§3.1 Second Order Linear Equations

Note: Exam 1 on Feb 1. Till 3.1, included

General second order equations.

A general second order equation is written in the form

$$y'' = f(t, y, y')$$

General second order equations.

A general second order equation is written in the form

$$y'' = f(t, y, y')$$

that is, it contains first *and* second order derivatives.

General second order equations.

A general second order equation is written in the form

$$y'' = f(t, y, y')$$

that is, it contains first *and* second order derivatives. For example

$$y'' = y^2 + y' + x$$

General second order equations.

A general second order equation is written in the form

$$y'' = f(t, y, y')$$

that is, it contains first *and* second order derivatives. For example

$$y'' = y^2 + y' + x$$

This is second order, nonlinear.

General second order equations.

A general second order equation is written in the form

$$y'' = f(t, y, y')$$

that is, it contains first *and* second order derivatives. For example

$$y'' = y^2 + y' + x$$

This is second order, nonlinear.

The general second order linear equation has the form

$$y'' + p(t)y' + q(t)y = g(t)$$

The general second order linear equation has the form

$$y'' + p(t)y' + q(t)y = g(t)$$

Even fewer second order equations can be solved with simple formulas. Not even linear equations can always be nicely solved.

Among simply solvable equations are homogeneous linear equations with constant coefficients.

Among simply solvable equations are homogeneous linear equations with constant coefficients.

This means

$$p(t), \quad q(t), \text{do not depend on } t, \quad g(t) = 0$$

Among simply solvable equations are homogeneous linear equations with constant coefficients.

This means

$$p(t), \quad q(t), \text{do not depend on } t, \quad g(t) = 0$$

(Nonhomogeneous ones can be solved too.)

Let's experiment.

Let's experiment. Take

$$y'' = y$$

Let's experiment. Take

$$y'' = y$$

can you guess a solution?

Let's experiment. Take

$$y'' = y$$

can you guess a solution?

$$y(x) = e^x$$

Let's experiment. Take

$$y'' = y$$

can you guess a solution?

$$y(x) = e^x$$

or

$$y(x) = e^{-x} \quad !$$

Is this all?

Let's experiment. Take

$$y'' = y$$

can you guess a solution?

$$y(x) = e^x$$

or

$$y(x) = e^{-x} \quad !$$

Is this all? No!

$$y(x) = C_1 e^x + C_2 e^{-x}$$

is a solution too! Is this all?

is a solution too! Is this all?Yes!

is a solution too! Is this all?Yes!How do we check this?
Let $e^{-x}y(x) = f(x)$.

is a solution too! Is this all?Yes!How do we check this?
Let $e^{-x}y(x) = f(x)$. Then, $y(x) = f(x)e^{x}$

is a solution too! Is this all?Yes!How do we check this?
Let $e^{-x}y(x) = f(x)$. Then, $y(x) = f(x)e^x$

$$y' = f'(x)e^x + f(x)e^x$$

is a solution too! Is this all?Yes!How do we check this?
Let $e^{-x}y(x) = f(x)$. Then, $y(x) = f(x)e^x$

$$y' = f'(x)e^x + f(x)e^x$$

$$y''(x) = f''(x)e^x + f'(x)e^x + f'(x)e^x + f(x)e^x$$

is a solution too! Is this all?Yes!How do we check this?
Let $e^{-x}y(x) = f(x)$. Then, $y(x) = f(x)e^x$

$$y' = f'(x)e^x + f(x)e^x$$

$$y''(x) = f''(x)e^x + f'(x)e^x + f'(x)e^x + f(x)e^x$$

and thus

is a solution too! Is this all?Yes!How do we check this?
Let $e^{-x}y(x) = f(x)$. Then, $y(x) = f(x)e^x$

$$y' = f'(x)e^x + f(x)e^x$$

$$y''(x) = f''(x)e^x + f'(x)e^x + f'(x)e^x + f(x)e^x$$

and thus

$$e^x(f'' + 2f' + f) = e^x f$$

is a solution too! Is this all?Yes!How do we check this?
Let $e^{-x}y(x) = f(x)$. Then, $y(x) = f(x)e^x$

$$y' = f'(x)e^x + f(x)e^x$$

$$y''(x) = f''(x)e^x + f'(x)e^x + f'(x)e^x + f(x)e^x$$

and thus

$$e^x(f'' + 2f' + f) = e^x f$$

or

$$(f'' + 2f' + f) = f$$

$$(f'' + 2f' + f) = f$$

$$(f'' + 2f' + f) = f$$

that is,

$$f'' + 2f' = 0 \quad (*)$$

$$(f'' + 2f' + f) = f$$

that is,

$$f'' + 2f' = 0 \quad (*)$$

Integrate once,

$$f' + 2f = C$$

$$(f'' + 2f' + f) = f$$

that is,

$$f'' + 2f' = 0 \quad (*)$$

Integrate once,

$$f' + 2f = C$$

This is first order linear. One solution is $f = C/2$.

$$(f'' + 2f' + f) = f$$

that is,

$$f'' + 2f' = 0 \quad (*)$$

Integrate once,

$$f' + 2f = C$$

This is first order linear. One solution is $f = C/2$. The general solution is $f = C/2$ plus the general solution of the homogeneous equation,

$$h' + 2h = 0$$

$$h' + 2h = 0$$

that is $h'/h = -2$; $\ln h = -2x + C$; $h = C_2 e^{-2x}$

$$h' + 2h = 0$$

that is $h'/h = -2$; $\ln h = -2x + C$; $h = C_2 e^{-2x}$ Thus the general solution of (*) is

$$f = C/2 + C_1 e^{-2x} = C_1 + C_2 e^{-2x}$$

$$h' + 2h = 0$$

that is $h'/h = -2$; $\ln h = -2x + C$; $h = C_2 e^{-2x}$ Thus the general solution of (*) is

$$f = C/2 + C_1 e^{-2x} = C_1 + C_2 e^{-2x}$$

but we had denoted $y = e^x f$

$$h' + 2h = 0$$

that is $h'/h = -2$; $\ln h = -2x + C$; $h = C_2 e^{-2x}$ Thus the general solution of (*) is

$$f = C/2 + C_1 e^{-2x} = C_1 + C_2 e^{-2x}$$

but we had denoted $y = e^x f$, and thus

$$y = C_1 e^x + C_2 e^{-x}$$

Example: solve

$$y'' = y; \quad y(0) = 0; \quad y'(0) = 1$$

Example: solve

$$y'' = y; \quad y(0) = 0; \quad y'(0) = 1$$

Note that we now need two conditions, or initial values: the value of y *and* the value of y'!

Example: solve

$$y'' = y; \quad y(0) = 0; \quad y'(0) = 1$$

Note that we now need two conditions, or initial values: the value of y *and* the value of y'! That is because we have to determine *two* constants! ($y = C_1 e^x + C_2 e^{-x}$. Then

$$= C_1 e^0 + C_2 e^{-0} = 0; \quad \text{thus } C_1 = -C_2$$

Example: solve

$$y'' = y; \quad y(0) = 0; \quad y'(0) = 1$$

Note that we now need two conditions, or initial values: the value of y *and* the value of y'! That is because we have to determine *two* constants! ($y = C_1 e^x + C_2 e^{-x}$. Then

$$= C_1 e^0 + C_2 e^{-0} = 0; \quad \text{thus } C_1 = -C_2$$

but

$$y' = C_1 e^x - C_2 e^{-x} = C_1 (e^x + e^{-x})$$

$$y' = C_1 e^x - C_2 e^{-x} = C_1(e^x + e^{-x})$$

$$y'(0) = 2C_1 = 1$$

$$y' = C_1 e^x - C_2 e^{-x} = C_1(e^x + e^{-x})$$

$$y'(0) = 2C_1 = 1$$

thus

$$C_1 = \frac{1}{2} \quad \text{thus } y(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$y' = C_1 e^x - C_2 e^{-x} = C_1(e^x + e^{-x})$$
$$y'(0) = 2C_1 = 1$$

thus

$$C_1 = \frac{1}{2} \text{ thus } y(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$

Now it is easy to see that any second order homogeneous ODE with constant coefficients

$$y'' + Ay' + By = 0$$

$$y'' + Ay' + By = 0$$

$$y'' + Ay' + By = 0$$

can be solved by exponentials

$$y'' + Ay' + By = 0$$

can be solved by exponentials , provided that the characteristic polynomial

$$r^2 + Ar + B = 0$$

$$y'' + Ay' + By = 0$$

can be solved by exponentials , provided that the characteristic polynomial

$$r^2 + Ar + B = 0$$

(replace y'' by r^2, y' by r and y by 1) has real distinct roots, r_1 and r_2 (we'll see later that we can deal with imaginary roots, equal roots, etc.).

Indeed, take $y_1 = C_1 e^{r_1 x}$

Indeed, take $y_1 = C_1 e^{r_1 x}$. Then, $y_1' = C_1 e^{r_1 x}$, $y_1'' = C_1 r_1^2 e^{r_1 x}$

Indeed, take $y_1 = C_1 e^{r_1 x}$. Then, $y_1' = C_1 e^{r_1 x}$, $y_1'' = C_1 r_1^2 e^{r_1 x}$ and

$$y_1'' + Ay_1' + By_1 = C_1 r_1^2 e^{r_1 x} + AC_1 e^{r_1 x} + BC_1 e^{r_1 x}$$

Indeed, take $y_1 = C_1 e^{r_1 x}$. Then, $y_1' = C_1 e^{r_1 x}$, $y_1'' = C_1 r_1^2 e^{r_1 x}$ and

$$y_1'' + Ay_1' + By_1 = C_1 r_1^2 e^{r_1 x} + AC_1 e^{r_1 x} + BC_1 e^{r_1 x}$$

$$= C_1 e^{r_1 x}(r_1^2 + Ar_1 + B) = 0$$

Indeed, take $y_1 = C_1 e^{r_1 x}$. Then, $y_1' = C_1 e^{r_1 x}$, $y_1'' = C_1 r_1^2 e^{r_1 x}$ and

$$y_1'' + Ay_1' + By_1 = C_1 r_1^2 e^{r_1 x} + AC_1 e^{r_1 x} + BC_1 e^{r_1 x}$$

$$= C_1 e^{r_1 x}(r_1^2 + Ar_1 + B) = 0$$

Similarly, if $y_2 = C_2 e^{r_2 x}$, then

$$y_2'' + Ay_2' + by_2 = C_2 e^{r_2 x}(r_2^2 + Ar_2 + Br_2) = 0$$

and

$$y_1'' + y_2'' + A(y_1' + y_2') + B(y_1 + y_2) = 0$$

As we have done with $y'' = y$ we can show, here too, that the general solution of

$$y'' + Ay' + By = 0$$

is $C_1 e^{r_1 x} + C_2 e^{r_2 x}$

Example

Solve the equation

$$4y'' - 8y' + 3y = 0; \quad y(0) = 2, y'(0) = 3$$

Example

Solve the equation

$$4y'' - 8y' + 3y = 0; \quad y(0) = 2, y'(0) = 3$$

that is,

$$y'' - 2y' + \frac{3}{4}y = 0$$

Example

Solve the equation

$$4y'' - 8y' + 3y = 0; \quad y(0) = 2, y'(0) = 3$$

that is,

$$y'' - 2y' + \frac{3}{4}y = 0$$

The characteristic polynomial is

$$r^2 - 2r + \frac{3}{4}$$

$$r^2 - 2r + \frac{3}{4}$$

$$r^2 - 2r + \frac{3}{4}$$

which has the same roots as

$$4r^2 - 8r + 3 = 0$$

$$r^2 - 2r + \frac{3}{4}$$

which has the same roots as

$$4r^2 - 8r + 3 = 0$$

(Note that we did not, in fact, need tio divide by 4 to start with!) with roots

$$r = \frac{4 \pm \sqrt{16 - 12}}{4} = 1 \pm 1/2$$

$$4y'' - 8y' + 3y = 0; \quad y(0) = 2, y'(0) = 3$$

$$r = \frac{4 \pm \sqrt{16 - 12}}{4} = 1 \pm 1/2$$

Thus the general solution is

$$y = C_1 e^{\frac{t}{2}} + C_2 e^{\frac{3t}{2}}$$

$$4y'' - 8y' + 3y = 0; \quad y(0) = 2, y'(0) = 3$$

$$r = \frac{4 \pm \sqrt{16 - 12}}{4} = 1 \pm 1/2$$

Thus the general solution is

$$y = C_1 e^{\frac{t}{2}} + C_2 e^{\frac{3t}{2}}$$

where we need $C_1 + C_2 = 2$ and $C_1/2 + 3C_2/2 = 3$ and thus $C_1 = 0$, $C_2 = 2$.

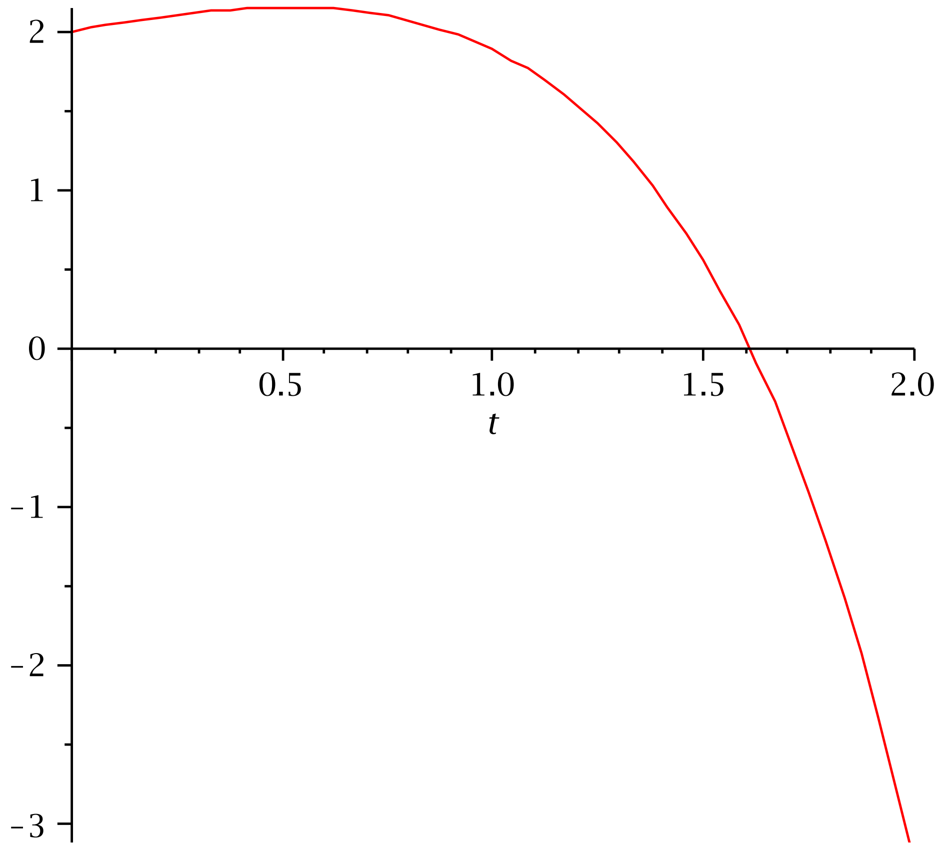
$$4y'' - 8y' + 3y = 0; \quad y(0) = 2, y'(0) = 3$$

$$r = \frac{4 \pm \sqrt{16 - 12}}{4} = 1 \pm 1/2$$

Thus the general solution is

$$y = C_1 e^{\frac{t}{2}} + C_2 e^{\frac{3t}{2}}$$

where we need $C_1 + C_2 = 2$ and $C_1/2 + 3C_2/2 = 3$ and thus $C_1 = 0$, $C_2 = 2$. Let's plot the solution:


2
1
0
-1
-2
-3
0.5
1.0
1.5
2.0
t