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When is this decomposition possible?

Piecewise differentiable functions

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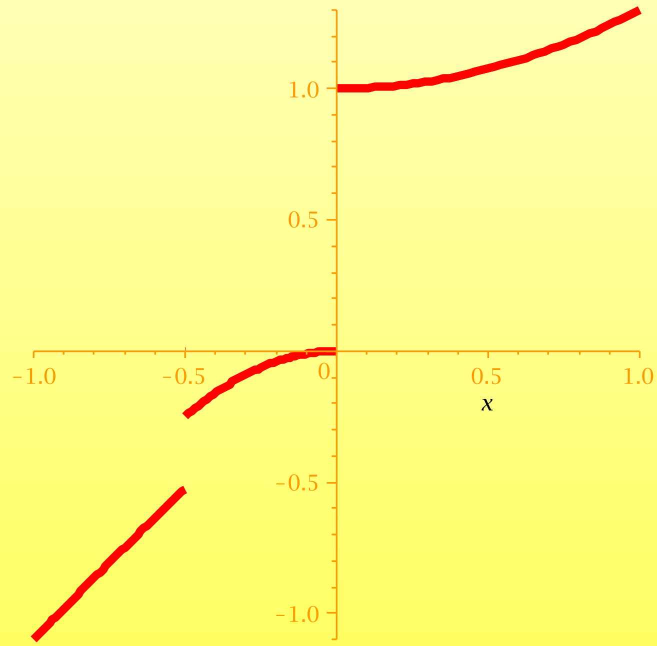

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Then

$$\frac{1}{2}(f(x_+) + f(x_-)) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L}$$

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at **all points**. Note again that $\frac{1}{2}(f(x_+) + f(x_-))$ is simply $f(x)$ at all ordinary points. Furthermore, a_m, b_m are given by (2).

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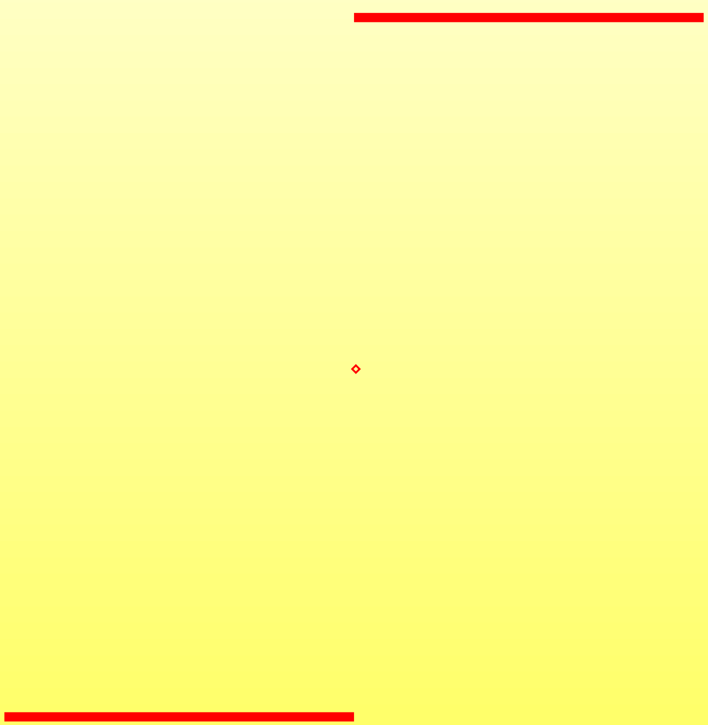

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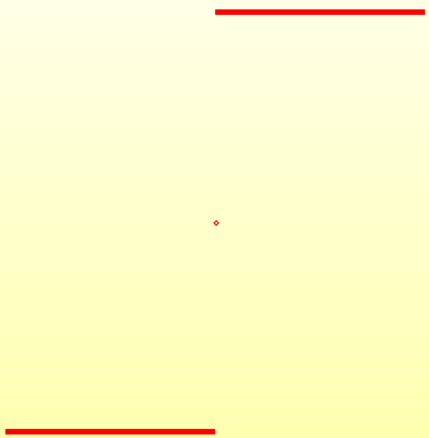
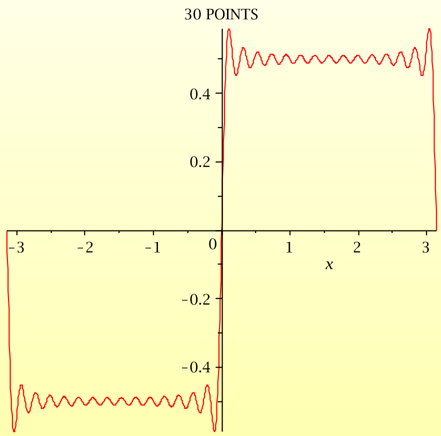
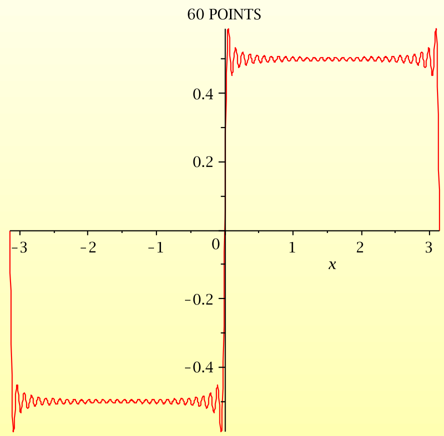
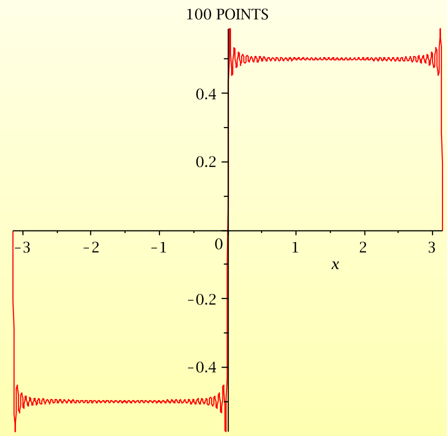
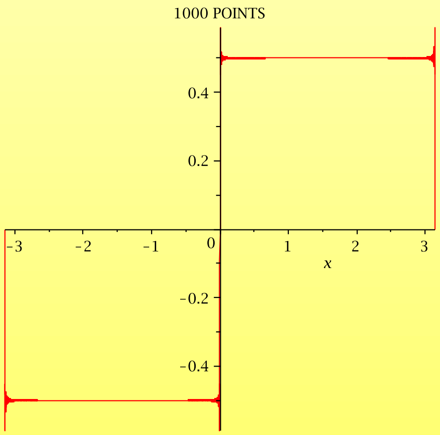


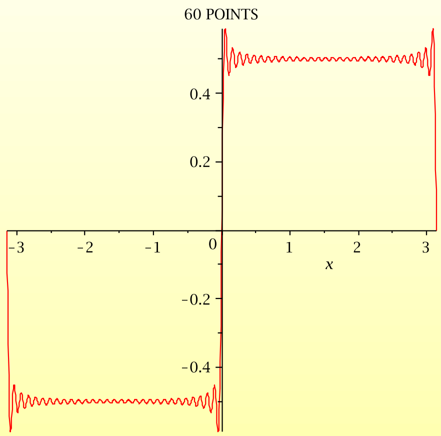


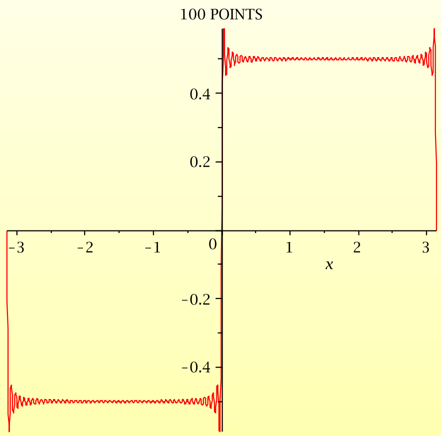




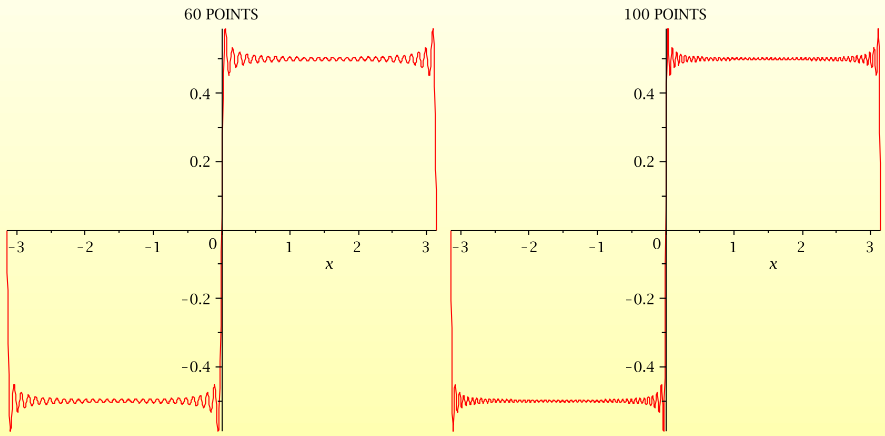


30 points means largest m is 29.





Note that there is overshoot at the jumps. This is the Gibbs phenomenon. It always occurs at discontinuities and it is about 8%.



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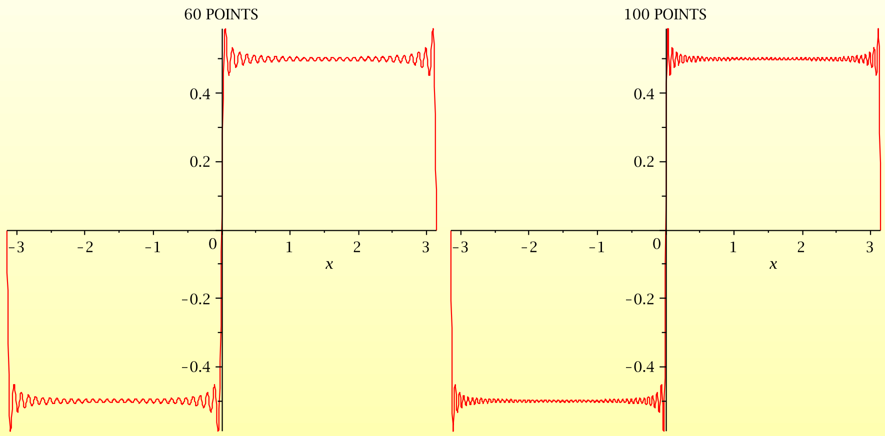

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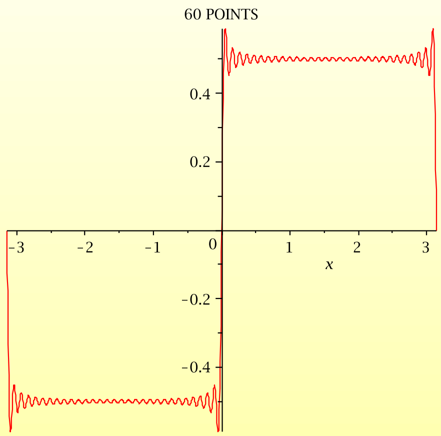

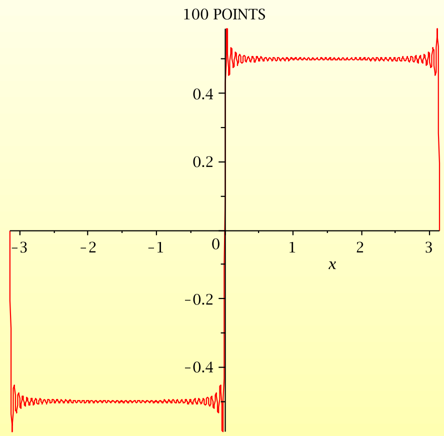

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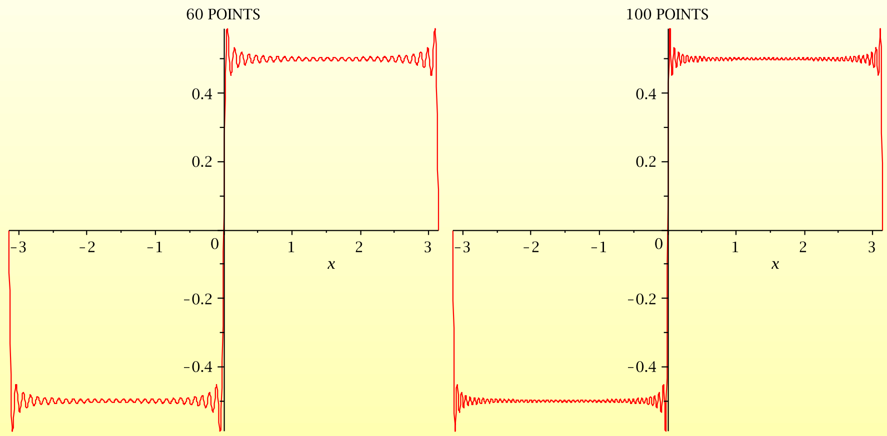

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Odd and even functions The formulas can be substantially simplified if the functions are even, or if they are odd.

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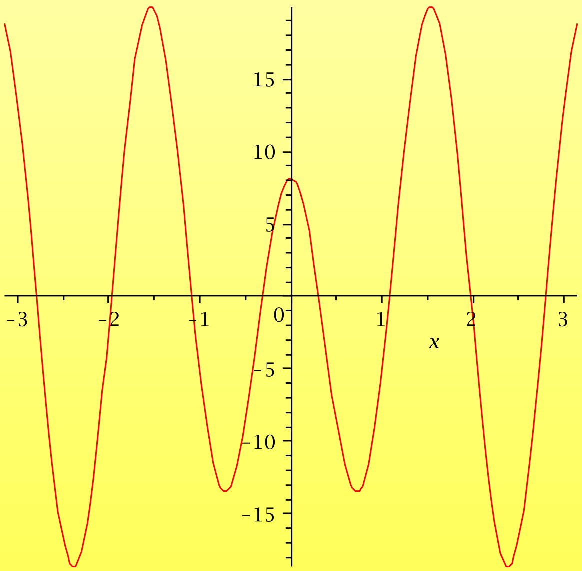

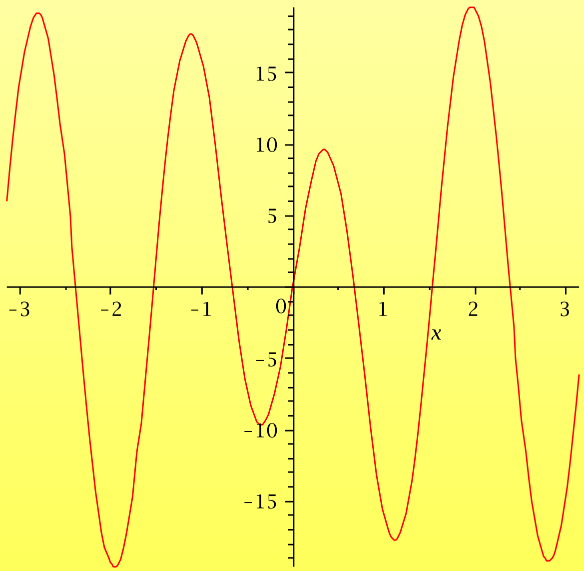

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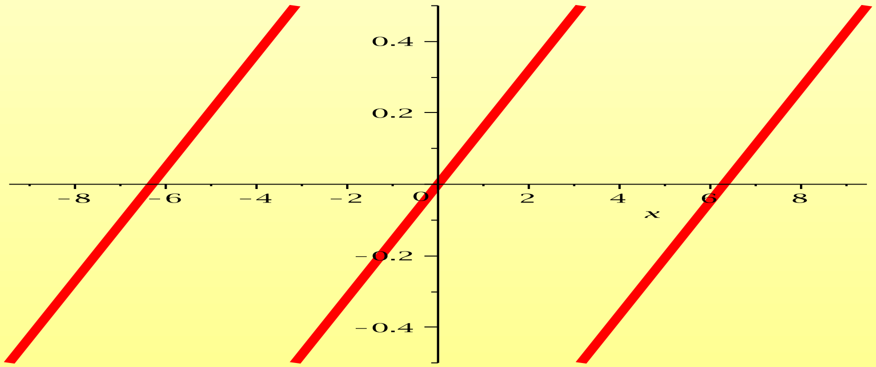

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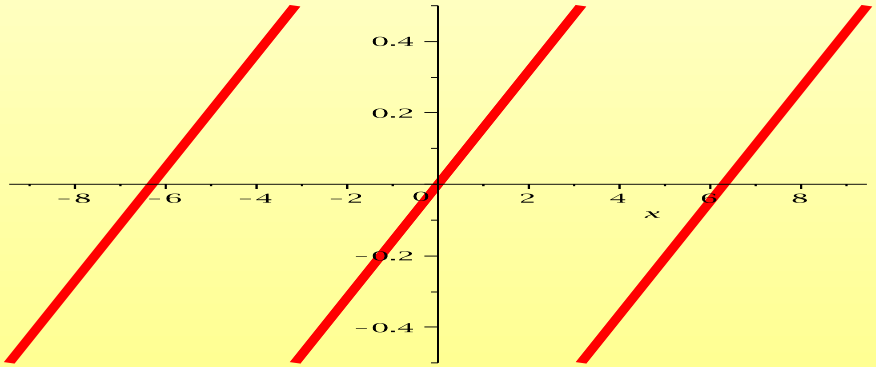

We have: f is odd.

Thus

$$a_m = 0; \quad b_m = \frac{2}{\pi}\int_0^\pi x \sin(m\pi x)dx = \frac{2}{m}(-1)^{m+1} \tag{8}$$

```
> assume(m,integer);
> F:=(x+Pi)/Pi/2-floor((x+Pi)/Pi/2)-1/2;
```

$$F := \frac{1}{2}\,\frac{x+\pi}{\pi} - \mathrm{floor}\left(\frac{1}{2}\,\frac{x+\pi}{\pi}\right) - \frac{1}{2} \tag{1}$$

```
> plot(F,x=-3*Pi..3*Pi,discont=true);
> cm:=2/Pi*int(F*sin(m*x),x=0..Pi);
```

$$cm := \frac{(-1)^{1+m\sim}}{\pi\, m\sim} \tag{2}$$

```
> S:=sum(cm*sin(m*x),m=1..10);
```

$$S := \frac{\sin(x)}{\pi} - \frac{1}{2}\,\frac{\sin(2\,x)}{\pi} + \frac{1}{3}\,\frac{\sin(3\,x)}{\pi} - \frac{1}{4}\,\frac{\sin(4\,x)}{\pi} + \frac{1}{5}\,\frac{\sin(5\,x)}{\pi} - \frac{1}{6}\,\frac{\sin(6\,x)}{\pi} + \frac{1}{7}\,\frac{\sin(7\,x)}{\pi} - \frac{1}{8}\,\frac{\sin(8\,x)}{\pi} + \frac{1}{9}\,\frac{\sin(9\,x)}{\pi} - \frac{1}{10}\,\frac{\sin(10\,x)}{\pi} \tag{3}$$

```
> plot(S,x=0..Pi);
>
```

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3. Extend it in many other number of ways.

4. Then, the Fourier series, calculated on $[-L, L]$ will converge to f

on $[0, L]$ by the general theorem (and to whatever we extended it with elsewhere).