RANK OF A MATRIX

The **row rank** of a matrix is the maximum number of rows, thought of as vectors, which are linearly independent. Similarly, the **column rank** is the maximum number of columns which are linearly independent. It is an important result, not too hard to show that the row and column ranks of a matrix are equal to each other. Thus one simply speaks of the **rank of a matrix**.

We will show this for 3×2 matrices – essentially without relying on linear algebra. Let

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \tag{1}$$

If the column rank is zero, clearly all entries are zero and the statement is obvious.

If the column rank is one, it means that $\mathbf{a} = (a_1, a_2, a_3)$ is a multiple of $\mathbf{b} = (b_1, b_2, b_3)$ (or vice-versa), or $a_1 = \lambda b_1$, $a_2 = \lambda b_2$ and $a_3 = \lambda b_3$ for some nonzero λ and

$$A = \begin{pmatrix} \lambda b_1 & b_1 \\ \lambda b_2 & b_2 \\ \lambda b_3 & b_3 \end{pmatrix} \tag{2}$$

Now, the row vectors are: $b_1(\lambda, 1)$, $b_2(\lambda, 1)$ and $b_3(\lambda, 1)$, all multiple of the same nonzero vector $(\lambda, 1)$, so there is one and only one linearly independent row.

Finally, assume that the column rank is 2, meaning $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Then both are nonzero and the angle θ between them is not 0 or π, or equivalently, $\sin\theta \neq 0$ or, equivalently still, $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$. But by definition

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \tag{3}$$

where $|\ |$ denotes as usual the determinant.

For $\mathbf{a} \times \mathbf{b}$ to be nonzero, at least one of the determinants above has to be nonzero. Say the last one is nonzero. But then we cannot have $a_1 = \lambda a_2$ and $b_1 = \lambda b_2$ since otherwise the determinant would be $a_1 b_2 - a_2 b_1 = \lambda a_2 b_2 - \lambda b_2 a_2 = 0$. Thus (a_1, b_1) is linearly independent from (a_2, b_2). Of course no more than two vecors with 2 components (that is, two vecors in $\mathbb{R}^2$ can be linearly independent, and thus the row rank is also 2.