5101 (AU 2013) **Homework 2** Name(s):

*Justify all your answers!*

**1.** Let $A_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}$, $Z = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$.

a1) True or False? $Y \in Sp(A_1, A_2, A_3)$.
a2) True or False? $Z \in Sp(A_1, A_2, A_3)$.
b) Find a basis for $Sp(A_1, A_2, A_3)$.

**2.** Consider the following polynomials in $\mathcal{P}_3$ (the linear space of polynomials of degree at most three) $\phi_1(t) = t^3$, $\phi_2(t) = t^2(1-t)$, $\phi_3(t) = t(1-t)^2$, $\phi_4(t) = (1-t)^3$.

Show that every polynomial $p \in \mathcal{P}_3$ has a unique representation $p(t) = \sum_{j=1}^{4} c_j \phi_j(t)$ with $c_1, \ldots, c_4$ constants.

**3.** Let $\mathcal{B}_N =$ the set of all linear combinations of $e^{ikt}, k = -N, \ldots, 0, \ldots N$, with complex coefficients. (Such linear combinations form a set of "band limited" functions.)

It is easy to see that $\mathcal{B}_N$ is a linear space of functions over $\mathbb{C}$, and it can be shown that $e^{ikt}, k = -N, \ldots, 0, \ldots N$ are linearly independent. Assume these are true.

a) What is $\dim \mathcal{B}_N$?
b) Show that $\{1, \cos t, \ldots, \cos Nt, \sin t, \ldots, \sin Nt\}$ is a basis for $\mathcal{B}_N$.

**4.** Consider the transformation of $\mathbb{R}^3$ given by the matrix multiplication $\mathbf{x} \to M\mathbf{x}$ where

$$M = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

a) Find the range and the null space of this transformation, describing these subspaces by giving a basis and as geometrical objects in $\mathbb{R}^3$.
b) Find all the vectors $\mathbf{b} \in \mathbb{R}^3$ for which the system $M\mathbf{x} = \mathbf{b}$ is soluble.
c) Find the general solution to $M\mathbf{x} = \mathbf{0}$.

*(More problems on next page)*

**5.** Let $\mathcal{M}_{2,2}(\mathbb{R})$ be the linear space of all $2 \times 2$ matrices with the real entries.

a) Prove that $\mathcal{M}_{2,2}(\mathbb{R})$ has dimension 4 and find a basis.

b) Show that the trace

$$Tr : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}, \quad Tr(A) = A_{11} + A_{22}$$

is a linear functional and use this to show that the set of matrices of zero trace form a subspace in $\mathcal{M}_{2,2}(\mathbb{R})$.

c) What is the dimension of the subspace of matrices of zero trace?

**6.** Let $V$ and $W$ both be subspaces of a given vector space.

True or False? Their intersection $V \cap W = \{x : x \in V \text{ and } x \in W\}$ is also a subspace. (To justify, you need to prove it if True, or find a counterexample if False.)