AREAS AND THE JORDAN MEASURE

To calculate areas we must first, of course, define the notion. The definition will attempt to capture our intuitive notion of an area, or, more precisely, the properties that we expect from it. Following the lines below, we essentially define the Jordan area of a set. We let $\mathcal{J}$ contain all sets subsets of the plane which have a Jordan measure and we let $\mathcal{A}$ be the area function, which associates to any set in $A \in \mathcal{J}$ its area, $\mathcal{A}(A)$.

Note 1. We will not show that an area function $\mathcal{A}$ with the properties below exists. Though this is not very difficult, the proper place to do that is in calculus in $\mathbb{R}^n$.

Properties of the area (take them for granted).

(1) $A \in \mathcal{J} \Rightarrow \mathcal{A}(A) \in [0, \infty]$ (the area can be infinite: this is the case of the area of the whole plane).
(2) $A \in \mathcal{J}, B \in \mathcal{J} \Rightarrow A \cup B, A \setminus B \in \mathcal{J}$
(3) If $A \in \mathcal{J}, B \in \mathcal{J}$ are disjoint, then $\mathcal{A}(A \cup B) = \mathcal{A}(A) + \mathcal{A}(B)$.
(4) $\mathbb{R}^2 \in \mathcal{J}$; any rectangle R is in $\mathcal{J}$ and $\mathcal{A}(R) = ab$, where a, b are the sides of the rectangle.

Exercises

Exercise 1 (5p)**.** *Show that $A \in \mathcal{J}, B \in \mathcal{J} \Rightarrow A \cap B \in \mathcal{J}$. Show that if $A \in \mathcal{J}, B \in \mathcal{J}$ have finite area, then $\mathcal{A}(A) + \mathcal{A}(B) - \mathcal{A}(A \cap B)$. (A picture would help you figure out why.)*

Exercise 2 (10p)**.** *Show that if $B \subset A$ and both sets are in $\mathcal{J}$ with finite area, then $\mathcal{A}(A) \geqslant \mathcal{A}(B)$.*

Let f be a nonnegative bounded function on $[a, b]$, let P be a partition of $[a, b]$ and define $\mathcal{G} = \{(x, y) | x \in [a, b]$ and $y \in [0, f(x)]$ (this is clearly the region *under the graph of* f).

Exercise 3 (15p)**.** *Assume that $\mathcal{G} \in \mathcal{J}$. Show that $L(f, P) \leqslant \mathcal{A}(\mathcal{G}) \leqslant U(f, P)$.*