Series

§5.1

Series: short review. Please brush up

Power series are used to solve differential equations, when explicit solutions are hard to find.

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- There is always a symmetric interval of convergence, $[x_0 - r, x_0 + r]$. r is called radius of convergence. r can be zero, finite, or infinity.The series converges absolutely if $\sum_{k=0}^{\infty} |a_k||x - x_0|^k$ converges.

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- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
- Convenient way to calculate functions: $\sqrt{e} = e^{1/2} \approx 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{6 \cdot 8} = 1.64 \cdots$

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Why?(2) $z = -1$. We get $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ convergent. Why?

Other examples: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ convergent for $|z| < 1$, divergent otherwise.

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Ratio test $\frac{\frac{1}{k!}}{\frac{1}{(k+1)!}} \to \infty$ thus $r' = \infty$ that is, the series converges everywhere.

$\sum_{k=0}^{\infty} k!x^k$ Ratio test $\frac{k!}{(k+1)!} = 0$, $R = 0$. This last series will not be too useful for us...

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They are multiplied as though they were polynomials. Ex: $(1 + ax + bx^2 + \cdots)(1 + Ax + Bx^2 + \cdots) = 1 + (a + A)x + (b + aA + B)x^2 + \cdots$.

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- The sum of a convergent power series, for $|x - x_0| < r$ is called analytic.

- If a series $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ converges for $|x - x_0| < r$, then, for $|x - x_0| < r$ it can be differentiated term by term.

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$$f'(x) = \sum_{k=1}^{\infty} k c_k (x - x_0)^{k-1} \overset{k=m+1}{=} \sum_{m=0}^{\infty} (m+1) c_{m+1} (x - x_0)^m$$

- Two power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ and $\sum_{k=0}^{\infty} b_k(x-x_0)^k$ are equal to each other if and only if $a_k = b_k$ for all k.

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  Note first that $\sum_{k=0}^{\infty} k c_k x^{k-1} = \sum_{k=1}^{\infty} k c_k x^{k-1}$ since the term with $k = 0$ is zero.

We change the index of summation $k = m + 1$:

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Thus,

$$f = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{c_0}{k!} x^k = c_0 \sum_{k=0}^{\infty} \frac{1}{k!} x^k = c_0 e^x \quad !$$

and we have solved the differential equation!