Nonhomogeneous Equations and the method of undetermined coefficients

§3.6
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Remember, (2) it is a different equation altogether, from (1). The solutions of (1) are not solutions of (2). The point is that the solutions of (2) help us in finding the solutions of (1).
\[ y'' + p(t)y' + q(t)y = 0 (*) \]
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Let \( Y_0 \) be any solution of \((\ast\ast)\).
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Let \( Y_0 \) be any solution of (\ast\ast). Then, the general solution of (\ast\ast) is \( Y_0 + c_1y_1 + c_2y_2 \) where \( y_1 \) and \( y_2 \) are two linearly independent solutions of (\ast).
\[ y'' + p(t)y' + q(t)y = 0(*) \]

\[ y'' + p(t)y' + q(t)y = g(t)(**) \]

Let \( Y_0 \) be any solution of (**). Then, the general solution of (**) is \( Y_0 + c_1 y_1 + c_2 y_2 \) where \( y_1 \) and \( y_2 \) are two linearly independent solutions of (*).

Thus if we know how to find some particular solution and how to solve the homogeneous equation, both simpler tasks, the equation (***) has been solved.
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Thus if we know how to find some particular solution and how to solve the homogeneous equation, both simpler tasks, the equation \( (\ast\ast) \) has been solved.
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Our equation is $L[y] = g$. Let now $y = Y_0 + h$. Then $L[y] = L[Y_0] + L[h]$. We must have $L[Y_0] + L[h] = g$. Using (*) we have $g + L[h] = g$ or $L[h] = g - g = 0$. Thus $h$ must be a solution of the homogeneous equation, and clearly any solution of the homogeneous equation would do.
Important remark

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- then $AY_1 + BY_2$ is a particular solution when $g(t) = Ag_1(t) + Bg_2(t)$. 
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- then $AY_1 + BY_2$ is a particular solution when $g(t) = Ag_1(t) + Bg_2(t)$.

This is because

$L[AY_1 + BY_2] = AL[Y_1] + BL[Y_2] = Ag_1(t) + Bg_2(t)$. This
allows us to break the nonhomogeneity $g$ into simpler pieces.

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★ So suppose $g = t^m$. The solution is sought in the form $Y_1 = C t^m + V$. In the equation for $V$, the degree of the new $g$ should be lower than $m$. This most often works. If not, try $C t^{m+1}$. If that fails too, then necessarily $C t^{m+2}$ should work.
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If \( g \) is a more complicated polynomial, it may be useful to try a polynomial solution with undetermined coefficients
\[ Y = a_0 + a_1 t + \ldots + a_n t^n, \]
substitute and solve a system of eqns. for \( a_0, \ldots, a_n \). Read textbook!!
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• **Trig functions** $g = \sin at$ or $g = \cos at$. 
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• Trig functions \( g = \sin at \) or \( g = \cos at \). Try first a particular solution \( Y_1 = A \sin at + B \cos at \). If, rarely, this fails, try \( t(A \sin at + B \cos at) \). If, rarely, this fails too, then \( t^2(A \sin at + B \cos at) \) must work.
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• **Linear** combinations of the above.
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• **Linear** combinations of the above. Break the linear combination into components and proceed as above with each of the pieces. Then add together all the $Y$’s thus
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Now, examples.
Try $Y = C t^2 + V$. 

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\[ 2C + V'' - 3Ct^2 - 3V = 5t^2 \]

Then take \( C = -\frac{5}{3} \).
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\[ -\frac{10}{3} + V'' - 3V = 0, \quad \text{or } V'' - 3V = \frac{10}{3} \]
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We now try \( V = A \).
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We now try \( V = A \). We get \(-3A = \frac{10}{3} \) thus \( A = -\frac{10}{9} \). Now add the pieces together: \( Y = -\frac{5t^2}{3} - \frac{10}{9} \).
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\[ -\frac{10}{3} + V'' - 3V = 0, \quad \text{or} \quad V'' - 3V = \frac{10}{3} \]

We now try \( V = A \). We get \(-3A = 10/3\) thus \( A = -\frac{10}{9} \). Now add the pieces together: \( Y = -\frac{5t^2}{3} - \frac{10}{9} \). Next. To find general solution, add general solu-
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\[-4A \sin 2t - 4B \cos 2t + 2A \cos 2t - 2B \sin 2t + 3(A \sin 2t + B \cos 2t) = \cos 2t \quad (3)\]
Thus \[-A - 2B = 0 - B + 2A = 1\]
Thus \(- A - 2B = 0 - B + 2A = 1\)

\(A = \frac{2}{5}, \quad B = -\frac{1}{5}\)
General solution.
General solution. Must solve homog. eq. Characteristic poly: \( r^2 + r + 3 = 0 \). We find

\[
y(t) = c_1 e^{-\frac{t}{2}} \sin \left( \frac{1}{2} \sqrt{11} t \right) + c_2 e^{-\frac{t}{2}} \cos \left( \frac{1}{2} \sqrt{11} t \right) - \frac{1}{5} \cos (2t) + \frac{2}{5} \sin (2t)
\]