ONE MORE PROPERTY OF REAL NUMBERS

1. Dedekind cuts

Once the natural numbers $\mathbb{N}$, then the signed integers $\mathbb{Z}$ and finally the rationals, $\mathbb{Q}$ have been constructed, Dedekind cuts are a common way to define the real numbers, $\mathbb{R}$. [1]

Definition 1. *A Dedekind cut is a partition of the rational numbers into two non-empty sets A and B, such that all elements of A are less than all elements of B, and A contains no greatest element.*

A real number is then identified with a Dedekind cut[2].

We don't aim at constructing $\mathbb{R}$ here[3]. Instead we assume $\mathbb{R}$ exists as a set of objects (numbers) satisfying (P1),...,(P12) as well as (P13') below. (P13') is essentially equivalent to the property (P13) discussed later in the course.

Property (P13')**.** *For any two non-empty sets A and B of rational numbers such that $\mathbb{Q} = A \cup B$, if all elements of A are less than all elements of B, then there is a unique $x \in \mathbb{R}$ s.t. for any $a \in A$ and any $b \in B$ we have $a \leqslant x \leqslant b$.*

Note first that A and B consist of rational numbers only, since $\mathbb{Q} = A \cup B$.

Exercise 1. *Show that for any non-empty sets A and B as in (P13') and for any $n \in \mathbb{N}$, there is a pair (a, b) with $a \in A$ and $b \in B$ such that $|b - a| = b - a < 1/n$.*

Denote $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$, *i.e.* $\mathbb{R}^+$ is the set of positive real numbers. This is a common notation, much more so than "P".

Exercise 2. *Show that for any $\varepsilon \in \mathbb{R}^+$ there is an $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$. Relatedly, show that for any $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ s.t. $n > |x|$.*

Exercise 3. *Show that the sets defined by $A = \{x \in \mathbb{Q} : x^2 < 2\}$ and $B = \{x \in \mathbb{Q} : x^2 > 2\}$ satisfy the conditions of (P13'), and that the corresponding x has the property $x^2 = 2$.*

Note 2. Since we proved that a number s.t. $x^2 = 2$ cannot be rational, this shows that the existence of $\sqrt{2}$ (*a fortiori* (P13')) does not follow from (P1),. . .,(P12). Furthermore, though the claims in Exercise 2 seem obvious, they do *not* follow solely from (P1),. . .,(P12) either!

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[1]The construction of integers is an "easier" task, once the foundations of math have been established say based on sets and their properties. If you are interested about this set-theoretical foundation, this article on Wikipedia is pretty easy to follow:

http://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory

To see how integers are defined in terms of sets see

http://en.wikipedia.org/wiki/Set-theoretic_definition_of_natural_numbers

[2]Intuitively, a Dedekind cut is a pair (A, B) where of the form $(-\infty, x) \cap \mathbb{Q}$, $\langle x, \infty) \cap \mathbb{Q}$ where $\langle$ can be [or (, and x is a real number. This singles out the real number x. *Of course, it would be circular to define the cut using this type of intervals, since we don't have real numbers before we construct them.* But this intuition suggests that, if we don't write A, B as intervals, the pair (A, B) can be taken as the "name of x".

[3]A good book where you can see the details is "A Course in Modern Analysis" by E. T. Whittaker and G. N. Watson, Cambridge University Press (2002).