MIDTERM 2: SOME PRACTICE PROBLEMS

Exercise 1. Find $\sqrt{101}$ with two decimal places without using a calculator. Justify your result.

Exercise 2. Assume f is continuous on $[0,1]$ and define the function g by

$$g(x) = \int_0^1 f(xt)dt$$

(a) Is the function g necessarily continuous on $(0,1)$?
(b) Is the function g necessarily differentiable on $(0,1)$?
(c) [Bonus] Does the limit $\lim_{x\to 0^+} x^{-1}[g(x) - g(0)]$ (the "right right derivative of g " at 0) exist ?

Exercise 3. Let

$$f(t) = \frac{t + e^{-t} - 1}{t^{5/2}}; \quad g(t) = \frac{1}{t^{1/2}(1+t)}; \quad h(t) = \frac{f(t)}{g(t)}$$

(a) Prove that $h(t)$ has a limit as $t \to 0$ and thus there exists a c such that the function H defined as $H(t) = h(t)$ for $t \neq 0$ and $H(0) = c$ is continuous on $[0,\infty)$ (this is expressed as "h extends by continuity to $[0,\infty)$").
(b) Prove that $\sup_{x\in(0,\infty)} |h(x)| < \infty$
(c)[Bonus] More precisely, show that $\forall t \in \mathbb{R}^+, h(t) \in (\frac{1}{2}, 1)$

Exercise 4. It is known that the integral $f(x) = \int_0^x e^{s^2} ds$ *cannot* be expressed in terms of elementary functions.

Decide which of the following functions can and which ones cannot be expressed in terms of elementary functions.
(a) $\displaystyle\int_1^x e^{s^2} ds$
(b) $\displaystyle\int_1^x e^s s^{-1/2} ds$
(c) $\displaystyle\int_1^x s e^{s^2} ds$
(d) $\int_1^x (1+s)e^{s^2} ds$

Exercise 5. Let P be the polynomial given by

$$P(x) = x - 5x^2 + 8x^3 - 4x^4$$

Show that

$$x \in [0,1] \Rightarrow \; P(x) \in [0, \tfrac{1}{16}]$$

(Assume we don't know how to explicitly solve general cubic equations. There are simple solutions to this problem. Plotting the function may give you an idea.)