1. Consider the heat equation

$$u_t = u_{xx};$$

with the following initial and boundary conditions:

$$u(x,0) = f(x); \quad u_x(0,t) = 0; \quad u(\pi,t) = 0 \tag{1}$$

(Note that the **derivative** of u is zero at one end while u itself vanishes at the other. That is, the left endpoint is thermally isolated while the right endpoint is kept at $u = 0$.)

Find the general solution by separation of variables. (It is **not** required that you solve for the coefficients c_n in the series solution).

Solution. Separation of variables means looking for particular solutions in the form $X(x)T(t)$. We get

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

This implies that for some constant $\lambda \in (-\infty, \infty)$ we have

$$T' + \lambda T = 0 \quad (a); \quad X'' + \lambda X = 0 \quad (b)$$

(why)? The boundary conditions $u_x(0,t) = 0; \quad u(\pi,t) = 0$ imply $X'(0) = 0, X(\pi) = 0$. From (b), we have, for $\lambda > 0$

$$X = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

Since $X'(0) = A$ (check) it follows that $A = 0$ and $X = B\cos(\sqrt{\lambda}x)$. We must have $X(\pi) = 0$ thus $\cos(\sqrt{\lambda}\pi) = 0$, thus

$$\sqrt{\lambda}\pi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ..., (2k+1)\frac{\pi}{2}, ...$$

(explain) so

$$\lambda = (k+1/2)^2$$

Then $T = Ce^{-(k+1/2)t}$. There are no eigenfunctions for $\lambda \leq 0$ (check!). Thus the general solution obtained in this way is

$$u(x,t) = \sum_{k=0}^{\infty} c_k e^{-(k+1/2)t} \cos(k+1/2)t$$

The initial condition reads:

$$u(x,0) = f(x) = \sum_{k=0}^{\infty} c_k \cos(k+1/2)t$$

we are not required to solve for c_n.

(a) In the equations below, find the general solution:

(b)

$$4y'' + 4y' + y = e^{-x/2}$$

(c)

$$4y'' + 4y' + y = xe^{-x/2}$$

Solution. We solve only (c), as (b) is similar. The characteristic roots are $r = -1/2, -1/2$ (equal roots). The general solution of the associated homogeneous equation is $Ae^{-x/2} + Bxe^{-x/2}$. By the general formula, the solution of (c) is

$$Ae^{-x/2} + Bxe^{-x/2} - \frac{1}{4}e^{-x/2}\int_0^x sf(s)e^{s/2}ds + \frac{1}{4}xe^{-x/2}\int_0^x f(s)e^{s/2}ds$$

where $f(s) = se^{-s/2}$. Thus, the solution of (c) is

$$Ae^{-x/2} + Bxe^{-x/2} + \frac{1}{24}x^3e^{-x/2}$$

2. Find all the solutions to the initial value problem:

$$yy' = 1; \;\; x > 0 \;\; y(0) = 0$$

Solution. This is a separable equation. Integrating both sides we get

$$\frac{y^2(x)}{2} = x + C$$

The initial condition implies

$$0 = 0 + C$$

thus $C = 0$. So any solution has the property

$$y(x)^2 = 2x$$

Thus there are exactly two solutions, $y(x) = \pm\sqrt{2x}$; both indeed satisfy the equation for $x > 0$ and the initial condition $y(0) = 0$.