STIRLING'S FORMULA

The Gaussian integral. We will use the Gaussian integral

$$I = \int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \tag{1}$$

There are many ways to derive this equality; an elementary but computationally heavy one is outlined in Problem 42, Chap. 19. One of the easiest ways is to evaluate the double integral $\int_0^\infty \int_0^\infty e^{-x^2-y^2} dx dy$ in polar coordinates, but that's for Semester 2... Another one is to change variables $x = \sqrt{u}$, $dx = \frac{1}{2} u^{-\frac{1}{2}}$, to get

$$I = \frac{1}{2} \int_0^\infty e^{-u} u^{-\frac{1}{2}} du = \tfrac{1}{2} \Gamma\left(\tfrac{1}{2}\right) \tag{2}$$

Note that the integral in (2) is improper at both endpoints of the interval of integrations! Show that it is well-defined. We can now use Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{3}$$

which is proved in Complex Analysis. With $z = 1/2$ we get

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \pi \tag{4}$$

which, together with (2) implies (1).

Laplace's method. Let g be a twice continuously differentiable function on $[-a, a]$ with a strict and absolute *maximum* at x_0 (say $x_0 = 0$), with $g''(0) < 0$, and let f be continuous and s.t. $f(0) \neq 0$. Consider the integral

$$\int_{-a}^{a} f(s) e^{ng(s)} ds \tag{5}$$

for large n. A good number of functions arising in applications can be brought to integrals of this type after suitable changes of variables, so it is useful to understand how to estimate them.

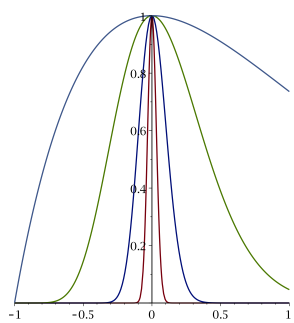

FIGURE 1. $e^{-n(u-\ln(1+u))}$ for $n = 1, 10, 100, 1000$. In this figure, larger n corresponds to a narrower maximum.

The function e^{ng} has a sharper and sharper maximum at $s = 0$ as $n \to \infty$. Indeed, if $x \neq 0 \Rightarrow g(0) - g(x) > \delta(x) > 0$, we have $e^{g(0)-g(x)} = \alpha(x) > 1$. Now, $e^{n[g(0)-g(x)]} =$

$\alpha(x)^{-n} \to 0$ as $n \to \infty$. Thus, only a tiny neighborhood of the origin is visible on a large n graph. (See Fig. 1 for $g = -u + \ln(1+u)$ relevant for the Gamma function.) For the same reason, for large n, the bulk of the integral (5) is expected to come from a small neighborhood of 0. We approximate

$$f(x) \approx f(0) \text{ and } g(x) \approx g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 = g(0) + \frac{1}{2}g''(0)x^2$$

Thus, we expect

$$\int_{-a}^{a} f(s)e^{ng(s)}ds \approx f(0)e^{ng(0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}n|g''(0)|x^2} du = f(0)\frac{\sqrt{2\pi}}{\sqrt{n|g''(0)|}} \tag{6}$$

where extending the integral to the whole of $\mathbb{R}$ is motivated by the same observation: for large n, only a small neighborhood of zero contributes to the integral. Of course, **all this needs to be justified!**

Lemma 1. *Assume $f \in C[-a,a]$, that $g \in C^2[-a,a]$ has a unique absolute maximum at $x = 0$ and that $f(0) \neq 0$ (say $f(0) > 0$) and $g''(0) < 0$. Then, there is a $\delta > 0$ such that (equivalently, for all $\delta > 0$ small enough, or for $\delta = a$, we have)*

$$\int_{-\delta}^{\delta} f(s)e^{ng(s)}ds = \sqrt{\frac{2\pi}{n|g''(0)|}} f(0)e^{ng(0)}(1 + h(n)) \quad \textit{where } \lim_{n\to\infty} h(n) = 0 \tag{7}$$

Proof. Choose $0 < \varepsilon < |g''(0)/2|$ and let δ be such that $|s| < \delta$ implies $|g''(s) - g''(0)| < \varepsilon$ and also $|f(s) - f(0)| < \varepsilon$. Then,

$$\begin{aligned} \int_{-\delta}^{\delta} e^{ng(s)} f(s)ds &\leqslant (f(0)+\varepsilon) \int_{-\delta}^{\delta} e^{ng(0)+\frac{n}{2}(g''(0)+\varepsilon)s^2} ds \\ &\leqslant (f(0)+\varepsilon) \int_{-\infty}^{\infty} e^{ng(0)+\frac{n}{2}(g''(0)+\varepsilon)s^2} ds = \sqrt{\frac{2\pi}{n|g''(0)| - \varepsilon}}(f(0)+\varepsilon)e^{ng(0)} \end{aligned} \tag{8}$$

where we used (2) and the change of variable $-\frac{n}{2}(g''(0)+\varepsilon)s^2 = u^2$ An inequality in the opposite direction follows in the same way, replacing $\leqslant$ with $\geqslant$ and ε with $-\varepsilon$ in the first line of (8), and then noting that

$$\frac{\int_{-\delta}^{\delta} e^{-ns^2} ds}{\int_{-\infty}^{\infty} e^{-ns^2} ds} \to 1 \quad \text{as } n \to \infty \tag{9}$$

as can be seen by changing variables to $u = sn^{-\frac{1}{2}}$. (Check the details!) ∎

We return to $n! = \Gamma(n+1)$ for large n. We write

$$\begin{aligned} n! = \Gamma(n+1) &= \int_0^{\infty} e^{-t}t^n dt = n^{n+1} \int_0^{\infty} e^{-nu}u^n du = n^{n+1} \int_0^{\infty} e^{-n(u-\ln u)} du \\ &= n^{n+1}e^{-n} \int_{-1}^{\infty} e^{nG(t)}dt = n^{n+1}e^{-n} \left(\int_{-1}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} \right) e^{nG(t)}dt \end{aligned} \tag{10}$$

with $G(t) = -t + \ln(1+t)$. On $[0,\infty)$, the function G takes a maximum value of 0 at $t = 0$, $G''(0) = -1$ (check!). Note that $G'(s) = -1 + \frac{1}{1+s} = -\frac{s}{1+s}$. By the Taylor theorem

$$G(s) = G(0) + G'(c)s = -\frac{cs}{1+c}; \quad [c \in (0,s) \text{ if } s > 0 \text{ and } c \in (s,0) \text{ if } s < 0] \tag{11}$$

and thus

$$G(s) < -\beta|s|; \ \beta = \frac{|\delta|}{1+|\delta|} \quad \text{if } s \in (-1,-|\delta|) \cup (|\delta|,\infty) \tag{12}$$

Using the lemma with $f = 1, g = G$ in the middle integral in (10) and (12) to estimate the other two we get

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + h(n) + R);\ |R| < 2\frac{e^{-n\beta\delta}}{n\beta} < \frac{2}{n\beta} \tag{13}$$

and thus

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + h_1(n));\quad \lim_{n\to\infty} h_1(n) = 0 \tag{14}$$

A sharper formula is

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots + \frac{a_k}{n^k} + R_k\right);\ |R_k| \leqslant \frac{c_k}{n^{k+1}} \tag{15}$$

The series in parentheses may be thought of as the Taylor series of $n!/\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right]$ at $n = +\infty$.

We could get (15) from Laplace's method by using more terms in the expansion around zero, but this would involve long calculations, and there are much better ways (beyond the scope of this course though).

The approximation by a truncated expansion gets more and more accurate as n grow. But even for $n = 5$, using only the first 4 terms we get

$$5! \approx 120.000001$$