ination steps produced the exact answer in a finite time. (Or equivalently, Cramer's rule gave an exact formula for the solution.) In the case of eigenvalues, no such steps and no such formula can exist, or Galois would turn in his grave. The characteristic polynomial of a 5 by 5 matrix is a quintic, and he proved that there can be no algebraic formula for the roots of a fifth degree polynomial. All he will allow is a few simple checks on the eigenvalues, *after* they have been computed, and we mention two of them:

5B The ***sum*** of the n eigenvalues equals the sum of the n diagonal entries:

$$\lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}. \tag{15}$$

This sum is known as the ***trace*** of A. Furthermore, the ***product*** of the n eigenvalues equals the ***determinant*** of A.

The projection matrix P had diagonal entries $\frac{1}{2}, \frac{1}{2}$ and eigenvalues $1, 0$—and $\frac{1}{2} + \frac{1}{2}$ agrees with $1 + 0$ as it should. So does the determinant, which is $0 \cdot 1 = 0$. We see again that a singular matrix, with zero determinant, has one or more of its eigenvalues equal to zero.

There should be no confusion between the diagonal entries and the eigenvalues. For a triangular matrix they are the same—but that is exceptional. Normally the pivots and diagonal entries and eigenvalues are completely different. And for a 2 by 2 matrix, we know everything:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has trace } a + d, \text{ determinant } ad - bc$$

$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (\text{trace})\lambda + \text{determinant}$$

$$\lambda = \frac{\text{trace} \pm [(\text{trace})^2 - 4 \det]^{1/2}}{2}.$$

Those two λ's add up to the trace; Exercise 5.1.9 gives $\sum \lambda_i = $ trace for all matrices.

EXERCISES

5.1.1 Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that *the* trace equals the sum of the eigenvalues, and the determinant equals their product.

5.1.2 With the same matrix A, solve the differential equation $du/dt = Au$, $u_0 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$. What are the two pure exponential solutions?

5.1.3 Suppose we shift the preceding A by subtracting $7I$:

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

What are the eigenvalues and eigenvectors of B, and how are they related to those of A?

5.1.4 Solve $du/dt = Pu$ when P is a projection:

$$\frac{du}{dt} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} u \quad \text{with} \quad u_0 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

The column space component of u_0 increases exponentially while the nullspace component stays fixed.

5.1.5 Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1\lambda_2\lambda_3$ equals the determinant.

5.1.6 Give an example to show that the eigenvalues can be changed when a multiple of one row is subtracted from another.

5.1.7 Suppose that λ is an eigenvalue of A, and x is its eigenvector: $Ax = \lambda x$.
(a) Show that this same x is an eigenvector of $B = A - 7I$, and find the eigenvalue. This should confirm Exercise 5.1.3.
(b) Assuming $\lambda \neq 0$, show that x is also an eigenvector of A^{-1}—and find the eigenvalue.

5.1.8 Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda), \tag{15}$$

and making a clever choice of λ.

5.1.9 Show that the trace equals the sum of the eigenvalues, in two steps. First, find the coefficient of $(-\lambda)^{n-1}$ on the right side of (15). Next, look for all the terms in

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

which involve $(-\lambda)^{n-1}$. Explain why they all come from the product down the main diagonal, and find the coefficient of $(-\lambda)^{n-1}$ on the left side of (15). Compare.

5.1.10 (a) Construct 2 by 2 matrices such that the eigenvalues of AB are not the products of the eigenvalues of A and B, and the eigenvalues of $A + B$ are not the sums of the individual eigenvalues.

(b) Verify however that the sum of the eigenvalues of $A + B$ equals the sum of all the individual eigenvalues of A and B, and similarly for products. Why is this true?

5.1.11 Prove that A and A^{T} have the same eigenvalues, by comparing their characteristic polynomials.

5.1.12 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

5.1.13 If B has eigenvalues 1, 2, 3 and C has eigenvalues 4, 5, 6, and D has eigenvalues 7, 8, 9, what are the eigenvalues of the 6 by 6 matrix $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$?

5.1.14 Find the rank and all four eigenvalues for both the matrix of ones and the checkerboard matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Which eigenvectors correspond to nonzero eigenvalues?

5.1.15 What are the rank and eigenvalues when A and C in the previous exercise are n by n? Remember that the eigenvalue $\lambda = 0$ is repeated $n - r$ times.

5.1.16 If A is the 4 by 4 matrix of ones, find the eigenvalues and the determinant of $A - I$ (compare Ex. 4.3.10).

5.1.17 Choose the third row of the "companion matrix"

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

so that its characteristic polynomial $|A - \lambda I|$ is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

5.1.18 Suppose the matrix A has eigenvalues 0, 1, 2 with eigenvectors v_0, v_1, v_2. Describe the nullspace and the column space. Solve the equation $Ax = v_1 + v_2$. Show that $Ax = v_0$ has no solution.

Remark In quantum mechanics it is matrices that don't commute—like position P and momentum Q—which suffer from Heisenberg's ***uncertainty principle.*** Position is symmetric, momentum is skew-symmetric, and together they satisfy $QP - PQ = I$. The uncertainty principle comes directly from the Schwarz inequality $(Qx)^{\mathrm{T}}(Px) \le \|Qx\|\,\|Px\|$ of Section 3.2:

$$\|x\|^2 = x^{\mathrm{T}}x = x^{\mathrm{T}}(QP - PQ)x \le 2\|Qx\|\,\|Px\|.$$

The product of $\|Qx\|/\|x\|$ and $\|Px\|/\|x\|$—which can represent momentum and position errors, when the wave function is x—is at least $\frac{1}{2}$. It is impossible to get both errors small, because when you try to measure the position of a particle you change its momentum.

At the end we come back to $A = S\Lambda S^{-1}$. That is the factorization produced by the eigenvalues. It is particularly suited to take powers of A, and the simplest case A^2 makes the point. The LU factorization is hopeless when squared, but $S\Lambda S^{-1}$ is perfect. The square is $S\Lambda^2 S^{-1}$, the eigenvectors are unchanged, and by following those eigenvectors we will solve difference equations and differential equations.

EXERCISES

5.2.1 Factor the following matrices into $S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

5.2.2 Find the matrix A whose eigenvalues are 1 and 4, and whose eigenvectors are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively. (*Hint*: $A = S\Lambda S^{-1}$.)

5.2.3 Find *all* the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and write down two different diagonalizing matrices S.

5.2.4 If a 3 by 3 upper triangular matrix has diagonal entries 1, 2, 7, how do you know it can be diagonalized? What is Λ?

5.2.5 Which of these matrices cannot be diagonalized?

$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}.$$

5.2.6 (a) If $A^2 = I$ what are the possible eigenvalues of A?
(b) If this A is 2 by 2, and not I or $-I$, find its trace and determinant.
(c) If the first row is $(3, -1)$ what is the second row?

5.2.7 If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ find A^{100} by diagonalizing A.

5.2.8 Suppose $A = uv^{\mathrm{T}}$ is a column times a row (a rank-one matrix).
(a) By multiplying A times u show that u is an eigenvector. What is λ?
(b) What are the other eigenvalues (and why)?
(c) Compute trace$(A) = v^{\mathrm{T}}u$ in two ways, from the sum on the diagonal and the sum of λ's.

5.2.9 Show by direct calculation that AB and BA have the same trace when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Deduce that $AB - BA = I$ is impossible. (It only happens in infinite dimensions.)

5.2.10 Suppose A has eigenvalues 1, 2, 4. What is the trace of A^2? What is the determinant of $(A^{-1})^{\mathrm{T}}$?

5.2.11 If the eigenvalues of A are 1, 1, 2, which of the following are certain to be true? Give a reason if true or a counterexample if false:
(1) A is invertible
(2) A is diagonalizable
(3) A is not diagonalizable

5.2.12 Suppose the only eigenvectors of A are multiples of $x = (1, 0, 0)$:

T F A is not invertible
T F A has a repeated eigenvalue
T F A is not diagonalizable

5.2.13 Diagonalize the matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots—a matrix such that $R^2 = A$. How many square roots will there be?

5.2.14 If A is diagonalizable, show that the determinant of $A = S\Lambda S^{-1}$ is the product of the eigenvalues.

5.2.15 Show that every matrix is the sum of two nonsingular matrices.