FURTHER RESULTS

1. An alternative proof to Theorem 6, p. 561.

Assume $\sum_{k=0}^{\infty} a_k z^k$ converges for $|z| \in (0, R)$. We know that it then converges absolutely for $|z| \in (0, R)$.

Let $x = |z|, \varepsilon = |h|$. Let $G(y) = \sum_{k=0}^{\infty} |a_k| y^k$; then G is an absolutely and uniformly convergent series in $(-R, R)$. By Theorem 6, p. 511, in calculus over $\mathbb{R}$, if $x, x + \varepsilon \in (-R, R)$ then

$$\lim_{\varepsilon \to 0} \frac{G(x+\varepsilon) - G(x)}{\varepsilon} - \sum_{k=1}^{\infty} |a_k| k x^{k-1} = 0 \tag{1}$$

Lemma 1.

$$\left| \frac{(z+h)^k - z^k}{h} - k z^{k-1} \right| \leqslant \frac{(x+\varepsilon)^k - x^k}{\varepsilon} - k x^{k-1} \tag{2}$$

Proof of Lemma 1. This can be proved in a number of ways, including induction. Perhaps the simplest is to note that for v, w real or complex, the bionomial formula gives

$$\frac{(v+w)^k - w^k}{w} - k v^{k-1} = \sum_{j=2}^{k} v^j w^{k-j-1}$$

and the rest follows since

$$\left| \sum_{j=2}^{k} v^j w^{k-j-1} \right| \leqslant \sum_{j=2}^{k} \binom{k}{j} |v|^j |w|^{k-j-1} = \frac{(|v| + |w|)^k - |v|^k}{|w|} - k|v|^{k-1}$$

∎

Proof of Theorem 6, p. 561. The proof now is an easy consequence of (1), Lemma 1 and the triangle inequality. ∎

2. Further results in complex analysis

We have shown that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges for $|z| \in (0, R)$, then $a_n = f^{(n)}(0)/n!$.

Proposition 2. *Assume $R > 0$ or $R = \infty$ and that $f(z) = \sum a_k z^k$ converges in $\mathbb{D}_R = \{z : |z| < R\}$ (where by convention $\mathbb{D}_\infty = \mathbb{C}$). Let $z_0 \in \mathbb{D}_R$ and $h \in \mathbb{D}_{R-|z_0|}$. Then*

$$f(z_0 + h) = f(z_0) + \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} h^k$$

Proof. Let $P_N(z) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} z^k$. You can check that $P_N^{(k)}(0) = f^{(k)}(0)$.Since P_N is a polynomial, we have, for any z_0

$$P_N(z_0 + h) = P_N(z_0) + \sum_{k=0}^{N} \frac{P_N^{(k)}(z_0)}{k!} h^k$$

Since $\sum_{k=0}^{\infty} a_k(z_0+h)^k$ converges for $h \in \mathbb{D}_{R-|z_0|}$, we have

$$\lim_{N\to\infty}\left(f(z_0+h)-\sum_{k=0}^{N}\frac{f^{(k)}(0)}{k!}(z_0+h)^k\right)=\lim_{N\to\infty}\left(f(z_0+h)-\sum_{k=0}^{N}\frac{f^{(k)}(z_0)}{k!}h^k\right)=0$$

$\blacksquare$

Note 3. *The radius of convergence of the power series in h may be larger than $R-|z_0|$. The series for $1/(1+z)$ centered at $1/2$ has radius of convergence $3/2$.*

Proposition 4. *Assume $f(z)=\sum_{k=0}^{\infty} f_k z^k$ and $g(z)=\sum_{k=0}^{\infty} g_k z^k$ converge in some $\mathbb{D}_R, R>0$ and coincide on a sequence of points $z_k \in \mathbb{D}_R$ s.t. $z_k \to 0$. Then $f=g$ in $\mathbb{D}_R$ and $f_k=g_k \forall k \geqslant 0$. ($z_k \to 0$ can be replaced by $z_k \to \alpha \in \mathbb{D}_R$: why?)*

Proof. Let $h_k=f_k-g_k$. With $h(z)=\sum_{j=0}^{\infty} h_j z^j$ we have $h(z_k)=0$, thus $\lim_{k\to\infty} h(z_k)=0=h_0$ the latter being true because h is continuous at zero. Thus $h_0=0$. Now take $h(z)/z=h_1+h_2 z+\ldots.$ The same argument applies, giving $h_1=0$. The general case is easy induction. $\blacksquare$

Corollary 5. *If $f=\sum_{k=0}^{\infty} c_k z^k$ and $f'=0$ on $\mathbb{D}_R, R>0$, then $f=c_0$ in n $\mathbb{D}_R$.*

Proof. Indeed, $f'=0$ on $[0,R)$. The rest follows from real calculus and the previous proposition. $\blacksquare$

Examples

Since

$$\sin z = z-\frac{z^3}{3!}+\ldots;\ \cos z = 1-\frac{z^2}{2}+\frac{z^4}{4!}+\ldots$$

converge for all $z\in\mathbb{C}$ and $\sin^2 x+\cos^2 x=1$ on $\mathbb{R}$, we have $\sin^2 z+\cos^2 z=1$ in $\mathbb{C}$. (Fill in the details). Another proof relies on the corollary and the fact that $(\sin^2+\cos^2)'=2\sin\cos-2\sin\cos=0$ in $\mathbb{C}$. Similarly, $e^{-z}=1/e^z$ in $\mathbb{C}$, $e^{z_1+z_2}=e^{z_1}e^{z_2}$ in $\mathbb{C}$ (fill in the details).

By Euler's identity, we have

$$e^{2k\pi i}=\cos(2k\pi)+i\sin(2k\pi)=1,\quad \forall k\in\mathbb{Z}$$

and thus by the above $e^{z+2k\pi i}=e^z$ for all $k\in\mathbb{Z}$ and thus e^z is periodic in $\mathbb{C}$ with period $2\pi i$.

Note 6. *You see that e^z is thus not one-to-one, though $(e^z)'=e^z$ is never zero.*

However, e^z is one-to-one in any strip of the form

$$\mathbb{S}_a=\{x+iy : x\in\mathbb{R}, y\in(a,a+2\pi)\}$$

and it is onto

$$\mathbb{C}\setminus\{z:\arg z=a\}$$

Proof.

$$e^{z_1}=e^{z_2}\Leftrightarrow e^{z_3}=1,\ z_3=z_2-z_1$$

If $z_3=x+iy$, then we have

$$e^x(\cos y+i\sin y)=1\Rightarrow x=0\&y=2k\pi$$

(why?). Since $|y_1-y_2|<2\pi$ we have $k=0$ and thus $z_1=z_2$ in $\mathbb{S}_a$. Now let $w\neq 0,\varphi=\arg w\in(a,a+2\pi)$. Then

$$e^z=w\Leftrightarrow e^x(\cos y+i\sin y)=|w|(cos\varphi+i\varphi)\Rightarrow x=\ln|w|\ \&y=\varphi$$

(why?) $\blacksquare$

Proposition 7. e^z *is invertible from* $\mathbb{S}_a$ *to* $\mathbb{C} \setminus \{z : z = 0$ *or* $\arg z = a\}$.

Proof. This follows from the argument above. $\blacksquare$

Note that the inverse of exp *depends on the choice of* a.

If $a = -\pi$ *we say that the corresponding inverse* ln *is the* principal branch of the log. *It is defined in* $\mathbb{C} \setminus (-\infty, 0]$. *Another relatively common branch is that with* $a = 0$.

Proposition 8. *For the* **natural branch of the log**,

$$\ln(\pm i) = \pm \pi i/2$$

Proof. Check that $\exp(\pm \pi i/2) = \pm i$.

Proposition 9. *Assume* $\arg z_1 = \varphi_1 \& \arg z_2 = \varphi_2 \in (-\pi, \pi)$ *and* $\varphi_1 + \varphi_2 \in (-\pi, \pi)$. *Then, for the natural branch of the log we have*

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

*(***Note that the restriction** $\varphi_1 + \varphi_2 \in (-\pi, \pi)$ **is needed. Otherwise we may overshoot the strip.** $\ln e^{3\pi i/4} + \ln e^{3\pi i/4} = 3\pi i/4 + 3\pi i/4 = 3\pi i/2 \neq -\pi i/2 = \ln e^{3\pi i/2} = \ln(-i)$**.**

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