1. a) For the basis dual to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ give explicit formulas for $\mathbf{e}_1'(x_1, x_2, x_3)$, $\mathbf{e}_2'(x_1, x_2, x_3)$, $\mathbf{e}_3'(x_1, x_2, x_3)$.

b) Write the linear functional $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x_1, x_2, x_3) = a_1 x_1 + a_2 x_2 + a_3 x_3 \tag{1}$$

as a linear combination of $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$.

2. Consider the basis for $\mathbb{R}^3$: $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

a) Find the dual basis, giving formulas in the form (1).

b) Find the row vector representation of the dual basis in the standard basis of $\mathbb{R}^3$.

Hint: you can do calculations based on definitions, or you can use the result of Section 1.4.1 of the notes on The Dual Space.

3. *Numerical quadrature.* Numerical integration of functions uses only the values of functions at certain sample points. In the space $\mathcal{P}_n$ of polynomials of degree at most n there is an *exact* numerical quadrature: show that if $s_0, s_1, \ldots, s_n$ are some sample points in an interval $[a, b]$ then there are some numbers $c_0, c_1, \ldots, c_n$ so that

$$\int_a^b p(x)\, dx = c_0 p(s_0) + a_1 p(s_1) + \ldots + c_n p(s_n) \quad \text{for all polynomials } p \in \mathcal{P}_n$$

Specify a way to calculate the numbers $c_0, c_1, \ldots, c_n$, but do not calculate them.

(continued on next page)

4. Consider the function $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}^2$ given by the formula

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2d \\ b-c \end{bmatrix}$$

a) Show that L is a linear transformation.
b) Give formulas for all the matrices in $\mathcal{N}(L)$.
c) Find a basis for $\mathcal{N}(L)$.
d) Determine the nullity and the rank of L.

5. a) Show that 1, $x-a$, $(x-a)^2$ is a basis for $\mathcal{P}_2$ and that E_a, $\frac{d}{dx}\big|_{x=a}$, $\frac{1}{2}\frac{d^2}{dx^2}\big|_{x=a}$ is the dual basis; the notation is the usual one:

$$(E_a, p) = p(a), \quad (\frac{d}{dx}\big|_{x=a}, p) = p'(a), \quad (\tfrac{1}{2}\frac{d^2}{dx^2}\big|_{x=a}, p) = \tfrac{1}{2}p''(a)$$

b) Let $T : \mathcal{P}_2 \to \mathcal{P}_4$ be a linear transformation such that

$$T(1) = x^4, \ T(x+1) = x^3 - 2x, \ T((x+1)^2) = x$$

Find $T(p)$ and $T(q)$ where $p = x^2 + 5x - 1$ and $q = a_0 + a_1 x + a_2 x^2$.

6. Let $T : \mathcal{P}_4 \to \mathcal{P}_3$ be given by

$$\begin{aligned} (2) \quad T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) &= (a_0 - a_1 + 2a_2 - a_3 + a_4) \\ &+ (-a_0 + 3a_1 - 2a_2 + 3a_3 - a_4)x \\ &+ (2a_0 - 3a_1 + 5a_2 - a_3 + a_4)x^2 \\ &+ (3a_0 - a_1 + 7a_2 + 2a_3 + 2a_4)x^3 \end{aligned}$$

a) Find a basis for $\mathcal{R}(T)$.
b) Show that T is not one-to-one.