1. \( y(x) = \frac{1}{2} \sin 2x \)

2. \( e^{-1}; \) The solution goes to zero.

3. (a) \( R \in (2, \infty); \) (b) the solution is \(-\frac{32}{9} \sin(t/8) \cos(9t/8)\) and thus the frequency is \( \frac{1}{16\pi} \).

4. \( \frac{1}{4} e^x + \frac{x^3}{6} e^{-x} + C_1 e^{-x} + C_2 xe^{-x} \)

5. (a) \((1/2, \infty); \) (b) a second solution is \( e^{x^2} \). (c) The actual interval of existence is \((-\infty, \infty)\), since \( e^x \) and \( e^{x^2} \) are smooth on \((-\infty, \infty)\); (d): \( x + \frac{1}{2} + C_1 e^x + C_2 e^{x^2} \)

6. \( c_0 y_0 + c_1 y_1 \) where \( y_0 = \sum_{k=0}^{\infty} b_{4k} x^{4k} \) and \( y_1 = \sum_{k=0}^{\infty} b_{4k+1} x^{4k+1} \), \( c_0, c_1 \) are arbitrary. The recurrence relation is \( b_{k+2} = -\frac{b_{k-2}}{(k+1)(k+2)} \), \( b_0 = 1, b_1 = 1, b_2 = 0, b_3 = 0, \) thus \( b_{4k+2} = b_{4k+3} = 0 \). The radius of convergence is infinite, by the ratio test.

7. Written in the form required by Theorem 3.2.1, \( y'' + \frac{1}{1 + x^2} y = 0 \), we see that the coefficients are continuous everywhere and thus by Theorem 3.2.1 the solutions exist on \((-\infty, \infty)\).

8. \( y = c_0 y_0 + c_1 y_1 \) where \( y_0 = \sum_{k=0}^{\infty} b_{2k} x^{2k} \), \( y_0 = \sum_{k=0}^{\infty} b_{2k+1} x^{2k+1} \) and \( c_0, c_1 \) are arbitrary. The recurrence relation is \( b_{k+2} = -b_{k} \frac{k^2 - k + 1}{(k+1)(k+2)} \), \( b_0 = 1, b_1 = 1 \). The radius of convergence is one, and it is calculated for instance by the ratio test.