MORE REVIEW PROBLEMS

Exercise 1. Prove or disprove:

(a) "Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with the property that any *sub*sequence $\{x_{n_k}\}$ is convergent. Then $\{x_n\}$ itself is a convergent sequence"

(b) "Let f be defined on $(0,1]$ and assume that for any sequence $\{x_n\}_{n\in\mathbb{N}}$ s.t. $x_n > 0$ and $x_n \to 0$ as $n \to \infty$ the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is also convergent (the limit might depend on $\{x_n\}_{n\in\mathbb{N}}$). Then $\lim_{x\to 0^+} f(x)$ exists."

(c) "Assume f is Lipschitz continuous on $[0,1]$, that is, there exists an $\alpha > 0$ s.t. for all $x \in [0,1]$ we have

$$|f(x) - f(y)| \leqslant \alpha |x - y| \tag{1}$$

Then

$$\int_0^1 \frac{f(x) - f(0)}{x} dx \tag{2}$$

is well defined."

(c') "Assume f is Hölder continuous of exponent $r \in (0,1)$ on $[0,1]$, that is, there exists an $r \in (0,1)$ and an $\alpha > 0$ s.t. for all $x \in [0,1]$ we have

$$|f(x) - f(y)| \leqslant \alpha |x - y|^r \tag{3}$$

Then

$$\int_0^1 \frac{f(x) - f(0)}{x} dx \tag{4}$$

is well defined."

(d) "Assume f is continuous on $[0,1]$. Then

$$\int_0^1 \frac{f(x) - f(0)}{x} dx \tag{5}$$

is well defined."

(e) "Assume $f \in C^\infty(-a,a)$. Then,

$$\frac{f(x) - f(0)}{x} \in C^\infty(-a,a) \tag{6}$$

"