Chapter 10: Partial differential equations.

§10.1: Two-point boundary value problems

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Alternatively, any two conditions could, we may think, determine the solution. For instance we can give $y(0)$ and $y(1)$ or $y(0)$ and $y'(1)$ etc. Such conditions are called two-point boundary conditions.

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If g is zero and the boundary values are zero, then the problem is called homogeneous, otherwise it is called nonhomogeneous.

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Thus, $y = \dfrac{\sin \sqrt{2}x}{\sin \sqrt{2}}$.

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By the same calculation, $B = 0$. But then, $y(\pi) = 1$ means $A\sin\pi = 1$ which is impossible since $\sin\pi = 0$. The inhomogeneous equation has **no** solution.

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This equation has no solution either.

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All **eigenvalues** are: $\lambda = 1, 4, 9, 16, ..., n^2, ...$