2}\phi_{s-2}(\mathbf{u})$ and requiring analytic linearization on all the domains containing pairs of singularities: $x_j$ and $x_{j+1}$, $j = 1, \ldots, s - 1$.

4. Appendices

4.1. Regular singular points of a linear differential equation. Consider a linear differential equation

$$\frac{dw}{dx} = M(x)w \quad w \in \mathbb{C}^d, \ x \in \mathbb{C}$$

A point $x = x_0$ is called a regular point of (35) if the matrix $M(x)$ is analytic at $x = x_0$.

A point $x = x_0$ is called a regular singular point if $M(x)$ has an isolated singularity at $x = x_0$ and there exists a constant $k$ so that all solutions of the system, in every sector of the complex $x$-plane with vertex at $x_0$ grow no faster than $x^k$ as $x \to x_0$ (see [19], Ch.7, §2.1).

The equation (35) is called Fuchsian if

$$M(x) = \sum_{j=1}^{r} \frac{1}{x - a_j} A_j$$

in which case solutions of (35) are analytic at all points except $a_1, \ldots, a_r$ and $\infty$ and these points are regular singular points. It turns out that in this case all solutions have formal asymptotic expansions as series in powers and logs, as $x \to a_j$, or $x \to \infty$ [21]. Moreover, these series converge.

For generic nonlinear perturbations of systems (35) near a regular singular point $x_0$, solutions also have convergent asymptotic expansions in powers of $x$ as $x \to x_0$ by Theorem II. For simplicity, the point $x = x_0$ will be also called a regular singular point (of a nonlinear system).

4.2. The matrix $M(x)$ can be assumed constant in (7). Indeed, if the eigenvalues of $M(0)$ are nonresonant (in the sense that no two eigenvalues differ by an integer) then the linear part of (7):

$$\frac{dw}{dx} = \frac{1}{x} M(x)w$$

is analytically equivalent to

$$\frac{d\tilde{w}}{dx} = \frac{1}{x} M(0)\tilde{w}$$
for small $x$ and $w$ [19]. Denoting $w = H(x)\tilde{w}$, the analytic equivalence map between (36) and (37), the same map transforms (7) into a system with the same regularity, but with $M$ constant: the map $u = H(x)\tilde{u}$ is an analytic equivalence of (7) and

$$\frac{d\tilde{u}}{dx} = \frac{1}{x} M(0)\tilde{u} + \frac{1}{x} H^{-1}\tilde{f}(x, H\tilde{u})$$

References

[17] R. D. Costin, M. D. Kruskal, Nonintegrability criteria for a class of differential equations with two regular singular points, Nonlinearity, 16 (2003), no. 4, 1295–1317