Series solutions to ODEs

§5.2

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- Shift in index

$$\sum_{k=1}^{\infty} c_k (x - x_0)^k = c_1 (x - x_0) + c_2 (x - x_0)^2 + \cdots$$

$$= \sum_{k=0}^{\infty} c_{k+1} (x - x_0)^{k+1}$$

$$f'(x) = \sum_{k=1}^{\infty} k c_k (x - x_0)^{k-1} \overset{k=m+1}{=} \sum_{m=0}^{\infty} (m+1) c_{m+1} (x - x_0)^m$$

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$$y'' = xy \Longrightarrow (k+1)(k+2)c_{k+2} = c_{k-1};\ \text{and } c_2 = 0$$

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The general solution!. y_0 and y_1 are 2 particular solutions

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- They are linearly independent! (Why?)

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$$\frac{\dfrac{x^{3k+3}}{(3k+3)(3k+2)3k(3k-1)\cdots}}{\dfrac{x^{3k}}{3k(3k-1)\cdots}} = \frac{x^3}{(3k+3)(3k+2)} \to 0 < 1; \quad \forall x$$

Thus the series converges for all x.

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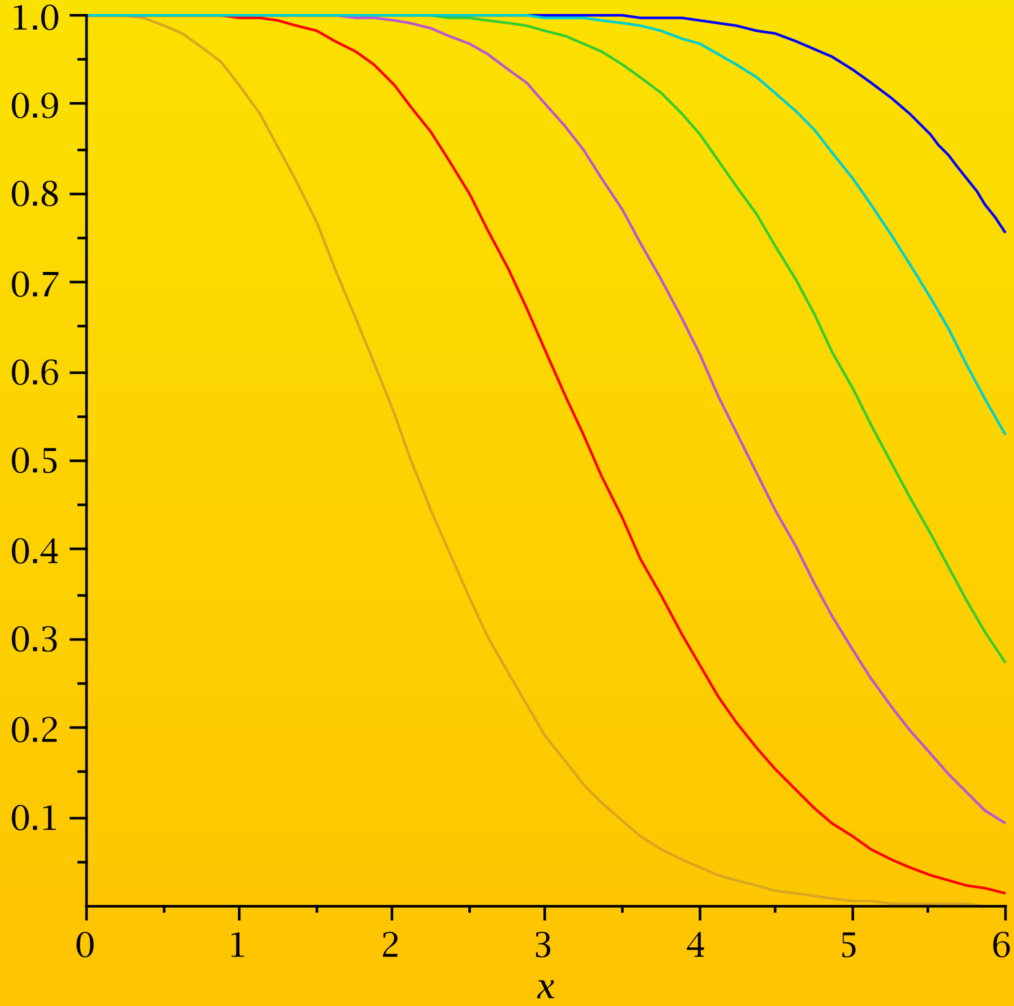

Ratio between solution and power series with: 1 term (yellow), 2 terms (red) 3 terms (mag.) etc. The graphs which are closer to 1 mean better approximations

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