Nonhomogeneous equations and the method of variation of parameters

§3.7

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This can be used to our advantage, in selecting a relation between u_1 and u_2 that simplifies the calculations.

$$y'' + p(t)y' + q(t)y = g(t)$$

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$$= u_1 \left(y_1'' + p(t) y_1' + q(t) y_1 \right) + u_2 \left(y_2'' + p(t) y_2' + q(t) y_2 \right)$$

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$$+ u_1' y_1' + u_2' y_2' = u_1' y_1' + u_2' y_2' = g(t) \; ! \; (6)$$

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Examples

$$y'' + y = \tan x; \quad \text{homog. eq.}: \quad y'' + y = 0$$

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$$Y = -y_1 \int \frac{g(t) y_2(t)}{W[y_1, y_2]} dt + y_2 \int \frac{g(t) y_1(t)}{W[y_1, y_2]} dt$$

$$= \sin t \int \cos t \frac{\sin t}{\cos t} dt - \cos t \int \sin t \frac{\sin t}{\cos t} dt \quad (11)$$

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$$
\begin{aligned}
&\sin t \int \overset{\downarrow}{\cos t}\, \frac{\sin t}{\underset{\uparrow}{\cos t}} dt - \cos t \int \sin t \frac{\sin t}{\cos t} dt \\
&= \sin t \int \sin t\, dt - \cos t \int \frac{\sin^2 t}{\cos t} dt \\
&= -\sin t \cos t - \cos t \int \frac{1 - \cos^2 t}{\cos t} dt \\
&= -\sin \overset{\downarrow}{t} \cos t + \cos \overset{\downarrow}{t} \sin t - \cos t \int \sec t\, dt \\
&= -\cos t \ln(\sec t + \tan t)
\end{aligned}
\tag{12}
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$$A \sin t + B \cos t - \cos t \ln(\sec t + \tan t) \tag{14}$$

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Particular solution

$$Y = -y_1 \int \frac{g(t) y_2(t)}{W[y_1, y_2]} dt + y_2 \int \frac{g(t) y_1(t)}{W[y_1, y_2]} dt$$

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$$= \frac{1}{2} e^x \int e^{-x} e^x dx - \frac{1}{2} e^{-x} \int e^x e^x dx$$

$$= \frac{1}{2}\mathrm{e}^{x}\int \mathrm{e}^{-x}\mathrm{e}^{x}dx - \frac{1}{2}\mathrm{e}^{-x}\int \mathrm{e}^{x}\mathrm{e}^{x}dx$$

$$= \frac{1}{2}x\mathrm{e}^{x} - \frac{1}{4}\mathrm{e}^{-x}\mathrm{e}^{2x}$$

$$= \frac{1}{2}e^{x}\int e^{-x}e^{x}dx - \frac{1}{2}e^{-x}\int e^{x}e^{x}dx$$

$$= \frac{1}{2}xe^{x} - \frac{1}{4}e^{-x}e^{2x}$$

General solution:

$$y = \frac{1}{2}xe^{x} + c_1e^{x} + c_2e^{-x}$$

Now we have to impose $y(0) = 0, y'(0) = 1$

$$= \frac{1}{2}e^x \int e^{-x}e^x dx - \frac{1}{2}e^{-x} \int e^x e^x dx$$

$$= \frac{1}{2}xe^x - \frac{1}{4}e^{-x}e^{2x}$$

General solution:

$$y = \frac{1}{2}xe^x + c_1 e^x + c_2 e^{-x}$$

Now we have to impose $y(0) = 0, y'(0) = 1,$

$$c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$= \frac{1}{2}e^{x}\int e^{-x}e^{x}dx - \frac{1}{2}e^{-x}\int e^{x}e^{x}dx$$

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General solution:

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$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{4}$$

$$\frac{1}{2} + c_1 - c_2 = 1$$

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$$c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{4}$$

$$y = \frac{1}{2}x\mathrm{e}^x + \frac{1}{4}\mathrm{e}^x - \frac{1}{4}\mathrm{e}^{-x}$$