MORE REVIEW PROBLEMS

Exercise 1. Let

$$a_0 = 1/2 \text{ and for any } n \geqslant 1,\ a_n = a_{n-1} - a_{n-1}^2$$

(a) Show that $a_n \to 0$ as $n \to \infty$.
(b) Show that, in fact, a_n goes to zero like $1/n$. More precisely,

$$n \geqslant 1 \Rightarrow \frac{1}{n+4} \leqslant a_n \leqslant \frac{1}{n+2}$$

(c) Show that

$$n \geqslant 1 \Rightarrow a_n = \frac{1}{n} + \frac{b_n}{n^2} \text{ where the sequence } \{b_n\}_{n\in\mathbb{N}} \text{ is bounded}$$

Exercise 2. Assume f is continuous on $[0,1]$ and define the function g by

$$g(x) = \int_0^1 f(xt)dt$$

(a) Is the function g necessarily continuous on $(0,1)$?
(b) Is the function g necessarily differentiable on $(0,1)$?
(c) Does g necessarily have a right derivative at 0?

Exercise 3. Let

$$f(t) = \frac{t + e^{-t} - 1}{t^{5/2}}; \quad g(t) = \frac{1}{t^{1/2}(1+t)}; \quad h(t) = \frac{f(t)}{g(t)}$$

(a) Prove that $h(t)$ has a limit as $t \to 0$ and thus h extends to a continuous function on $[0, \infty)$.
(b) Prove that $\sup_{x\in(0,\infty)} |h(x)| < \infty$
(c) More precisely, show that $\forall t \in \mathbb{R}^+, h(t) \in (\frac{1}{2}, 1)$

Exercise 4. Recall that

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$$

Is the improper integral

$$\int_0^\infty \frac{1 - t - e^{-t}}{t^{5/2}} dt$$

well defined? If it is, calculate its value. You can use the previous exercise (though it is not necessary).

Exercise 5. Let P be the polynomial given by

$$P(x) = x - 5x^2 + 8x^3 - 4x^4$$

Show that

$$x \in [0,1] \Rightarrow \; P(x) \in [0, \tfrac{1}{16}]$$

(Assume we don't know how to explicitly solve general cubic equations. There are simple solutions to this problem. Plotting the function may give you an idea.)

Exercise 6. Show that the function f given by

$$f(x) = \frac{x}{\sin x} \text{ for } x \neq 0 \;\text{ and } f(0) = 0$$

is in $C^\infty(\mathbb{R})$ and find its Taylor series at $x = 0$ with remainder of the form $C(x)x^4$ with C bounded for small x.

Exercise 7. For what values of $a > 0$ is the series

$$\sum_{n=1}^{\infty} \sin(n^{-a})$$

convergent?