BONUS PROBLEMS

Remember that the rational number p/q is written in irreducible form if p and q have no common factors.

1. [10p]

Let

$$P(x) = a_n x^n + \cdots + a_0, \quad a_j \in \mathbb{Z}, j = 0, 1, ..., n$$

Show that any rational root p/q (irreducible) of P has the property that a_0 is an integer multiple of p and a_n is an integer multiple of q. (Thus, one can after finitely many trials, find all rational roots of polynomials with rational coefficients.)

2. [20p]

Show that for any $x, y \in \mathbb{C}$

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$$

Let $P_m = \cos(mx)$. Show that P_m satisfies the recurrence relation

$$P_{m+1} = 2P_1 P_m - P_{m-1}$$

Show by induction that $\cos(mx)$ is a polynomial in $y = \cos x$ of the form

$$P_m(y) = 2^{m-1} y^m + a_{m-1,m} y^{m-2} + \cdots + a_{0,m}$$

where $a_{0,m} = \mathrm{Im}(i^{m+1})$ and all coefficients $a_{m,j}$ are integers.

3. [30p]

Use problem 2 to show that if $r \in \mathbb{Q}$ and $\cos \pi r = r' \in \mathbb{Q}$, then $r' = 2^{-m}\mathrm{Im}(i^k)$ for some $m \in \{0, 1, 2, ...\}$ and $k \in \{1, 2, 3\}$.

4. [100p without any hint, 50p with a hint]

For which $r \in \mathbb{Q}$ is $\cos(r\pi) \in \mathbb{Q}$? Are there regular polygons inscribed in the unit circle with the property that the distance from the center to one side is rational? For which $r \in \mathbb{Q}$ is $\sin(r\pi) \in \mathbb{Q}$?