Nonhomogeneous Equations and the method of undetermined coefficients

§3.6

General linear nonhomogeneous equation:

General linear nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

General linear nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

The associated homogeneous equation or the homogeneous equation corresponding to (1) is

General linear nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

The associated homogeneous equation or the homogeneous equation corresponding to (1) is

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{2}$$

General linear nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

The associated homogeneous equation or the homogeneous equation corresponding to (1) is

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{2}$$

Remember, (2) it is a different equation altogether, from (1).

General linear nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

The associated homogeneous equation or the homogeneous equation corresponding to (1) is

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{2}$$

Remember, (2) it is a different equation altogether, from (1). The solutions of (1) are not solutions of (2).

General linear nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

The associated homogeneous equation or the homogeneous equation corresponding to (1) is

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{2}$$

Remember, (2) it is a different equation altogether, from (1). The solutions of (1) are not solutions of (2).The point is that the solutions of (2) help us in finding the solutions of (1).

$$y'' + p(t)y' + q(t)y = 0 (*)$$

$$y'' + p(t)y' + q(t)y = 0(*)$$

$$y'' + p(t)y' + q(t)y = g(t)(**)$$

Let Y_0 be any solution of (**).

$$y'' + p(t)y' + q(t)y = 0(*)$$

$$y'' + p(t)y' + q(t)y = g(t)(**)$$

Let Y_0 be any solution of (**). Then, the general solution of (**) is $Y_0 + c_1y_1 + c_2y_2$ where y_1 and y_2 are two linearly independent solutions of (*).

$$y'' + p(t)y' + q(t)y = 0(*)$$

$$y'' + p(t)y' + q(t)y = g(t)(**)$$

Let Y_0 be any solution of (**). Then, the general solution of (**) is $Y_0 + c_1y_1 + c_2y_2$ where y_1 and y_2 are two linearly independent solutions of (*).

Thus if we know how to find some particular solution and how to solve the homogeneous equation, both simpler tasks, the equation (**) has been solved.

$$y'' + p(t)y' + q(t)y = 0(*)$$

$$y'' + p(t)y' + q(t)y = g(t)(**)$$

Let Y_0 be any solution of (**). Then, the general solution of (**) is $Y_0 + c_1y_1 + c_2y_2$ where y_1 and y_2 are two linearly independent solutions of (*).

Thus if we know how to find some particular solution and how to solve the homogeneous equation, both simpler tasks, the equation (**) has been solved.

Shapes of g for which particular solutions can be easily found if L is linear with constant coefficients

- Polynomials

Shapes of g for which particular solutions can be easily found if L is linear with constant coefficients

- Polynomials
- Exponentials

Shapes of g for which particular solutions can be easily found if L is linear with constant coefficients

- Polynomials
- Exponentials
- Trig functions

Shapes of g for which particular solutions can be easily found if L is linear with constant coefficients

- Polynomials
- Exponentials
- Trig functions
- Combinations of the above.

Shapes of g for which particular solutions can be easily found if L is linear with constant coefficients

- Polynomials
- Exponentials
- Trig functions
- Combinations of the above.

Proof of the fact that the general solution of (**) is $Y_0 + c_1 y_1 + c_2 y_2$ where y_1 and y_2 are two linearly independent solutions of (*).

Proof of the fact that the general solution of (**) is $Y_0 + c_1 y_1 + c_2 y_2$ where y_1 and y_2 are two linearly independent solutions of (*).We have

$$L[Y_0] = g \quad (*)$$

Proof of the fact that the general solution of (**) is $Y_0 + c_1y_1 + c_2y_2$ where y_1 and y_2 are two linearly independent solutions of (*).We have

$$L[Y_0] = g \quad (*)$$

Our equation is $L[y] = g$.

Proof of the fact that the general solution of (**) is $Y_0 + c_1 y_1 + c_2 y_2$ where y_1 and y_2 are two linearly independent solutions of (*).We have

$$L[Y_0] = g \quad (*)$$

Our equation is $L[y] = g$. Let now $y = Y_0 + h$.

Proof of the fact that the general solution of (**) is $Y_0 + c_1 y_1 + c_2 y_2$ where y_1 and y_2 are two linearly independent solutions of (*).We have

$$L[Y_0] = g \quad (*)$$

Our equation is $L[y] = g$. Let now $y = Y_0 + h$. Then $L[y] = L[Y_0] + L[h]$.

Proof of the fact that the general solution of (**) is $Y_0 + c_1 y_1 + c_2 y_2$ where y_1 and y_2 are two linearly independent solutions of (*).We have

$$L[Y_0] = g \quad (*)$$

Our equation is $L[y] = g$. Let now $y = Y_0 + h$. Then $L[y] = L[Y_0] + L[h]$. We must have $L[Y_0] + L[h] = g$.

Proof of the fact that the general solution of (**) is $Y_0 + c_1 y_1 + c_2 y_2$ where y_1 and y_2 are two linearly independent solutions of (*).We have

$$L[Y_0] = g \quad (*)$$

Our equation is $L[y] = g$. Let now $y = Y_0 + h$. Then $L[y] = L[Y_0] + L[h]$. We must have $L[Y_0] + L[h] = g$. Using (*) we have $g + L[h] = g$ or $L[h] = g - g = 0$.

Proof of the fact that the general solution of (**) is $Y_0 + c_1y_1 + c_2y_2$ where y_1 and y_2 are two linearly independent solutions of (*).We have

$$L[Y_0] = g \quad (*)$$

Our equation is $L[y] = g$. Let now $y = Y_0 + h$. Then $L[y] = L[Y_0] + L[h]$. We must have $L[Y_0] + L[h] = g$. Using (*) we have $g + L[h] = g$ or $L[h] = g - g = 0$. Thus h must be a solution of the homogeneous equation, and clearly any solution of the homogeneous equation would do.

Important remark If

- $g(t) = Ag_1(t) + Bg_2(t)$

Important remark If

- $g(t) = Ag_1(t) + Bg_2(t)$
- and Y_1 is a particular solution when $g = g_1$

Important remark If

- $g(t) = Ag_1(t) + Bg_2(t)$
- and Y_1 is a particular solution when $g = g_1$
- while Y_2 is a particular solution when $g = g_2$,

Important remark If

- $g(t) = Ag_1(t) + Bg_2(t)$
- and Y_1 is a particular solution when $g = g_1$
- while Y_2 is a particular solution when $g = g_2$,
- then $AY_1 + BY_2$ is a particular solution when $g(t) = Ag_1(t) + Bg_2(t)$.

Important remark If

- $g(t) = Ag_1(t) + Bg_2(t)$
- and Y_1 is a particular solution when $g = g_1$
- while Y_2 is a particular solution when $g = g_2$,
- then $AY_1 + BY_2$ is a particular solution when $g(t) = Ag_1(t) + Bg_2(t)$.

This is because
$L[AY_1 + BY_2] = AL[Y_1] + BL[Y_2] = Ag_1(t) + Bg_2(t)$. This

allows us to break the nonhomogeneity g into simpler pieces.

How to proceed? We now consider equations with constant coefficients .

allows us to break the nonhomogeneity g into simpler pieces.

How to proceed? We now consider equations with constant coefficients . Suppose g is a

- Polynomial

allows us to break the nonhomogeneity g into simpler pieces.

How to proceed? We now consider equations with constant coefficients . Suppose g is a

- Polynomial It is enough to look at the case $g = t^m$!

allows us to break the nonhomogeneity g into simpler pieces.

How to proceed? We now consider equations with constant coefficients . Suppose g is a

- Polynomial It is enough to look at the case $g = t^m$! This is because any polynomial can be broken linearly into a sum of monomials of the form $a_m t^m$.

allows us to break the nonhomogeneity g into simpler pieces.

How to proceed? We now consider equations with constant coefficients . Suppose g is a

- Polynomial It is enough to look at the case $g = t^m$! This is because any polynomial can be broken linearly into a sum of monomials of the form $a_m t^m$.

- $g = t^m$.

  ⋆ So suppose $g = t^m$.

- $g = t^m$.
  - ⋆ So suppose $g = t^m$. The solution is sought in the form $Y_1 = Ct^m + V$. In the equation for V, the degree of the new g should be lower than m. This most often works.

- $g = t^m$.
  - ⋆ So suppose $g = t^m$. The solution is sought in the form $Y_1 = Ct^m + V$. In the equation for V, the degree of the new g should be lower than m. This most often works. If not, try Ct^{m+1}.

- $g = t^m$.
  - ⋆ So suppose $g = t^m$. The solution is sought in the form $Y_1 = Ct^m + V$. In the equation for V, the degree of the new g should be lower than m. This most often works. If not, try Ct^{m+1}.If that fails too, then necessarily Ct^{m+2} should work.

- $g = t^m$.
  - ⋆ So suppose $g = t^m$. The solution is sought in the form $Y_1 = Ct^m + V$. In the equation for V, the degree of the new g should be lower than m. This most often works. If not, try Ct^{m+1}.If that fails too, then necessarily Ct^{m+2} should work. Repeat the trick on the V equation etc. until you bring down the equation to one in which the rhs, g, is zero.

- $g = t^m$.

  ⋆ So suppose $g = t^m$. The solution is sought in the form $Y_1 = Ct^m + V$. In the equation for V, the degree of the new g should be lower than m. This most often works. If not, try Ct^{m+1}.If that fails too, then necessarily Ct^{m+2} should work. Repeat the trick on the V equation etc. until you bring down the equation to one in which the rhs, g, is zero. If you found yourself having an exceptional equation where you had to try an increased power, then the same increment should be applied to all later monomials.

If g is a more complicated polynomial, it may be useful to try a polyn. solution with undetermined coefficients $Y = a_0 + a_1 t + \ldots + a_n t^n$, substitute and solve a system of eqns. for $a_0, \ldots, a_n$. Read textbook!!

- Exponentials $g = e^{at}$.

- Exponentials $g = e^{at}$. Try a particular solution in the form Ce^{at}. If, rarely, this fails, try Cte^{at}.

- Exponentials $g = e^{at}$. Try a particular solution in the form Ce^{at}. If, rarely, this fails, try Cte^{at}. If, rarely, this fails too, then Ct^2e^{at} must work.

- Exponentials $g = e^{at}$. Try a particular solution in the form Ce^{at}. If, rarely, this fails, try Cte^{at}. If, rarely, this fails too, then Ct^2e^{at} must work.

- Trig functions $g = \sin at$ or $g = \cos at$.

- Exponentials $g = e^{at}$. Try a particular solution in the form Ce^{at}. If, rarely, this fails, try Cte^{at}. If, rarely, this fails too, then Ct^2e^{at} must work.

- Trig functions $g = \sin at$ or $g = \cos at$. Try first a particular solution $Y_1 = A\sin at + B\cos at$. If, rarely, this fails, try $t(A\sin at + B\cos at)$. If, rarely, this fails too, then $t^2(A\sin at + B\cos at)$ must work.

- Exponentials $g = e^{at}$. Try a particular solution in the form Ce^{at}. If, rarely, this fails, try Cte^{at}. If, rarely, this fails too, then Ct^2e^{at} must work.

- Trig functions $g = \sin at$ or $g = \cos at$. Try first a particular solution $Y_1 = A \sin at + B \cos at$. If, rarely, this fails, try $t(A \sin at + B \cos at)$. If, rarely, this fails too, then $t^2(A \sin at + B \cos at)$ must work.

- Linear combinations of the above.

- Exponentials $g = e^{at}$. Try a particular solution in the form Ce^{at}. If, rarely, this fails, try Cte^{at}. If, rarely, this fails too, then Ct^2e^{at} must work.

- Trig functions $g = \sin at$ or $g = \cos at$. Try first a particular solution $Y_1 = A \sin at + B \cos at$. If, rarely, this fails, try $t(A \sin at + B \cos at)$. If, rarely, this fails too, then $t^2(A \sin at + B \cos at)$ must work.

- Linear combinations of the above. Break the linear combination into components and proceed as above with each of the pieces. Then add together all the $Y's$ thus obtained.

obtained.

Now, examples.

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$.

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$. We get

$$2C + V'' - 3Ct^2 - 3V = 5t^2$$

Then take $C = -5/3$.

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$. We get

$$2C + V'' - 3Ct^2 - 3V = 5t^2$$

Then take $C = -5/3$. We then get

$$-10/3 + V'' - 3V = 0, \quad \text{or} \, V'' - 3V = 10/3$$

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$. We get

$$2C + V'' - 3Ct^2 - 3V = 5t^2$$

Then take $C = -5/3$. We then get

$$-10/3 + V'' - 3V = 0, \quad \text{or}\, V'' - 3V = 10/3$$

We now try $V = A$.

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$. We get

$$2C + V'' - 3Ct^2 - 3V = 5t^2$$

Then take $C = -5/3$. We then get

$$-10/3 + V'' - 3V = 0, \quad \text{or} \, V'' - 3V = 10/3$$

We now try $V = A$. We get $-3A = 10/3$ thus $A = -10/9$.

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$. We get

$$2C + V'' - 3Ct^2 - 3V = 5t^2$$

Then take $C = -5/3$. We then get

$$-10/3 + V'' - 3V = 0, \quad \text{or} V'' - 3V = 10/3$$

We now try $V = A$. We get $-3A = 10/3$ thus $A = -10/9$. Now add the pieces together: $Y = -5t^2/3 - 10/9$.

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$. We get

$$2C + V'' - 3Ct^2 - 3V = 5t^2$$

Then take $C = -5/3$. We then get

$$-10/3 + V'' - 3V = 0, \quad \text{or} V'' - 3V = 10/3$$

We now try $V = A$. We get $-3A = 10/3$ thus $A = -10/9$. Now add the pieces together: $Y = -5t^2/3 - 10/9$. Next.

$$y'' - 3y = 5t^2$$

Try $Y = Ct^2 + V$. We get

$$2C + V'' - 3Ct^2 - 3V = 5t^2$$

Then take $C = -5/3$. We then get

$$-10/3 + V'' - 3V = 0, \quad \text{or}\, V'' - 3V = 10/3$$

We now try $V = A$. We get $-3A = 10/3$ thus $A = -10/9$. Now add the pieces together: $Y = -5t^2/3 - 10/9$. Next. To find general solution, add general solution of the homogeneous equation.

tion of the homogeneous equation. Characteristic polynomial equation $r^2 - 3 = 0$ $r = \pm\sqrt{3}$.

tion of the homogeneous equation. Characteristic polynomial equation $r^2 - 3 = 0$ $r = \pm\sqrt{3}$. Gen. sol. of homog. equation is $c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}$.

tion of the homogeneous equation. Characteristic polynomial equation $r^2 - 3 = 0$ $r = \pm\sqrt{3}$. Gen. sol. of homog. equation is $c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}$. Finally thus, $y = c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t} - 5t^2/3 - 10/9$.

$$y'' + y' + 3y = \cos 2t$$

tion of the homogeneous equation. Characteristic polynomial equation $r^2 - 3 = 0$ $r = \pm\sqrt{3}$. Gen. sol. of homog. equation is $c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}$. Finally thus, $y = c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t} - 5t^2/3 - 10/9$.

$$y'' + y' + 3y = \cos 2t$$

Try $Y = A \sin 2t + B \cos 2t$. We get

tion of the homogeneous equation. Characteristic polynomial equation $r^2 - 3 = 0$ $r = \pm\sqrt{3}$. Gen. sol. of homog. equation is $c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}$. Finally thus, $y = c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t} - 5t^2/3 - 10/9$.

$$y'' + y' + 3y = \cos 2t$$

Try $Y = A\sin 2t + B\cos 2t$. We get

$$-4A\sin 2t - 4B\cos 2t + 2A\cos 2t - 2B\sin 2t + 3(A\sin 2t + B\cos 2t) = \cos 2t \quad (3)$$

Thus $\quad -A - 2B = 0 - B + 2A = 1$

$$\text{Thus} \quad -A - 2B = 0 \quad -B + 2A = 1$$

$$A = 2/5, \quad B = -1/5$$

General solution.

General solution. Must solve homog. eq. Characteristic poly: $r^2 + r + 3 = 0$. We find

$$y(t) = c_1 e^{-\frac{t}{2}} \sin\left(\frac{1}{2}\sqrt{11}t\right) + c_2 e^{-\frac{t}{2}} \cos\left(\frac{1}{2}\sqrt{11}t\right) - 1/5 \cos(2t) + 2/5 \sin(2t) \quad (4)$$