# (A VERSION OF) THE HEINE-BOREL THEOREM IN $\mathbb{R}$ AND UNIFORM CONTINUITY

**Theorem 1.** Let $a < b$ be real numbers and $\mathcal{O}$ be a set of open intervals. If $[a,b]$ is contained in the union of the intervals in $\mathcal{O}$, then there exists $n \in \mathbb{N}$ and $O_1, ..., O_n \in \mathcal{O}$ s.t. $[a,b] \subset \cup_{i=1}^n O_i := O_1 \cup \cdots \cup O_n$. ("If there is an open interval covering, then there is a finite subcovering")

*Proof.* Let

$$S = \{x \in [a,b] : \exists n \in \mathbb{N}, O_1, ..., O_n \in \mathcal{O} \text{ s.t } [a,x] \subset \cup_{i=1}^n O_i\}$$

$S$ is nonempty, since $a$ must be contained in at least one $O \in \mathcal{O}$. $S$ is bounded above by $b$, and thus it has a least upper bound $\alpha$. We show that $\alpha = b$ and that $b \in S$, completing the proof. Since $\alpha \in [a,b]$, there is an interval $(c,d) \in \mathcal{O}$ s.t. $\alpha \in (c,d)$. Then for some $\varepsilon > 0$ s.t. $(\alpha - \varepsilon, \alpha + \varepsilon) \subset (c,d)$ (find one!). Since $\alpha - \varepsilon/2 < \alpha$ we have $\alpha - \varepsilon/2 \in S$, thus there is an $N$ and $O_i, i = 1...N$ s.t. $[a, \alpha - \varepsilon/2] \subset \cup_{i=1}^N O_i$. But then, if we choose $O_{N+1}$ to be $(c,d)$, then $[a, \alpha + \varepsilon/2] \subset \cup_{i=1}^{N+1} O_i$. ∎

**Definition 2.** A function is uniformly continuous on an interval if for any $\varepsilon$ there is a $\delta$ s.t. *for all $x, y$ s.t.* $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. The key property here is that $\delta$ does not depend on $x$ or $y$.

**Theorem 3.** *Let $a, b \in \mathbb{R}$ and $f$ be continuous on $[a,b]$. Then $f$ is uniformly continuous on $[a,b]$.*

*Proof.* For each $x$ there is a $\delta(x)$ s.t. any $y$ in $(x - 2\delta(x), x + 2\delta(x))$ satisfies $|f(y) - f(x)| < \varepsilon/2$. But the intervals $(x - \delta(x), x + \delta(x))$ cover $[a,b]$. Thus there is a finite set $S$ of intervals $\{(x_k - \delta(x_k), x_k + \delta(x_k)), k = 1, ..., n\}$ which cover $[a,b]$. Let $\delta = \min(\delta_1, ..., \delta_n)$. Let $x, y \in [a,b]$ with $|x - y| < \delta$. Since $S$ covers $[a,b]$, then $|x - x_k| < \delta_k$ for some $k$. Then by the triangle inequality, $|y - x_k| = |y - x + x - x_k| < 2\delta_k$. Still by the triangle inequality, $|f(x) - f(y)| = |f(x) - f(x_k) + f(x_k) - f(y)| < \varepsilon$. ∎