November 5, 2014

EIGENVALUES AND EIGENVECTORS

BASED ON NOTES BY RODICA D. COSTIN

Contents

1. Motivation

1.1. Diagonal matrices. Perhaps the simplest type of linear transformations are those whose matrix is diagonal (in some basis). Consider for example the matrices

$$M = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \ N = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \tag{1}$$

It can be easily checked that

$$\alpha M + \beta N = \begin{bmatrix} \alpha a_1 + \beta b_1 & 0 \\ 0 & \alpha a_2 + \beta b_2 \end{bmatrix}$$

and

$$M^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 \\ 0 & \frac{1}{a_2} \end{bmatrix}, \ M^k = \begin{bmatrix} a_1^k & 0 \\ 0 & a_2^k \end{bmatrix}, \ MN = \begin{bmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{bmatrix}$$

Diagonal matrices behave like the bunch of numbers on their diagonal!

The linear transformation consisting of multiplication by the matrix M in (1) dilates a_1 times vectors in the direction of $\mathbf{e}_1$ and a_2 times vectors in the direction of $\mathbf{e}_2$.

In this chapter we will see that most linear transformations do have diagonal matrices in a special basis, whose elements are called the eigenvectors of the transformation. We will learn how to find these bases. Along the special directions of the eigenvectors the transformation just dialation by a factor, called eigenvalue.

1.2. Example: solving linear differential equations. Consider the simple equation

$$\frac{du}{dt} = \lambda u$$

which is linear, homogeneous, with constant coefficients, and unknown function $u(t) \in \mathbb{R}$ (or in $\mathbb{C}$). Its general solution is, as it is well known, $u(t) = Ce^{\lambda t}$.

Consider now a similar equation, but where the unknown $\mathbf{u}(t)$ is a vector valued function:

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u}, \quad \mathbf{u}(t) \in \mathbb{R}^n, \quad M \text{ is an } n \times n \text{ constant matrix} \tag{2}$$

Inspired by the one dimensional case we look for exponential solutions. Substituting in (2) $\mathbf{u}(t) = e^{\lambda t}\mathbf{v}$ (where λ is a number and $\mathbf{v}$ is a constant vector, both be determined) and dividing by $e^{\lambda t}$, we obtain that the scalar λ and the vector $\mathbf{v}$ must satisfy

$$\lambda \mathbf{v} = M\mathbf{v} \tag{3}$$

or

$$(M - \lambda I)\mathbf{v} = \mathbf{0} \tag{4}$$

If the null space of the matrix $M - \lambda I$ is zero, then the only solution of (4) is $\mathbf{v} = \mathbf{0}$ which gives the (trivial!) solution $\mathbf{u}(t) \equiv \mathbf{0}$.

If however, we can find special values of λ for which $\mathcal{N}(M - \lambda I)$ is not null, then we found a nontrivial solution of (2). Such values of λ are called **eigenvalues** of the matrix M, and vectors $\mathbf{v} \in \mathcal{N}(M - \lambda I)$, $\mathbf{v} \neq \mathbf{0}$, are called **eigenvectors corresponding to the eigenvalue** λ.

Of course, the necessary and sufficient condition for $\mathcal{N}(M - \lambda I) \neq \{\mathbf{0}\}$ is that

$$\det(M - \lambda I) = 0 \tag{5}$$

Example. Let us calculate the exponential solutions for

$$M = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} \tag{6}$$

Looking for eigenvalues of M we solve equation (5), which for (6) is

$$\det\left(\begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} -1-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda)$$

with solutions $\lambda_1 = -1$ and $\lambda_2 = 2$.

We next determine an eigenvector corresponding to the eigenvalue $\lambda = \lambda_1 = -1$: looking for a nozero vector $\mathbf{v}_1$ such that $(M - \lambda_1 I)\mathbf{v}_1 = 0$ we solve

$$\begin{bmatrix} 0 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

giving $x_2 = 0$ and x_1 arbitrary; therefore the first eigenvector is any scalar multiple of $\mathbf{v}_1 = (1, 0)^T$.

Similarly, for the eigenvalue $\lambda = \lambda_2 = 2$ we solve $(M - \lambda_2 I)\mathbf{v}_2 = 0$:

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $y_2 = -y_1$ and y_1 arbitrary, and the second eigenvector is any scalar multiple of $\mathbf{v}_2 = (1, -1)^T$.

We found two particular solutions of (2), (6), namely $\mathbf{u}_1(t) = e^{-t}(1, 0)^T$ and $\mathbf{u}_2(t) = e^{2t}(1, -1)^T$. These are functions belonging to the null space of the linear operator $L\mathbf{u} = \frac{d\mathbf{u}}{dx} - M\mathbf{u}$, therefore any linear combination of these two solutions also belongs to the null space: any $C_1\mathbf{u}_1(t) + C_2\mathbf{u}_2(t)$ is also a solution, for and constants C_1, C_2.

A bit later we will show that these are all the solutions.

2. Eigenvalues and eigenvectors: definition and calculation

2.1. **Definitions.**

Denote the set of $n \times n$ (square) matrices with entries in F ($= \mathbb{R}$ or $\mathbb{C}$)

$$\mathcal{M}_n(F) = \{M \mid M = [M_{ij}]_{i,j=1,\dots n},\ M_{ij} \in F\}$$

A matrix $M \in \mathcal{M}_n(F)$ defines an endomorphism the vector space F^n(over the scalars F) by usual multiplication $\mathbf{x} \mapsto M\mathbf{x}$.

Note that a matrix with real entries can also act on $\mathbb{C}^n$, since for any $\mathbf{x} \in \mathbb{C}^n$ also $M\mathbf{x} \in \mathbb{C}^n$. But a matrix with complex non real entries cannot act on $\mathbb{R}^n$, since for $\mathbf{x} \in \mathbb{R}^n$ the image $M\mathbf{x}$ may not belong to $\mathbb{R}^n$ (while certainly $M\mathbf{x} \in \mathbb{C}^n$).

Definition 1. *Let M be an $n \times n$ matrix acting on the vector space $V = F^n$.*

A scalar $\lambda \in F$ is an **eigenvalue** *of M if for some* nonzero *vector $\mathbf{v} \in V$, $\mathbf{v} \neq \mathbf{0}$ we have*

$$M\mathbf{v} = \lambda\mathbf{v} \tag{7}$$

The vector $\mathbf{v}$ is called **eigenvector corresponding to the eigenvalue** λ*.*

Of course, if $\mathbf{v}$ is an eigenvector corresponding to λ, then so is any scalar multiple $c\mathbf{v}$ (for $c \neq 0$).

2.2. **The characteristic equation.** Equation (7) can be rewritten as $M\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$, or $(M - \lambda I)\mathbf{v} = \mathbf{0}$, which means that the nonzero vector $\mathbf{v}$ belongs to the null space of the matrix $M - \lambda I$, and in particular this matrix is not invertible. Using the theory of matrices, we know that this is equivalent to

$$\det(M - \lambda I) = 0$$

The determinant has the form

$$\det(M - \lambda I) = \begin{vmatrix} M_{11} - \lambda & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} - \lambda & \dots & M_{2n} \\ \vdots & \vdots & & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} - \lambda \end{vmatrix}$$

This is a polynomial in λ, having degree n. To understand why this is the case, consider first $n = 2$ and $n = 3$.

For $n = 2$ the characteristic polynomial is

$$\begin{vmatrix} M_{11} - \lambda & M_{12} \\ M_{21} & M_{22} - \lambda \end{vmatrix} = (M_{11} - \lambda)\,(M_{22} - \lambda) - M_{12}\,M_{21}$$

$$= \lambda^2 - (M_{11} + M_{22})\lambda + (M_{11}M_{22} - M_{12}\,M_{21})$$

which is a quadratic polynomial in λ; the dominant coefficient is 1.

For $n = 3$ the characteristic polynomial is

$$\begin{vmatrix} M_{11} - \lambda & M_{12} & M_{13} \\ M_{21} & M_{22} - \lambda & M_{23} \\ M_{13} & M_{23} & M_{33} - \lambda \end{vmatrix}$$

and expanding along say, row 1,

$$= (-1)^{1+1}\,(M_{11}-\lambda) \begin{vmatrix} M_{22} - \lambda & M_{23} \\ M_{23} & M_{33} - \lambda \end{vmatrix} + (-1)^{1+2}\,M_{12} \begin{vmatrix} M_{21} & M_{23} \\ M_{13} & M_{33} - \lambda \end{vmatrix}$$

$$+ (-1)^{1+3}\,M_{13} \begin{vmatrix} M_{21} & M_{22} - \lambda \\ M_{13} & M_{23} \end{vmatrix}$$

$$= -\lambda^3 + (M_{11} + M_{22} + M_{33})\lambda^2 + \dots$$

which is a cubic polynomial in λ; the dominant coefficient is -1.

It is easy to show by induction that $\det(M - \lambda I)$ is polynomial in λ, having degree n, and that the coefficient of λ^n is $(-1)^n$.

Definition 2. *The polynomial* $\det(M - \lambda I)$ *is called the* **characteristic polynomial** *of the matrix M, and the equation* $\det(M - \lambda I) = 0$ *is called the* **characteristic equation** *of M.*

Remark. Some authors refer to the characteristic polynomial as $\det(\lambda I - M)$; the two polynomial are either equal or a -1 multiple of each other, since $\det(\lambda I - M) = (-1)^n \det(M - \lambda I)$.

2.3. Geometric interpretation of eigenvalues and eigenvectors.

Let M be an $n \times n$ matrix, and $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = M\mathbf{x}$ be the corresponding linear transformation.

If $\mathbf{v}$ is an eigenvector corresponding to an eigenvalue λ of M: $M\mathbf{v} = \lambda\mathbf{v}$, then T expands or contracts $\mathbf{v}$ (and any vector in its direction) λ times (and it does not change its direction!).

If the eigenvalue/vector are not real, a similar fact is true, only that multiplication by a complex (not real) scalar cannot be easily called an expansion or a contraction (there is no ordering in complex numbers), see the example of rotations, §2.13.1.

The special directions of the eigenvectors are called **principal axes** of the linear transformation (or of the matrix).

2.4. Digression: the fundamental theorem of algebra.

2.4.1. *Polynomials of degree two: roots and factorization.*

Consider polynomials of degree two, with real coefficients: $p(x) = ax^2 + bx + c$. It is well known that $p(x)$ has real solutions $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if $b^2 - 4ac \geq 0$ (where $x_1 = x_2$ when $b^2 - 4ac = 0$), and $p(x)$ has no real solutions if $b^2 - 4ac < 0$.

When the solutions are real, then the polynomial factors as

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

In particular, if $x_1 = x_2$ then $p(x) = a(x - x_1)^2$ and x_1 is called a *double root*; x_1 is said to have *multiplicity two*. It is convenient to say that also in this case $p(x)$ has two roots.

If, on the other hand, if $p(x)$ has no real roots, then p cannot be factored within real numbers, and it is called *irreducible* (over the real numbers).

2.4.2. *Complex numbers and factorization of polynomials of degree two.*

If $p(x) = ax^2 + bx + c$ is irreducible this means that $b^2 - 4ac < 0$ and we cannot take the square root of this quantity in order to calculate the two roots of

$p(x)$. However, writing $b^2 - 4ac = (-1)(-b^2 + 4ac)$ and introducing the symbol i for $\sqrt{-1}$ we can write the zeroes of $p(x)$ as

$$x_{1,2} = \frac{-b \pm i\sqrt{-b^2 + 4ac}}{2a} = \frac{-b}{2a} \pm i\frac{\sqrt{-b^2 + 4ac}}{2a} \in \mathbb{R} + i\mathbb{R} = \mathbb{C}$$

Considering the two roots x_1, x_2 complex, we can still factor $ax^2+bx+c = a(x - x_1)(x - x_2)$, only now the factors have complex coefficients. Within complex numbers every polynomial of degree two is reducible!

Note that the two roots of a quadratic polynomial with real coefficients are complex conjugate: if $a, b, c \in \mathbb{R}$ and $x_{1,2} \notin \mathbb{R}$ then $x_2 = \overline{x_1}$.

2.4.3. *The fundamental theorem of algebra.* It is absolutely remarkable that *any polynomial* can be completely factored using complex numbers:

Theorem 3. The fundamental theorem of algebra

Any polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ *with coefficients* $a_j \in \mathbb{C}$ *can be factored*

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = a_n(x - x_1)(x - x_2)\ldots(x - x_n) \tag{8}$$

for a unique set of complex numbers $x_1, x_2, \ldots, x_n$ *(not necessarily distinct), called the roots of the polynomial* $p(x)$.

Remark. With probability one, the zeroes $x_1, \ldots, x_n$ of polynomials $p(x)$ are distinct. Indeed, if x_1 is a double root (or has higher multiplicity) then both relations $p(x_1) = 0$ and $p'(x_1) = 0$ must hold. This means that there is a relation between the coefficients $a_0, \ldots a_n$ of $p(x)$ (the multiplet $(a_0, \ldots a_n)$ belongs to an n dimensional surface in $\mathbb{C}^{n+1}$).

2.4.4. *Factorization within real numbers.* If we want to restrict ourselves only within real numbers then we can factor any polynomial into factors of degree one or two:

Theorem 4. Factorization within real numbers

Any polynomial of degree n with real coefficients can be factored into factors of degree one or two with real coefficients.

Theorem 4 is an easy consequence of the (deep) Theorem 3. Indeed, first, factor the polynomial in complex numbers (8). Then note that the zeroes $x_1, x_2, \ldots, x_n$ come in pairs of complex conjugate numbers, since if z satisfies $p(z) = 0$, then also its complex conjugate $\overline{z}$ satisfies $p(\overline{z}) = 0$. Then each pair of factors $(x - z)(x - \overline{z})$ must be replaced in (8) by its expanded value:

$$(x - z)(x - \overline{z}) = x^2 - (z + \overline{z})\,x + |z|^2$$

which is an irreducible polynomial of degree 2, with real coefficients. $\square$

2.5. **Diagonal matrices.** Let D be a diagonal matrix:

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} \tag{9}$$

To find its eigenvalues, calculate

$$\det(D-\lambda I) = \begin{vmatrix} d_1-\lambda & 0 & \dots & 0 \\ 0 & d_2-\lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n-\lambda \end{vmatrix} = (d_1-\lambda_1)(d_2-\lambda_2)\dots(d_n-\lambda_n)$$

The eigenvalues are precisely the diagonal elements, and the eigenvector corresponding to d_j is $\mathbf{e}_j$ (as it is easy to check). The principal axes of diagonal matrices the coordinate axes. Vectors in the direction of one of these axes preserve their direction and are stretched or compressed: if $\mathbf{x} = c\mathbf{e}_k$ then $D\mathbf{x} = d_k\mathbf{x}$.

Diagonal matrices are easy to work with: what was noted for the 2×2 matrices in §1 is true in general, and one can easily check that any power D^k is the diagonal matrix having d_j^k on the diagonal.

If $p(x)$ is a polynomial

$$p(t) = a_k t^k + a_{k-1}t^{k-1} + \dots + a_1 t + a_0$$

then for any square matrix M one can define $p(M)$ as

$$p(M) = a_k M^k + a_{k-1}M^{k-1} + \dots + a_1 M + a_0 I \tag{10}$$

If D is a diagonal matrix (9) then $p(D)$ is the diagonal matrix having $p(d_j)$ on the diagonal. (Check!)

Diagonal matrices can be viewed as the collection of their eigenvalues!

Exercise. Show that the eigenvalues of an upper (or lower) triangular matrix are the elements on the diagonal.

2.6. **Similar matrices have the same eigenvalues.** It is very easy to work with diagonal matrices and a natural question arises: which linear transformations have a diagonal matrix in a well chosen basis? This is the main topic we will be exploring for many sections to come.

Recall that if the matrix M represents the linear transformation $L : V \to V$ in some basis $\mathcal{B}_V$ of V, and the matrix $\tilde{M}$ represents the same linear transformation L, only in a different basis $\tilde{\mathcal{B}}_V$, then the two matrices are similar: $\tilde{M} = S^{-1}MS$ (where S the the matrix of change of basis).

Eigenvalues are associated to the linear transformation (rather than its matrix representation):

Proposition 5. *Two similar matrices have the same eigenvalues: if $M, \tilde{M}, S$ are $n \times n$ matrices, and $\tilde{M} = S^{-1}MS$ then the eigenvalues of M and of $\tilde{M}$ are the same.*

This is very easy to check, since

$$\det(\tilde{M} - \lambda I) = \det(S^{-1}MS - \lambda I) = \det\left[S^{-1}(M - \lambda I)S\right]$$

$$= \det S^{-1}\, \det(M - \lambda I)\, \det S = \det(M - \lambda I)$$

so M and $\tilde{M}$ have the same characteristic equation. □

2.7. **Projections.** Recall that projections do satisfy $P^2 = P$ (we saw this for projections in dimension two, and we will prove it in general).

Proposition 6. *Let P be a square matrix satisfying $P^2 = P$. Then the eigenvalues of P can only be 0 or 1.*

Proof. Let λ be an eigenvalue; this means that there is a nonzero vector $\mathbf{v}$ so that $P\mathbf{v} = \lambda\mathbf{v}$. Applying P to both sides of the equality we obtain $P^2\mathbf{v} = P(\lambda\mathbf{v}) = \lambda P\mathbf{v} = \lambda^2\mathbf{v}$. Using the fact that $P^2\mathbf{v} = P\mathbf{v} = \lambda\mathbf{v}$ it follows that $\lambda\mathbf{v} = \lambda^2\mathbf{v}$ so $(\lambda - \lambda^2)\mathbf{v} = \mathbf{0}$ and since $\mathbf{v} \neq \mathbf{0}$ then $\lambda - \lambda^2 = 0$ so $\lambda \in \{0, 1\}$. □

Example. Consider the projection of $\mathbb{R}^3$ onto the x_1x_2 plane. Its matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is diagonal, with eigenvalues $1, 1, 0$.

2.8. **Trace, determinant and eigenvalues.**

Definition 7. *Let M be an $n \times n$ matrix, $M = [M_{ij}]_{i,j=1,\ldots,n}$. The* **trace** *of M is the sum of its elements on the principal diagonal:*

$$\operatorname{Tr} M = \sum_{j=1}^{n} M_{jj}$$

The following theorem shows that what we noticed in §2.2 for $n = 2$ is true for any n:

Theorem 8. *Let M be an $n \times n$ matrix, an let $\lambda_1, \ldots, \lambda_n$ be its n eigenvalues (complex, not necessarily distinct). Then*

$$\det M = \lambda_1 \lambda_2 \ldots \lambda_n \tag{11}$$

and

$$\operatorname{Tr} M = \lambda_1 + \lambda_2 + \ldots + \lambda_n \tag{12}$$

In particular, the traces of similar matrices are equal, and so are their determinants.

Proof.

We expand the determinant $\det(M-\lambda I)$ using minors and cofactors keeping track of the coefficient of λ^{n-1}. As seen on the examples in §2.2, only the first term in the expansion contains the power λ^{n-1}, and continuing to expand to lower and lower dimensional determinants, we see that the only term containing λ^{n-1} is

(13)
$$p(\lambda) = \det(M-\lambda I) = (M_{11}-\lambda)(M_{22}-\lambda)\dots(M_{nn}-\lambda) + \text{ lower powers of } \lambda$$
$$= (-1)^n\lambda^n - (-1)^n(M_{11}+M_{22}+\dots+M_{nn})\lambda^{n-1} + \text{ lower powers of } \lambda$$

On the other hand, this polynomial has roots $\lambda_1, ..., \lambda_n$ so it must be of the form

$$A(\lambda-\lambda_1)\cdots(\lambda-\lambda_n) = A[\lambda^n+(\lambda_1+\cdots+\lambda_n)\lambda^{n-1}+\cdots+(-1)^n\lambda_1\cdots\lambda_n \tag{14}$$

If we expand (14) and compare with (13), we get that $A = (-1)^n$. On the other hand, from (13) we have $p(0) =$det(M) by definition as seen from (13)

$$p(0) = (-1)^n\lambda_1\cdots\lambda_n \Rightarrow \det(M) = \lambda_1\cdots\lambda_n \tag{15}$$

The proof of the trace is very similar. □

2.9. **The eigenvalue zero.** As an immediate consequence of Theorem 8, we can recognize invertible matrices by looking at their eigenvalues:

Corollary 9. *A matrix M is invertible if and only if all its eigenvalues are nonzero.*

Note that a matrix M has an eigenvalue equal to zero if and only if its null space $\mathcal{N}(M)$ is nontrivial. Moreover, the matrix M has $\dim\mathcal{N}(M)$ eigenvectors linearly independent which correspond to the eigenvalue zero.

2.10. **Eigenvectors corresponding to different eigenvalues are independent.**

Theorem 10. *Let M be an $n\times n$ matrix.*

Let $\lambda_1, \dots \lambda_k$ a set of distinct *eigenvalues of M and $\mathbf{v}_1, \dots, \mathbf{v}_k$ be corresponding eigenvectors.*

Then the set $\mathbf{v}_1, \dots, \mathbf{v}_k$ is linearly independent.

In particular, if M has entries in $F = \mathbb{R}$ or $\mathbb{C}$, and all the eigenvalues of M are in F and are distinct, then the set of corresponding eigenvectors form a basis for F^n.

Proof.

Assume, to obtain a contradiction, that the eigenvectors are linearly dependent: there are $c_1, \dots, c_k \in F$ not all zero such that

$$c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{0} \tag{16}$$

Step I. We can assume that all c_j are not zero, otherwise we just remove those $\mathbf{v}_j$ from (16) and we have a similar equation with a smaller k.

If after this procedure we are left with $k = 1$, then this implies $c_1\mathbf{v}_1 = \mathbf{0}$ which contradicts the fact that not all c_j are zero or the fact that $\mathbf{v}_1 \neq \mathbf{0}$.

Otherwise, for $k \geq 2$ we continue as follows.

Step II. Then we can solve (16) for $\mathbf{v}_k$:

$$\mathbf{v}_k = c_1'\mathbf{v}_1 + \ldots + c_{k-1}'\mathbf{v}_{k-1} \tag{17}$$

where $c_j' = -c_j/c_k$.

Applying M to both sides of (17) we obtain

$$\lambda_k\mathbf{v}_k = c_1'\lambda_1\mathbf{v}_1 + \ldots + c_{k-1}'\lambda_{k-1}\mathbf{v}_{k-1} \tag{18}$$

Multiplying (17) by λ_k and subtracting from (18) we obtain

$$\mathbf{0} = c_1'(\lambda_1 - \lambda_k)\mathbf{v}_1 + \ldots + c_{k-1}'(\lambda_{k-1} - \lambda_k)\mathbf{v}_{k-1} \tag{19}$$

Note that all $c_j'(\lambda_j - \lambda_k)$ are non-zero (since all c_1' are non-zero, and $\lambda_j \neq \lambda_k$).

If k=2, then this implies $\mathbf{v}_1 = \mathbf{0}$ which is a contradiction.

If $k > 2$ we go to *Step I.* with a lower k.

The procedure decreases k, therefore it must end, and we have a contradiction. □

2.11. **Diagonalization of matrices with linearly independent eigenvectors.** Suppose that the M be an $n \times n$ matrix has n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Note that, by Theorem 10, this is the case if we work in $F = \mathbb{C}$ and all the eigenvalues are distinct (recall that this happens with probability one). Also this is the case if we work in $F = \mathbb{R}$ and all the eigenvalues are real and distinct.

Let S be the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$$

which is invertible, since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Since $M\mathbf{v}_j = \lambda_j\mathbf{v}_j$ then

$$M[\mathbf{v}_1, \ldots, \mathbf{v}_n] = [\lambda_1\mathbf{v}_1, \ldots, \lambda_n\mathbf{v}_n] \tag{20}$$

The left side of (20) equals MS. To identify the matrix on the right side of (20) note that since $S\mathbf{e}_j = \mathbf{v}_j$ then $S(\lambda_j\mathbf{e}_j) = \lambda_j\mathbf{v}_j$ so

$$[\lambda_1\mathbf{v}_1, \ldots, \lambda_n\mathbf{v}_n] = S[\lambda_1\mathbf{e}_1, \ldots, \lambda_n\mathbf{e}_n] = S\Lambda, \quad \text{where } \Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Relation (20) is therefore

$$MS = S\Lambda, \quad \text{or} \quad S^{-1}MS = \Lambda = \text{diagonal}$$

Note that the matrix S which diagonalizes a matrix **is not unique**. For example, we can replace any eigenvector by a scalar multiple of it. Also, we can use different orders for the eigenvectors (this will result on a diagonal matrix with the same values on the diagonal, but in different positions).

Example 1. Consider the matrix (6) for which we found the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$ and the corresponding eigenvectors $\mathbf{v}_1 = (1,0)^T$. $\mathbf{v}_2 = (1,-1)^T$.

$$S = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

we have

$$S^{-1}MS = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Not all matrices are diagonalizable, certainly those with distinct eigenvalues are, and some matrices with multiple eigenvalues.

Example 2. The matrix

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{21}$$

has eigenvalues $\lambda_1 = \lambda_2 = 0$ but only one (up to a scalar multiple) eigenvector $\mathbf{v}_1 = \mathbf{e}_1$.

Multiple eigenvalues are not guaranteed to have an equal number of independent eigenvectors!

N is not diagonalizable. Indeed, assume the contrary, to arrive at a contradiction. Suppose there exists an invertible matrix S so that $S^{-1}NS = \Lambda$ where Λ is diagonal, hence it has the eigenvalues of N on its diagonal, and therefore it is the zero matrix: $S^{-1}NS = 0$, which multiplied by S to the left and S^{-1} to the right gives $N = 0$, which is a contradiction.

Some matrices with multiple eigenvalues may still be diagonalized; next section explores when this is the case.

2.12. **Eigenspaces.** Consider an $n \times n$ matrix M with entries in F, with eigenvalues $\lambda_1, \ldots, \lambda_n$ in F.

Definition 11. *The set*

$$V_{\lambda_j} = \{\mathbf{x} \in F^n \mid M\mathbf{x} = \lambda_j \mathbf{x}\}$$

is called **the eigenspace** *of M associated to the eigenvalue λ_j.*

Exercise. Show that V_{λ_j} is the null space of the transformation $M - \lambda I$ and that V_{λ_j} is a subspace of F^n.

Note that all the *nonzero* vectors in V_{λ_j} are eigenvectors of M corresponding to the eigenvalues λ_j.

Definition 12. *A subspace V is called an* **invariant subspace** *for M if $M(V) \subset V$ (which means that if $\mathbf{x} \in V$ then $M\mathbf{x} \in V$).*

The following Remark gathers main features of eigenspaces; their proof is left to the reader.

Remark. 1. Each V_{λ_j} is an invariant subspace for M.

2. $V_{\lambda_j} \cap V_{\lambda_l} = \{\mathbf{0}\}$ if $\lambda_j \neq \lambda_l$.

3. Denote by $\lambda_1, \ldots, \lambda_k$ the distinct eigenvalues of M and by r_j the multiplicity of the eigenvalue λ_j, for each $j = 1, \ldots, k$; it is clear that

$$\det(M - \lambda I) = \prod_{j=1}^{k} (\lambda_j - \lambda)^{r_j} \text{ and } r_1 + \ldots + r_k = n$$

Then

$$\dim V_{\lambda_j} \leq r_j$$

4. M is diagonalizable in F^n if and only if $\dim V_{\lambda_j} = r_j$ for all $j = 1, \ldots, k$ and then

$$V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_k} = F^n$$

Example. Consider the matrix

$$M := \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \tag{22}$$

Its characteristic polynomial is

$$\det(M - \lambda I) = -\lambda^3 + 3\,\lambda^2 - 4 = -\,(\lambda + 1)\,(\lambda - 2)^2$$

so $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = 2$ is a double eigenvalue. The eigenspace V_{λ_1} is one dimensional, spanned by an eigenvector, which, after a simple calculation turns out to be $\mathbf{v}_1 = (0, 1, 1)^T$. If the eigenspace V_{λ_2} is two-dimensional (which is not guaranteed) then the matrix M is diagonalizable. A simple calculation shows that there are two independent eigenvectors corresponding to the eigenvalue $\lambda_2 = 2$, for example $\mathbf{v}_2 = (1, 0, 1)^T$ and $\mathbf{v}_3 = (2, 1, 0)^T$ (the null space of $M - \lambda_2 I$ is two-dimensional). Let

$$S = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then

$$S^{-1} M S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.13. **Real matrices with complex eigenvalues; decomplexification.**

2.13.1. *Complex eigenvalues of real matrices.* For an $n \times n$ matrix with real entries, if we want to have n guaranteed eigenvalues, then we have to accept working in $\mathbb{C}^n$. Otherwise, if we want to restrict ourselves to working only with real vectors, then we have to accept that we may have fewer (real) eigenvalues, or perhaps none.

Complex eigenvalues of real matrices come in pairs: if λ is an eigenvalue of M, then so is its complex conjugate $\overline{\lambda}$ (since the characteristic equation has real coefficients). Also, if $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ, then $\overline{\mathbf{v}}$ is eigenvector corresponding to the eigenvalue $\overline{\lambda}$ (check!). The real and imaginary parts of $\mathbf{v}$ span a plane where the linear transformation acts by rotation, and a possible dilation. Simple examples are shown below.

Example 1: rotation in the xy-plane. Consider a rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{23}$$

To find its eigenvalues calculate

$$\det(R_\theta - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1$$

hence the solutions of the characteristic equations $\det(R_\theta - \lambda I) = 0$ are $\lambda_{1,2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$. It is easy to see that $\mathbf{v}_1 = (i, 1)^T$ is the eigenvector corresponding to $\lambda_1 = e^{i\theta}$ and $\mathbf{v}_2 = (-i, 1)^T$ is the eigenvector corresponding to $\lambda_2 = e^{-i\theta}$.

Example 2: complex eigenvalues in $\mathbb{R}^3$. Consider the matrix

$$M = \begin{bmatrix} 1 - \frac{1}{2}\sqrt{3} & -\frac{5}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & 1 + \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Its characteristic polynomial is

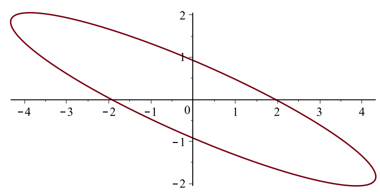
$$\det(M - \lambda I) = -\lambda^3 - 2\lambda^2 + 4\lambda - 16 = -(\lambda + 4)\left(\lambda^2 - 2\lambda + 4\right)$$

and its eigenvalues are: $\lambda_{1,2} = 1 \pm i\sqrt{3} = 2e^{\pm i\pi/3}$ and $\lambda_3 = -4$, and corresponding eigenvectors $\mathbf{v}_{1,2} = (-1 \pm 2i, 1, 0)^T$, and $\mathbf{v}_3 = \mathbf{e}_3$. The matrix $S = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ diagonalized the matrix: $S^{-1}MS$ is the diagonal matrix, having the eigenvalues on the diagonal, but all these are complex matrices.

To understand how the matrix acts on $\mathbb{R}^3$, we consider the real and imaginary parts of $\mathbf{v}_1$: let $\mathbf{x}_1 = \Re\mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) = (-1, 1, 0)^T$ and $\mathbf{y}_1 = \Im\mathbf{v}_1 = \frac{1}{2i}(\mathbf{v}_1 - \mathbf{v}_2) = (2, 0, 0)^T$. Since the eigenspaces are invariant under M, then so is $Sp(\mathbf{x}_1, \mathbf{y}_1)$, over the complex and even over the real numbers (since M has real elements). The span over the real numbers is the xy-plane, and it is invariant under M. The figure shows the image of the unit circle in the xy-plane under the matrix M: it is an ellipse.

Along the direction of the third eigenvector (the z-axis) the matrix multiples any $c\,\mathbf{e}_3$ by -4.

FIGURE 1. The image of the unit circle in the xy-plane.



In the basis $\mathbf{x}_1, \mathbf{y}_1, \mathbf{v}_3$ the matrix of the linear transformation has its simplest form: using $S_{\mathbb{R}} = [\mathbf{x}_1, \mathbf{y}_1, \mathbf{v}_3]$ we obtain the matrix of the transformation in this new basis as

$$S_{\mathbb{R}}^{-1} M S_{\mathbb{R}} = \begin{bmatrix} 1 & \sqrt{3} & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

and the upper 2×2 block represents the rotation and dilation $2R_{-\pi/3}$.

2.13.2. *Decomplexification.* Suppose the $n \times n$ matrix M has real elements, eigenvalues $\lambda_1, \ldots, \lambda_n$ and n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then M is diagonalizable: if $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ then $S^{-1}MS = \Lambda$ where Λ is a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on its diagonal.

Suppose that some eigenvalues are not real. Then the matrices S and Λ are not real either, and the diagonalization of M must be done in $\mathbb{C}^n$.

Suppose that we want to work in $\mathbb{R}^n$ only. Recall that the nonreal eigenvalues and eigenvectors of real matrices come in pairs of complex-conjugate ones. In the complex diagonal form Λ one can replace diagonal 2×2 blocks

$$\begin{bmatrix} \lambda_j & 0 \\ 0 & \overline{\lambda_j} \end{bmatrix}$$

by a 2×2 matrix which is *not diagonal*, but has real entries.

To see how this is done, suppose $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_2 = \overline{\lambda_1}$, $\mathbf{v}_2 = \overline{\mathbf{v}_1}$. Splitting into real and imaginary parts, write $\lambda_1 = \alpha_1 + i\beta_1$ and $\mathbf{v}_1 = \mathbf{x}_1 + i\mathbf{y}_1$. Then from $M(\mathbf{x}_1 + i\mathbf{y}_1) = (\alpha_1 + i\beta_1)(\mathbf{x}_1 + i\mathbf{y}_1)$ identifying the real and imaginary parts, we obtain

$$M\mathbf{x}_1 + iM\mathbf{y}_1 = (\alpha_1 \mathbf{x} - \beta_1 \mathbf{y}) + i(\beta_1 \mathbf{x} + \alpha_1 \mathbf{y})$$

In the matrix $S = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ composed of independent eigenvectors of M, replace the first two columns $\mathbf{v}_1, \mathbf{v}_2 = \overline{\mathbf{v}_1}$ by $\mathbf{x}_1, \mathbf{y}_1$ (which are vectors in $\mathbb{R}^n$): using the matrix $\tilde{S} = [\mathbf{x}_1, \mathbf{y}_1, \mathbf{v}_3, \ldots, \mathbf{v}_n]$ instead of S we have $M\tilde{S} = \tilde{S}\tilde{\Lambda}$ where

$$\tilde{\Lambda} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \ldots & 0 \\ -\beta_1 & \alpha_1 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_m \end{bmatrix}$$

We can similarly replace any pair of complex conjugate eigenvalues with 2×2 real blocks.

Exercise. Show that each 2×2 real block obtained through decomplexification has the form

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \rho\, R_\theta$$

for a suitable $\rho > 0$ and R_θ rotation matrix (23).

2.14. **Jordan normal form.** We noted in §2.12 that a matrix is similar to a diagonal matrix if and only if the dimension of each eigenspace V_{λ_j} equals the order of multiplicity of the eigenvalue λ_j. Otherwise, there are fewer than n independent eigenvectors; such a matrix is called *defective*.

2.14.1. *Jordan blocks.* Defective matrices can not be diagonalized, but we will see that they are similar to block diagonal matrices, called Jordan normal forms; these are upper triangular, have the eigenvalues on the diagonal, 1 in selected placed above the diagonal, and zero in the rest. After that, in section §2.14.3 it is shown how to construct the transition matrix S, which conjugates a defective matrix to its Jordan form; its columns are made of *generalized eigenvectors.*

The Jordan blocks which appear on the diagonal of a Jordan normal form are as follows.

1×1 *Jordan blocks* are just $[\lambda]$.

2×2 *Jordan blocks* have the form

$$J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \tag{24}$$

For example, the matrix (21) is a Jordan block $J_2(0)$.

3×3 *Jordan blocks* have the form

$$J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \tag{25}$$

In general, a $k \times k$ Jordan block, $J_k(\lambda)$, is a matrix having the same number, λ, on the diagonal, 1 above the diagonal and 0 everywhere else.

Note that Jordan blocks $J_k(\lambda)$ have the eigenvalue λ with multiplicity k, and the dimension of the eigenspace is one.

Example of a matrix in Jordan normal form:

$$\left[\begin{array}{c|c|ccc} 3 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

which is block-diagonal, having two 1×1 Jordan blocks and one 3×3 Jordan block along its diagonal. The eigenvalue 3 is simple, while 2 has multiplicity four. The eigenspace corresponding to 2 is two-dimensional ($\mathbf{e}_2$ and $\mathbf{e}_3$ are eigenvectors).

Note how Jordan blocks act on the vectors of the basis. For (24): $J_2(\lambda)\mathbf{e}_1 = \lambda\mathbf{e}_1$, so $\mathbf{e}_1$ is an eigenvector. Also

$$J_2(\lambda)\mathbf{e}_2 = \mathbf{e}_1 + \lambda\mathbf{e}_2 \tag{26}$$

which implies that $(J_2(\lambda) - \lambda I)^2\mathbf{e}_2 = (J_2(\lambda) - \lambda I)\mathbf{e}_1 = \mathbf{0}$.

Similarly, for (32): $J_3(\lambda)\mathbf{e}_1 = \lambda\mathbf{e}_1$ so $\mathbf{e}_1$ is an eigenvector. Then

$$J_3(\lambda)\mathbf{e}_2 = \mathbf{e}_1 + \lambda\mathbf{e}_2 \tag{27}$$

implying that $(J_3(\lambda) - \lambda I)^2\mathbf{e}_2 = (J_3(\lambda) - \lambda I)\mathbf{e}_1 = \mathbf{0}$. Finally,

$$J_3(\lambda)\mathbf{e}_3 = \mathbf{e}_2 + \lambda\mathbf{e}_3 \tag{28}$$

implying that $(J_3(\lambda) - \lambda I)^3\mathbf{e}_3 = (J_3(\lambda) - \lambda I)^2\mathbf{e}_2 = \mathbf{0}$. This illuminates the idea behind the notion of generalized eigenvectors defined in the next section.

2.14.2. *The generalized eigenspace.* Defective matrices are similar to a matrix which is block-diagonal, having Jordan blocks on its diagonal. An appropriate basis is formed using generalized eigenvectors:

Definition 13. *A* **generalized eigenvector** *of M corresponding to the eigenvalue λ is a vector $\mathbf{x} \neq \mathbf{0}$ so that*

$$(M - \lambda I)^k\mathbf{x} = \mathbf{0} \tag{29}$$

for some positive integer k.

Examples.

1) Eigenvectors are generalized eigenvectors (take $k = 1$ in (29)).

2) Vectors in the standard basis are generalized eigenvectors for Jordan blocks.

Definition 14. *The* **generalized eigenspace** *of M corresponding to the eigenvalue λ is the subspace*

$$E_\lambda = \{\mathbf{x} \,|\, (M - \lambda I)^k \mathbf{x} = \mathbf{0} \,\text{for some } k \in \mathbb{Z}_+\}$$

Sometimes we want to refer to only at the distinct eigenvalues of a matrix, this set is called "the spectrum":

Definition 15. *The* **spectrum** *$\sigma(M)$ of a matrix M is the set of its eigenvalues.*

Theorem 16. *For any $n \times n$ matrix M the following hold:*
(i) $V_\lambda \subset E_\lambda$;
(ii) E_λ is a subspace;
(iii) E_λ is an invariant subspace under M;
(iv) $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$ for $\lambda_1 \neq \lambda_2$.
(v) dim E_λ=the multiplicity of λ.
(vi)The set of eigenvectors and generalized eigenvectors of M span the whole space $\mathbb{C}^n$:

$$\oplus_{\lambda \in \sigma(M)} \, E_\lambda = \mathbb{C}^n$$

The proofs of (i)-(iv) are simple exercises.

The proofs of (v), (vi) are by splitting first the space into spaces generated by eigenvectors. So we may assume M has only one eigenvalue λ and only one eigenvector, $\mathbf{e}_1$. The proof from here on is by induction on the dimension, and follow the strategy below, illustrated in the 2×2 case.

Assume M has only one eigenvalue λ and only one eigenvector, $\mathbf{e}_1$.We want to show that $(M - \lambda I)^2 = 0$. Let $\mathbf{e}_2$ be any vector linearly independent from $\mathbf{e}_1$. Then,

$$(M - \lambda I)\mathbf{e}_2 = \alpha \mathbf{e}_1 + \beta \mathbf{e}_2$$

for some α, β. Now,

$$(M - \lambda I)^2 \mathbf{e}_2 = (M - \lambda I)(\alpha \mathbf{e}_1 + \beta \mathbf{e}_2) = \beta (M - \lambda I)\mathbf{e}_2$$

which shows that $(M - \lambda I)\mathbf{e}_2$ is an eigenvector for $M - \lambda I$. But the only eigenvector of $M - \lambda I$ is zero. Thus

$$(M - \lambda I)^2 \mathbf{e}_2 = 0 (M - \lambda I)\mathbf{e}_2 = 0$$

while indeed $\mathbf{e}_1, \mathbf{e}_2$ satisfy $(M - \lambda I)^2 \mathbf{x} = 0$, and span $\mathbb{R}^2$ by construction, which we wanted to show.

2.14.3. *How to find a basis for each E_λ that can be used to conjugate a matrix to a Jordan normal form.*

Example 1. The matrix

$$M = \begin{bmatrix} 1 + a & -1 \\ 1 & a - 1 \end{bmatrix} \tag{30}$$

is defective: it has eigenvalues a, a and only one independent eigenvector, $(1,1)^T$. It is therefore similar to $J_2(a)$. To find a basis $\mathbf{x}_1, \mathbf{x}_2$ in which the matrix takes this form, let $\mathbf{x}_1 = (1,1)^T$ (the eigenvector); to find $\mathbf{x}_2$ we solve $(M - aI)\mathbf{x}_2 = \mathbf{x}_1$ (as seen in (26) and in (27)). The solutions are $\mathbf{x}_2 \in (1,0)^T + \mathcal{N}(M - aI)$, and any vector in this space works, for example $\mathbf{x}_2 = (1,0)^T$. For

$$S = [\mathbf{x}_1, \mathbf{x}_2] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{31}$$

we have $S^{-1}MS = J_2(a)$.

Example 2.
The matrix

$$M = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

has eigenvalues $2, 2, 2$ and only one independent eigenvector $\mathbf{v}_1 = (1,1,1)^T$.

Let $\mathbf{x}_1 = \mathbf{v}_1 = (1,1,1)^T$. Solving $(M - 2I)\mathbf{x}_2 = \mathbf{x}_1$ we obtain $\mathbf{x}_2 = (1,-1,0)^T$ (plus any vector in $\mathcal{N}(M-2I) = V_{\lambda_1}$). Next solve $(M-2I)\mathbf{x}_3 = \mathbf{x}_2$ which gives $\mathbf{x}_3 = (0,1,1)^T$ (plus any vector in the null space of $M - 2I$). For $S = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$ we have

$$S^{-1}MS = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \tag{32}$$

In general, if λ is an eigenvalue of M for which $\dim V_\lambda$ is less than the multiplicity of λ, we do the following. Choose a basis for V_λ. For each eigenvector $\mathbf{v}$ in this basis set $\mathbf{x}_1 = \mathbf{v}$ and solve recursively

$$(M - \lambda I)\mathbf{x}_{k+1} = \mathbf{x}_k, \ \ k = 1, 2, \ldots \tag{33}$$

Note that each $\mathbf{x}_1$ satisfies (29) for $k = 1$, $\mathbf{x}_2$ satisfies (29) for $k = 2$, etc.

At some step k_1 the system $(M - \lambda I)\mathbf{x}_{k_1+1} = \mathbf{x}_{k_1}$ will have no solution; we found the generalized eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_{k_1}$ (which will give a $k_1 \times k_1$ Jordan block). We then repeat the procedure for a different eigenvector in the chosen basis for V_λ, and obtain a new set of generalized eigenvectors, corresponding to a new Jordan block.

Note: Jordan form is not unique.

2.14.4. *Real Jordan normal form.* If a real matrix has multiple complex eigenvalues and is defective, then its Jordan form can be replaced with an upper block diagonal matrix in a way similar to the diagonal case illustrated in §2.13.2, by replacing the generalized eigenvectors with their real and imaginary parts.

For example, a real matrix which can be brought to the complex Jordan normal form

$$\begin{bmatrix} \alpha+i\beta & 1 & 0 & 0 \\ 0 & \alpha+i\beta & 1 & 0 \\ 0 & 0 & \alpha-i\beta & 1 \\ 0 & 0 & 0 & \alpha-i\beta \end{bmatrix}$$

can be conjugated (by a real matrix) to the real matrix

$$\begin{bmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix}$$

In general, the real normal Jordan form is

$$J_i = \begin{bmatrix} \Lambda_i & I & & \\ & \Lambda_i & \ddots & \\ & & \ddots & I \\ & & & \Lambda_i \end{bmatrix}$$

where

$$\Lambda_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.15. **Nilpotents.** Note that a complex Jordan block is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} = \lambda_i I + N;\ N := \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

Note that I commutes with any matrix, in particular with N. Note also that

$$N^2 := \begin{bmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & \ddots & \\ & & \ddots & 0 & 0 \\ & & & 0 & 0 \end{bmatrix}$$

and so on, with $N^k = 0$.

2.16. **Block matrices.**

2.16.1. *Multiplication of block matrices.* It is sometimes convenient to work with matrices split in blocks. We have already used this when we wrote

$$M[\mathbf{v}_1, \ldots, \mathbf{v}_n] = [M\mathbf{v}_1, \ldots, M\mathbf{v}_n]$$

More generally, if we have two matrices M, P with dimensions that allow for multiplication (i.e. the number of columns of M equals the number of rows of P) and they are split into blocks:

$$M = \begin{bmatrix} M_{11} & \mid & M_{12} \\ --- & - & --- \\ M_{21} & \mid & M_{22} \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & \mid & P_{12} \\ --- & - & --- \\ P_{21} & \mid & P_{22} \end{bmatrix}$$

then

$$MP = \begin{bmatrix} M_{11}P_{11} + M_{12}P_{21} & \mid & M_{11}P_{12} + M_{12}P_{22} \\ ------ & - & ------ \\ M_{21}P_{11} + M_{22}P_{21} & \mid & M_{21}P_{12} + M_{22}P_{22} \end{bmatrix}$$

if the number of columns of M_{11} equals the number of rows of P_{11}.

Exercise. Prove that the block multiplication formula is correct.

More generally, one may split the matrices M and P into many blocks, so that the number of block-columns of M equal the number of block-rows of P and so that all products $M_{jk}P_{kl}$ make sense. Then MP can be calculated using blocks by a formula similar to that using matrix elements.

In particular, if M, P are block diagonal matrices, having the blocks M_{jj}, P_{jj} on the diagonal, then MP is a block diagonal matrix, having the blocks $M_{jj}P_{jj}$ along the diagonal.

For example, if M is a matrix in Jordan normal form, then it is block diagonal, with Jordan blocks M_{jj} along the diagonal. Then the matrix M^2 is block diagonal, having M_{jj}^2 along the diagonal, and all powers M^k are block diagonal, having M_{jj}^k along the diagonal. Furthermore, any linear combination of these powers of M, say $c_1M+c_2M^2$ is block diagonal, having the corresponding $c_1M_{jj} + c_2M_{jj}^2$ along the diagonal.

2.16.2. *Determinant of block matrices.*

Proposition 17. *Let M be a square matrix, having a triangular block form:*

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \quad \text{or } M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

where A and D are square matrices, say A is $k \times k$ and D is $l \times l$.

Then $\det M = \det A \det D$.

Moreover, if $a_1, \ldots, a_k$ are the eigenvalues of A, and $d_1, \ldots, d_l$ are the eigenvalues of D, then the eigenvalues of M are $a_1, \ldots, a_k, d_1, \ldots, d_l$.

The proof is left to the reader as an exercise.[1]

[1] *Hint:* bring A, D to Jordan normal form, then M to an upper triangular form.

For a more general 2×2 block matrix, with D invertible[2]

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the identity

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix}$$

together with Proposition 17 implies that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C) \det D = \det(AD - BD^{-1}CD)$$

For larger number of blocks, there are more complicated formulas.

[2] *References:* J.R. Silvester, Determinants of block matrices, Math. Gaz., 84(501) (2000), pp. 460-467, and P.D. Powell, Calculating Determinants of Block Matrices, http://arxiv.org/pdf/1112.4379v1.pdf

3. Solutions of linear differential equations with constant coefficients

In §1.2 we saw an example which motivated the notions of eigenvalues and eigenvectors. General linear first order systems of differential equations with constant coefficients can be solved in a quite similar way. Consider

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u} \tag{34}$$

where M is an $m \times m$ *constant* matrix and $\mathbf{u}$ in an m-dimensional vector. Here, as usual

$$\frac{d\mathbf{u}}{dt} = \lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h} = \left(\frac{du_1}{dt}, ..., \frac{du_n}{dt}\right)$$

3.1. **The case when M is diagonalizable.**

Let's see if there is a change of basis that would simplify the system.

Let

$$\mathbf{u} = S\mathbf{v}$$

then, it is easy to see from the definition of the derivative above, that we get

$$S\mathbf{v}' = MS\mathbf{v} \Rightarrow \mathbf{v}' = S^{-1}MS\mathbf{v}$$

Clearly, we should choose S to be the matrix that diagonalizes M. In this basis we get

$$= \begin{pmatrix} v_1' \\ ... \\ v_n' \end{pmatrix} = \begin{bmatrix} \lambda_1 & 0... & 0 \\ ... & ... & \\ 0 & ... & \lambda_n \end{bmatrix}\begin{pmatrix} v_1 \\ ... \\ v_n \end{pmatrix}$$

or

$$v_i' = \lambda_i v_i, i = 1, ..., n$$

which implies that

$$v_i(t) = C_i e^{\lambda_i t}$$

and then clearly $C_i = v_i(0)$ which means that

$$\mathbf{u} = S\mathbf{v} = S(v_i(0)e^{\lambda_i t})^T = \begin{bmatrix} e^{\lambda_1 t} & 0... & 0 \\ ... & ... & \\ 0 & ... & e^{\lambda_n t} \end{bmatrix}\mathbf{v} = e^{\Lambda t}\mathbf{v}(0)$$

Why this notation? We'll see in the next section.

3.1.1. *The matrix e^{Mt}.*

It is often preferable to work with a matrix of independent solutions $U(t)$ rather than with a set of independent solutions.

In dimension one this equation reads $\frac{du}{dt} = \lambda u$ having its general solution $u(t) = Ce^{\lambda t}$. Let us check this fact based on the fact that the exponential is the sum of its Taylor series:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n + \ldots = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

where the series converges for all $x \in \mathbb{C}$. Then

$$e^{\lambda t} = 1 + \frac{1}{1!}\,\lambda t + \frac{1}{2!}\,\lambda^2 t^2 + \ldots + \frac{1}{n!}\,\lambda^n x^n + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\,\lambda^n t^n$$

and the series can be differentiated term-by-term, giving

$$\frac{d}{dt} e^{\lambda t} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{n!}\,\lambda^n t^n = \sum_{n=0}^{\infty} \frac{1}{n!}\,\lambda^n \frac{d}{dt} t^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\,\lambda^n t^{n-1} = \lambda e^{\lambda t}$$

Perhaps one can define, similarly, the exponential of a matrix and obtain solutions to (42)?

For any square matrix M, one can define polynomials, as in (10), and it is natural to define

$$e^{tM} = 1 + \frac{1}{1!}\,tM + \frac{1}{2!}\,t^2 M^2 + \ldots + \frac{1}{n!}\,t^n M^n + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\,t^n M^n \tag{35}$$

provided that the series converges. If, furthermore, the series can differentiated term by term, then this matrix is a solution of (42) since

$$\frac{d}{dt} e^{tM} = \frac{d}{dt} \sum_{n=0}^{N} \frac{1}{n!}\,t^n M^n = \sum_{n=0}^{N} \frac{1}{n!}\,\frac{d}{dt} t^n M^n = \sum_{n=1}^{N} \frac{n}{n!}\,t^{n-1} M^n = M e^{tM} \tag{36}$$

Thus we want to study

$$\sum_{n=0}^{N} \frac{1}{n!}\,t^n M^n$$

as $N \to \infty$.

Take first $M = \Lambda$, a diagonal matrix:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_m \end{bmatrix} \tag{37}$$

It is easy to see that

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 & \ldots & 0 \\ 0 & \lambda_2^n & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_m^n \end{bmatrix} \quad \text{for } n = 1, 2, 3 \ldots \tag{38}$$

therefore

$$\sum_{n=1}^{N} \frac{1}{n!}\,t^n \Lambda^n = \begin{bmatrix} \sum_{n=1}^{N} \frac{1}{n!}\,t^n \lambda_1^n & 0 & \ldots & 0 \\ 0 & \sum_{n=1}^{N} \frac{1}{n!}\,t^n \lambda_2^n & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \sum_{n=1}^{N} \frac{1}{n!}\,t^n \lambda_m^n \end{bmatrix} \tag{39}$$

$$\underset{N\to\infty}{\to} \begin{bmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{t\lambda_m} \end{bmatrix} = e^{t\Lambda} \tag{40}$$

Convergence and term-by-term differentiation can be justified by diagonalizing M:

Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be independent eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_m$ of M, let $S = [\mathbf{v}_1, \dots, \mathbf{v}_m]$. Then $M = S\Lambda S^{-1}$ with Λ the diagonal matrix with entries $\lambda_1, \dots, \lambda_m$.

Note that

$$M^2 = \left(S\Lambda S^{-1}\right)^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

then

$$M^3 = M^2 M = \left(S\Lambda^2 S^{-1}\right)\,\left(S\Lambda S^{-1}\right) = S\Lambda^3 S^{-1}$$

and so on; for any power

$$M^n = S\Lambda^n S^{-1}$$

Then

$$\begin{aligned} \sum_{n=0}^{N} \frac{1}{n!}\, t^n M^n &= \sum_{n=0}^{N} \frac{1}{n!}\, t^n S\Lambda^n S^{-1} \\ &= S\left(\sum_{n=0}^{N} \frac{1}{n!}\, t^n \Lambda^n\right) S^{-1} \underset{N\to\infty}{\to} S e^{t\Lambda} S^{-1} =: e^{tM} \end{aligned} \tag{41}$$

Therefore $U(T) = e^{tM}$ is a solution of the differential equation

$$\frac{d}{dt}U(t) = M\, U(t) \tag{42}$$

and is a *fundamental solution* of (42).

3.2. **The most general solution of** (42)**.** Let $N(t)$ be an $n \times k$ matrix, $k \le n$ which solves (42). The differential equation (42) makes sense for N. Since $e^{Mt}e^{-Mt} = I$ (check!), we define $Q(t) = e^{-Mt}N(t) \Rightarrow N(t) = e^{Mt}Q(t)$.

We have

$$\begin{aligned} (e^{Mt}Q(t))' = Me^{Mt}Q(t) \Rightarrow N(t)Me^{Mt}Q + e^{Mt}Q' = Me^{Mt}Q \\ \Rightarrow Q' = 0 \Rightarrow Q = \text{const.} = Q(0) \end{aligned} \tag{43}$$

Then, the general solution of the equation

$$U' = MU; \quad U(0) = u_0 \text{ is } U(t) = e^{Mt}U_0 \tag{44}$$

We note that Q could be an $n \times 1$ matrix, a vector thus, and the general solution of the system (34) is

$$\mathbf{u}(t) = e^{Mt}\mathbf{u}(0) \tag{45}$$

the matrix e^{Mt} is called the **fundamental matrix** of the matrix equation.

$$\mathbf{u}(t) = Se^{\Lambda t}S^{-1}\mathbf{u}(0) = Se^{\Lambda t}\mathbf{v}(0) \tag{46}$$

$$U(t) = Se^{t\Lambda} = e^{tM}S \tag{47}$$

Expanded out,

$$\mathbf{u}(t) = a_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + a_m e^{\lambda_m t}\mathbf{v}_m, \quad a_j \text{ arbitrary constants} \tag{48}$$

In conclusion:

Proposition 18. *If the $m \times m$ constant matrix M has has m independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_m$, then equation (34) has m linearly independent solutions $\mathbf{u}_j(t) = e^{\lambda_j t}\mathbf{v}_j$, $j = 1, \ldots, m$ and any solution of (34) is a linear combination of them.*

Example. Solve the initial value problem

$$\begin{array}{ll} \frac{dx}{dt} = x - 2y, & x(0) = \alpha \\ \frac{dy}{dt} = -2x + y, & y(0) = \beta \end{array} \tag{49}$$

Denoting $\mathbf{u} = (x, y)^T$, problem (49) is

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u}, \quad \text{where } M = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \quad \text{with } \mathbf{u}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{50}$$

Calculating the eigenvalues of M, we obtain $\lambda_1 = -1$, $\lambda_2 = 3$, and corresponding eigenvectors $\mathbf{v}_1 = (1, 1)^T$, $\mathbf{v}_2 = (-1, 1)^T$. There are two independent solutions of the differential system:

$$\mathbf{u}_1(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2(t) = e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and a fundamental matrix solution is

$$U(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t)] = \begin{bmatrix} e^{-t} & -e^{3t} \\ e^{-t} & e^{3t} \end{bmatrix} \tag{51}$$

The general solution is a linear combination of the two independent solutions

$$\mathbf{u}(t) = a_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = U(t)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

This solution satisfies the initial condition if

$$a_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

which is solved for a_1, a_2: from

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

it follows that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\alpha+\beta}{2} \\ \frac{-\alpha+\beta}{2} \end{bmatrix}$$

therefore

$$\mathbf{u}(t) = \tfrac{\alpha+\beta}{2} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{-\alpha+\beta}{2} e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{52}$$

so

$$\begin{aligned} x(t) &= \tfrac{\alpha+\beta}{2} e^{-t} - \tfrac{-\alpha+\beta}{2} e^{3t} \\ y(t) &= \tfrac{\alpha+\beta}{2} e^{-t} + \tfrac{-\alpha+\beta}{2} e^{3t} \end{aligned}$$

Example. For the example (50) we have

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad e^{t\Lambda} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

and

$$\mathbf{u}_1(t) = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{u}_2(t) = e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The fundamental matrix $U(t)$ is given by (51).

Using (44)

$$e^{tM} = Se^{t\Lambda}S^{-1} = \begin{bmatrix} \frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t} & \frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t} \\ \frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t} & \frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t} \end{bmatrix}$$

and the solution to the initial value problem is

$$e^{tM} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\alpha + \left(\frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\beta \\ \left(\frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\alpha + \left(\frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\beta \end{bmatrix}$$

which, of course, is the same as (52).

3.3. **Non-diagonalizable matrices.** The exponential e^{tM} is defined similarly, only a Jordan normal form must be used instead of a diagonal form: writing $S^{-1}MS = J$ where S is a matrix formed of generalized eigenvectors of M, and J is a Jordan normal form, then

$$e^{tM} = Se^{tJ}S^{-1} \tag{53}$$

It only remains to check that the series defining the exponential of a Jordan form converges, and that it can be differentiated term by term.

Also to be determined are m linearly independent solutions, since if M is not diagonalizable, then there are fewer than m independent eigenvectors, hence fewer than m independent solutions of pure exponential type. This can be done using the analogue of (47), namely by considering the matrix

$$U(t) = Se^{tJ} = e^{tM}S \tag{54}$$

The columns of the matrix (54) are linearly independent solutions, and we will see that among them are the purely exponential ones multipling the eigenvectors of M.

Since J is block diagonal (with Jordan blocks along its diagonal), then its exponential will be block diagonal as well, with exponentials of each Jordan block (see §2.16.1 for multiplication of block matrices).

3.3.1. *Example: 2×2 blocks:* for

$$J=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} \tag{55}$$

direct calculations give

$$J^2=\begin{bmatrix}\lambda^2 & 2\,\lambda\\ 0 & \lambda^2\end{bmatrix},\ J^3=\begin{bmatrix}\lambda^3 & 3\,\lambda^2\\ 0 & \lambda^3\end{bmatrix},\ldots,J^k=\begin{bmatrix}\lambda^k & k\,\lambda^{k-1}\\ 0 & \lambda^k\end{bmatrix},\ldots \tag{56}$$

and then

$$e^{tJ}=\sum_{k=0}^{\infty}\frac{1}{k!}t^kJ^k=\begin{bmatrix}e^{t\lambda} & te^{t\lambda}\\ 0 & e^{t\lambda}\end{bmatrix} \tag{57}$$

More abstractly,

$$e^{tJ}=\sum_{k=0}^{\infty}\frac{1}{k!}t^k(\lambda I+N)^k=\sum_{k=0}^{\infty}\frac{1}{k!}t^k(\lambda^kI+Nk\lambda^{k-1})=e^{t\lambda I}+tNe^{\lambda tI} \tag{58}$$

For the equation (34) with the matrix M is similar to a 2×2 Jordan block: $S^{-1}MS=J$ with J as in (55), and $S=[\mathbf{x}_1,\mathbf{x}_2]$ a fundamental matrix solution is $U(t)=Se^{tJ}=[e^{t\lambda}\mathbf{x}_1,e^{t\lambda}(t\mathbf{x}_1+\mathbf{x}_2)]$ whose columns are two linearly independent solutions

$$\mathbf{u}_1(t)=e^{t\lambda}\mathbf{x}_1,\quad \mathbf{u}_2(t)=e^{t\lambda}(t\mathbf{x}_1+\mathbf{x}_2) \tag{59}$$

and any linear combination is a solution:

$$\mathbf{u}(t)=a_1e^{t\lambda}\mathbf{x}_1+a_2e^{t\lambda}(t\mathbf{x}_1+\mathbf{x}_2) \tag{60}$$

Example. Solve the initial value problem

$$\begin{aligned}\tfrac{dx}{dt}&=(1+a)x-y, &\qquad x(0)&=\alpha\\ \tfrac{dy}{dt}&=x+(a-1)y, &\qquad y(0)&=\beta\end{aligned} \tag{61}$$

Denoting $\mathbf{u}=(x,y)^T$, the differential system (61) is $\frac{d\mathbf{u}}{dt}=M\mathbf{u}$ with M given by (30), matrix for which we found that it has a double eigenvalue a and only one independent eigenvector $\mathbf{x}_1=(1,1)^T$.

Solution 1. For this matrix we already found an independent generalized eigenvector $\mathbf{x}_2=(1,0)^T$, so we can use formula (60) to write down the general solution of (61).

Solution 2. We know one independent solution to the differential system, namely $\mathbf{u}_1(t)=e^{at}\mathbf{x}_1$. We look for a second independent solution as the same exponential multiplying a polynomial in t, of degree 1: substituting $\mathbf{u}(t)=e^{at}(t\mathbf{b}+\mathbf{c})$ in $\frac{d\mathbf{u}}{dt}=M\mathbf{u}$ we obtain that $a(t\mathbf{b}+\mathbf{c})+\mathbf{b}=M(t\mathbf{b}+\mathbf{c})$ holds for all t, therefore $M\mathbf{b}=a\mathbf{b}$ and $(M-aI)\mathbf{c}=\mathbf{b}$ which means that $\mathbf{b}$

is an eigenvector of M (or $\mathbf{b} = \mathbf{0}$), and $\mathbf{c}$ is a generalized eigenvector. We have re-obtained the formula (59).

By either method it is found that a fundamental matrix solution is

$$U(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t)] = e^{at} \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix}$$

and the general solution has the form $\mathbf{u}(t) = U(t)\mathbf{c}$ for an arbitrary constant vector $\mathbf{c}$. We now determine $\mathbf{c}$ so that $\mathbf{u}(0) = (\alpha, \beta)^T$, so we solve

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{c} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

which gives

$$\mathbf{c} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha - \beta \end{bmatrix}$$

and the solution to the initial value problem is

$$\mathbf{u}(t) = e^{at} \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix} \begin{bmatrix} \beta \\ \alpha - \beta \end{bmatrix} = e^{at} \begin{bmatrix} t\,(\alpha-\beta)+\alpha \\ \beta + t\,(\alpha - \beta) \end{bmatrix}$$

or

$$x(t) = e^{at}\left(t\,(\alpha-\beta)+\alpha\right), \; y(t) = e^{at}\left(t\,(\alpha-\beta)+\beta\right)$$

3.3.2. *Example:* 3×3 *blocks:* for

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \tag{62}$$

direct calculations give

$$J^2 = \begin{bmatrix} \lambda^2 & 2\,\lambda & 1 \\ 0 & \lambda^2 & 2\,\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}, \; J^3 = \begin{bmatrix} \lambda^3 & 3\,\lambda^2 & 3\,\lambda \\ 0 & \lambda^3 & 3\,\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}, \; J^4 = \begin{bmatrix} \lambda^4 & 4\,\lambda^3 & 6\,\lambda^2 \\ 0 & \lambda^4 & 4\,\lambda^3 \\ 0 & 0 & \lambda^4 \end{bmatrix}$$

Higher powers can be calculated by induction; it is clear that

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix} \tag{63}$$

Then

$$e^{tJ} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k J^k = \begin{bmatrix} e^{t\lambda} & te^{t\lambda} & \frac{1}{2}t^2 e^{t\lambda} \\ 0 & e^{t\lambda} & te^{t\lambda} \\ 0 & 0 & e^{t\lambda} \end{bmatrix} \tag{64}$$

For $M = SJS^{-1}$ with J as in (62) and $S = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$, a fundamental matrix solution for (34) is

$$Se^{tJ} = [\mathbf{x}_1 e^{\lambda t}, (t\mathbf{x}_1 + \mathbf{x}_2)e^{\lambda t}, (\frac{1}{2}t^2\mathbf{x}_1 + t\mathbf{x}_2 + \mathbf{x}_3)e^{\lambda t}]$$

3.3.3. *In general,* if an eigenvalue λ has multiplicity r, but there are only $k < r$ independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then, besides the k independent solutions $e^{\lambda t}\mathbf{v}_1, \ldots, e^{\lambda t}\mathbf{v}_k$ there are other $r - k$ independent solutions in the form $e^{\lambda t}\mathbf{p}(t)$ with $\mathbf{p}(t)$ polynomials in t of degree at most $r - k$, with vector coefficients (which turn out to be generalized eigenvectors of M).

Then *the solution of the initial value problem (34),* $\mathbf{u}(0) = \mathbf{u}_0$ *is*

$$\mathbf{u}(t) = e^{tM}\mathbf{u}_0$$

Combined with the results of uniqueness of the solution of the initial value problem (known from the general theory of ordinary differential equations) it follows that:

Theorem 19. *Any linear differential equation* $\mathbf{u}' = M\mathbf{u}$ *where* M *is an* $m \times m$ *constant matrix, and* $\mathbf{u}$ *is an* m*-dimensional vector valued function has* m *linearly independent solutions, and any solution is a linear combination of these. In other words, the solutions of the equation form a linear space of dimension* m*.*

3.4. **Fundamental facts on linear differential systems.**

Theorem 20. *Let* M *be an* $n \times n$ *matrix (diagonalizable or not).*

(i) The matrix differential problem

$$\frac{d}{dt}U(t) = M\,U(t), \quad U(0) = U_0 \tag{65}$$

has a unique solution, namely $U(t) = e^{Mt}U_0$.

(ii) Let $W(t) = \det U(t)$*. Then*

$$W'(t) \;=\; \mathrm{Tr}M\; W(t) \tag{66}$$

therefore

$$W(t) = W(0)\, e^{t\mathrm{Tr}M} \tag{67}$$

(iii) If U_0 *is an invertible matrix, then the matrix* $U(t)$ *is invertible for all* t*, called* **a fundamental matrix solution***; the columns of* $U(t)$ *form an independent set of solutions of the system*

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u} \tag{68}$$

(iv) Let $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ *be solutions of the system (68). If the vectors* $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ *are linearly independent at some* t *then they are linearly independent at any* t*.*

Proof.

(i) Clearly $U(t) = e^{Mt}U_0$ is a solution, and it is unique by the general theory of differential equations: (65) is a linear system of n^2 differential equation in n^2 unknowns.

(ii) Using (53) it follows that

$$W(t) = \det U(t) = \det(Se^{tJ}S^{-1}U_0) = \det e^{tJ} \det U_0 = e^{t\sum_{j=1}^{n}\lambda_j} \det U_0$$
$$= e^{t\mathrm{Tr}M} \det U_0 = e^{t\mathrm{Tr}M} W(0)$$

which is (67), implying (66).

(iii), (iv) are immediate consequences of (67). □

3.5. **Eigenvalues and eigenvectors of the exponential of a matrix.** It is not hard to show that

$$(e^M)^{-1} = e^{-M}, \quad (e^M)^k = e^{kM}, \quad e^{M+cI} = e^c e^M$$

More generally, it can be shown that if $MN = NM$, then $e^M e^N = e^{M+N}$. *Warning:* if the matrices do not commute, this may not be true!

Recall that if M is diagonalizable, in other words if $M = S\Lambda S^{-1}$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, then $e^M = Se^\Lambda S^{-1}$ where $e^\Lambda = \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$. If follows that the eigenvalues of e^M are $e^{\lambda_1}, \ldots, e^{\lambda_n}$ and the columns of S are eigenvectors of M, and also of e^M.

If M is not diagonalizable, let J be its Jordan normal form. Recall that if $M = SJS^{-1}$ then $e^M = Se^JS^{-1}$ where e^J is an upper triangular matrix, with diagonal elements still being exponentials of the eigenvalues of M. The matrix e^J is not a Jordan normal form; however, generalized eigenvectors of M are also of e^M.

Exercise.

1. Show that if $M\mathbf{x} = \mathbf{0}$ then $e^M\mathbf{x} = \mathbf{x}$.
2. Show that if $\mathbf{v}$ is an eigenvector of M corresponding to the eigenvalues λ, then $\mathbf{v}$ is also an eigenvector of e^M corresponding to the eigenvalues e^λ.
3. Show that if $M\mathbf{v} = \lambda\mathbf{v}$ then $e^M\mathbf{v} = e^\lambda[\mathbf{v} + (M - \lambda I)\mathbf{v}]$.

Note that if $(M - \lambda I)^2\mathbf{x} = 0$ then $(e^M - e^\lambda)^2\mathbf{x} = 0$. Indeed, $(e^M - e^\lambda)^2\mathbf{x} = (e^{2M} - 2e^\lambda e^M + e^{2\lambda}I)^2\mathbf{x} = e^{2\lambda}e^{2(M-\lambda)}\mathbf{x} - 2e^{2\lambda}e^{M-\lambda}\mathbf{x} + e^{2\lambda}\mathbf{x} = e^{2\lambda}[\mathbf{x} + 2(M - \lambda)\mathbf{x}] - 2e^{2\lambda}[\mathbf{x} + (M - \lambda)\mathbf{x}] + e^{2\lambda}\mathbf{x} = \mathbf{0}$.

In general, if $\mathbf{x}$ is a generalized eigenvector of M corresponding to the eigenvalues λ, then $\mathbf{x}$ is also a generalized eigenvector of e^M corresponding to the eigenvalues e^λ.

3.6. **Higher order linear differential equations; companion matrix.** Consider scalar linear differential equations, with constant coefficients, of order n:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \tag{69}$$

where $y(t)$ is a scalar function and $a_1, \ldots, a_{n-1}$ are constants.

Such equations can be transformed into systems of first order equations: the substitution

$$u_0 = y,\ u_1 = y',\ \ldots,\ u_n = y^{(n-1)} \tag{70}$$

transforms (69) into the system

$$\mathbf{u}' = M\mathbf{u}, \quad \text{where } M = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \tag{71}$$

The matrix M is called *the companion matrix* to the differential equation (69).

To find its eigenvalues an easy method is to search for λ so that the linear system $M\mathbf{x} = \lambda\mathbf{x}$ has a solution $\mathbf{x} \neq \mathbf{0}$:

$$x_2 = \lambda x_1,\ x_3 = \lambda x_2, \ldots,\ x_n = \lambda x_{n-1},\ -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n = \lambda x_n$$

which implies that

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0 \tag{72}$$

which is the characteristic equation of M.

Note that the characteristic equation (72) can also be obtained by searching for solutions of (69) which are purely exponential: $y(t) = e^{\lambda t}$.

3.6.1. *Linearly independent sets of functions.* We are familiar with the notion of linear dependence or independence of functions belonging to a given linear space. In practice, functions arise from particular problems, or classes of problems, for example as solutions of equations and only a posteriori we find a linear space to accommodates them. A natural definition of linear dependence or independence which can be used in most usual linear space of functions is:

Definition 21. *A set of function* $f_1, \ldots, f_n$ *are called* linearly dependent on an interval I *if there are constants* $c_1, \ldots, c_n$*, not all zero, so that*

$$c_1 f_1(t) + \ldots + c_n f_n(t) = 0 \quad \textit{for all } t \in I \tag{73}$$

A set of functions which are not linearly dependent on I are called *linearly independent on* I. This means that if, for some constants $c_1, \ldots, c_n$ relation (73) holds, then necessarily all $c_1, \ldots, c_n$ are zero.

If all functions $f_1, \ldots, f_n$ are enough many times differentiable then there is a simple way to check linear dependence or independence:

Theorem 22. *Assume functions $f_1, \ldots, f_n$ are $n-1$ times differentiable on the interval I. Consider their Wronskian*

$$W[f_1, \ldots, f_n](t) = \begin{vmatrix} f_1(t) & \ldots & f_n(t) \\ f_1'(t) & \ldots & f_n'(t) \\ \vdots & & \vdots \\ f_1^{(n-1)}(t) & \ldots & f_n^{(n-1)}(t) \end{vmatrix}$$

(i) If the functions $f_1, \ldots, f_n$ are linearly dependent then

$$W[f_1, \ldots, f_n](t) = 0 \quad \textit{for all } t \in I$$

(ii) If there is some point $t_0 \in I$ so that $W[f_1, \ldots, f_n](t_0) \neq 0$ then the functions are linearly independent on I.

Indeed, to show (i), assume that (73) holds for some constants $c_1, \ldots, c_n$, not all zero; then by differentiation, we see that the columns of $W(t)$ are linearly dependent for each t, hence $W(t) = 0$ for all t.

Part (ii) is just the negation of (i). □

Example 1. To check if the functions $1, t^2, e^t$ are linearly dependent we calculate their Wronskian

$$W[1, t^2, e^t] = \begin{vmatrix} 1 & t^2 & e^t \\ 0 & 2t & e^t \\ 0 & 2 & e^t \end{vmatrix} = 2\,\mathrm{e}^t\,(t-1) \text{ is not identically } 0$$

so they are linearly independent (even if the Wronskian happens to be zero for $t = 1$).

Example 2. If the numbers $\lambda_1, \ldots, \lambda_n$ are all distinct then the functions $e^{t\lambda_1}, \ldots, e^{t\lambda_n}$ are linearly independent.

Indeed, their Wronskian equals the product $e^{t\lambda_1} \ldots e^{t\lambda_n}$ multiplied by a Vandermonde determinant which equals $\prod_{i<j}(\lambda_j - \lambda_i)$ which is never zero if $\lambda_1, \ldots, \lambda_n$ are all distinct, or identically zero if two of λs are equal.

I what follows we will see that if the functions $f_1, \ldots, f_n$ happen to be solutions of the same linear differential equation, then their Wronskian is either identically zero, or never zero.

3.6.2. *Linearly independent solutions of nth order linear differential equations.* Using the results obtained for first order linear systems, and looking just at the first component $u_1(t)$ of the vector $\mathbf{u}(t)$ (since $y(t) = u_1(t)$) we find:

(i) if the characteristic equation (72) has n distinct solutions $\lambda_1, \ldots, \lambda_n$ then the general solution is a linear combination of purely exponential solutions

$$y(t) = a_1 e^{\lambda_1 t} + \ldots + a_n e^{\lambda_n t}$$

(ii) if λ_j is a repeated eigenvalue of multiplicity r_j then there are r_j independent solutions of the type $e^{\lambda_j t} q(t)$ where $q(t)$ are polynomials in t of degree at most r_j, therefore they can be taken to be $e^{\lambda_j t}, te^{\lambda_j t}, \ldots, t^{r_j - 1} e^{\lambda_j t}$.

Example. Solve the differential equation

$$y''' - 3y'' + 4y = 0$$

The characteristic equation, obtained by substituting $y(t) = e^{\lambda t}$, is $\lambda^3 - 3\lambda^2 + 4 = 0$ which factored is $(\lambda - 2)^2(\lambda + 1) = 0$ giving the simple eigenvalue -1 and the double eigenvalue 2. There are tree independent solutions $y_1(t) = e^{-t}$, $y_2(t) = e^{2t}$, $y_3(t) = te^{2t}$ and any solution is a linear combination of these.

3.6.3. *The Wronskian.* Let $y_1(t), \ldots, y_n(t)$ be n solutions of the equation (69). The substitution (70) produces n solutions $\mathbf{u}_1, \ldots \mathbf{u}_n$ of the system (71). Let $U(t) = [\mathbf{u}_1(t), \ldots \mathbf{u}_n(t)]$ be the matrix solution of (71). Theorem 20 applied to the companion system yields:

Theorem 23. *Let $y_1(t), \ldots, y_n(t)$ be solutions of equation (69).*

(i) Their Wronskian $W(t) = W[y_1, \ldots, y_n](t)$ satisfies

$$W(t) = e^{-ta_{n-1}} W(0)$$

(ii) $y_1(t), \ldots, y_n(t)$ are linearly independent if and only if their Wronskian at some t is not zero.

3.6.4. *Decomplexification.* Suppose equation (69) has real coefficients, $a_j \in \mathbb{R}$, but there are nonreal eigenvalues, say $\lambda_{1,2} = \alpha_1 \pm i\beta_1$. Then there are two independent solutions $y_{1,2}(t) = e^{t(\alpha_1 \pm i\beta_1)} = e^{t\alpha_1}[\cos(t\beta_1) \pm i \sin(t\beta_1)]$. If real valued solutions are needed (or desired), note that $Sp(y_1, y_2) = Sp(y_c, y_s)$ where

$$y_c(t) = e^{t\alpha_1} \cos(t\beta_1), \;\; y_s(t) = e^{t\alpha_1} \sin(t\beta_1)$$

and y_s, y_c are two independent solutions (real valued).

Furthermore, any solution in $Sp(y_c, y_s)$ can be written as

$$C_1 e^{t\alpha_1} \cos(t\beta_1) + C_2 e^{t\alpha_1} \sin(t\beta_1) = A e^{t\alpha_1} \sin(t\beta_1 + B) \tag{74}$$

where

$$A = \sqrt{C_1^2 + C_2^2}$$

and B is the unique angle in $[0, 2\pi)$ so that

$$\cos B = \frac{C_2}{\sqrt{C_1^2 + C_2^2}}, \;\; \sin B = \frac{C_1}{\sqrt{C_1^2 + C_2^2}}$$

Example 1. Solve the equation of the *harmonic oscillator*

$$x' = -y, \;\; y' = k^2 x \quad \text{ where } k \in \mathbb{R} \tag{75}$$

giving both complex and real forms.

In this example it is quicker to solve by turning the system into a second order scalar equation: taking the derivative in the first equation we obtain $x'' = -y'$ and using the second equation it follows that $x'' + k^2 x = 0$, with characteristic equation $\lambda^2+k^2=0$ and eigenvalues $\lambda_{1,2}=\pm ik$. The general solution is $x(t)=c_1e^{ikt}+c_2e^{-ikt}$. Then $y(t)=-x'(t)=ikc_1e^{ikt}-ikc_2e^{-ikt}$.

In real form

$$x(t)=A\,\sin(kt+B),\;\; y(t)=-Ak\,\cos(kt+B) \tag{76}$$

Example 2. Solve the differential equation $y^{(iv)}+y=0$. Find four real valued independent solutions.

The characteristic equation is $\lambda^4+1=0$ with solutions $\lambda_k=e^{i\pi(2k+1)/4}$, $k=0,1,2,3$. The equation has four independent solutions $y_k(t)=\exp(i\pi(2k+1)/4\,t)$, $k=0,1,2,3$.

To identify the real and imaginary parts of the eigenvalues, note that $\lambda_0=\exp(\frac{i\pi}{4})=\frac{\sqrt{2}}{2}+i\frac{\sqrt{3}}{2}$, $\lambda_3=\overline{\lambda_0}$, $\lambda_2=-\lambda_0$, $\lambda_1=-\lambda_3$. (Alternatively, one can factor $\lambda^4+1=(\lambda^2+\sqrt{2}\lambda+1)(\lambda^2-\sqrt{2}\lambda+1)$ then solve.) We have the four independent solutions $\exp(\pm t\frac{\sqrt{2}}{2})\cos(t\frac{\sqrt{3}}{2})$, $\exp(\pm t\frac{\sqrt{2}}{2})\sin(t\frac{\sqrt{3}}{2})$.

3.7. **Systems of second order equations.** Systems of higher order linear equations, with constant coefficients can de solved using similar ideas. Consider for example

$$\frac{d^2\mathbf{u}}{dt^2}=M\mathbf{u} \tag{77}$$

Such systems can be reduced to a first order system by introducing new variables: denoting $\mathbf{v}=\frac{d\mathbf{u}}{dt}$ the n-dimensional system of second order equations (77) becomes the $2n$-dimensional system of first order equations

$$\frac{d}{dt}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}=\mathcal{M}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}\quad\text{where } \mathcal{M}=\begin{bmatrix}0 & I\\ M & 0\end{bmatrix} \tag{78}$$

To find the eigenvalues μ of $\mathcal{M}$ we solve $\det(\mathcal{M}-\mu I)=0$, and using Proposition 17 we find

$$\det(\mathcal{M}-\mu I)=\begin{vmatrix}-\mu I & I\\ M & -\mu I\end{vmatrix}=\det((-\mu I)^2-(-\mu I)^{-1}M(-\mu I))$$
$$=\det(\mu^2 I-M)$$

therefore μ^2 is an eigenvalue of M. It follows that if $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of M, then the eigenvalues of $\mathcal{M}$ are $\pm\sqrt{\lambda_1},\ldots,\pm\sqrt{\lambda_n}$. It can be checked that if $M\mathbf{u}_j=\lambda_j\mathbf{u}_j$ then

$$\begin{bmatrix}0 & I\\ M & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}_j\\ \pm\sqrt{\lambda_j}\mathbf{u}_j\end{bmatrix}=\pm\sqrt{\lambda_j}\begin{bmatrix}\mathbf{u}_j\\ \pm\sqrt{\lambda_j}\mathbf{u}_j\end{bmatrix}$$

giving the eigenvectors of $\mathcal{M}$. With this information solutions can be readily found.

3.8. **Stability in differential equations.**

3.8.1. *Stable versus unstable equilibrium points.* A linear, first order system of differential equation

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u} \tag{79}$$

always has the zero solution: $\mathbf{u}(t) = \mathbf{0}$ for all t. The point $\mathbf{0}$ is called an **equilibrium point** of the system (79). More generally,

Definition 24. *An* **equilibrium point** *of a differential equation* $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ *is a point* $\mathbf{u}_0$ *for which the constant function* $\mathbf{u}(t) = \mathbf{u}_0$ *is a solution, therefore* $\mathbf{f}(\mathbf{u}_0) = \mathbf{0}$.

It is important in applications to know how solutions behave near an equilibrium point.

An equilibrium point $\mathbf{u}_0$ is called **stable** if any solutions which start close enough to $\mathbf{u}_0$ remain close to $\mathbf{u}_0$ for all $t > 0$. (This definition can be made more mathematically precise, but it will not be needed here, and it is besides the scope of these lectures.)

Definition 25. *An equilibrium point* $\mathbf{u}_0$ *is called* **asymptotically stable** *if*

$$\lim_{t\to\infty} \mathbf{u}(t) = \mathbf{u}_0 \quad \textit{for any solution } \mathbf{u}(t)$$

It is clear that an asymptotically stable point is stable, but the converse is not necessarily true. For example, the harmonic oscillator (75) has solutions confined to ellipses, since from (76) it follows that $x^2+y^2/k^2 = A^2$. Solutions are close to the origin if A is small, and they go around the origin along an ellipse, never going too far, and not going towards the origin: the origin is a stable, but not asymptotically stable equilibrium point.

An equilibrium point which is not stable is called **unstable**.

Suppose one is interested in the stability of an equilibrium point of an equation $\mathbf{u}' = \mathbf{f}(\mathbf{u})$. By a change of variables the equilibrium point can be moved to $\mathbf{u}_0 = \mathbf{0}$, hence we assume $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. It is natural to approximate the equation by its linear part: $\mathbf{f}(\mathbf{u}) \approx M\mathbf{x}$, where the matrix M has the elements $M_{ij} = \partial \mathbf{f}_i/\partial \mathbf{x}_j(\mathbf{0})$, and expect that the stability (or instability) of the equilibrium point of $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ to be the same as for its linear approximation $\mathbf{u}' = M\mathbf{u}$.

This is true for asymptotically stable points, and for unstable points, under fairly general assumptions on $\mathbf{f}$. But it is not necessarily true for stable, not asymptotically stable, points as in this case the neglected terms of the approximation may change the nature of the trajectories.

Understanding the stability for linear systems helps understand the stability for many nonlinear equations.

3.8.2. *Characterization of stability for linear systems.* The nature of the equilibrium point $\mathbf{u}_0 = \mathbf{0}$ of linear differential equations depends on the eigenvalues of the matrix M as follows.

We saw that solutions of a linear system (79) are linear combinations of exponentials $e^{t\lambda_j}$ where λ_j are the eigenvalues of the matrix M, and if M is not diagonalizable, also of $t^k e^{t\lambda_j}$ for $0 < k \leq$(multiplicity of $\lambda_j) - 1$.

Recall that

$$\lim_{t\to\infty} t^k e^{t\lambda_j} = 0 \ \text{ if and only if } \Re\lambda_j < 0$$

Therefore:

(i) if all λ_j have negative real parts, then any solution $\mathbf{u}(t)$ of (79) converge to zero: $\lim_{t\to\infty} \mathbf{u}(t) = \mathbf{0}$, and $\mathbf{0}$ is asymptotically stable.

(ii) If all $\Re\lambda_j \leq 0$, and some real parts are zero, and eigenvalues with zero real part have the dimension of the eigenspace equal to the multiplicity of the eigenvalue[3] then $\mathbf{0}$ is stable.

(iii) If any eigenvalue has a positive real part, then $\mathbf{0}$ is unstable.

As examples, let us consider 2 by 2 systems with real coefficients.

[3]This means that if $\Re\lambda_j = 0$ then there are no solutions $\mathbf{q}(t)e^{t\lambda_j}$ with nonconstant $\mathbf{q}(t)$.

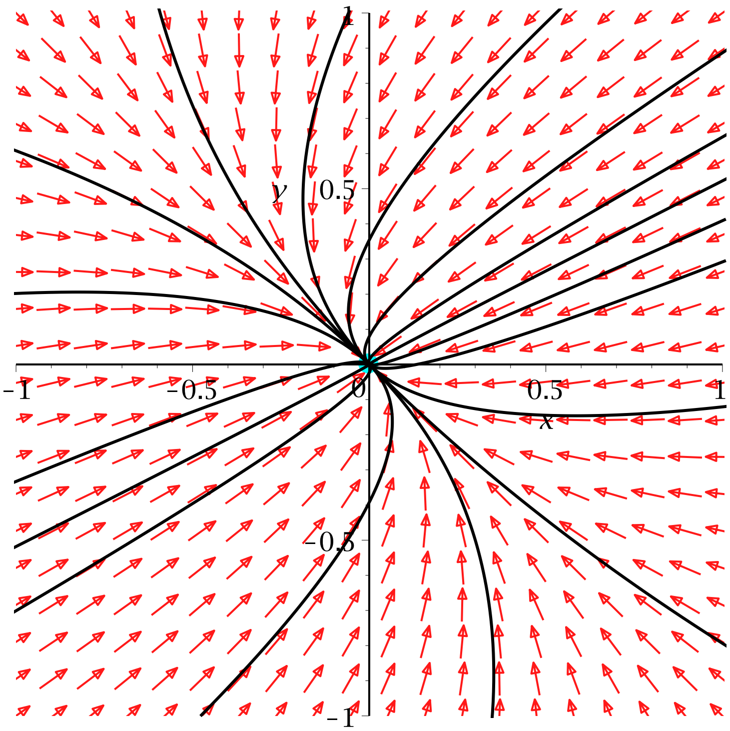


FIGURE 2. Asymptotically stable equilibrium point, negative eigenvalues.

Example 1: an asymptotically stable case, with all eigenvalues real.
For

$$M = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix}, \; M = S\Lambda S^{-1}, \text{ with } \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -6 \end{bmatrix}, \; S = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

The figure shows the field plot (a representation of the linear transformation $\mathbf{x} \to M\mathbf{x}$ of $\mathbb{R}^2$). The trajectories are tangent to the line field, and they are going towards the origin. Solutions with initial conditions along the directions of the two eigenvectors of M are straight half-lines (two such solutions are shown in the picture); these are the solutions $\mathbf{u}(t) = e^{\lambda_j t} c\mathbf{v}_j$. (Solutions with any other initial conditions are not straight lines.)

Example 1': an asymptotically unstable case, with all eigenvalues real. For

$$M = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix},\ M = S\Lambda S^{-1},\ \text{with } \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix},\ S = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

The figure shows the field plot (a representation of the linear transformation $\mathbf{x} \to M\mathbf{x}$ of $\mathbb{R}^2$). The trajectories are tangent to the line field, and they are going away from the origin. Solutions with initial conditions along the directions of the two eigenvectors of M are straight half-lines (two such solutions are shown in the picture); these are the solutions $\mathbf{u}(t) = e^{\lambda_j t} c\mathbf{v}_j$. (Solutions with any other initial conditions are not straight lines.)

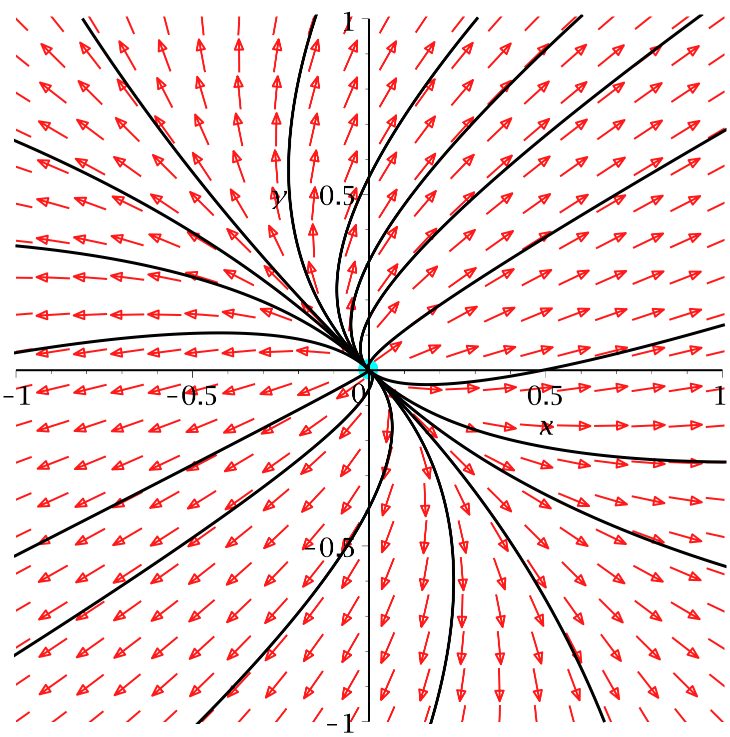


FIGURE 3. Asymptotically stable equilibrium point, nonreal eigenvalues.

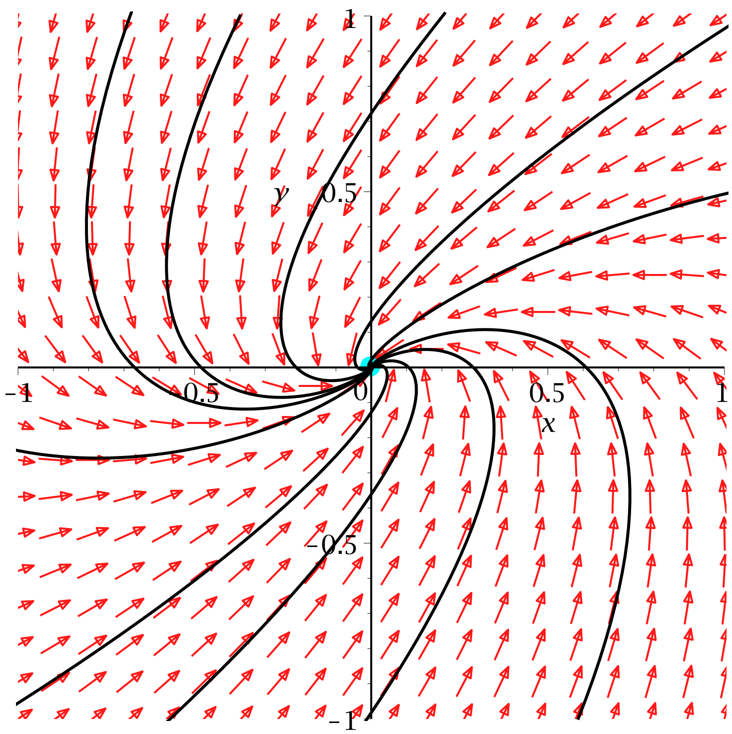


FIGURE 4. Asymptotically stable equilibrium point, nonreal eigenvalues.

Example 2: an asymptotically stable case, with nonreal eigenvalues.
For

$$M = \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} \text{ with } \Lambda = \begin{bmatrix} -2+i & 0 \\ 0 & -2-i \end{bmatrix}, \; S = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}$$

The figure shows the field plot and two trajectories. All trajectories are going towards the origin, though rotating around it. The equilibrium point $\mathbf{0}$ is hyperbolic.

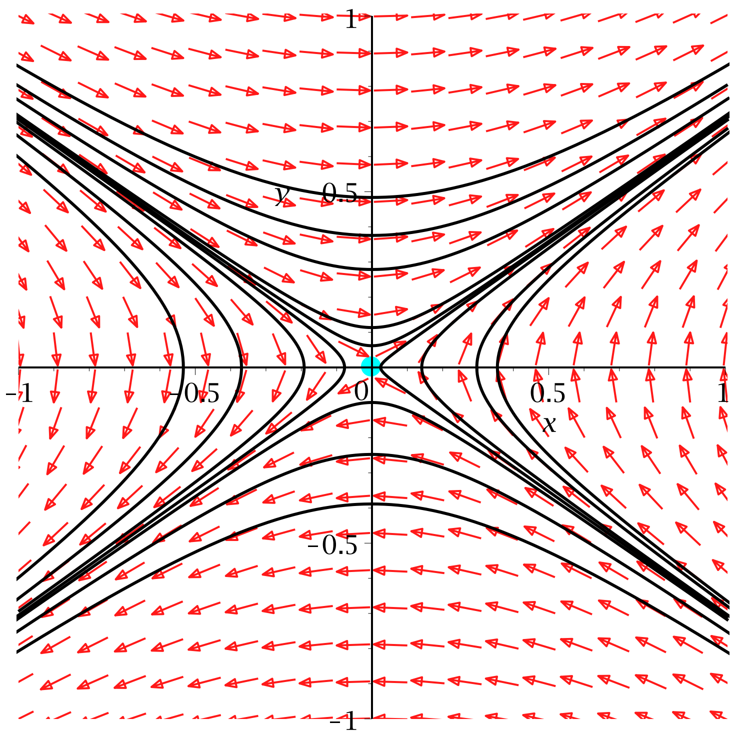


FIGURE 5. Unstable equilibrium point, one eigenvalue with positive real part and one with negative real part.

Example 3: an unstable case, with one negative eigenvalue, and a positive one.

For

$$M = \begin{bmatrix} 3 & 6 \\ 3 & 0 \end{bmatrix} \text{ with } \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix},\ S = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

The figure shows the field plot. Note that there is a stable direction (in the direction of the eigenvector corresponding to the negative eigenvalue), and an unstable one (the direction of the second eigenvector, corresponding to the positive eigenvalue). Any trajectory starting at a point not on the stable direction has infinite limit as $t \to \infty$.

The equilibrium point $\mathbf{0}$ is a saddle point.

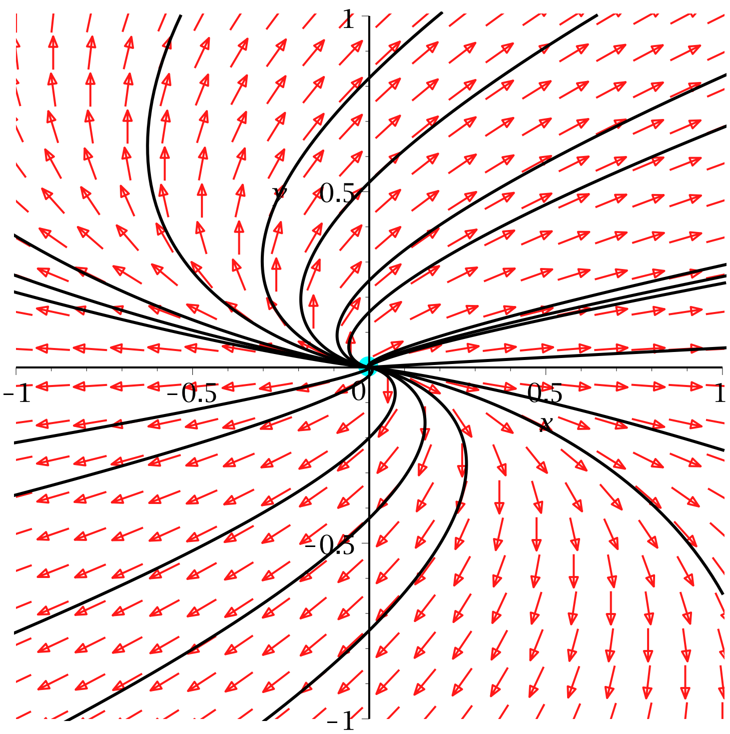


FIGURE 6. Unstable equilibrium point, one positive eigenvalue, nontrivial Jordan form.

Example 4: Unstable equilibrium, nontrivial Jordan form

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

What is the eigenvector? What is the generalized eigenvector?

Example 5: the equilibrium point $\mathbf{0}$ is stable, not asymptotically stable. For

$$M = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \text{ with } \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ S = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}$$

The trajectories rotate around the origin on ellipses, with axes determined by the real part and the imaginary part of the eigenvectors.

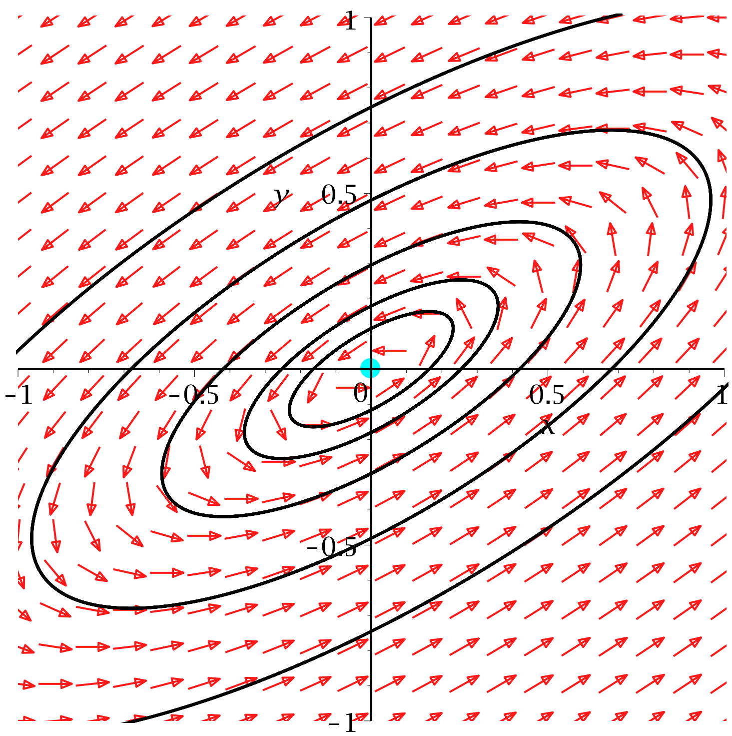


FIGURE 7. Stable, not asymptotically stable, equilibrium point.

4. Difference equations (Discrete dynamical systems)

4.1. **Linear difference equations with constant coefficients.** A first order difference equation, linear, homogeneous, with constant coefficients, has the form

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \tag{80}$$

where M is an $n \times n$ matrix, and $\mathbf{x}_k$ are n-dimensional vectors. Given an initial condition $\mathbf{x}_0$ the solution of (80) is uniquely determined: $\mathbf{x}_1 = M\mathbf{x}_0$, then we can determine $\mathbf{x}_2 = M\mathbf{x}_1$, then $\mathbf{x}_3 = M\mathbf{x}_2$, etc. Clearly the solution of (80) with the initial condition $\mathbf{x}_0$ is

$$\mathbf{x}_k = M^k\mathbf{x}_0 \tag{81}$$

A second order difference equation, linear, homogeneous, with constant coefficients, has the form

$$\mathbf{x}_{k+2} = M_1\mathbf{x}_{k+1} + M_0\mathbf{x}_k \tag{82}$$

A solution of (82) is uniquely determined if we give two initial conditions, $\mathbf{x}_0$ and $\mathbf{x}_1$. Then we can find $\mathbf{x}_2 = M_1\mathbf{x}_1 + M_0\mathbf{x}_0$, then $\mathbf{x}_3 = M_1\mathbf{x}_2 + M_0\mathbf{x}_1$ etc.

Second order difference equations can be reduced to first order ones: let $\mathbf{y}_k$ be the $2n$ dimensional vector

$$\mathbf{y}_k = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k+1} \end{bmatrix}$$

Then $\mathbf{y}_k$ satisfies the recurrence

$$\mathbf{y}_{k+1} = M\mathbf{y}_k \quad \text{where } M = \begin{bmatrix} 0 & I \\ M_0 & M_1 \end{bmatrix}$$

which is of the type (80), and has a unique solution if $\mathbf{y}_0$ is given.

More generally, a difference equation of order p which is linear, homogeneous, with constant coefficients, has the form

$$\mathbf{x}_{k+p} = M_{p-1}\mathbf{x}_{k+p-1} + \ldots + M_1\mathbf{x}_{k+1} + M_0\mathbf{x}_k \tag{83}$$

which has a unique solution if the initial p values are specified $\mathbf{x}_0, \mathbf{x}_1 \ldots, \mathbf{x}_{p-1}$. The recurrence (83) can be reduced to a first order one for a vector of dimension np.

To understand the solutions of the linear difference equations it then suffices to study the first order ones, (80).

4.2. **Solutions of linear difference equations.** Consider the equation (80). If M has n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (i.e. M is diagonalizable) let $S = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$ and then $M = S\Lambda S^{-1}$ with Λ the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. The solution (81) can be written as

$$\mathbf{x}_k = M^k\mathbf{x}_0 = S\Lambda^k S^{-1}\mathbf{x}_0$$

and denoting $S^{-1}\mathbf{x}_0 = \mathbf{b}$,

$$\mathbf{x}_k = S\Lambda^k\mathbf{b} = b_1\lambda_1^k\mathbf{v}_1 + \ldots + b_n\lambda_n^k\mathbf{v}_n$$

hence solutions $\mathbf{x}_k$ are linear combinations of λ_j^k multiples of the eigenvectors $\mathbf{v}_j$.

Example. Solve the recurrence relation $z_{n+2} = 3z_{n+1} - 2z_n$ if $z_0 = \alpha$, $z_1 = \beta$.

This is a scalar difference equation, and we could turn it into a first order system. But, by analogy to higher order scalar differential equations, it may be easier to work directly with the scalar equation. We know that there are solutions of the form $z_n = \lambda^n$, and substituting this in the recurrence we get $\lambda^{n+2} = 3\lambda^{n+1} - 2\lambda^n$ therefore $\lambda^2 - 3\lambda + 2 = 0$, implying $\lambda_1 = 1$, $\lambda_2 = 2$, or $\lambda = 0$. We found the solutions $z_n = 1$ and $z_n = 2^n$. We can always discard the value $\lambda = 0$ since it corresponds to the trivial zero solution. The general solution is $z_n = c1 + 2^n c_2$. The constants c_1, c_2 can be determined from the initial conditions: $z_0 = c_1 + c_2 = \alpha$, $z_1 = c_1 + 2c_2 = \beta$, therefore $z_n = (2\alpha + \beta) + (\beta - \alpha)2^n$.

If M is not diagonalizable, just as in the case of differential equations, then consider a matrix S so that $S^{-1}MS$ is in Jordan normal form.

Consider the example of a 2×2 Jordan block: $M = SJS^{-1}$ with J given by (62). As in Let $S = [\mathbf{y}_1, \mathbf{y}_2]$ where $\mathbf{y}_1$ is the eigenvector of M corresponding to the eigenvalue λ and $\mathbf{y}_2$ is a generalized eigenvector. Using (56) we obtain the general solution

$$\mathbf{x}_k = [\mathbf{y}_1, \mathbf{y}_2] \begin{bmatrix} \lambda^k & k\,\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1\lambda^k\mathbf{y}_1 + b_2\left(k\lambda^k\mathbf{y}_1 + \lambda^k\mathbf{y}_2\right)$$

and the recurrence has two linearly independent solutions of the form $\lambda^k\mathbf{y}_1$ and $\mathbf{q}(k)\lambda^k$ where $\mathbf{q}(k)$ is a polynomial in k of degree one.

In a similar way, for $p \times p$ Jordan blocks there are p linearly independent solutions of the form $\mathbf{q}(k)\lambda^k$ where $\mathbf{q}(k)$ are polynomials in k of degree at most $p - 1$, one of then being constant, equal to the eigenvector.

Example. Solve the recurrence relation $y_{n+3} = 9\,y_{n+2} - 24\,y_{n+1} + 20y_n$.

Looking for solutions of the type $y_n = \lambda^n$ we obtain $\lambda^{n+3} = 9\,\lambda^{n+2} - 24\,\lambda^{n+1} + 20\lambda^n$ which implies (disregarding $\lambda = 0$) that $\lambda^3 - 9\,\lambda^2 + 24\,\lambda - 20 = 0$ which factors $(\lambda - 5)(\lambda - 2)^2 = 0$ therefore $\lambda_1 = 5$ and $\lambda_2 = \lambda_3 = 2$. The general solution is $z_n = c_1 5^n + c_2 2^n + c_3 n 2^n$.

4.3. **Stability.** Clearly the constant zero sequence $\mathbf{x}_k = \mathbf{0}$ is a solution of any linear homogeneous discrete equation (80): $\mathbf{0}$ is an **equilibrium point** (a *steady state*).

As in the case of differential equations, an equilibrium point of a difference equation is called *asymptotically stable*, or an **attractor**, if solutions starting close enough to the equilibrium point converge towards it.

For linear difference equations this means that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{0}$ for all solutions $\mathbf{x}_k$. This clearly happens if and only if all the eigenvalues λ_j of M satisfy $|\lambda_j| < 1$.

If all eigenvalues have either $|\lambda_j| < 1$ or $|\lambda_j| = 1$ and for eigenvalues of modulus 1, the dimension of the eigenspace equals the multiplicity of the eigenvalue, $\mathbf{0}$ is a *stable* point (or neutral).

In all other cases the equilibrium point is called *unstable*.

4.4. Example: Fibonacci numbers.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

this is one of the most famous sequence of numbers, studied for more that 2 milennia (it fist appeared in ancient Indian mathematics), which describes countless phenomena in nature, in art and in sciences.

The Fibonacci numbers are defined by the recurrence relation

$$F_{k+2} = F_{k+1} + F_k \tag{84}$$

with the initial condition $F_0 = 0,\ F_1 = 1$.

Substituting $F_k = \lambda^k$ into the recurrence (84) it follows that $\lambda^2 = \lambda + 1$ with solutions

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \phi = \textbf{the golden ratio}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} = -1/\phi$$

F_k is a linear combination of λ_1^k and λ_2^k: $F_k = c_1\lambda_1^k + c_2\lambda_2^k$. The values of c_1, c_2 can be found from the initial conditions: $c_1 + c_2 = 1$ and $c_1\phi_1 + c_2/phi_1 = 1$ implying

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$

Note that the ratio of two consecutive Fibonacci numbers converges to the golden ratio:

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \phi$$

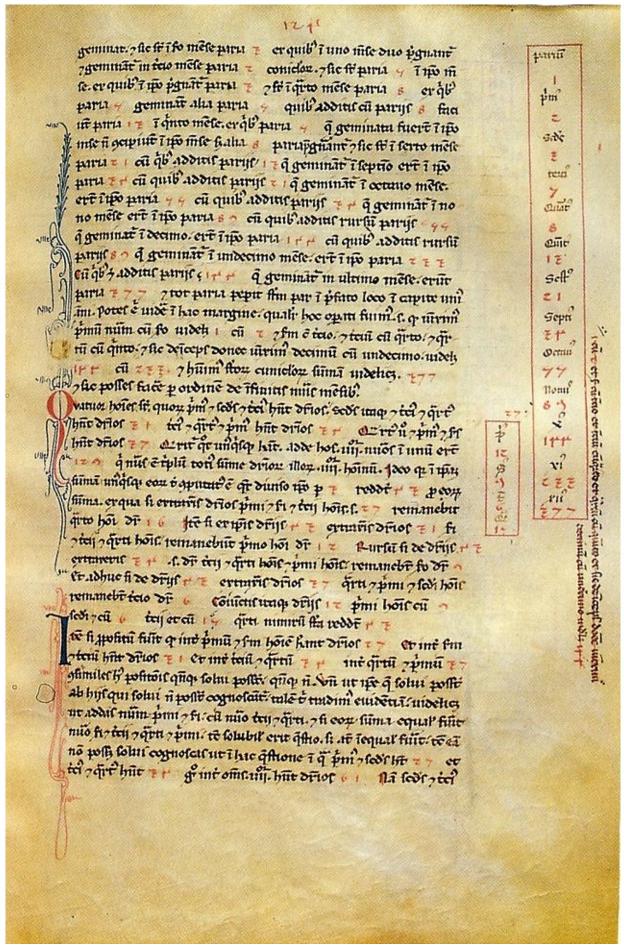


FIGURE 8. A page of Fibonacci's Liber Abaci from the Biblioteca Nazionale di Firenze showing (in box on right) the Fibonacci sequence with the position in the sequence labeled in Latin and Roman numerals and the value in Hindu-Arabic numerals (from Wiki).

4.5. **Positive matrices.**

Definition 26. *A* **positive matrix** *is a square matrix whose entries are all positive numbers.*

Caution: this is not to be confused with *positive definite* self adjoint matrices, which will be studied later.

Positive matrices have countless applications and very special properties.
Notations.

$$\mathbf{x} \geq 0 \text{ denotes a vector with all components } x_j \geq 0$$

$$\mathbf{x} > 0 \text{ denotes a vector with all components } x_j > 0$$

Theorem 27. Perron-Frobenius Theorem

Let P be a positive matrix: $P = [P_{ij}]_{i,j=1,\ldots,n}$, $P_{ij} > 0$.

P has a dominant eigenvalue *(or,* Perron root*, or* Perron-Frobenius eigenvalue*) $r(P) = \lambda_1$ with the following properties:*

(i) $\lambda_1 > 0$ and the associated eigenvector $\mathbf{v}_1$ is positive: $\mathbf{v}_1 > 0$.

(ii) λ_1 is a simple eigenvalue.

(iii) All other eigenvalues have smaller modulus: if $|\lambda_j| < \lambda_1$ for all eigenvalues λ_j of P, $j > 1$.

(iv) All other eigenvectors of P are not nonnegative, $\mathbf{v}_j \not\geq 0$ (they have have at least one negative or nonreal entry).

(v) λ_1 satisfies the following maximin property: $\lambda_1 = \max T$ where

$$T = \{t \geq 0 \,|\, P\mathbf{x} \geq t\mathbf{x}, \text{ for some } \mathbf{x} \geq 0, \ \mathbf{x} \neq \mathbf{0}\}$$

(v') λ_1 satisfies the following minimax property: $\lambda_1 = \min S$ where

$$S = \{t \geq 0 \,|\, P\mathbf{x} \leq t\mathbf{x}, \text{ for all } \mathbf{x} \geq 0, \mathbf{x} \neq \mathbf{0}\}$$

(vi) Also

$$\min_i \sum_j P_{ij} \leq \lambda_1 \leq \max_i \sum_j P_{ij}$$

The proof of the Perron theorem will not be given here.

4.6. **Markov chains.** Markov processes model random chains of events events whose likelihood depends on, and only on, what happened last. Step $n+1$ depends only on the current state: the process has no memory.

4.6.1. *Example.* Suppose that it was found that every year 1% of the US population living in coastal areas moves inland, and 2% of the US population living inland moves to coastal areas[4]. Denote by x_k and y_k the number of people living in coastal areas, respectively inland, at year k. We are interested to understand how the population distribution among these areas evolves in the future.

Assuming the US population remains the same, in the year $k+1$ we find that $x_{k+1} = .99x_k + .02y_k$ and $y_{k+1} = .01x_k + .98y_k$ or

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \tag{85}$$

where

$$\mathbf{x}_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \ M = \begin{bmatrix} .99 & .02 \\ .01 & .98 \end{bmatrix}$$

Relation (85) modeling our process is a first order difference equation. Note that the entries of the matrix M are nonnegative (they represent a percentage, or a probability), and that its columns add up to 1, since the whole population is subject to the process: any person of the US population is in one of the two regions.

[4]These are not real figures. Unfortunately, I could not find real data on this topic.

Question: what happens in the long run, as $k \to \infty$? Would the whole population eventually move to coastal areas?

To find the solution $\mathbf{x}_k$ of (85) we need the eigenvalues and eigenvectors of M: it is easily calculated that there is one eigenvalue equal to 1 corresponding to $\mathbf{v}_1 = (2,1)^T$, and an eigenvalue .97, corresponding to $\mathbf{v}_2 = (-1,1)^T$. (Note that M is a positive matrix, and the Perron-Frobenius Theorem applies: the dominant eigenvalue is 1, and its eigenvector has positive components, while the other eigenvector has both positive and nonpositive components.)

Then

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2\, .97^k\mathbf{v}_2$$

and

$$\mathbf{x}_\infty = \lim_{k\to\infty} \mathbf{x}_k = c_1\mathbf{v}_1$$

The limit is an eigenvector corresponding to the eigenvalue 1!

In fact this is not a big surprise if we reason as follows : assuming that $\mathbf{x}_k$ converges (which is not guaranteed without information on the eigenvalues of M) then taking the limit $k \to \infty$ in the recurrence relation (85) we find that $\mathbf{x}_\infty = M\mathbf{x}_\infty$ hence the limit $\mathbf{x}_\infty$ is an eigenvector of M corresponding to the eigenvalue 1, or the limit is $\mathbf{0}$ - which is excluded by the interpretation that $x_k + y_k = const$=the total population.

Note: *all the eigenvectors corresponding to the eigenvalue 1 are steady-states*: if the initial population distribution was $\mathbf{x}_0 = a\mathbf{v}_1$ then the population distribution remains the same: $\mathbf{x}_k = \mathbf{x}_0$ for all k (since $M\mathbf{v}_1 = \mathbf{v}_1$).

Exercise. What is the type of the equilibrium points $c_1\mathbf{v}_1$ (asymptotically stable, stable, unstable)?

In conclusion, in the long run the population becomes distributed with twice as many people living in coastal area than inland.

4.6.2. *Markov matrices.* More generally, a Markov process is governed by an equation (85) where the matrix M has two properties summarized as follows.

Definition 28. *An $n \times n$ matrix $M = [M_{ij}]$ is called a* **Markov matrix** *(or a* **Stochastic matrix***) if:*
(i) all $M_{ij} \geq 0$, and
(ii) each column adds up to 1: $\sum_i M_{ij} = 1$.

Theorem 29. Properties of Markov matrices

If M is a Markov matrix then:
(i) $\lambda = 1$ is an eigenvalue.
(ii) All the eigenvalues satisfy $|\lambda_j| \leq 1$. If all the entries of M are positive, then $|\lambda_j| < 1$ for $j > 1$.
(iii) If for some k all the entries of M^k are positive, then $\lambda_1 = 1$ has multiplicity 1 and all the other eigenvalues satisfy $|\lambda_j| < 1$ for $j = 2, \ldots, n$.

Proof.
(i) The matrix $M - I$ is not invertible, since all the columns of M add up to 1, and therefore the columns of $M - I$ add up to zero. Therefore $\det(M - I) = 0$ and 1 is an eigenvalue.
(ii) and (iii) follow from Perron-Frobenius Theorem 27. □

Note that for general Markov matrices all eigenvectors corresponding to the eigenvalue 1 are steady states.

5. More functional calculus

5.1. **Discrete equations versus differential equations.**

Suppose a function $y(t)$ satisfies a the differential equation

$$\frac{dy}{dt} = ay(t),\ y(0) = y_0 \tag{86}$$

with solution $y(t) = e^{at}y_0$

Discretize the equation: fix some small h and consider only the values $t = t_k = kh$. Using the linear approximation

$$y(t_{k+1}) = y(t_k) + y'(t_k)h + O(h^2) \tag{87}$$

then

$$hy'(t_k) \approx y(t_{k+1}) - y(t_k)$$

which used in (88) gives the difference equation

$$\tilde{y}(t_{k+1}) - \tilde{y}(t_k) = ah\tilde{y}(t_k),\ \tilde{y}(0) = y_0 \tag{88}$$

with solution $\tilde{y}(t_k) = (1 + ah)^k y_0$

The discrete equation (88) is an approximation of the continuous equation (88).

We can get better approximations in two ways. Taking h smaller and smaller in (88), $\tilde{y}(t_k)$ approaches $y(t)$:

$$\tilde{y}(t_k) = (1 + ah)^{t_k/h}\, y_0 \to e^{at_k}y_0 \ \text{ as } h \to 0$$

.

Another way to improve the approximation is to retain more terms in the approximation (87):

$$y(t_{k+1}) = y(t_k) + y'(t_k)h + \tfrac{1}{2}y''(t_k)h^2 + \tfrac{1}{6}y'''(t_k)h^3 + \ldots$$

"The limit" is the Taylor series

$$y(t_{k+1}) = y(t_k + h) = \sum_{k=0}^{\infty} \frac{1}{n!}\frac{d^n}{dt^n}y(t_k)\, h^n$$

(this is **complete hand-waving at this stage**. The series may not even exist for a fixed y let alone converge to it!) Noting that $\frac{d^n}{dt^n}$ is the n-th power of the linear operator $\frac{d}{dt}$ one can formally write

$$y(t+h) = \left(\sum_{k=0}^{\infty} \frac{1}{n!}\, \frac{d^n}{dt^n}\, h^n\right) y(t) = e^{h\frac{d}{dt}} y(t)$$

therefore

$$y(t+h) = e^{h\frac{d}{dt}} y(t) \tag{89}$$

which a remarkable formula: *the exponential of differentiation is a shift.*

Note that the fact that $y(t)$ solves (88) means that $y(t)$ is an eigenfunction of the operator $\frac{d}{dt}$ corresponding to the eigenvalue a. By §3.5 this means that $y(t)$ is an eigenfunction of $e^{h\frac{d}{dt}}$ corresponding to the eigenvalue e^{ah}. Therefore $e^{h\frac{d}{dt}} y(t) = e^{ah} y(t)$

5.2. **Functional calculus for digonalizable matrices.** Let M be a square $n \times n$ matrix, assumed diagonalizable: it has n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ and if $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ then $S^{-1}MS = \Lambda$ a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on its diagonal.

5.2.1. *Polynomials of M.* We looked at positive integer powers of M, and we saw that $M^k = S\Lambda^k S^{-1}$, where the power k is applied to each diagonal entry of Λ. To be consistent we clearly need to define $M^0 = I$.

Recall that M is invertible if and only if all its eigenvalues are not zero. Assume this is the case. Then we can easily check that $M^{-1} = S\Lambda^{-1}S^{-1}$ where the power k is applied to each diagonal entry of Λ. We can then define any negative integer power of M.

If $p(t) = a_n t^n + \ldots + a_1 t + a_0$ is a polynomial in t, we can easily define

$$p(M) = a_n M^n + \ldots + a_1 M + a_0 I = S\, p(\Lambda)\, S^{-1}$$

where $p(\Lambda)$ is the diagonal matrix with $p(\lambda_j)$ on the diagonal.

5.2.2. *The exponential e^M.* We defined the exponential e^M using its Taylor series

$$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$$

and $e^M = Se^{\Lambda}S^{-1}$ where e^{Λ} is the diagonal matrix with e^{λ_j} on the diagonal.

5.2.3. *The resolvent.* For which numbers $z \in \mathbb{C}$ the matrix $zI - M$ has an inverse, and what are its eigenvalues? Clearly the matrix $zI - M$ is invertible for all z which differ from the eigenvalues of M (in the infinite dimensional case things are not quite so).

The matrix valued function $R(z) = (zI - M)^{-1}$, defined for all $z \neq \lambda_1, \ldots \lambda_n$ is called the resolvent of M. The resolvent has many uses, and is particularly useful in infinite dimensions.

Let $z = 1$. If M is diagonalizable then $(zI - M)^{-1} = S\,(zI - \Lambda)^{-1}\,S^{-1}$ where $(zI - \Lambda)^{-1}$ is the diagonal matrix with $(z - \lambda_j)^{-1}$ on the diagonal.

Here is another formula, very useful for the infinite dimensional case: if

$$M \text{ is diagonalizable, with all the eigenvalues satisfying } |\lambda_j| < 1$$

then

$$(I - M)^{-1} = I + M + M^2 + M^3 + \ldots \tag{90}$$

which follows from the fact that

$$\frac{1}{1-\lambda_j} = 1 + \lambda_j + \lambda_j^2 + \lambda_j^3 + \ldots \quad \text{if } |\lambda_j| < 1$$

The resolvent is extremely useful for nondiagonalizable cases as well. In infinite dimensions the numbers z for which the resolvent does not exist, **the spectrum** of the linear transformation, (they may or may nor be eigenvalues) play the role of the eigenvalues in finite dimensions.

We will see that (90) is true for matrices of norm less than 1 - this is a good motivation for introducing the notion of norm of a matrix later on.

5.2.4. *The square root of M.* Given the diagonalizable matrix M can we find matrices R so that $R^2 = M$, and what are they?

Using the diagonal form of M we have: $R^2 = S\Lambda S^{-1}$ which is equivalent to $S^{-1}R^2S = \Lambda$ and therefore $(S^{-1}RS)^2 = \Lambda$.

Assuming that $S^{-1}RS$ is diagonal, then $S^{-1}RS =$diag$(\pm\lambda_1^{1/2}, \ldots, \pm\lambda_n^{1/2})$ and therefore

$$R = S \begin{bmatrix} \sigma_1 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & \sigma_n \end{bmatrix} S^{-1}, \quad \sigma_j \in \{1, -1\} \tag{91}$$

There are 2^n such matrices!

But there are also matrices with $S^{-1}RS$ not diagonal. Take for example $M = I$, and find all the matrices R with $R^2 = I$. Then besides the four diagonal solutions

$$R = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad \sigma_{1,2} \in \{1, -1\}$$

there is the two parameter family of the solutions

$$R = \begin{bmatrix} \pm\sqrt{1-ab} & a \\ b & \mp\sqrt{1-ab} \end{bmatrix}$$

Some of these matrices have nonreal entries!

Can you find *all* matrices s.t. $M^2 = I$ in any number of dimensions?

5.2.5. *Functional calculus for diagonalizable matrices.* What other functions of M can we define? *If M is diagonalizable* it seems that given a function $f(t)$ we can define $f(M)$ provided that all $f(\lambda_j)$ are defined (a careful construction is needed).

Diagonalizable matrices are thus very "user friendly". Later on we will see that there is a quick test to see which matrices are diagonalizable, and which are not.

5.2.6. *Working with Jordan blocks.* The calculations done for 2 and 3 dimensional Jodan blocks in §3.3 can be done in a tidy way for the general $n \times n$ blocks using functional calculus.

First note that any $n \times n$ Jordan block, with eigenvalue λ can be written as

$$J = \lambda I + N$$

where N is a matrix whose only nonzero entries are 1 above the diagonal. A short calculation shows that N^2 only nonzero entries are a sequence of 1's at a distance two above diagonal, and so on: each additional power of N pushes the slanted line of 1 moves toward the upper right corner. Eventually $N^n = 0$. For example in dimension four:

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} N^4 = 0$$

Since I and N commute we can use the binomial formula which gives

$$J^k = (\lambda I + N)^k = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \lambda^{k-j} N^j$$

which for $k > n-2$ equals $\sum_{j=0}^{n-1} \begin{pmatrix} k \\ j \end{pmatrix} \lambda^{k-j} N^j$. See (56), (63) for $n = 2$ and $n = 3$.

Also *because I and N commute*

$$e^J = e^{\lambda I + N} = e^{\lambda I} e^N = e^{\lambda} \sum_{k=0}^{n-1} \frac{1}{k!} N^k$$

See (57), (64) for $n = 2$ and $n = 3$.

Exercise. What is $(I - J)^{-1}$ for J an $n \times n$ Jordan block with eigenvalue λ?

5.2.7. *The Cayley-Hamilton Theorem.* Here is a beautiful fact:

Theorem 30. The Cayley-Hamilton Theorem. *Let M be a square matrix, and $p(\lambda) = \det(M - \lambda I)$ be its characteristic polynomial.*

Then $p(M) = 0$.

Note that if M is $n \times n$ then it follows in particular that M^n is a linear combinations of earlier powers $I, M, M^2, \ldots, M^{n-1}$.

Proof of the Cayley-Hamilton Theorem.

Assume first that M is diagonalizable: $M = S\Lambda S^{-1}$. Then $p(M) = p(S\Lambda S^{-1}) = S\,p(\Lambda)\,S^{-1}$ where $p(\Lambda)$ is the diagonal matrix having $p(\lambda_j)$ on the diagonal. Since $p(\lambda_j) = 0$ for all j then $p(\Lambda) = 0$ and the theorem is proved.

In the general case $M = SJS^{-1}$ where J is a Jordan normal form. Then $p(J)$ is a block diagonal matrix, the blocks being p applied to standard Jordan blocks. Let J_1 be any one of these blocks, with eigenvalue λ_1 and dimension p_1. Then the characteristic polynomial of M contains the factor $(\lambda_1 - \lambda)^{p_1}$. Since $(\lambda_1 - J_1)^{p_1} = (-N_1)^{p_1} = 0$ then $p(J_1) = 0$. As this is true for each Jordan block composing J, the theorem follows. $\square$

5.3. **Commuting matrices.** The beautiful world of functional calculus with matrices is marred by noncommuatitivity. For example $e^A e^B$ equals e^{A+B} *only if* A and B commute, and the square $(A+B)^2 = A^2 + AB + BA + B^2$ cannot be simplified to $A^2 + 2AB + B^2$ unless A and B commute.

When do two matrices commute?

Theorem 31. *Let A and B be two diagonalizable matrices.*

Then $AB = BA$ if and only if they have the same matrix matrix of eigenvectors S (they are simultaneously *diagonalizable).*

Proof. Assume that $A = S\Lambda S^{-1}$ and $B = S\Lambda' S^{-1}$ where Λ, Λ' diagonal. Then, since diagonal matrices commute,

$$AB = S\Lambda S^{-1} S\Lambda' S^{-1} = S\Lambda\Lambda' S^{-1} = S\Lambda'\Lambda S^{-1} = S\Lambda' S^{-1} S\Lambda S^{-1} = BA$$

Conversely, assume $AB = BA$ and let $S = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be the matrix diagonalizing A, with $A\mathbf{v}_j = \alpha_j \mathbf{v}_j$. Then $BA\mathbf{v}_j = \alpha_j B\mathbf{v}_j$ so $AB\mathbf{v}_j = \alpha_j B\mathbf{v}_j$ which means that both $\mathbf{v}_j$ and $B\mathbf{v}_j$ are eigenvectors of A corresponding to the same eigenvalue α_j.

If all the eigenvalues of A are simple, then this means that $B\mathbf{v}_j$ is a scalar multiple of $\mathbf{v}_j$ so S diagonalizes B as well.

If A has multiple eigenvalues then we may need to change S a little bit (each set of eigenvectors of A corresponding to the same eigenvalue), to accommodate B.

First replace A by a diagonal matrix: $AB = BA$ is equivalent to $S\Lambda S^{-1}B = BS\Lambda S^{-1}$ therefore $\Lambda S^{-1}BS = S^{-1}BS\Lambda$. Let $C = S^{-1}BS$, satisfying $\Lambda C = C\Lambda$.

We can assume that the multiple eigenvalues of Λ are grouped together, so that Λ is built of diagonal blocks of the type $\alpha_j I$ of dimensions d_j, with distinct α_j.

A direct calculation shows that $\Lambda C = C\Lambda$ is equivalent to the fact that C is block diagonal with blocks of dimensions d_j. Since C is diagonalizable, each block can be diagonalized: $C = T\Lambda' T^{-1}$, and this conjugation leaves Λ invariant (why?): $T\Lambda T^{-1} = \Lambda$.

Then the matrix ST diagonalizes both A and B. $\square$

Examples. Any two functions of a matrix M, $f(M)$ and $g(M)$, commute.