Nonintegrability criteria for a class of differential equations with two regular singular points

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Abstract.

Criteria for nonintegrability (in the sense of inexistence of continuous, single-valued first integrals in the complex domain) are given for first order differential equations whose linear part is a homogeneous equation with several regular singular points (in a bounded domain of the complex plane). The method used is the poly-Painlevé test. Local equivalence maps between the nonlinear equations and their linear part are used in the proofs, and the (noncommutative) group of monodromy maps is studied to establish dense branching of solutions, hence nonintegrability.

A dicussion concerning the notion of first integral is included.

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1. Introduction

The present paper gives criteria for nonintegrability (in the sense of non-existence of continuous first integrals in the complex domain) for the class of first order differential equations having 0 as a particular solution, and whose linearization has two regular singular points (in a bounded domain of the complex plane).

More precisely, let $D \subset \mathbb{C}$ be a relatively compact, connected, simply connected open set and equations

$$p(x)\frac{du}{dx} = q(x)u + f(x,u) \tag{1.1}$$

where p, q are analytic on $\overline{D}$, $p(x)$ with only two zeroes in $\overline{D}$, both simple: $p(x) = (x-s_1)(x-s_2)p_1(x)$, where $s_1 \neq s_2$, $s_{1,2} \in D$, and $p_1(x) \neq 0$ and analytic in $\overline{D}$. The function f (which collects the nonlinear terms) satisfies $f(x,u) = O(u^2)$ $(u \to 0)$ and is assumed analytic for $x \in \overline{D} \setminus \{s_1, s_2\}$ and $|u| < R_f$ $(R_f > 0)$.

Equation (1.1) can be brought to a canonical form as follows. Dividing the equation by $p_1(x)$, it may be assumed that p is a polynomial

$$p(x) = (x-s_1)(x-s_2) \tag{1.2}$$

Then write $q(x)/p(x) = \mu_1/(x-s_1) + \mu_2/(x-s_2) + r(x)$ with $\mu_{1,2} \in \mathbb{C}$, and $r(x)$ analytic on $\overline{D}$. Then the analytic transformation $u = u_1 \exp(\int r(x)dx)$ of the dependent variable transforms (1.1) into a similar equation, but with $q(x)$ of the form

$$q(x) = p(x)\left(\frac{\mu_1}{x-s_1} + \frac{\mu_2}{x-s_2}\right) \ , \ \mu_{1,2} \in \mathbb{C} \tag{1.3}$$

Hence, without loss of generality, it may be assumed that the coefficients of (1.1) satisfy (1.2), (1.3).

The integrability properties of equations of type (1.1) with only one singular point in D ($p(x) = x - s_1$, $q(x) = \mu_1$) were studied in [1]. It was shown that if $\mu_1 \in \mathbb{C} \setminus \mathbb{R}$ then there exists a single-valued first integral (for x in an annulus around s_1 and small u) and that necessarily this integral is singular (not meromorphic) for $u = 0$. In the case when $\mu_1 \in \mathbb{R} \setminus \mathbb{Q}$, there is no analytic first integral (generically), but there is one real-valued, real-analytic integral (for x in an annulus around s_1 and small u).

The present paper addresses the case when two singularities s_1, s_2 are present. The methods used can be generalized in a straightforward way to the case when D contains more than two simple zeroes of $p(x)$, but since it turns out that two are (generically) sufficient to ensure nonexistence of continuous first integrals of (1.1), the paper addresses, for simplicity, only this case.

Criteria of nonintegrability for (1.1) are obtained using the poly-Painlevé test [2], [3] (see also [1], [4], [5]). The basic idea is that branching of solutions may obstruct the existence of single-valued first integrals. Indeed, let $u = \mathbf{u}(x)$ be a multivalued solution of a given differential equation. A single-valued first integral is a continuous, nonconstant function $F(x,u)$ which is constant on the trajectories. Suppose that a solution has dense branching, in the sense that (for generic x) the set of values of $\mathbf{u}(x)$ on all the branches

$$S_{u,x} = \{u(x)\,;\, u \in \mathbf{u}\} \tag{1.4}$$

is dense in $\mathbb{C}$. Since $F(x,u)$ is continuous, then it can not depend on u, hence it is a constant function: there is no first integral.

In order to find the branching of solutions, expressed as the set $S_{u,x}$, (hard to determine unless the equation can be solved explicitly), the poly-Painlevé test proposes the use of appropriately chosen asymptotic expansions of solutions, and the idea that if an asymptotic approximation of a solution has dense branching, then the true solution also has dense branching (at least in generic cases).

For equations (1.1) the test can proceed as follows. Denote

$$f(x,u) = f_2(x)u^2 + f_3(x)u^3 + ...$$

Choosing to study (1.1) for small u, introduce a small parameter ϵ using the substitution $u = \epsilon U$. The equation becomes

$$p(x)\frac{dU}{dx} = q(x)U + \epsilon^{-1}f(x,\epsilon U) \tag{1.5}$$

where

$$\epsilon^{-1}f(x,\epsilon U) = \epsilon f_2(x)U^2 + \epsilon^2 f_3(x)U^3 + ...$$

Solutions of (1.5) have a power series expansion in ϵ (convergent for x in appropriate domains)

$$U(x) = U_0(x) + \epsilon U_1(x) + \epsilon^2 U_2(x) + ... \tag{1.6}$$

whose terms can be calculated order by order. The first approximation, $U_0(x)$, satisfies $p(x)U_0'(x) = q(x)U_0(x)$ hence $U_0(x) = K_0 A(x)$ where

$$A(x) = \exp\left(\int_a^x q(t)/p(t)\,dt\right) \tag{1.7}$$

$$= (x-s_1)^{\mu_1}(x-s_2)^{\mu_2}(a-s_1)^{-\mu_1}(a-s_2)^{-\mu_2} \tag{1.8}$$

for some $a \in D, a \neq s_{1,2}$. (Initial branches of the powers are chosen to fix the constant K_0.) Denote

$$\theta_j = \exp(2\pi i \mu_j) \tag{1.9}$$

The set of values of $U_0(x)$ on all branches is $S_{U_0,x} = \{\theta_1^{n_1}\theta_2^{n_2}U_0(x)\,;\, n_{1,2} \in \mathbb{Z}\}$. For generic pairs $(\mu_1,\mu_2) \in \mathbb{C}^2$ the set $S_{U_0,x}$ is dense in $\mathbb{C}$, hence $U_0(x)$ has dense branching (Proposition 3 (i)).

There are important cases when $S_{U_0,x}$ is not dense (e.g. equations with real coefficients, when s_1 and μ_1 are the complex conjugates of s_2, respectively μ_2). Higher approximations of $U(x)$ can then be studied.

The term $U_1(x)$ satisfies

$$p(x)\frac{dU_1}{dx} = q(x)U_1 + f_2(x)U_0^2$$

with the general solution

$$U_1(x) = K_1 A(x) + K_0^2 A(x) \int_a^x A(\xi) f_2(\xi)(\xi - s_1)^{-1}(\xi - s_2)^{-1}\, d\xi \tag{1.10}$$

It should be noted that the constants $\epsilon, K_0, K_1, ...$ are not independent. The Appendix §3.3 shows this, and gives the connection between these explanatory calculations and the notations used in the proofs of the present paper.

Upon analytic continuation on a closed path π_1, winding counter-clockwise around $x = s_1$, the value of $U_1(x)$ becomes (see §3.1 for calculations)

$$AC_{\pi_1} U_1(x) = \left[\theta_1 K_1 + \alpha_1 \left(\theta_1 - \theta_1^2\right) K_0^2\right] A(x) \tag{1.11}$$

$$+\theta_1^2 K_0^2 A(x) \int_a^x A(\xi) f_2(\xi)(\xi - s_1)^{-1}(\xi - s_2)^{-1}\, d\xi \tag{1.12}$$

Going back to the variable $u = \epsilon U_0 + \epsilon^2 U_1 + O(\epsilon^3)$, the initial value of $u(x)$ is

$$u(a) = \epsilon K_0 + \epsilon^2 K_1 + O(\epsilon^2) \tag{1.13}$$

and, after analytic continuation around $x = s_1$

$$AC_{\pi_1} u(a) = \epsilon \theta_1 K_0 + \epsilon^2 \left[\theta_1 K_1 + \alpha_1 \left(\theta_1 - \theta_1^2\right) K_0^2\right] + O(\epsilon^2) \tag{1.14}$$

A similar formula holds for analytic continuation around s_2:

$$AC_{\pi_2} u(a) = \epsilon \theta_2 K_0 + \epsilon^2 \left[\theta_2 K_1 + \alpha_2 \left(\theta_2 - \theta_2^2\right) K_0^2\right] + O(\epsilon^2) \tag{1.15}$$

Consider the closed path Δ starting at x which goes counterclockwise around s_1, then around s_2, then clockwise around s_1, then s_2. After analytic continuation on Δ, the value of u at a becomes (see §3.2 for calculations)

$$AC_{\Delta} u(a) = \epsilon K_0 + \epsilon^2 \left[K_1 + (\alpha_1 - \alpha_2)(1 - \theta_1)(1 - \theta_2) K_0^2\right] + O(\epsilon^2) \tag{1.16}$$

After analytic continuation, p times counterclockwise around s_1, followed by n times along Δ, and finally p times clockwise around s_1 the value of u at $x = a$ is

$$AC_{\pi_1^{-p}\Delta^n \pi_1^p} u(a) = \epsilon K_0 + \epsilon^2 \left[K_1 + n\theta_1^p (\alpha_1 - \alpha_2)(1 - \theta_1)(1 - \theta_2) K_0^2\right] + O(\epsilon^2) \tag{1.17}$$

The set $\{n\theta_1^p\,;\, n, p \in \mathbb{Z}\}$ is (generically) dense in the unit disk in $\mathbb{C}$ (Lemma 14), hence if $\alpha_1 \neq \alpha_2$ then the approximation of u modulo $O(\epsilon^2)$ terms has dense branching.

If $\alpha_1 = \alpha_2$ higher order terms of (1.6) should be studied. Proposition 4 (b) contains the criteria for dense branching in some higher order. In fact $\alpha_1 \neq \alpha_2$, generically for the nongeneric case when U_0 is not densely branched, (Proposition 1 (ii)).

Therefore $U_0(x) + \epsilon U_1(x)$ is densely branched (generically).

It remains to show that dense branching of a truncate of the series solution (1.6) (e.g. $U_0(x)$, or $U_0(x) + \epsilon U_1(x)$) implies dense branching of the true solution $u(x)$. The proof will be done using local analytic equivalence maps of (1.1) to its linearization.

Solutions of (1.1) may be expected to be well approximated by solutions of the linearized equation

$$p(x) \frac{dw}{dx} = q(x) w \tag{1.18}$$

for small u, and x not too close to a singular point s_j. This idea may be formalized as the existence of an equivalence map $u = h(x, w)$ (for small w) between (1.18) and (1.1) which is close to the identity: $h(x, w) = w + O(w^2)$.

For x in a small regular region of (1.1) the existence of such equivalence maps follows from the general theory of ordinary differential equations. For x in an annular region centered at a singular point s_j, $j \in \{1, 2\}$, such an equivalence map also exists (generically) — it is a problem closely related to the equivalence to normal forms for periodic equations [6], [7] (see also [1]). The question that needs to be addressed in the present paper is a more global one: finding equivalences in a domain of x which surrounds both singular points s_1, s_2, making thus possible to follow solutions upon analytic continuations around both singularities.

It turns out that such global equivalence maps exist (generically): local equivalence maps can be glued together, yielding global maps. These maps may be branched at the points s_1, s_2. However, for each $j \in \{1, 2\}$ there is one equivalence map which is single-valued around s_j and branched at the other singular point. Most importantly, different maps are related through analytic transformations.

Solutions of (1.1) are expressed, using equivalence maps, in terms of solutions of (1.18), whose analytic continuation can be easily written.

Moreover, solutions of (1.1) which are initially small remain in the domain of equivalence to (1.18) upon analytic continuation on certain paths. As a consequence, analytic continuation of all terms of (1.6) is found and dense branching of the full series (hence of the true solution) is established, based on dense branching of a truncate (Proposition 4).

2. Main Results

2.1. Notations and Assumptions

In the present paper only equivalence maps which are close to the identity (as functions of the dependent variable) are considered, and, in order to simplify the exposition, the statement that a map $u = h(x, w)$ is an *equivalence map* will be taken to imply that h satisfies $h(x, w) = w + O(w^2)$ $(w \to 0)$.

For each s_j $(j = 1, 2)$, we use results on analytic equivalence of (1.1) to (1.18) for x in an annulus around s_j and small u. This holds if the numbers μ_j satisfy a condition similar to requirements needed in similar problems of analytic equivalence, when small denominators appear in the Taylor series of the equivalence map (for example, "condition C" [6], "condition ω" [7], "Diophantine condition" [8]).

More precisely, equations (1.1) and (1.18) are analytically equivalent for x in an annulus around s_j and small u (cf. [1]) if the number μ_j is *not* nearly-rational, in the following sense: there exist constants $C > 0$ and $\nu > 0$ such that for all integers $k \geq 1$ and l

$$|l + k\mu_j| > C(k + |l|)^{-\nu} \tag{2.19}$$

(Note that any nonreal number is not nearly-rational, and in fact condition (2.19) this is nontrivial only for real numbers.)

In the case when the nonlinear term $f(\cdot,u)$ is in fact, analytic at s_j, the condition (2.19) can be weakened to include all positive numbers (the property reduces to Poincaré's or Siegal's theorems [21]).

Assumption: In the present paper we assume that μ_1,μ_2 are *not* nearly-rational. If $f(\cdot,u)$ is analytic at s_j for some $j\in\{1,2\}$, then μ_j can also be a positive number.

Notation:

Denote

$$D_\eta = D\cap\{|x-s_1|>\eta\}\cap\{|x-s_2|>\eta\} \tag{2.20}$$

We assume $\eta>0$ is small enough, so that the set D_η is connected.

Assumption:

The point a satisfies $a\in D_\eta$.

Notation:

We choose some generators of the homotopy group of D_η in a standard way. More precisely, let $a\in D_\eta$. Denote by π_1, respectively π_2, a closed curve in D_η, starting at a, winding once, counterclockwise, around the point s_1, respectively s_2 (and not winding around s_2, respectively s_1).

2.2. Normal form of (1.1) and equivalence maps

Proposition 1 states the properties of equivalence maps between (1.1) and (1.18). They are (generically) branched when x encircles the singular points s_1 and s_2; but there is one equivalence map which is single-valued upon analytic continuation around one (and, generically, only one) point s_j.

Let $\mathcal{R}$ be the Riemann surface above $D\setminus\{s_1,s_2\}$ and $\mathcal{R}_\eta\subset\mathcal{R}$ relatively compact, connected open subset whose projection on $\mathbb{C}$ is D_η (cf. (2.20)).

Proposition 1 *(i) There exists $\delta=\delta_\eta>0$ and equivalence maps $u=h^{(1)}(x,w)$, $u=h^{(2)}(x,w)$ of (1.18) to (1.1) which are analytic on $\mathcal{R}_\eta\times\{|w|<\delta\}$ and such that $h^{(j)}$ is analytic on a domain $\{\eta<|x-s_j|<\eta'\}\times\{|w|<\delta\}$ $(j=1,2)$.*

The maps $h^{(1)},h^{(2)}$ are related by $h^{(2)}=h^{(1)}_\phi$ where

$$h_\phi(x,w)=h\left(x,A(x)\phi\left(A(x)^{-1}w\right)\right) \tag{2.21}$$

and $\phi(z)=z+O(z^2)$ is a function analytic at $z=0$.

(ii) Assume that μ_1,μ_2 are nonintegers and $\mu_1+\mu_2\not\in\{0,1,2,3,...\}$. Then for generic $f(x,u)$ there is no analytic equivalence map on $D_\eta\times\{|w|<\delta\}$ (i.e. the maps $h^{(j)}$ are not single-valued upon analytic continuation around both $x=s_1$ and $x=s_2$).

Remark 2 *The domain of analyticity of an equivalence map $h^{(j)}$ is in fact larger (a straightforward consequence of the regularity of $h^{(j)}$ near $x=s_j$ and of the matching relation (2.21)). However, for integrability analysis we only need a domain which contains closed curves around s_1 and s_2.*

It is interesting to note that the existence of equivalence maps which are single-valued around one of the singular points s_1, s_2 can be generalized in the following way. There exits a unique equivalence map $u = h(x, w)$ of (1.18) to (1.1) which is analytic on $\mathcal{R}_\eta$, and such that $h(\cdot, w)$ returns to the initial values after analytic continuation along any chosen (generic) path. The proof is given in the Appendix §4.

2.3. Criteria for nonintegrability of (1.1)

Let $\alpha_j = \Re\mu_j,\ \beta_j = \Im\mu_j,\ (j = 1, 2)$. If one of the numbers μ_1, μ_2 is not real, e.g. $\mu_1 \in \mathbb{C} \setminus \mathbb{R}$, denote

$$\beta = \beta_2/\beta_1 \quad , \quad \alpha = \alpha_2 - \alpha_1\beta_2/\beta_1 \tag{2.22}$$

The next Proposition summarizes the integrability properties of the linear equation (1.18).

Proposition 3 *Consider equation (1.18) with (1.2), (1.3), and $\mu_{1,2} \in \mathbb{C}$.*

Denote $\mathcal{F} = \mathbb{C}^2 \setminus \{(x, u)\,;\, x = s_1 \text{ or } x = s_2 \text{ or } u = 0\}$.

(i) If $\mu_1, \mu_2 \in \mathbb{C} \setminus \mathbb{R}$ and the numbers α, β, 1 are linearly independent over $\mathbb{Z}$, then any nonzero solution of (1.18) has dense branching, hence there is no continuous first integral on $\mathcal{F}$.

If:

(ii) one of the numbers μ_1, μ_2 is not real (e.g. $\mu_1 \in \mathbb{C} \setminus \mathbb{R}$), and α, β, 1 are linearly dependent over $\mathbb{Z}$, and also: $\alpha \notin \mathbb{Q}$ or $\beta \notin \mathbb{Q}$,

or, if

(ii') μ_1, μ_2 are both real, at least one of them irrational,

then the closure of the set $S_{w,x}$ (cf. (1.4)) of values at x (on all branches) of a nonzero solution, has real dimension 1 in $\mathbb{C}$. There is a real-analytic, real-valued, first integral of (1.18) (and no analytic integral), defined for $(x, u) \in \mathcal{F}$.

(iii) In all other cases the multivaluedness of solutions forms a discrete set and there is a single-valued, locally analytic, first integral of (1.18).

Note. The locally analytic first integrals in case (iii) with $\mu_{1,2} \notin \mathbb{R}$ are *not* meromorphic on $\mathbb{C}^2$ cf. §6.4.

For nonlinear equations there are more cases with dense branching:

Proposition 4 *Consider equation (1.1) with (1.2), (1.3), under the assumptions of §2.1.*

(a) If the numbers μ_1, μ_2 satisfy the condition (i) of Proposition 3 then there exists $r_0 > 0$ such that any $u(x)$ of (1.1) with $0 < |u(a)| < r_0$ has dense branching: $\overline{S_{u,x}}$ contains a disk centered at $u = 0$.

(b) Furthermore, if equation (1.1) is not *analytically equivalent to (1.18), and*

(i) $\mu_1 \notin \mathbb{R},\ \alpha_1 \notin \mathbb{Q},\ \mu_2 \notin \mathbb{Q}$, or

(ii) $\mu_1 \in \mathbb{R} \setminus \mathbb{Q}$ and $\mu_2 \notin \mathbb{Q}$

then there exists $r_0 > 0$ such that any solution $u(x)$ of (1.1) with $0 < |u(a)| < r_0$ has dense branching: $\overline{S_{u,x}}$ contains a disk centered at $u = 0$.

Note. The condition $\mu_2 \notin \mathbb{Q}$ of (b)(ii) can be weakened, cf. Remark 15.

The proof of Proposition 4 relies on writing convergent expansions for the monodromy maps corresponding to analytic continuation of solutions around s_1, respectively s_2. Following the values of some solution upon analytic continuation is equivalent to looking at the group generated by the two monodromy maps. This group is (generically) noncommutative, and the techniques used in the proofs are generalizations of the study of iterations of maps to the case of a noncommutative group of maps.

2.4. Nonexistence of first integrals

There is a vast amount of literature proving, or disproving, the existence of first integrals for ordinary differential equations. Different authors consider, in various contexts, integrals defined on the whole, or just part, of the phase space, and various types of regularity is assumed analytic, meromorphic, algebraic, differentiable, etc. An overview with a wealth of references is found in [9]. Searching for meromorphic first integrals of Hamiltonian systems, Ziglin's approach [10] of investigating a linearization of the equation in order to conclude about integrability properties of the full equation lead to an approach based on differential Galois theory [11]-[13] and was recently developed yielding comprehensive results on nonexistence of meromorphic integrals for Hamiltonian systems (see [14]-[16] and references therein). Other approaches are in [17], the collection of papers [18] and the references therein; see also [19], [3], [5].

Equations not having first integrals in one class of functions, may have in a broader class. The question then arises: what is a natural class of first integrals to be considered? A very interesting discussion, for equations in the real domain, is found in [20].

The following remarks concern first integrals of (locally) analytic differential equations in the complex domain. For simplicity only first order equations are discussed (but the remarks can be generalized in a natural way to differential systems).

To the extent that first integrals are used to describe the geometric properties of solutions of a given differential equation, results like those of Propositions 3 and 4 on the density properties of trajectories provide such a description. We propose below a way of defining first integrals that would provide this type of global information for equations in the complex domain.

2.4.1. Domains of first integrals Existence or nonexistence of first integrals depends decisively on the domain.

In a neighborhood of a point of the phase space where the differential equation is regular there always are analytic first integrals (by the basic local rectifiability theorem, see e.g. [21], also [22]).

On the other hand, if the domain is too large, first integrals may not exist, in spite of a very "ordered", and "predictable" structure of the trajectories. Consider as example the equation

$$xu'(x) = u(x) \tag{2.23}$$

with the general solution $u(x;C) = Cx$. There is no continuous first integral on $\mathbb{C}^2$ since a continuous, nonconstant function $F(x,u)$ which is constant on the trajectories ($F(x,Cx) = \text{const}$) does not have a limit as $(x,u) \to (0,0)$. It would be, however, uninformative to call the simple equation (2.23) "nonintegrable". But there are first integrals defined *almost* everywhere on the phase space: $F_1(x,u) = u/x$, which is meromorphic, singular only on the manifold $x = 0$ (a singular manifold of the equation), or $F_2(x) = x/u$ which is meromorphic (singular on the trajectory $u = 0$) or $F_3(x,u) = x^2/(x^2+u^2)$ which is meromorphic, singular on the trajectories of the solutions $u(x) = \pm ix$). The fact that any first integral has singularities indicates a singularity of the manifold of solutions of the differential equation (all trajectories pass through the origin).

Then how large would a domain of an "informative" first integral be? From a practical point of view, it may be necessary to let the independent variable x vary only on a subdomain $D_x \subset \mathbb{C}$, so that local techniques of study can be used. Since first integrals are constant on the trajectories, it would be natural to require that if a piece of a trajectory is in the domain of the first integral, then the whole trajectory (for $x \in D_x$) is in the domain. It may be useful to remove from the domain§ manifolds along which the differential equation is singular (and local existence, uniqueness, or regularity of solutions fail), thus allowing for singularities of first integrals. The study in regions surrounding singular manifolds (in the extended complex phase space) may be particularly useful in establishing absence of first integrals.

A domain of a first integral may be defined as follows. Fix a domain D_x of variation of the independent variable x. Consider an initial point $a \in D_x$ and an open set of initial values U_a such that all (a,u_a) with $u_a \in U_a$ are regular points of the equation. Then for each $u_a \in U_a$ there exists a unique solution $u(\cdot;u_a)$ of the differential equation satisfying $u(a;u_a) = u_a$, and it is analytic in a neighborhood of $x = a$. Let $\mathcal{F}(D_x,a,U_a)$ the set of points (x,u) with $x \in D_x$ having the following property: there exists a smooth path $\ell : [0,1] \to D_x$ in D_x joining a with x, and there exists $u_a \in U_a$, such that the solution $u(\cdot;u_a)$ can be continued along ℓ and after continuation the value of the solution at x is u: $AC_\ell u(x;u_a) = u$.

A domain of a first integral would be a set $\mathcal{F} = \mathcal{F}(D_x,a,U_0) \setminus \mathcal{M}$, where $\mathcal{M}$ is a lower dimensional closed set consisting of singularities of the equation.

2.4.2. The regularity of a first integral Existence of a more regular a first integral is expected to entail smoother properties for the family of trajectories.

For analytic equations it is natural to inquire about existence of analytic first integrals over extended regions, since there are families of local analytic first integrals (away from the singular points). But real-analytic integrals may also be of interest. For example, equation

$$xu' = \mu u \quad , \quad \mu \in \mathbb{R} \setminus \mathbb{Q} \tag{2.24}$$

§ Relatedly, see also the discussion on holomorphic direction fields in [21].

has the general solution $u(x) = Cx^{\mu}$ and has (in the complex phase space) a single-valued, real-analytic, and real-valued first integral $F(x,u) = |ux^{-\mu}|$, which counts, dimension-wise, as a "half integral".

Equation (2.24) fails to have an analytic integral on a too extended domain. Indeed, let D_x be any domain encircling $x = 0$, for example the annulus $D_x = \{0 < |x| < 1\}$, and U_a be a small ball in $\mathcal{D}_x$. Let $a \in D_x$ and a solution $u = Cx^{\mu}$ with initial value in U_a. Continuing the solution on paths going around $x = 0$ the values of the solution at x form the set $\{Cx^{\mu}\exp(2\pi i n\mu)\,;\, n \in \mathbb{Z}\}$ which is dense in a circle (since μ is irrational), hence any analytic function constant on the trajectory must be constant.

An interesting question, which however we do not investigate here, is whether considering nonanalytic, but continuous, first integrals of an analytic equation may lead to qualitatively different answers.

Another question is whether we should require first integrals to be single-valued, or may be allowed to be multivalued. Algebraic first integrals have been considered by many authors (see, e.g. references in [9], [20]). Consider, as an example, the simple equation

$$(u')^2 = 4u \tag{2.25}$$

with the general solution $u(x;C) = (x - C)^2$, and the special solution $u \equiv 0$. In spite of the simplicity of the manifold of solutions, there is no single-valued first integral on $\mathbb{C}^2$. The manifold $u = 0$ is a singularity of the equation (a branch point of the equation when written in normal form), but even confining ourselves to a smaller domain (by removing the singular part) $\mathbb{C}^2 \setminus \{(x,0)\,;\, x \in \mathbb{C}\}$ there is still no analytic integral. Indeed, let $F(x,u)$ be a *single-valued* analytic function which is constant on the general solution $u(x;C) = (x - C)^2$. Then for all x,C, with $x \neq \pm C$,

$$F(x,(x-C)^2) = F(0,C^2) = F(0,(-C)^2) = F(x,(x+C)^2) \tag{2.26}$$

Taking the derivative with respect to x in (2.26)

$$F_x(x,(x-C)^2) + 2(x-C)F_u(x,(x-C)^2) = F_x(x,(x+C)^2) + 2(x+C)F_u(x,(x+C)^2)$$

and evaluating at $x = 0$ yields $F_u(0,C^2) = 0$, which means that $F(0,C^2)$ does not depend on C, hence $F(x,(x-C)^2) = \text{const}$ for all x and C so F is constant.

However, equation (2.25) has an algebraic first integral $F(x,u) = x - \sqrt{u}$ (singular on the manifold $u = 0$).

It may be desirable to consider some multivalued (more general than algebraic) first integrals. It was proposed in [3] that an integral can be allowed (mentaining its usefulness) to be multivalued as long as is not densely branched. We will not pursue this issue here.

2.4.3. Criteria for nonexistence of first integrals The result on nonexistence of single-valed first integrals of (1.1) is:

Proposition 5 *Under the assumptions of Propositions 3 (i) and 4, there is no continuous, nowhere constant, (single-valued) first integral of (1.1) on $\mathcal{F}(D_\eta, a, U_a)$ where $a \in D_\eta$, and $U_a = \{0 < |u| < \delta\}$ for all $\eta > 0$ (such that D_η is connected) and $\delta = \delta(\eta) > 0$ small enough.*

Conclusions. If equation (1.1) is analytically equivalent to its linear part then its integrability properties are, of course, the same as those of its linear part, and integrability criteria are given by Proposition 3. Otherwise (which is the generic case), conclusions have been obtained in the present paper only under the assumption that the numbers μ_1, μ_2 are not nearly rational (in the sense that they satisfy (2.19)). Then the existence of a first integral has *not* been ruled out if one of the numbers $\alpha_j = \Re\mu_j$, $j = 1, 2$ is rational and, at the same time, the numbers $\alpha_2\beta_1 - \alpha_1\beta_2,\ \beta_1,\ \beta_2$ (where $\beta_j = \Im\mu_j$) are linearly dependent over $\mathbb{Z}$.

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APPENDIX: Proofs

3. Calculations in the poly-Painlevé test

3.1. Analytic continuation of $U_1(x)$

The following decomposition (see e.g. [24]) is used. Let $g(x)$ be a function analytic at $x = 0$ and $a \neq 0$ small. For $\mu \in \mathbb{C} \setminus \mathbb{Z}_-$ there exists a constant α and a function $\mathcal{H}(x)$, analytic at $x = 0$, such that

$$I(x) \equiv x^{-\mu} \int_a^x \xi^{\mu-1} g(\xi)\, d\xi = \alpha x^{-\mu} + \mathcal{H}(x) \tag{3.27}$$

Indeed, if $g(x) = \sum_{n\geq 0} g_n x^n$ then

$$\mathcal{H}(x) = \sum_{n\geq 0} g_n/(n+\mu) x^n \quad , \quad \alpha = -a^\mu \mathcal{H}(a) \tag{3.28}$$

Writing (cf. (1.10), (1.8))

$$U_1(x) = K_1 A(x) + K_0^2 A(x)(x - s_1)^{\mu_1} \left[\alpha_1 (x - s_1)^{-\mu_1} + \mathcal{H}_1(x)\right]$$

where $\mathcal{H}_1(x)$ is analytic at $x = s_1$, the value of $U_1(x)$ upon analytic continuation on a closed path, winding p times around $x = s_1$ is

$$AC_{\pi_1^p} U_1(x) = \theta_1^p K_1 A(x) + \theta_1^{2p} K_0^2 A(x)(x - s_1)^{\mu_1} \left[\alpha_1 \theta_1^{-p} (x - s_1)^{-\mu_1} + \mathcal{H}_1(x)\right]$$

$$= \left[\theta_1^p K_1 + \alpha_1 \left(\theta_1^p - \theta_1^{2p}\right) K_0^2\right] A(x) + \theta_1^{2p} K_0^2 A(x) \int_a^x A(\xi) f_2(\xi)(\xi - s_1)^{-1}(\xi - s_2)^{-1}\, d\xi$$

yielding (1.12) for $p = 1$.

3.2. Repeated analytic continuations

The constants of integration of $U_0(x)$, $U_1(x)$ are, initially, K_0, K_1. After analytic continuation around $x = s_j$, $(j = 1, 2)$ K_0 becomes $\theta_j K_0$ and K_1 becomes $\theta_j K_1 + \alpha_j(\theta_j - \theta_j^2)K_0^2$ (cf. (1.14), (1.15)). To determine the values of K_0 and K_1 after repeated analytic continuations, one only needs to compose the corresponding maps. For example, after analytic continuation around s_1, followed by s_2, the new value of K_0 is $\theta_1\theta_2 K_0$, and of K_1 is $\theta_1\theta_2 K_1 + \alpha_1\theta_1\theta_2(1-\theta_1)K_0^2 + \alpha_2\theta_1^2\theta_2(1-\theta_2)K_0^2$. Finally, analytic continuation on clockwise path around s_1 brings K_0 to $\theta_2 K_0$, and K_1 to $\theta_2 K_1 + [\alpha_1\theta_2(1-\theta_1) + \alpha_2\theta_1\theta_2(1-\theta_2) + \alpha_1\theta_2^2(\theta_1 - 1)]\, K_0^2$. Further analytic continuation clockwise around s_2 yields (1.16).

3.3. Going to the notations of the proofs

Denote $z = u(a)$ and eliminate ϵ between (1.14), (1.15):

$$\gamma_1(z) \equiv AC_{\pi_1} u(a) = \theta_1 z + \alpha_1 \left(\theta_1 - \theta_1^2\right) z^2 + O(z^3) \tag{3.29}$$

A formula similar to (3.29) holds for the monodromy map $\gamma_2(z)$ which takes the initial value z of a solution of (1.1) at a into the value at a after analytic continuation around $x = s_2$:

$$\gamma_2(z) \equiv AC_{\pi_2} u(a) = \theta_2 z + \alpha_2 \left(\theta_2 - \theta_2^2\right) z^2 + O(z^3) \tag{3.30}$$

We note that γ_1, γ_2 are analytic at $z = 0$. If $u(x)$ is a solution of (1.1) with initial value $u(a) = z$ then its value at a after repeated analytic continuations is obtained by composition of corresponding monodromy maps γ_1, γ_2.

Formulas (7.63), (7.70) represent the monodromy maps written in coordinates in which γ_1 has the normal form (7.63). Such coordinates exist under the assumptions of §2.1 on $\mu_{1,2}$, as a consequence of Theorem 10, and Proposition 11.

4. A remark on analytic continuation of equivalence maps

Remark 6 *Let $\pi = \prod_{p=1}^N \pi_{j_p}^{n_p}$ ($j_p \in \{1, 2\}$, $n_p \in \mathbb{Z}$) be a path (cf. §2.1) such that the number $\mu_\pi = \sum_{p=1}^N n_p \mu_{j_p}$ satisfies:*

$$\text{there are } C, \nu > 0 \text{ such that } |n\mu_\pi + k| > Cn^{-\nu} \text{ for all } n, k \in \mathbb{Z},\ n \geq 1 \tag{4.31}$$

There exists a unique equivalence map $u = h(x, w)$ of (1.18) to (1.1) which is analytic on $\mathcal{R}_\eta$, and such that $h(\cdot, w)$ returns to the initial values after analytic continuation along π.

Proof

Fix h an analytic equivalence map near the point a and let ϕ_1 be an arbitrary function analytic at $z = 0$, with $\phi_1(z) = z + O(z^2)$.

To find the form of h_{ϕ_1} after analytic continuation along π we proceed by matching it with $h^{(1)}$ and $h^{(2)}$ (whose branching properties are known).

For $j=1,2$ let ψ_j be the function (analytic at the origin, close to the identity) such that $h=h^{(j)}_{\psi_j}$ (cf. Proposition 8).

Note the following (trivial in view of (2.21)) identities: if $h=h^{(j)}_{\psi_j}$ then $h_{\psi_j^{-1}}=h^{(j)}$ and also $h_\phi=h^{(j)}_{\psi_j\circ\phi}$ and $h^{(j)}_\lambda=h_{\psi_j^{-1}\circ\lambda}$.

Denote

$$M_\theta(z)=\theta z \quad , \quad N_\theta(\phi)=\phi^{-1}\circ M_\theta\circ\phi \tag{4.32}$$

To determine the analytic continuation of h_{ϕ_1} around s_1 we write it in terms of $h^{(j_1)}$: $h_{\phi_1}=h^{(j_1)}_{\psi_{j_1}\circ\phi_1}$. Hence

$$AC_{\pi_{j_1}^{n_1}}h_{\phi_1}(x,w)=AC_{\pi_{j_1}^{n_1}}h^{(j_1)}\left(x,A(x)\left(\psi_{j_1}\circ\phi_1\right)\left(A(x)^{-1}w\right)\right)$$

$$=h^{(j_1)}\left(x,\theta_{j_1}^{n_1}A(x)\left(\psi_{j_1}\circ\phi_1\right)\left(\theta_{j_1}^{-n_1}A(x)^{-1}w\right)\right)=h^{(j_1)}_{M_{\theta_{j_1}^{n_1}}\circ\psi_{j_1}\circ\phi_1\circ M_{\theta_{j_1}^{-n_1}}}(x,w)=h_{\phi_2}(x,w)$$

where

$$\phi_2=\psi_{j_1}^{-1}\circ M_{\theta_{j_1}^{n_1}}\circ\psi_{j_1}\circ\phi_1\circ M_{\theta_{j_1}^{-n_1}}=N_{\theta_{j_1}^{n_1}}\left(\psi_{j_1}\right)\phi_1\circ M_{\theta_{j_1}^{-n_1}}$$

Then

$$AC_{\pi_{j_2}^{n_2}\pi_{j_1}^{n_1}}h_{\phi_1}=AC_{\pi_{j_2}^{n_2}}h_{\phi_2}=h_{\phi_3}$$

where

$$\phi_3=N_{\theta_{j_2}^{n_2}}\left(\psi_{j_2}\right)\phi_2\circ M_{\theta_{j_2}^{-n_2}}$$

and so on. After the last continuation, along π_N, h_{ϕ_1} becomes $h_{\phi_{N+1}}$ where

$$\phi_{N+1}=N_{\theta_N^{n_N}}(\psi_N)\circ\phi_N\circ M_{\theta_N^{-n_N}}$$
$$=N_{\theta_N^{n_N}}(\psi_N)\circ\ldots\circ N_{\theta_1^{n_1}}(\psi_1)\circ\phi_1\circ M_{\theta_1^{-n_1}}\circ\ldots M_{\theta_N^{-n_N}}$$

The condition $h_{\phi_1}=h_{\phi_{N+1}}$ is

$$N_{\theta_N^{n_N}}(\psi_N)\circ\phi_N\circ M_{\theta_N^{-n_N}}=N_{\theta_N^{n_N}}(\psi_N)\circ\ldots\circ N_{\theta_1^{n_1}}(\psi_1)\circ\phi_1(z)$$
$$=\phi_1(\theta_1^{n_1}\ldots\theta_N^{n_N}z)$$

which is an equation for ϕ_1 of the form

$$H\left(\phi_1(z)\right)=\phi_1(\theta z) \tag{4.33}$$

where $\theta=\theta_1^{n_1}\ldots\theta_N^{n_N}$ and H is an analytic function at the origin, with $H(z)=\theta z+O(z^2)$. Siegel's theorem [6] insures that if μ satisfies condition (4.31) then there exists ϕ_1 which satisfies (4.33). ∎

5. Proof of Proposition 1

5.1. Local results

We establish a family of local equivalences, which can be glued together. Remarks 7 and 8 show the existence of such a family in small regular regions of the equations, and Remark 9 presents results on existence of local equivalence maps in (small) regions surrounding one singular point.

Remark 7 *Let $x_0 \in D_\eta$. There exist equivalence maps $u = h(x, w)$ of (1.18) to (1.1) which are analytic for x close to x_0 and small w.*

Proof

It is well known that in neighborhoods of regular points of the phase space any two differential equations are analytically equivalent. It only remains to show that there are equivalences which are close to the identity.

Consider the change of dependent variable $u = A(x)u_1$ (analytic for x near x_0). Then (1.1) becomes

$$\frac{du_1}{dx} = f_1(x, u_1) \tag{5.34}$$

where $f_1(x, u_1) = p(x)^{-1}A(x)^{-1}f(x, A(x)u_1)$ so f_1 analytic near $(x_0, 0)$ and $f_1(x, u_1) = O(u_1^2)$ $(u_1 \to 0)$.

By the local rectifiability theorem [21] there exists, near the point $(x, u_1) = (x_0, 0)$, a biholomorphic map which leaves unchanged the independent variable $w_1 = k_1(x, u_1),\ x = x$, which transforms (5.34) into its linearization $\frac{dw_1}{dx} = 0$, i.e. k_1 satisfies

$$\partial_x k_1 + f_1 \partial_{u_1} k_1 = 0 \tag{5.35}$$

If $k_1(x, u_1) = k_{1,0}(x) + k_{1,1}(x)u_1 + O(u_1^2)$ is the expansion of k_1 in powers of u_1, then from (5.35) $k'_{1,0}(x) = k'_{1,1}(x) = 0$. By replacing $k_1(x, u_1)$ by $k_1(x, u_1) - k_{1,0}$ we may assume $k_{1,0} = 0$. Since $k_1(x, \cdot)$ is invertible, then $k_{1,1} \neq 0$, and by multiplying k_1 by $k_{1,1}^{-1}$ we can arrange that $k_{1,1} = 1$, hence $k_1(x, \cdot)$ is close to the identity.

Then $k(x, u) = A(x)k_1(x, A(x)^{-1}u)$ is the w-inverse of an equivalence map $u = h(x, w)$ of (1.18) to (1.1). ∎

Remark 8 *Let $x_0 \in D_\eta$.*

(i) Let $\phi(z) = z + O(z^2)$ be a function analytic at $z = 0$ and let $u = h(x, w)$ be an equivalence map as in Remark 7, analytic for $|x - x_0| < r$ and small w.

Then the map h_ϕ given by (2.21) is also an equivalence map, analytic for $|x - x_0| < r$ and small enough w.

(ii) Conversely, let $h(x, w)$ and $\tilde{h}(x, w)$ be two equivalence maps as in Remark 7.

There exists a unique function ϕ, analytic at $z = 0$, $\phi(z) = z + O(z^2)$, such that

$$\tilde{h} = h_\phi \tag{5.36}$$

Proof

(i) Follows directly from the fact that a map $u = h(x, w)$ transforms (1.18) to (1.1) iff h satisfies

$$p\partial_x h + qw\partial_w h - qh = f(x, h) \tag{5.37}$$

(ii) Let $r > 0$ be small, such that $|x - x_0| < r$ implies $x \in D_\eta$ and such that $h, \tilde{h}$ are analytic for $|x - x_0| < r$ and $|w| < \delta$. The function $h(x, \cdot)$ invertible for small δ; denote by $k(x, \cdot)$ its inverse, analytic for $|x - x_0| < r$ and $|u| < \delta_u$.

Denote $A_0 = \sup_{|x-x_0|<r} |A(x)|$ and $\tilde{M} = \sup_{|x-x_0|<r, |w|<\delta} |\tilde{h}(x, w)/w|$. Let

$$G(z; x) = A(x)^{-1} k\left(x, \tilde{h}\left(x, A(x)z\right)\right) \tag{5.38}$$

analytic for $|x - x_0| < r$ and $|z| < \max\{\delta A_0^{-1}, \delta_u A_0^{-1} \tilde{M}^{-1}\}$.

The function G does not depend on x (it represents an analytic change of the constant of integration). Indeed, a straightforward calculation, using (5.37) and the fact that the map $w = k(x, u)$ (which takes (1.1) to (1.18)) satisfies $pk_x + k_u(qu + f) - qk = 0$, shows that $\partial_x G = 0$, hence G is constant in x for $|x - x_0| < r$. Then $\phi(z) = G(z; x)$. ∎

Remark 9 *(i) Let $j \in \{1, 2\}$ and $\eta > 0$. Assume that μ_j is not nearly-rational. There exists a unique equivalence map $u = h^{(j)}(x, w)$ of (1.18) to (1.1) which is analytic on an annulus $\eta < |x - s_j| < \eta'$ and small w.*

(ii) In particular, assume $f(\cdot, u)$ is analytic at $x = s_j$, and that μ_j is not nearly-rational, or is positive. There exists a unique equivalence map $u = h(x, w)$ of (1.18) to (1.1) which is analytic for x close to s_j and small w.

Remark 9 will follow from the following two results.

Theorem 10 *Consider the equation*

$$x\frac{du}{dx} = \mu u + f(x, u) \tag{5.39}$$

where $f(x, u) = O(u^2)$ $(u \to 0)$, and f is analytic for x in an annulus $r' < |x| < r''$ and $|u| < R_u$. Consider the linearization of (5.39)

$$x\frac{dw}{dx} = \mu w \tag{5.40}$$

If the number μ is not nearly-rational then there exist positive numbers $R_w, \rho' < \rho''$, and a unique equivalence map $u = h(x, w)$ which transforms (5.40) to (5.39) and which is analytic for $\rho' < |x| < \rho''$ and $|w| < R_w$.

The numbers ρ' and ρ'' (which satisfy $r' < \rho' < \rho'' < r''$) can be chosen arbitrarily close to r', respectively r'' (by lowering R_w).

This result is very close to the results on normal forms for periodic equations (cf. [6], [7]). A self-contained proof for n-dimensional systems was given in [1].

The following Proposition shows that in the particular case when $f(x, u)$ is analytic at $(0, 0)$, the equivalence map of Theorem 10 is also analytic at $(0, 0)$.

Proposition 11 *Consider the equation (5.39) where $f(x,u) = O(u^2)$ $(u \to 0)$, and f is analytic at $(0,0)$.*

If μ is not nearly rational, or is positive, then there exists a unique equivalence map $u = h(x,w)$ which transforms (5.40) to (5.39) and which is analytic at $(0,0)$.

Proof

Proposition 11 follows from Siegel's theorem by rewriting equation (5.39) as an autonomous system $\dot{u} = \mu u + f(x,u)$, $\dot{x} = x$ and noting that the resonant terms are not present and that x does not change.

Alternatively, a direct proof is obtained as a particular case of the proof of Theorem 10 in [1], by iterating analytic maps $u = w + H(x,w)$ with a Taylor series expansion in x (rather than a Laurent one in [1]). ∎

Proof of Remark 9

Denote

$$B_1(x) = (x - s_2)^{\mu_2}(a - s_2)^{-\mu_2} \tag{5.41}$$

For x close to s_1 the analytic change of variables $u = B_1(x)u_1$, respectively $w = B_1(x)w_1$, transforms equation (1.1) to

$$(x - s_1)\frac{du_1}{dx} = \mu_1 u_1 + f_1(x,u_1) \tag{5.42}$$

where

$$f_1(x,u_1) = (x - s_2)^{-1}B_1(x)^{-1}f(x, B_1(x)u_1)$$

and, respectively, transforms (1.18) to

$$(x - s_1)\frac{dw_1}{dx} = \mu_1 w_1 \tag{5.43}$$

Using Theorem 10, and, respectively Proposition 11, there exists an equivalence map $u_1 = h_1(x,w_1)$ of (5.43) to (5.42), analytic for x in an annulus around s_1, respectively near s_1. Going back to the variables u and w yields an equivalence map $h(x,w) = B_1(x)h_1(x, B_1(x)^{-1}w)$ of (1.18) to (1.1). ∎

5.2. Proof of Proposition 1 (i)

Is a direct consequence of Remarks 7, 8, 9.

5.3. Proof of Proposition 1 (ii)

The map $h^{(1)}$ has an expansion in powers of w

$$h^{(1)}(x,w) = w + \sum_{n\geq 2} h_n^{(1)}(x)w^n \tag{5.44}$$

Since $h^{(1)}$ satisfies (5.37) then h_n satisfy the recursive system of differential equations

$$p\frac{dh_n^{(1)}}{dx} + (n-1)qh_n = R_n\left(x, h_2^{(1)}, ..., h_{n-1}^{(1)}\right) \quad , \quad n \geq 2 \tag{5.45}$$

where R_n are the Taylor coefficients at $w=0$ of $f(x,h^{(1)}(x,w)) \equiv \sum_{n\geq 2} R_n w^n$.

In particular, for $n=2$

$$p\frac{dh_2^{(1)}}{dx} + qh_2^{(1)} = f_2(x) \tag{5.46}$$

Assume that $h_2^{(1)}$ is single-valued also around $x=s_2$. Let $\tilde{f}(x,w)=f(x,w)+\lambda w^2$ ($\lambda \in \mathbb{C}$) be a modification of f. Then the equivalence map $\tilde{h}^{(1)}$ of (1.18) to the new (modified) equation (1.1) satisfies

$$p\frac{d\tilde{h}_2^{(1)}}{dx} + q\tilde{h}_2^{(1)} = f_2(x) + \lambda \tag{5.47}$$

hence

$$p\frac{d(\tilde{h}_2^{(1)} - h_2^{(1)})}{dx} + q(\tilde{h}_2^{(1)} - h_2^{(1)}) = \lambda \tag{5.48}$$

which does not have a solution which is single-valued at both s_1 and s_2 if $\lambda \neq 0$ and if $\mu_1+\mu_2 \notin \{0,1,2,...\}$.

Indeed, the prove this last claim, denote $h_d = \tilde{h}_2^{(1)} - h_2^{(1)}$. The general solution of (5.48) is

$$h_d(x) = CA(x)^{-1} + \lambda I(x) \tag{5.49}$$

where

$$I(x) = A(x)^{-1}\int_a^x A(z)p(z)^{-1}\,dz \tag{5.50}$$

Analytic continuation of h_d around $x=s_1$ is determined as in §3.1: writing

$$I(x) = (x-s_2)^{-\mu_2}\left[\alpha_1(x-s_1)^{-\mu_1} + \mathcal{H}_1(x)\right]$$

where $\mathcal{H}_1$ is analytic at s_1 it follows that

$$AC_{\pi_1}h_d(x) = \left[\theta_1^{-1}C + \lambda\alpha_1\left(\theta_1^{-1}-1\right)\right]A(x)^{-1} + \lambda A(x)^{-1}\int_a^x A(z)p(z)^{-1}\,dz$$

Hence h_d is single-valued at s_1 if $C = \theta_1^{-1}C + \lambda\alpha_1\left(\theta_1^{-1}-1\right)$ and, similarly, is single-valued at s_2 if $C = \theta_2^{-1}C + \lambda\alpha_2\left(\theta_2^{-1}-1\right)$. Since $\mu_{1,2} \notin \mathbb{Z}$, then $\theta_{1,2} \neq 1$ so $C = \lambda\alpha_1 = \lambda\alpha_2$.

It remains to show that α_1 and α_2 cannot be equal for generic a. Assume the contrary: $\alpha_1=\alpha_2$ for generic a. Then from (3.28): $(a-s_1)^{\mu_1}\mathcal{H}_1(a) = (a-s_2)^{\mu_2}\mathcal{H}_2(a)$ for all a so that $(x-s_2)^{-\mu_2}\mathcal{H}_1(x) = (x-s_1)^{-\mu_1}\mathcal{H}_2(x) \equiv \mathcal{H}(x)$ where $\mathcal{H}$ is analytic at s_1 and at s_2, hence is entire. Since $I(x)$ (cf. (5.50)) has power increase as $|x|\to\infty$, it follows that $\mathcal{H}$ is a polynomial. But $\mathcal{H}$ satisfies

$$(x-s_1)(x-s_2)\mathcal{H}'(x) - \mathcal{H}(x)\left[\mu_1(x-s_2)+\mu_2(x-s_1)\right] = 1 \tag{5.51}$$

It is easy to check, using (5.51), that if d is the degree of $\mathcal{H}$, then $\mu_1+\mu_2=d$. ∎

6. Proof of Proposition 3

The solutions of (1.18) have the form $w(x) = zA(x)$ (cf. (1.8)), where $z \in \mathbb{C}$ is the (unusual notation for) the constant of integration: $w(a) = z$ (some branches of the powers are chosen to fix the constant z). After analytic continuation along π_j (cf. §2.1) the solution becomes $w(x) = \theta_j z A(x)$ (cf. (1.9)): the solution of (1.18) corresponding to the constant z changed to the solution corresponding to $\theta_1 z$. After analytic continuation on arbitrary closed paths around the branch points s_1 and s_2 the values of the solution at a point x form the set

$$S_{w,x} = \{\theta_1^n \theta_2^p w(x)\, ;\, n, p \in \mathbb{Z}\}$$

Any continuous first integral $F(x, w)$ of (1.18) (defined on the trajectory of $w(x)$) is a function of the constant of integration $F(x, w) = \Phi(z) = \Phi(wA(x)^{-1})$. Note that $F(a, w) = \Phi(w)$ so Φ is at least as regular as F is.

If F is single-valued, then it must satisfy

$$F(x, w(x)) = F\left(x, \theta_1^n \theta_2^p w(x)\right) \quad , \text{ for all } n, p \in \mathbb{Z}$$

therefore

$$\Phi(z) = \Phi\left(\theta_1^n \theta_2^p z\right) \quad , \text{ for all } n, p \in \mathbb{Z} \tag{6.52}$$

Hence, by continuity, $\Phi(z) = \Phi(\xi z)$ for all the points ξ in the closure of the set

$$\mathcal{M} = \{\theta_1^n \theta_2^p\, ;\, n, p \in \mathbb{Z}\}$$

6.1. Proof of (i)

Consider the case when one of μ_1, μ_2 is not real, e.g. $\mu_1 \in \mathbb{C} \setminus \mathbb{R}$. To determine the closure of the set $\mathcal{M}$ it is convenient to make some transformations. Let $\mathcal{N}$ be the set

$$\mathcal{N} = \{m + n\mu_1 + p\mu_2\, ;\, m, n, p \in \mathbb{Z}\} \subset \mathbb{C} \tag{6.53}$$

so that $\exp(2\pi i \mathcal{N}) = \mathcal{M}$. Using the identification $\mathbb{C} = \mathbb{R}^2$ we can write

$$\mathcal{N} = \left\{ m \begin{pmatrix} 1 \\ 0 \end{pmatrix} + n \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + p \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \; ; \; m, n, p \in \mathbb{Z} \right\} \subset \mathbb{R}^2$$

and using the linear transformation L of $\mathbb{R}^2$

$$L = \begin{pmatrix} 1 & \alpha_1 \\ 0 & \beta_1 \end{pmatrix} \tag{6.54}$$

we get (cf. (2.22))

$$L^{-1}(\mathcal{N}) = \left\{ m \begin{pmatrix} 1 \\ 0 \end{pmatrix} + n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + p \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \; ; \; m, n, p \in \mathbb{Z} \right\}$$

The density property of the trajectories of Proposition 3 in case (i) follows directly from part (i) of the following Lemma.

Lemma 12 *Let $\alpha, \beta \in \mathbb{R}$, and $p_0 \in \mathbb{Z}$.*

Consider the set

$$\mathcal{L} = \left\{ \begin{pmatrix} m + p\alpha \\ n + p\beta \end{pmatrix} \ ; \ m, n, p \in \mathbb{Z} \right\} \subset \mathbb{R}^2 \tag{6.55}$$

(i) If the numbers $\alpha, \beta, 1$ are linearly independent over $\mathbb{Z}$ then the set $\mathcal{L}$ is dense in $\mathbb{R}^2$.

(ii) Assume the condition of (i) fails, and that not both α and β are rational. Let M, N, P be the integers (where the nonnull ones do not have a common divisor) so that $M\alpha + N\beta - P = 0$.

Then the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ is a countable union of lines:

$$\overline{\mathcal{L}} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \, ; \, Mx + Ny \in \mathbb{Z} \right\} = \begin{pmatrix} 1/M \\ 0 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} -N \\ M \end{pmatrix} \mathbb{R}$$

if $M \neq 0$ and

$$\overline{\mathcal{L}} = \begin{pmatrix} 0 \\ 1/N \end{pmatrix} \mathbb{Z} + \begin{pmatrix} -N \\ M \end{pmatrix} \mathbb{R}$$

if $N \neq 0$.

(iii) If $\alpha, \beta \in \mathbb{Q}$ (not both zero) then $\mathcal{L}$ is the two-dimensional discrete lattice:

$$\mathcal{L} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \quad , \quad \text{where } \omega_1 = \begin{pmatrix} Du/M \\ Dv/B \end{pmatrix} \quad , \quad \omega_2 = \begin{pmatrix} P/(dM) \\ A/(dB) \end{pmatrix} \tag{6.56}$$

where $\alpha = P/M, \beta = A/B$ (A, B, P, M integers with $(A, B) = 1$, $(P, M) = 1$), $D = (M, B)$, $d = (A, P)$ and $Au - Pv = d$ for $u, v \in \mathbb{Z}$. (In the case $A = 0$ we take $d = P$, $u = 0$, $v = -1$, $D = -B$, and similarly, in the case $P = 0$, $d = A$, $v = 0$, $u = 1$, $D = M$.)

(iv) The same results hold if in the definition of $\mathcal{L}$ (6.55) we restrict the values of p to $p > p_0$, or to $p < p_0$ (for some arbitrary, fixed p_0).

Proof of Lemma 12

(i) follows from the fact that, under the given assumptions, the orbit of every point under the translation on the torus $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$ (mod 1) is dense on the torus [8].

To show (iv) under the assumptions of (i), let $p_0 \in \mathbb{Z}$. The subset of $\mathcal{L}$ where in (6.55) we consider only p such that $|p| > p_0$ differs from $\mathcal{L}$ by a discrete set, hence it is also dense in $\mathbb{C}$. Furthermore, we can take only $p > 0$ (or, similarly, $p < 0$) in (6.55) and the density property is preserved by the symmetry of the problem around the origin.

To prove (ii)...(iv), let $p_0 \in \mathbb{Z}$. We consider in (6.55) only values of p satisfying $p > p_0$; the case $p < p_0$ is analogue.

(ii) For $M \neq 0$ consider the invertible transformation B of $\mathbb{R}^2$

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Mx + Ny \\ y \end{pmatrix}$$

Then

$$B\mathcal{L} = \left\{ \begin{pmatrix} mM + nN + qP \\ n + q\beta \end{pmatrix} \ ; \ m, n, q \in \mathbb{Z}, q \geq p_0 \right\}$$

We need to show that the closure of $B\mathcal{L}$ is $\mathbb{Z} \times \mathbb{R}$.

Denote by d the greatest common divisor of N and P (if $N = 0$ take $d = P$, and if $P = 0$ take $d = N$); there are integers a, b, u, v such that $aN + bP = d$, $ud + vM = 1$. Let $(k, x) \in \mathbb{Z} \times \mathbb{R}$. Write $k = M\tilde{k} + r$, $(r \in \{0, 1, ..., M-1\})$. There are integers $\tilde{n}, \tilde{q}$ such that if $n = ur(M\tilde{n} + a)$ and $q = ur(M\tilde{q} + b)$, $\tilde{q} \geq p_0$ then $n + q\beta$ approximates x (if $r = 0$ write $n = Mn_1, q = Mq_1$ for appropriate n_1, q_1); choose then $m = (k - nN - qP)/M$ (which is an integer).

If $M = 0$ then $N \neq 0$ and the proof is similar.

(iii) If $\alpha = \beta = 0$ the claim is obvious. Otherwise, to show (6.56), let

$$x_{m,n,q} = \begin{pmatrix} m + q\frac{P}{M} \\ n + \frac{A}{B} \end{pmatrix} \in \mathcal{L} \quad , \quad y_{k,l} = k\begin{pmatrix} Du/M \\ Dv/B \end{pmatrix} + l\begin{pmatrix} P/(dM) \\ A/(dB) \end{pmatrix}$$

For any $m, n, q \in \mathbb{Z}$ there are $k, l \in \mathbb{Z}$ such that $x_{m,n,q} = y_{k,l}$: indeed, $l = uBn - vMm + qd$ and $k = (AMm - PBn)/(dD)$. Conversely, given $k, l \in \mathbb{Z}$, there are $k, l \in \mathbb{Z}$: let $a, b \in \mathbb{Z}$ such that $aAM - bBP = dD$. Denote $t = BM/D$. Since $(d, t) = 1$ there are integers q, γ such that $qd + \gamma t = l + k(vMa - uBb)$. Then $m = ka + \gamma PB/(dD)$ and $n = kb + \gamma AM/(dD)$. ▮

6.2. Proof of (ii)

In this case, a single-valued first integral is easily obtained by going to the coordinates of Lemma 12. Namely, let $\Phi(z)$ be a continuous, single-valued first integral of (1.18); denote

$$\Psi = \Phi \circ \exp \circ 2\pi i \circ L \quad , \quad t = L^{-1}\left(\ln z/(2\pi i)\right) \tag{6.57}$$

(where we used $\mathbb{C} \equiv \mathbb{R}^2$). From (6.52) it follows that Ψ must satisfy

$$\Psi(t) = \Psi(t + \mathcal{L}) \tag{6.58}$$

Then using Lemma 12 (ii) we get, for $M \neq 0$, $\Psi(t) = \exp[2\pi i(M\Re t + N\Im t)]$, which written in terms of z gives the single-valued, real-analytic first integral

$$\tilde{\Phi}(z) = \exp\left[iM \arg z + i\left(M\alpha_1 - N\right)/\beta_1 \ln|z|\right]$$

We note that its values are in the unit circle (it is “half-an-integral”); the integral is singular at $z = 0$.

6.3. Proof of (ii')

A single-valued first integral satisfies $\Phi(z) = \Phi(z\exp[2\pi i(n_1\alpha_1 + n_2\alpha_2)])$ for all $n_{1,2} \in \mathbb{Z}$. If α_1 or α_2 is irrational, then by continuity, Φ must satisfy $\Phi(z) = \Phi(z\xi)$ for all ξ with $|\xi| = 1$, hence $\Phi(z)$ is (a function of) $|z|$.

6.4. Proof of (iii)

Consider the remaining cases when $\mu_{1,2} \in \mathbb{C} \setminus \mathbb{R}$, therefore $\alpha, \beta \in \mathbb{Q}$. With the notations $\Psi_0 = \Phi \circ \exp$, $s = \ln z$, using (6.53), (6.54), we must have $\Psi_0(s) = \Psi_0(s + 2\pi i L(\mathcal{L}))$. The two generators ω_1, ω_2 of L in (6.56) are $\mathbb{R}$-linearly independent. Let then Ψ_0 be a doubly-periodic function, periods $2\pi i L(\omega_1)$, $2\pi i L(\omega_2)$. Then $\Psi_0(\ln z)$ is a single-valued, locally analytic first integral (with essential singularity at $z = 0$).

Consider the remaining cases when $\mu_1, \mu_2 \in \mathbb{R}$, therefore rational. Denote $\mu_j = A_j/B_j$ where $(A_j, B_j) = 1$ (for $j = 1, 2$). If $Q = B_1B_2/(B_1, B_2)$ then $\Phi(z) = z^Q$ is a polynomial first integral.

7. Proof of Proposition 4

7.1. The monodromy maps

Let d_j be cuts of the domain D_η: arcs starting at s_j ($j = 1, 2$) such that $D_\eta \setminus d_1 \setminus d_2$ is connected and simply connected.

We may assume that $\pi_1 \subset D_\eta \setminus d_2$ and $\pi_2 \subset D_\eta \setminus d_1$ (cf. §2.1).

Let $h^{(1)}(x, w)$ and $h^{(2)}(x, w)$ be the equivalence maps of Proposition 1 (i), analytic on $(D_\eta \setminus d_2) \times \{|w| < \delta\}$, respectively on $(D_\eta \setminus d_1) \times \{|w| < \delta\}$. Let $\phi(z) = z + O(z^2)$, analytic near $|z| \leq r$, such that $h^{(1)}(x, w) = h^{(2)}_\phi(x, w)$ (cf. (2.21)).

Assume $a \in D_\eta \setminus d_1 \setminus d_2$. For any number $z \in \mathbb{C}$ the function $w(x) = zA(x)$ (cf. (1.8)) is the solution of the linear equation (1.18), corresponding to the initial condition $w(a) = z$ (an initial branch of $A(x)$ is chosen on the first Riemann sheet $D_\eta \setminus d_1 \setminus d_2$ of the universal covering $\mathcal{R}_\eta$ of D_η, in order to fix z). Using the equivalence map $u = h^{(1)}(x, w)$ we get the general form of solutions of (1.1) with small enough initial conditions at $x = a$ for $x \in D_\eta \setminus d_1 \setminus d_2$:

$$u(x; z) = h^{(1)}(x, zA(x)) \quad , \quad \text{where } u(a; z) = h^{(1)}(a, z) \tag{7.59}$$

which is well defined if

$$|z|\, M < \delta \quad , \quad \text{where } M = \sup_{x \in D_\eta \setminus d_1 \setminus d_2} |A(x)| \tag{7.60}$$

The identity (7.59) implies that the solution $u(x; z)$ can be analytically continued along π_1. Similarly, it can be continued along π_2.

Assuming (7.60) satisfied, the solution $u(x; z)$ becomes after analytic continuation along π_1 (cf. (1.9))

$$AC_{\pi_1} u(x; z) = h^{(1)}(x, \theta_1 z A(x)) = u(x; \theta_1 z) \tag{7.61}$$

Relation (7.61) holds for all $x \in D_\eta \setminus d_1 \setminus d_2$ (on this new Riemann sheet of $\mathcal{R}_\eta$) if

$$|z|\,|\theta_1|M < \delta \tag{7.62}$$

Hence the monodromy map along π_1 (showing the change of the constant of integration z after analytic continuation of $u(x;z)$ along π_1) is

$$\mathcal{M}_{\pi_1}(z) = \theta_1 z \equiv M_{\theta_1}(z) \tag{7.63}$$

for z satisfying (7.60), (7.62).

To determine analytic continuation of $u(x;z)$ along π_2, we write $u(x;z)$ in terms of $h^{(2)}$:

$$u(x;z) = h^{(1)}\left(x, zA(x)\right) = h^{(2)}_\phi\left(x, zA(x)\right) = h^{(2)}\left(x, \phi(z)A(x)\right) \tag{7.64}$$

well defined for all $x \in D_\eta \setminus d_1 \setminus d_2$ (on the initial Riemann sheet) if

$$|z| \le r \quad , \quad |z|M < \delta \quad , \quad \text{and} \quad |\phi(z)|\, M < \delta \tag{7.65}$$

After analytic continuation along π_2 the solution $u(x;z)$ becomes

$$AC_{\pi_2}u(x;z) = AC_{\pi_2}h^{(2)}\left(x, \phi(z)A(x)\right) = h^{(2)}\left(x, \theta_2\phi(z)A(x)\right) \tag{7.66}$$

$$= h^{(1)}\left(x, \phi^{-1}(\theta_2\phi(z))A(x)\right) = u\left(x; \phi^{-1}(\theta_2\phi(z))\right) \tag{7.67}$$

where $AC_{\pi_2}u$ is analytic on this new Riemann sheet if

$$|\phi(z)|\,|\theta_2|M < \delta \tag{7.68}$$

Also (7.67) holds if, in addition,

$$|\phi^{-1}(\theta_2\phi(z))|M < \delta \tag{7.69}$$

Clearly, there is $r_0 > 0$ such that conditions (7.60), (7.62), (7.65), (7.68), (7.69) are satisfied for $|z| < r_0$.

Hence the monodromy map corresponding to π_2 (showing the change of the constant of integration z after analytic continuation of $u(x;z)$ along π_2) is

$$\mathcal{M}_{\pi_2}(z) = \phi^{-1}(\theta_2\phi(z)) \equiv N_{\theta_2}(z) \tag{7.70}$$

which is analytic for $|z| \le r_0$. Also, the solution $u(x;z)$ is in the domain of $h^{(1)}$ initially, and after analytic continuation once along π_1, or once along π_2.

7.2. Proof of Proposition 4(a)

Consider the monodromy maps $\gamma_{n,p} = \mathcal{M}^n_{\pi_1}\mathcal{M}^p_{\pi_2}$ for $n, p \in \mathbb{Z}$ (cf. (7.63), (7.70)). We only consider $p \ge 0$ if $|\theta_2| < 1$, respectively $p \le 0$ if $|\theta_2| > 1$ so that $\gamma_{n,p}$ are analytic for $|z| < r$.

From Lemma 12 (i),(iv), for any $\xi \in \mathbb{C}$ there are sequences of integers n_k, p_k with $p_k > 0$, respectively $p_k < 0$, such that $\lim_{k\to\infty} \theta_1^{n_k}\theta_2^{p_k} = \xi$. For generic ξ we have $\lim_{k\to\infty}|p_k| = \infty$. Since the function ϕ of (7.70) is close to the identity, denote $\phi^{-1}(z) = z + z^2\tilde{\phi}(z)$ (where $\tilde{\phi}$ is analytic). Then

$$\gamma_{n_k,p_k}(z) = \theta_1^{n_k}\phi^{-1}(\theta_2^{p_k}\phi(z))$$

$$= \theta_1^{n_k}\theta_2^{p_k}\phi(z) + \theta_1^{n_k}\theta_2^{2p_k}z^2\tilde{\phi}(z) \to \xi\phi(z) \quad (k \to \infty)$$

and the convergence is uniform on the disk $|z| < r$.

We need to show that there exists $r_0 > 0$ such that analytic continuation of solutions $u(x; z)$ with $|z| < r_0$ along the paths ${\pi_1}^{n_k}{\pi_2}^{p_k}$ (corresponding to the monodromy maps γ_{n_k,p_k}) remains in the domain of equivalence of (1.1) and (1.18). The conditions (7.60), (7.62), (7.65), (7.68), (7.69) are satisfied (upon repeated analytic continuations) if

$$|z|M < \delta \quad , \quad |\phi(z)|M < \delta \quad , \quad |\phi(z)||\theta_2|^{p_k}M < \delta \quad , \quad |\phi^{-1}\left(\theta_2^{p_k}z\right)| < \delta$$

and

$$|\theta_1^{n_k}\phi^{-1}\left(\theta_2^{p_k}z\right)| < \delta$$

for $|z| < r_0$ and all k. Such an $r_0 \in (0, r]$ clearly exists for $|\xi| < C_0$, since $|\theta_2|^{p_k} < 1$ for all k and we can assume $|\theta_1^{n_k}\theta_2^{p_k}| < 2C_0$ for all k.

Then

$$\lim_{k\to\infty} u\left(x; \gamma_{n_k,p_k}(z)\right) = u\left(x; \xi\phi(z)\right)$$

which shows that the values (for fixed x) of the solution $u(x; z)$ on all the branches form a set whose closure contains a disk centered at 0.

7.3. Two lemmas

We need two lemmas in the proof of Proposition 4(b),(i). The first one gives the uniform asymptotic behavior of iterations of analytic maps which are close to the identity.

Denote repeated composition by $\gamma^{\circ n} \equiv \gamma \circ \gamma \circ ... \circ \gamma$ (n times).

Lemma 13 *Let γ be a function analytic near $|z| \le r$, $\gamma(z) = z + \omega q^{-1}z^{q+1} + O(z^{q+2})$, $\omega \neq 0$, $q \ge 1$.*

Then there exists $\rho > 0$ such that the sequence of functions

$$n^{1/q}\gamma^{\circ n}\left(zn^{-1/q}\right) \quad , \; n \in \mathbb{N}$$

converges to the function

$$\gamma_\omega(z) = \frac{z}{\left(1 - \omega z^q\right)^{1/q}} \tag{7.71}$$

uniformly for $|z| < \rho$.

The second lemma is a density result.

Lemma 14 *Let $\theta \in \mathbb{C}$, with $0 < |\theta| < 1$, and $\arg\theta \notin \pi\mathbb{Q}$. Let $p_0 \in \mathbb{N}$.*

Then the set

$$\{n\theta^p \, ; \, n, p \in \mathbb{Z}, \, p > p_0, \, |n\theta^p| < 1\}$$

is dense in the unit disk in $\mathbb{C}$.

Proof of Lemma 13

Since

$$\gamma_\omega^{\circ n}(z) = \frac{z}{(1-n\omega z^q)^{1/q}}$$

and

$$\gamma^{\circ n}(z) = z + n\omega q^{-1} z^{q+1} + O(z^{q+2})$$

for all $n \geq 1$, we can write

$$\gamma^{\circ n}(z) = \gamma_\omega^{\circ n}(z) + z^{q+2}\Gamma_n(z) \quad , \quad \text{for } n \geq 1$$

where Γ_n is analytic at $z = 0$.

Using the identity $\gamma^{\circ n}(z) = \gamma^{\circ(n-1)}(\gamma(z))$ we get the recurrence

$$\Gamma_n(z) = (\gamma(z)/z)^{q+2}\,\Gamma_{n-1}(\gamma(z)) + v_n(z)$$

where

$$v_n(z) = \frac{1}{z^{q+2}}\left[\gamma_\omega^{\circ(n-1)}(\gamma(z)) - \gamma_\omega^{\circ n}(z)\right]$$

hence, with the notation $G_n(z) = \Gamma_n(zn^{-1/q})$, we have

$$G_n(z) = \left(\gamma(zn^{-1/q})z^{-1}n^{1/q}\right)^{q+2} G_{n-1}\left((n-1)^{1/q}\gamma(zn^{-1/q})\right) + v_n(zn^{-1/q}) \tag{7.72}$$

Let $\rho > 0$ such that

$$\rho < \min\left\{r, (2\omega)^{-1/q}, M^{-1}\right\} \quad , \quad \text{where } \; M = \sup_{|z|<r} |(\gamma(z) - z)/z^2| \tag{7.73}$$

We first estimate v_n. We have (cf. (7.73))

$$\sup_{|z|<\rho n^{-1/q}} |\gamma(z)| \leq \rho n^{-1/q} + \rho^2 M n^{-2/q} < \rho(n-1)^{-1/q} \tag{7.74}$$

therefore, if $|z| < \rho n^{-1/q}$ then

$$|v_n(z)| < \frac{|\gamma(z) - \gamma_\omega(z)|}{|z^{q+2}|} \sup_{|z|<\rho(n-1)^{-1/q}} \left|\frac{d}{dz}\gamma_\omega^{\circ(n-1)}(z)\right| < Const|\Gamma_1(z)| \tag{7.75}$$

so that

$$\sup_{|z|<\rho n^{-1/q}} |v_n(z)| < V \tag{7.76}$$

for some constant $V > 0$.

It will follow that the functions G_n are analytic for $|z| < \rho$. Indeed, the argument is done by induction, using estimate (7.76) and

$$|(n-1)^{1/q}\gamma\left(zn^{-1/q}\right)| < (n-1)^{1/q}|zn^{-1/q}|\left(1 + |z|n^{-1/q}M\right) < \rho$$

Since

$$|\gamma(z)z^{-1}| \leq 1 + \omega/q|z|^q + |z|^{q+1}|\Gamma_1(z)|$$

we have

$$\sup_{|z|<\rho} |\left(\gamma(zn^{-1/q})z^{-1}n^{1/q}\right)^{q+2}| \le 1 + Const\,\rho^q n^{-1} = 1 + \frac{A}{n} \tag{7.77}$$

where $A = A(\rho) > 0$ is a constant, and $A < 1$ for small ρ.

Denote $g_n = \sup_{|z|<\rho} |G_n(z)|$. Then from (7.72), (7.76), (7.77) we get

$$g_n \le \left(1 + \frac{A}{n}\right) g_{n-1} + V \tag{7.78}$$

With the substitution $g_n = x_n \prod_{k=2}^{n} \left(1 + \frac{A}{k}\right)$ the recursive inequality (7.78) becomes

$$x_n \le x_{n-1} + V \left[\prod_{k=2}^{n} \left(1 + \frac{A}{k}\right)\right]^{-1}$$

which yields

$$x_n \le x_1 + V \sum_{p=2}^{n} \left[\prod_{k=2}^{p} \left(1 + \frac{A}{k}\right)\right]^{-1}$$

$$< x_1 + V \sum_{p=2}^{n} (p+1)^{-A} e^{Const\,A^2} < x_1 + V e^{Const\,A^2} (1-A)^{-1} (n+1)^{1-A}$$

Using the inequality $\prod_{k=2}^{n} \left(1 + \frac{A}{k}\right) < e^{Const\,A}(n+1)^A$ we get

$$g_n < Const(A)\,n$$

Finally, since

$$n^{1/q} \gamma^{\circ n} \left(zn^{-1/q}\right) = \gamma_\omega(z) + z^{q+2} n^{-1-1/q} G_n(z)$$

the result of Lemma 13 follows. ∎

Proof of Lemma 14

Denote $\tau = |\theta|$, $\alpha = \arg\theta$. Let $se^{it} \in \mathbb{C}, 0 < s < 1$.

There exists an increasing sequence of integers p_k such that $\lim_{k\to\infty} e^{ip_k\alpha} = e^{it}$. The set $\{n\tau^{p_k}\,;\, n \in \mathbb{N}\}$ splits the interval $[0, 1]$ in intervals of length τ^{p_k} which goes to 0 as $k \to \infty$. Let n_k be such that $n_k\tau^{p_k} \le s < (n_k + 1)\tau^{p_k}$. Then $n_k\theta^{p_k} \to se^{it}$ as $k \to \infty$. ∎

7.4. Proof of Proposition 4(b),(i)

Consider the commutator of $\mathcal{M}_{\pi_1}$ and $\mathcal{M}_{\pi_2}$: the monodromy map

$$\gamma = \mathcal{M}_{\pi_2}^{-1} \circ \mathcal{M}_{\pi_1}^{-1} \circ \mathcal{M}_{\pi_2} \circ \mathcal{M}_{\pi_1} \tag{7.79}$$

The equivalence map $h^{(1)}(\cdot, w)$ analytic near $x = s_1$ is not analytic near $x = s_2$ if and only if the function ϕ is not the identity (cf. (7.64) and the uniqueness of $h^{(j)}$ cf. Theorems 10, and 11). Then there exists some $q \ge 1$ such that

$$\phi(z) = z + \phi_{q+1} z^{q+1} + O(z^{q+2}) \quad , \quad \text{with } \phi_{q+1} \ne 0 \tag{7.80}$$

It follows that (cf. (7.79), (7.63), (7.70))

$$\gamma(z) = z + \omega q^{-1} z^{q+1} + O(z^{q+2}) \tag{7.81}$$

where

$$\omega q^{-1} = -\phi_{q+1}\,(1-\theta_1^q)\,(1-\theta_2^q) \tag{7.82}$$

and $\omega \neq 0$.

Let $\xi \in \mathbb{C}$ with $|\xi| < 1$ and sequences of integers n_k, p_k such that $n_k\theta_1^{p_k} \to \xi$ (cf. Lemma 14), where $p_k > 0$ if $|\theta_1| < 1$ and $p_k < 0$ if $|\theta_1| > 1$. For generic ξ we have $\lim_{k\to\infty} |n_k| = \infty$, which we assume.

Then using Lemma 13

$$\Gamma_{n_k,p_k}(z) \equiv \mathcal{M}_{\pi_1}^{-p_k} \circ \gamma^{\circ n_k} \circ \mathcal{M}_{\pi_1}^{p_k}(z) = \theta_1^{-p_k}\gamma^{\circ n_k}\left(z\theta_1^{p_k}\right) \to \xi^{-1}\gamma_\omega(\xi z)\ \ (k\to\infty) \tag{7.83}$$

We need to show that the functions Γ_{n_k,p_k} represent the change of the constant of integration z of solutions $u(x;z)$ of (1.1) upon analytic continuation on the corresponding paths $\pi_1^{-p_k}\pi_\Delta^{n_k}\pi_1^{p_k}$ (where $\pi_\Delta = \pi_2^{-1}\pi_1^{-1}\pi_2\pi_1$) for $|z| < r_0$ (with r_0 independent of k)—in other words, that $u(x;z)$ remains in the domain of equivalence upon these analytic continuations. This is done by a repeated use of conditions (7.60), (7.62), (7.65), (7.68), (7.69) in the following way.

First, a solution $u(x;z)$ remains in the domain of equivalence (i.e. the domain of $h^{(1)}$ and $h^{(2)}$) upon analytic continuation around s_1 a number p_k of times if $|z||\theta_1|^j < \delta M^{-1}$ for all $j = 0,1,...,p_k$, so if $|z| < \delta M^{-1}$ and $|z||\theta_1|^{p_k} < \delta M^{-1}$.

Let $z_1 = z\theta_1^{p_k}$. Then $u(x;z_1)$ remains in the domain of equivalence when continued along π_Δ if the numbers: z_1 , $\theta_1 z_1$, $\phi(\theta_1 z_1) \equiv \theta_1\psi_1(z_1)$, $\theta_2\phi(\theta_1 z_1) \equiv \theta_1\theta_2\psi_1(z_1)$, $\phi^{-1}(\theta_2\phi(\theta_1 z_1)) \equiv \theta_1\theta_2\psi_2(z_1)$, $\theta_1^{-1}\phi^{-1}(\theta_2\phi(\theta_1 z_1)) \equiv \theta_2\psi_2(z_1)$, $\phi(\theta_1^{-1}\phi^{-1}(\theta_2\phi(\theta_1 z_1))) \equiv \theta_2\psi_3(z_1)$, $\theta_2^{-1}\phi(\theta_1^{-1}\phi^{-1}(\theta_2\phi(\theta_1 z_1))) \equiv \psi_3(z_1)$, $\phi^{-1}(\theta_2^{-1}\phi(\theta_1^{-1}\phi^{-1}(\theta_2\phi(\theta_1 z_1)))) \equiv \psi_4(z_1) \equiv \gamma(z_1)$ have absolute value less than δM^{-1}.

The functions ψ_j are close to the identity and analytic at the origin: $\psi_j(z) = z + \sum_{k>1}\psi_{j,k}z^k$. Consider the function $\Phi(t) = t + \sum_{k>1}(\sum_{j=1}^4 |\psi_{j,k}|)t^k$ well defined on some interval $t \in [0,\tau)$. If

$$\Phi(|z_1|) < \delta' = \delta M^{-1}\min\{1, |\theta_1|, |\theta_2|, |\theta_1\theta_2|\}$$

then the conditions for $u(x;z_1)$ to remain in the domain of $h^{(1)}$ upon analytic continuation on the path π_Δ (corresponding to γ) are satisfied.

These conditions must be iterated n_k times. Note that the function Φ satisfies $\Phi(|\gamma(z)|) \le \Phi(|\gamma|(|z|)) \le (\Phi\circ\Phi)(z)$ (for small z). Then $u(x;z_1)$ remains in the domain of equivalence upon analytic continuation on $\pi_\Delta^{n_k}$ if $\Phi^{\circ m}(|z_1|) < \delta'$ for all $m = 0,1,...,n_k$, and since $t < \Phi(t)$ the condition is $\Phi^{\circ n_k}(|z_1|) < \delta'$, which is implied by $|\theta_1|^{-p_k}\Phi^{\circ n_k}(|z\theta_1^{p_k}|) < \delta'$. There exists $r_0 > 0$ so that this condition is satisfied for all $|z| < r_0$ and all sequences n_k, p_k with $n_k\theta_1^{p_k} \to \xi$ with $n_k|\theta_1|^{p_k} > |\xi|/2$ for all k, in view of Lemma 13.

Therefore, if $|z| < r_0$, then after analytic continuation along the path $\pi_1^{-p_k}\pi_\Delta^{n_k}\pi_1^{p_k}$ corresponding to the monodromy map $\Gamma_{n_k,p_k}(z)$ the solution $u(x;z)$ takes values with

$$\lim_{k\to\infty} u\left(x;\Gamma_{n_k,p_k}(z)\right) = u\left(x; z(1-\omega(\xi z)^{q+1})^{-1/q}\right)$$

therefore its values (for fixed x) are dense in an open set.

7.5. Proof of Proposition 4(b), (ii)

For $m \in \mathbb{Z}$ consider the path $\pi_{[m]} = \pi_1^{-m}\pi_2^{-1}\pi_1^m\pi_2$ and the corresponding monodromy map

$$\mathcal{M}_{\pi_{[m]}} = \mathcal{M}_{\pi_1}^{-m} \circ \mathcal{M}_{\pi_2}^{-1} \circ \mathcal{M}_{\pi_1}^{m} \circ \mathcal{M}_{\pi_2} \equiv M_{\theta_1^{-m}} \circ N_{\theta_2^{-1}} \circ M_{\theta_1^m} \circ N_{\theta_2} \qquad (7.84)$$

(cf. (7.63), (7.70)); $\mathcal{M}_{\pi_{[m]}}$ is analytic on a disk $|z| < r$ (independent of m).

Using (7.80) and formulas analogue to (7.81), (7.82) for $\mathcal{M}_{\pi_{[m]}}$, $\mathcal{M}_{\pi_{[n]}}$ we get

$$\mathcal{M}_{\pi_{[n]}} \circ \mathcal{M}_{\pi_{[m]}}(z) = z + \phi_{q+1}\left(1-\theta_2^q\right)\left(2-\theta_1^{mq}-\theta_1^{nq}\right) z^{q+1} + O\left(z^{q+2}\right) \qquad (7.85)$$

Note that there is $r_0 > 0$ such that solutions $u(x;z)$ remain in the domain of equivalence to (1.18) upon analytic continuation on any path of the form $\pi_{[n]}\pi_{[m]}$ (since $|\theta_1| = 1$).

For any $s,t \in \mathbb{R}$ there exist sequences of integers m_k, n_k such that $\theta_1^{m_k q} \to e^{is}$ and $\theta_1^{n_k q} \to e^{it}$ as $k \to \infty$. Then the corresponding sequence of monodromy maps $\mathcal{M}_{\pi_{[n_k]}}\mathcal{M}_{\pi_{[m_k]}}$ converges uniformly to the map $\gamma(z;t,s) = M_{e^{-is}} \circ N_{\theta_2^{-1}} \circ M_{e^{is}} \circ N_{\theta_2} \circ M_{e^{-it}} \circ N_{\theta_2^{-1}} \circ M_{e^{it}} \circ N_{\theta_2}$ and (cf. (7.85))

$$\gamma(z;t,s) = z + \phi_{q+1}\left(1-\theta_2^q\right)\left(2-e^{it}-e^{is}\right) z^{q+1} + O\left(z^{q+2}\right)$$

Since, by assumption, $\phi_{q+1} \neq 0$ and $\theta_2^q \neq 1$ (see also Remark 15 bellow), the closure of the set of values of $u(x;z)$ upon analytic continuation on the paths $\pi_{[n]}\pi_{[m]}$ contains the values $u(x;\gamma(z;t,s))$ for all $t,s,\in \mathbb{R}$, hence an open set.

Remark 15 *The assumption $\mu_2 \notin \mathbb{Q}$ of Proposition 4(b) (i),(ii) can be weakened. If fact, it is enough that $\theta_2^q \neq 1$ where $q+1$ is the first term $h_{q+1}^{(1)}(x)$ of (5.44) which is not analytic at both points s_1 and s_2.*

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