§10.2-3: Fourier Series.

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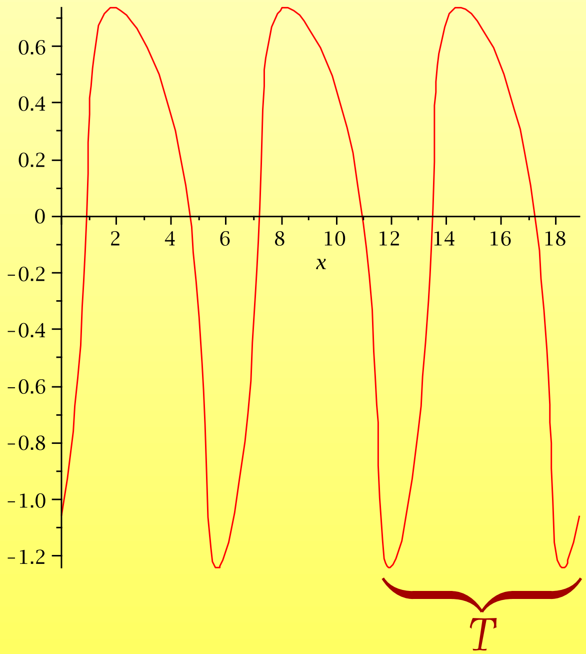
Fourier series are very useful in representing periodic functions. Examples of periodic functions.

A function is periodic with period T if $f(t+T) = f(t)$ **for any** t. The **fundamental period** is th smallest T with this property.



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A general Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$$

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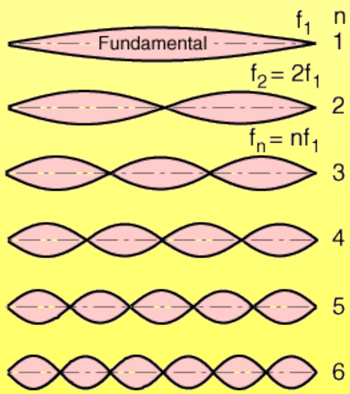

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This expansion is an expansion in terms of infinitely many independent functions, $\cos\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}$ or infinitely many "linearly independent vectors".

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These vectors are orthogonal, (7)

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Examples. Consider the function

$$G(x) = \begin{cases} 0 & -1 < x \; and \; x < -1/3 \\ 1 & -1/3 < x \; and \; x < 1/3 \\ 0 & 1/3 < x \; and \; x < 1 \end{cases}$$

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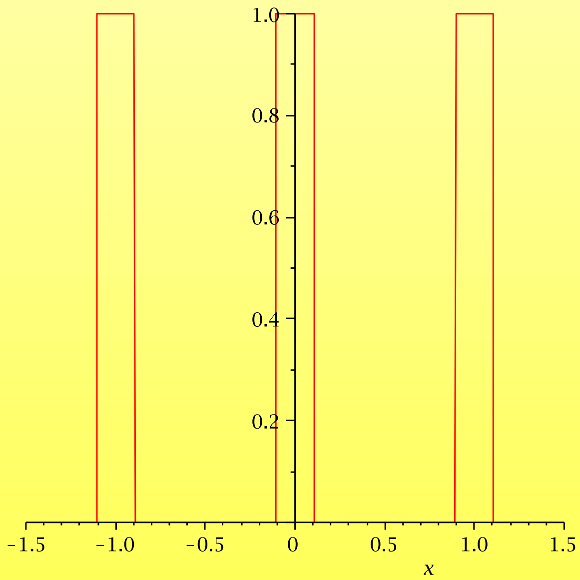

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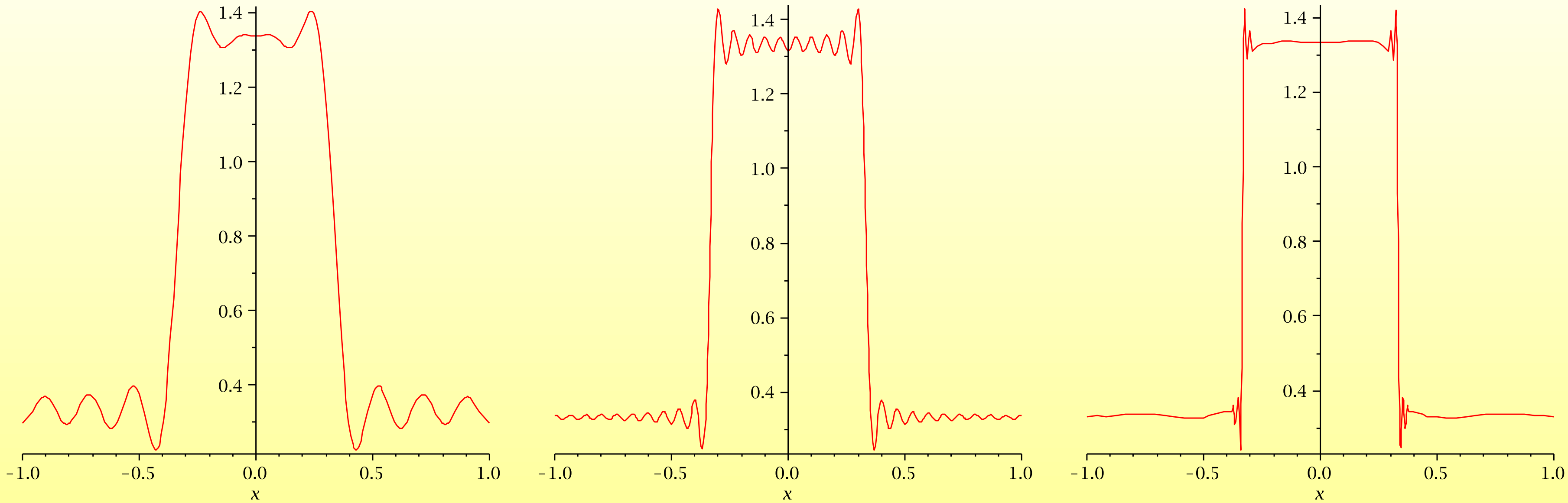
Maple 11

$> plot(2/3 + sum(2 * sin(n * Pi/3)/n/Pi * cos(n * Pi * x), n = 1..30), x = -1..1);$



1.4
1.2
1.0
0.8
0.6
0.4
-1.0
-0.5
0.0
0.5
1.0
x