COUNTABLE SETS, OPEN SETS, ZERO MEASURE SETS

One-one functions are also called "injective" and "surjective" stands for onto; if a function is both injective and surjective, it is called "bijective".

Clearly, two sets A and B containing finitely many elements have the same number of elements –the same *cardinality* $|A| = |B|$– iff there is a bijective function f from A to B.

We can extend the notion of cardinality to infinite sets:

Definition. Two sets have the same cardinality, $|A| = |B|$, if there is a bijective $f : A \to B$.

Exercise 1 (15p)**.** Show that if A is such that there is a surjective map $f : \mathbb{N} \to A$ and a surjective $g : A \to \mathbb{N}$, then there is a bijective $h : \mathbb{N} \to A$ and thus $|A| = |\mathbb{N}|$. (Hint: one possibility is to define h inductively, eliminating repetitions. Namely, you can try to make the following construction precise: "$h(1) = f(1)$; if $f(2) = f(1)$ then skip 2 and look for the smallest $n = n_2$ s.t. $f(n_2) \neq f(1)$ and define $h(2) = f(n_2)$ etc.".)

Countable sets

Sets A s.t. $|A| = |\mathbb{N}|$ are called countable. If there is a map from $\{1, 2, ..., n\}$ to A which is bijective, then A is a *finite* set, and n is its cardinality.

Sometimes, countable is understood as "finite or countable". Then, to distinguish finite sets from sets s.t. $|A| = |\mathbb{N}|$, one calls A in the latter case *countably infinite*. In this note, I will use this terminology.

We showed in class that there is a surjective map from $\mathbb{N}$ to $\mathbb{Q}$ (the two-dimensional spiraling count function) and also an injective one (the identity is injective from $\mathbb{N}$ to, in fact, $\mathbb{N} \subset \mathbb{Q}$). It follows from the definition and from Exercise 1 that $\mathbb{Q}$ is countably infinite.

Open sets

Definition A set $\mathcal{O} \subset \mathbb{R}$ is called open if for any $x \in \mathcal{O}$ there is a $\delta > 0$ s.t. $(x - \delta, x + \delta) \subset \mathcal{O}$. The empty set $\emptyset$ is open (it has no elements to speak of, so the condition holds "vacuously"). $\mathbb{R}$ is open, and for $b > a$, (a, b) is open (check!).

Exercise 2 (15p)**.** Show that any nonempty open set $\mathcal{O}$ in $\mathbb{R}$ can be written in a unique way as a countable (as we agreed, this could mean finite) union of nonempty disjoint intervals. One way to proceed is the following:

(a) For any $x \in \mathcal{O}$ take J_x to be the largest interval (a, b) containing it. Here, a is defined as $\inf\{a' : (a', x) \subset \mathcal{O}\}$ and $b = \sup\{b' : (x, b') \subset \mathcal{O}\}$. If the a's are unbounded below then we write $a = -\infty$ and similarly if the b' are unbounded, we write $b = +\infty$. Show that this construction makes sense and let $\mathcal{J}$ be the set of all intervals thus constructed. Let $\mathcal{J} = \{A : A = J_x \text{ for some } x \in \mathcal{O}\}$. that, if $x, y \in \mathcal{O}$, then

$$\text{Either } J_x = J_y \text{ or else } J_x \cap J_y = \emptyset \tag{1}$$

(b) Take a bijection f from $\mathbb{N}$ into $\mathbb{Q}$ and denote $f(i) = q_i$. If there is an $x \in \mathcal{O}$ s.t. $q_i \in J_x$ then let $\hat{J}_i = J_x$ and otherwise let $\hat{J}_i = \emptyset$. Show that $i \mapsto \hat{J}_i$ defines a surjective map from $\mathbb{N}$ to $\mathcal{J}$. This shows that the set $\mathcal{J}$ is countable.

(c) As in Exercise 1, show that there is a bijective f map defined either on $\{1, 2, ..., n\}$ or else on $\mathbb{N}$, with values in $\mathcal{J}$.

Lebesgue measure of open sets; zero measure sets

Let $\mathcal{O}$ be an open set and $\mathcal{J}$ its disjoint interval decomposition. We distinguish the following cases:

(i) Some interval in $\mathcal{J}$ is infinite (i.e., of the form $(-\infty, a), (a, \infty), (-\infty, \infty)$). Then we say that the measure of $\mathcal{O}$ is $\lambda(\mathcal{O}) = +\infty$.

(ii) there are finitely many disjoint intervals (a_i, b_i) in $\mathcal{J}$, each of them finite. Then $\mathcal{J} = \cup_{i=1}^{N}(a_i, b_i)$ and we define the measure of $\mathcal{J}$, $\lambda(\mathcal{J}) = \sum_{i=1}^{n}(b_i - a_i)$.

(iii) The last case is: there are countably infinitely many intervals (a_i, b_i) in $\mathcal{J}$, each of them finite. Note that, if there is an M s.t. $\forall N \; S_N = \sum_{i=1}^{N}(b_i - a_i) < M$, then S_N has a limit as $N \to \infty$ since the sequence $\{S_N\}$ is nondecreasing. If no such upper bound M exists, then $\sum_{i=1}^{N}(b_i - a_i) \to +\infty$.

Definition. We define $\lambda(\mathcal{J}) = \lim_{N\to\infty} \sum_{i=1}^{N}(b_i - a_i)$ if the limit exists, and $\lambda(\mathcal{J}) = +\infty$ otherwise.

Definition. The measure of an open set $\mathcal{O}$, $\lambda(\mathcal{O})$, is defined as the measure $\lambda(\mathcal{J})$ of its disjoint interval decomposition.

Definition. A set $S \subset \mathbb{R}$ has zero measure if for any ε there is an open set $\mathcal{O}_\varepsilon \supset S$ with $\lambda(\mathcal{O}_\varepsilon) < \varepsilon$.

Exercise 3 (15p). *Show that a finite or countably infinite set S has zero measure. Hint: one possibility is to note that*

$$S \subset \bigcup_{s_n \in S} (s_n - 2^{-n}\varepsilon, s_n + 2^{-n}\varepsilon)$$

and show that $\lambda\Big(\cup_{s_n \in S}(s_n - 2^{-n}\varepsilon, s_n + 2^{-n}\varepsilon)\Big) < 2\varepsilon$. The sets $(s_n - 2^{-n}\varepsilon, s_n + 2^{-n}\varepsilon)$ can be inductively chosen s.t. they are mutually disjoint.

In particular, $\mathbb{Q}$ has zero measure.

Notes. (i) Check that any nonempty open set has positive measure.

(ii) There are sets which are uncountable and yet have zero measure. An example is the following (example of a *Cantor set)*: the set of all $x \in (0, 1)$ for which no digit in the decimal representation equals 7 (of course 7 can be replaced with $1, 2, ..., 9$). (Don't try to prove (ii), unless you need something hard to work on.)