5101 (AU 2013) **Homework 1** Name(s):

1. Check that $\mathbb{R}^2$ with the usual, component-wise, addition and scalar multiplication is a vector space over the scalars $\mathbb{R}$.

2. a) Show that the set of continuous functions over the interval $[a, b]$

$$C[a,b] = \{f : [a,b] \to \mathbb{R} \,|\, f \text{ continuous on } [a,b]\}$$

is a linear space over the scalars $\mathbb{R}$.

b) Show that $C_0[a,b] := \{f \in C[a,b] \,|\, f(a) = f(b) = 0\}$ is a subspace of $C[a,b]$.

c) Is $\{f \in C[a,b] \,|\, \int_a^b f(x)\,dx = 0\}$ a subspace of $C[a,b]$? Justify.

3. a) Let X and Y be two vector spaces over F. Let

$$X \oplus Y = \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x} \in X,\, \mathbf{y} \in Y\}$$

Show that $X \oplus Y$ is a vector space over F with addition and scalar multiplication defined component-wise, as

$$(\mathbf{x}_1, \mathbf{y}_1) + (\mathbf{x}_2, \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2), \quad c(\mathbf{x}, \mathbf{y}) = (c\mathbf{x}, c\mathbf{y})$$

$X \oplus Y$ is called the (external) *direct sum* of the vector spaces X and Y.

b) Given $\{\mathbf{v}_i\}_{i=1,\ldots n}$ a basis for X, and $\{\mathbf{w}_j\}_{j=1,\ldots,k}$ a basis for Y, find a basis for $X \oplus Y$. Express the dimension of $X \oplus Y$ in terms of $\dim X$ and $\dim Y$.

4. In each of the following cases, establish whether or not the given set of vectors is linearly independent or linearly dependent in the given vector/linear space. Explain.

a) $1, \cos t, \cos 2t, \ldots, \cos nt$ in $C[0, 2\pi]$.

b) $p(t) = (t-1)(t-2)(t-3)$, $q(t) = t(t-2)(t-3)$, $r(t) = t(t-1)(t-3)$, $s(t) = t(t-1)(t-2)$ in $\mathcal{P}$.

c) $t^{\sqrt{2}}$, t^e, t^π in $C(0, \infty)$

d) $\cosh x$, $\cosh(x-1)$ in $C(\mathbb{R})$.

e) $(1, 1, 1, 1)$, $(0, 2, 1, -1)$, $(2, -4, -1, 5)$ in $\mathbb{R}^4$.

f) $(1, 1, 1, 1)$, $(0, 2, 1, -1)$, $(2, -1, 1, -1)$ in $\mathbb{R}^4$.

5. [Optional, **not** assigned] What is the span of the vectors

$$v_1 = (1, 0, 0, ..., 0), v_2 = (1, 1, 0, ..., 0), v_3 = (1, 1, 1, ..., 0) \cdots v_n = (1, 1, 1, ..., 1)$$

in $\mathbb{R}^n$?