August 29, 2014

LINEAR ALGEBRA (BASED ON RODICA D. COSTIN'S NOTES)

Contents

1. Vector spaces

1.1. Notations.

$x \in S$ denotes the fact that the element x belongs to the set S.
$A \subset B$ denotes that the set A is included in the set B (possibly $A = B$).
$\mathbb{Z}$ denotes the set of all integer numbers,
$\mathbb{Z}_+$ denotes the set of all positive integers: $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$,
$\mathbb{N}$ denotes the set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$,
$\mathbb{Q}$ denotes the set of rational numbers: $\mathbb{Q} = \{\frac{m}{n} \,|\, m, n \in \mathbb{Z},\, n \neq 0\}$,
$\mathbb{R}$ denotes the set of real numbers,
$\mathbb{C}$ denotes the set of complex numbers.
Of course, $\mathbb{R} \subset \mathbb{C}$ (the set of real numbers is included in the set of complex numbers).

While in planar geometry it is customary to denote the coordinates of points by (x, y), in linear algebra it is often preferable to use (x_1, x_2).

Similarly, instead of denoting generic coordinates in space by (x, y, z), it may be preferable using (x_1, x_2, x_3).

In linear algebra the components of vectors are listed vertically, so I should write them as $(x_1, x_2)^T$. I will start using the correct notation as soon as it matters, namely when they start being multiplied by matrices.

1.2. **The definition of vector spaces.**

1.2.1. *Classical examples.* The rules of operations with vectors originate from mechanics (working with forces, velocities) and from the usage of complex numbers.

Recall the vectors in the plane: one can do geometry with them by adding or subtracting, by multiplying with numbers.

Any vector $\mathbf{x}$ in the plane can be represented as an arrow starting at the origin O and ending at a point with coordinates, say, (x_1, x_2). When multiplying the vector $\mathbf{x}$ by a scalar c (which is a real number) the result is a vector $c\mathbf{x}$ staring at O and ending at the point (cx_1, cx_2).

When adding, using the parallelogram rule, two vectors $\mathbf{x}$, respectively $\mathbf{y}$, which start at O and end at (x_1, x_2), respectively (y_1, y_2), the result is a vector $\mathbf{x}+\mathbf{y}$, which starts at O and ends at (x_1+y_1, x_2+y_2). The result of the subtraction $\mathbf{x}-\mathbf{y}$, using the triangle rule, is a vector ending at (x_1-y_1, x_2-y_2).

Similarly for vectors in space (3 dimensions): when adding (using the parallelogram rule) two vectors $\mathbf{x}$, $\mathbf{y}$ (starting at O and) ending at (x_1, x_2, x_3), (y_1, y_2, y_3) we obtain a vector (starting at O and) ending at $(x_1+y_1, x_2+y_2, x_3+y_3)$. Multiplying the vector $\mathbf{x}$ by the scalar c we obtain a vector $c\mathbf{x}$ (starting at O and) ending at (cx_1, cx_2, cx_3).

Therefore: *to operate with vectors we don't need to draw them, we can just work with coordinates! We just write* $\mathbf{x} = (x_1, x_2, x_3)$, *and we can call this object a vector, or a point, whichever helps our intuition more.*

We denote by $\mathbb{R}^2$ the set of vectors in the plane:

$$\mathbb{R}^2 = \{\mathbf{x} = (x_1, x_2) \,|\, x_j \in \mathbb{R}\}$$

with addition defined as

$$\mathbf{x}+\mathbf{y} = (x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$$

and multiplication by scalars defined as:

$$c\mathbf{x} = c(x_1, x_2) = (cx_1, cx_2) \quad \text{for } c \in \mathbb{R}$$

Similarly $\mathbb{R}^3$ denotes the set of vectors in space:

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) \,|\, x_j \in \mathbb{R}\}$$

with operations defined as

$$\mathbf{x}+\mathbf{y} = (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$$

$$c\mathbf{x} = c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3) \quad \text{for } c \in \mathbb{R}$$

Similarly, we can define n-dimensional vectors by

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \,|\, x_j \in \mathbb{R}\}$$

which we can add by:

$$\mathbf{x}+\mathbf{y} = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$

and multiply by scalars:

$$c\mathbf{x} = c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n) \quad \text{for } c \in \mathbb{R}$$

1.2.2. *Complex vectors.* It turns out that there is a great advantage to allow for complex coordinates, in which case we consider $\mathbb{C}^2$:

$$\mathbb{C}^2 = \{\mathbf{z} = (z_1, z_2)\,;\, z_{1,2} \in \mathbb{C}\}$$

which can be added:

$$\mathbf{z} + \mathbf{w} = (z_1, z_2) + (w_1, w_2) = (z_1 + w_1, z_2 + w_2)$$

and multiplied by *scalars which are complex numbers*:

$$c\mathbf{z} = c(z_1, z_2) = (cz_1, cz_2) \quad \text{for } c \in \mathbb{C}$$

Similarly, we can consider $\mathbb{C}^3, \ldots, \mathbb{C}^n$.

1.2.3. *The Abstract Definition of a Vector Space.* The following definition summarizes some properties of addition and multiplication by scalars of the vector spaces listed above.

In the following F denotes $\mathbb{R}$ or $\mathbb{C}$. In fact, F can also be $\mathbb{Q}$, $\mathbb{Z}_p$, or any field[1].

Definition 1. *The set V is a* **vector space** *over the scalar field F if V is endowed with two operations, one between vectors:*

$$\text{for every } \mathbf{x}, \mathbf{y} \in V \quad \text{there is } \mathbf{x} + \mathbf{y} \in V \tag{1}$$

and one between scalars and vectors:

$$\text{for every } c \in F \text{ and } \mathbf{x} \in V \quad \text{there is } c\mathbf{x} \in V \tag{2}$$

having the following properties:

(i) commutativity of addition:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

(ii) associativity of addition:

$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$$

(iii) existence of zero: there is an element $\mathbf{0} \in V$ so that

$$\mathbf{x} + \mathbf{0} = \mathbf{x} \text{ for all } \mathbf{x} \in V$$

(iv) existence of the opposite: for any $\mathbf{x} \in V$ there is an opposite, denoted by $-\mathbf{x}$, so that

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$

(v) distributivity of scalar multiplication with respect to vector addition:

$$c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$$

(vi) distributivity of scalar multiplication with respect to field addition:

$$(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$$

[1] A field is a commutative ring where any nonzero element has an inverse.

(vii) compatibility of scalar multiplication with field multiplication:

$$c(d\mathbf{x}) = (cd)\mathbf{x}$$

(viii) identity element of scalar multiplication:

$$1\mathbf{x} = \mathbf{x}$$

Remark 2. *From the axioms above we can deduce other properties, which are obvious for $\mathbb{R}^n$ and $\mathbb{C}^n$, but we need to know them in general. For example, it follows that:*
(ix) the zero scalar multiplied by any vector is the zero vector:

$$0\mathbf{x} = \mathbf{0}$$

(x) the scalar -1 multiplying any vector equals the opposite of the vector:

$$(-1)\mathbf{x} = -\mathbf{x}$$

Proof:
To show *(ix)* note that $0\mathbf{x} = (0+0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$ (by *(vi)*) so $0\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$ and adding the opposite of $0\mathbf{x}$ we get $0\mathbf{x} = \mathbf{0}$ (using, in order, *(iv),(ii),(iv),(iii)*.

Then to show *(x)* note that $(-1)\mathbf{x}+\mathbf{x} = (-1)\mathbf{x}+1\mathbf{x} = (-1+1)\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ where we used, in order, *(viii),(vi),(ix)*. $\square$

Remark. Another name used for **vector space** is **linear space**. The latter name is preferable when the space V consists of functions, see examples **4.-8.** below.

Vector spaces over the scalars $F = \mathbb{R}$ are also called "vector spaces over the reals", or "real vector spaces", and similarly, for the complex case $F = \mathbb{C}$, one can say "vector spaces over the complex numbers", or "complex vector spaces".

1.2.4. **Examples.**
1. $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3, \ldots \mathbb{R}^n$ are vector spaces over the reals.
2. $\mathbb{C}$, $\mathbb{C}^2$, $\mathbb{C}^3, \ldots \mathbb{C}^n$ are vector spaces over the complex numbers.
2'. $\mathbb{C}^n$ is also vector spaces over the real numbers, but $\mathbb{R}^n$ is *not* a vector space over $\mathbb{C}$.
3. $\mathbb{R}^{\mathbb{Z}_+} = \{(x_1, x_2, x_3, \ldots, x_n, \ldots) \,|\, x_j \in \mathbb{R}\}$ is a vector space over the reals.
4. The set of all polynomials with real coefficients, of degree at most n

$$\mathcal{P}_n(\mathbb{R}) = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \,|\, a_j \in \mathbb{R}\} \tag{3}$$

is a linear space over $\mathbb{R}$. The zero element is the zero polynomials.
4'. The set of all polynomials with real coefficients, of degree *exactly* n is *not* a linear space over $\mathbb{R}$.
5. The set of all polynomials with real coefficients

$$\mathcal{P}(\mathbb{R}) = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \,|\, a_j \in F, n \in \mathbb{N}\} \tag{4}$$

is a linear space over $\mathbb{R}$.
6. The set of all polynomials with complex coefficients

$$\mathcal{P}(\mathbb{C}) = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \mid a_j \in \mathbb{C}, n \in \mathbb{N}\} \tag{5}$$

is a linear space over $\mathbb{C}$. It is also a linear space over $\mathbb{R}$.
7. The set of functions which are continuous on $[0,1]$ and have values in F:

$$C([0,1], F) = \{f : [0,1] \to F \mid f \text{ continuous }\}$$

is a linear space over F. The zero element is the function which is identically zero.
8. The set of all solutions of the linear differential equation $u''(t) = u(t)$ is a linear space.
Exercise. Justify the statements **3.-8.**

1.3. **Subspaces.** Let V be a vector space over the scalars F. If U is a subset in V we can add two elements of U, but there is no guarantee that the results will remain in U. Similarly, we can multiply by scalars elements of U, but there is no guarantee that the result will be also in U. But if these are true, then U is called a subspace of V:

Definition 3. *A subset $U \subset V$ is called a* **subspace** *of V if*

$$\text{for any } \mathbf{x}, \mathbf{y} \in U,\ c \in F \text{ we have } \mathbf{x} + \mathbf{y} \in U,\ c\mathbf{x} \in U$$

Note that the two properties above are sometime written more compactly as

$$\text{for any } \mathbf{x}, \mathbf{y} \in U,\ c, d \in F \text{ we have } c\mathbf{x} + d\mathbf{y} \in U$$

(but the two formulations are equivalent - why?).

Remarks.

1. Note that a subspace must contain the zero vector (multiply any vector in U by the scalar 0).

2. Moreover, a subspace U is a vector space in itself (with respect to the addition and scalar multiplication inherited from the bigger space V). Indeed, properties (1), (2) are guaranteed by the definition of the subspace, while all the other properties *(i)-(viii)* are automatically satisfied (they are true for all elements of V, in particular for those in U).

1.3.1. **Examples.**

Let V be a vector space.

1. The set $\{\mathbf{0}\}$, consisting of only the zero element, is a subspace of V.
2. V is a subspace of itself.
3. $U \subset \mathbb{R}^2$ given by $U = \{(x_1, x_2) \mid x_1 = 3x_2\}$ is a subspace of $\mathbb{R}^2$.
4. $U \subset \mathbb{R}^3$, $U = \{(x_1, x_2, x_3) \mid x_1 = 3x_2, x_3 = x_2\}$ is a subspace of $\mathbb{R}^3$.
5. $U \subset \mathbb{R}^3$, $U = \{(x_1, x_2, x_3) \mid x_2 = 1\}$ is not a subspace of $\mathbb{R}^3$.
6. Lines which pass through the origin are subspaces.
7. Any line which is a subspace must pass through O.

8. Planes passing through the origin are subspaces in $\mathbb{R}^3$.
Indeed, a plane in $\mathbb{R}^3$ is given be a linear equation: $Ax_1 + Bx_2 + Cx_3 = D$. The plane passes through the origin for $D = 0$. Consider then a plane

$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, Ax_1 + Bx_2 + Cx_3 = 0\}$$

It is now easy to show that U is a subspace.

9. $\mathcal{P}_n$ is a subspace of the space of $\mathcal{P}$ (see $(3), (4), (5)$).

10. $\mathcal{P}_n$ is a subspace of $C[0, 1]$.

11. The set $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_j \geq 0\}$ is not a subspace.

Exercise. Justify the statements above.

Definition 4. *Let U, W two subspaces of V. Then their intersection $U \cap W$ consists of all vectors $\mathbf{x}$ that belong to both U and W:*

$$U \cap W = \{\mathbf{x} \,|\, \mathbf{x} \in U \text{ and } \mathbf{x} \in W\}$$

Exercise. Show that *the intersection of two subspaces is also a subspace.*

Example. Consider two planes U, W in the space, containing the origin O. If $U \neq W$ then their intersection is a line containing O.

The union of two subspaces is not necessarily a subspace. But we may consider the smallest subspace containing them:

Definition 5. *Let U, W two subspaces of V. Their sum $U + W$ consists of all vectors $\mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in W$:*

$$U + W = \{\mathbf{u} + \mathbf{w} \,|\, \mathbf{u} \in U,\ \mathbf{w} \in W\}$$

Exercise. Show that *the sum $U + W$ of two subspaces is also a subspace.*

Examples.

1. Consider two lines U, W in the space $\mathbb{R}^3$, passing thorough the origin. If $U \neq W$ then $U + W$ is the plane containing the two lines.

2. Consider a line U and a plane W in the space $\mathbb{R}^3$, both passing thorough the origin. If U is not contained in W then $U + W$ is the whole space $\mathbb{R}^3$. But if $U \subset W$ then $U + W = W$.

Exercise. Give a justification for the statements above.

1.4. **Linear Span.** Let V be a vector space over the scalars F.

Definition 6. *The vector $\mathbf{u}$ is the* **linear combination** *of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ means that*

$$\mathbf{u} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r \qquad \textit{for some } c_1, \ldots, c_r \in F$$

Note that any linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ still belongs to V.

Exercise. Show that if all $\mathbf{v}_1, \ldots, \mathbf{v}_r$ belong to a subspace U of V, then any linear combination of these vectors also belongs to the subspace U.

Definition 7. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *be vectors in* V*. The set of* all *linear combinations of these vectors is called the subspace spanned by* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *(or simply the span of* $\mathbf{v}_1, \ldots, \mathbf{v}_r$*):*

$$Sp(\mathbf{v}_1, \ldots, \mathbf{v}_r) = \{c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r \mid c_1, \ldots, c_r \in F\}$$

Exercise. Show that $Sp(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ *is indeed a subspace.*

Exercise. Show that $Sp(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is the smallest subspace containing all $\mathbf{v}_1, \ldots, \mathbf{v}_r$ (in the sense that if all $\mathbf{v}_1, \ldots, \mathbf{v}_r$ belong to a subspace U, then necessarily $Sp(\mathbf{v}_1, \ldots, \mathbf{v}_r) \subset U$).

Note: it is sometimes useful to define the linear span of a possibly infinite set S of vectors ($S \subset V$). In this case, define

$$Sp(S) = \{c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \mid \mathbf{v}_j \in S, \ \ c_j \in F, \ \ k \in \mathbb{Z}_+\}$$

Warning: this formula does not hold for vector spaces with extra structure, such as Banach or Hilbert spaces. But here is an equivalent definition which works: $Sp(S)$ is the intersection of all subspaces containing S (this requires a proof, not included here).

1.5. **Linear dependence; linear independence.**

Definition 8. *A finite set of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ *are called* linearly dependent *if there is a nontrivial linear relation between them: there exist some scalars* $c_1, \ldots, c_r \in F$ not all zero *so that their linear combination is the zero vector:*

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r = \mathbf{0} \qquad \text{where at least one } c_j \neq 0$$

Note: this means that (at least) one of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ belongs to the span of the others. (Why?)

Examples.

1. If $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$ then $Sp(\mathbf{i}, \mathbf{j}) = \mathbb{R}^2$.
2. The vectors $\mathbf{i}, \mathbf{j}, \mathbf{i} - \mathbf{j}$ are linearly dependent, and so are $\mathbf{i}, \mathbf{j}, \mathbf{i} - \mathbf{j}, 2\mathbf{i}$. Then $Sp(\mathbf{i}, \mathbf{j}, \mathbf{i} - \mathbf{j}) = Sp(\mathbf{i}, \mathbf{j}, \mathbf{i} - \mathbf{j}, 2\mathbf{i}) = Sp(\mathbf{i}, \mathbf{j})$.
3. If $\mathbf{u}, \mathbf{v}$ are two nonzero vectors in $\mathbb{R}^3$ then $Sp(\mathbf{u}, \mathbf{v})$ is the plane determined by these two vectors if $\mathbf{u} \not\parallel \mathbf{v}$, and it is the line containing the vectors if $\mathbf{u} \| \mathbf{v}$ (in which case $\mathbf{u}, \mathbf{v}$ are linearly dependent).

Exercise. Prove the statements above.

Remark 9. *A useful observation: if two vectors are linearly dependent then either they are scalar multiples of each other, or one of them is zero.*

Indeed, let $\mathbf{u}, \mathbf{v} \in V$ with $c\mathbf{u} + d\mathbf{v} = 0$ with not both c, d zero. Say $c \neq 0$ then dividing by c we have $\mathbf{u} + \frac{d}{c}\mathbf{v} = 0$ and adding the opposite $\mathbf{u} = -\frac{d}{c}\mathbf{v}$ hence $\mathbf{u}$ is a multiple of $\mathbf{v}$. Furthermore, note that also $d \neq 0$ (otherwise we would have $\mathbf{u} = 0$) hence also $\mathbf{v} = -\frac{c}{d}\mathbf{u}$. $\square$

Definition 10. *A set of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ *are called* linearly independent *if they are not linearly dependent, or, in other words, if*

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r = \mathbf{0} \ \text{ implies } c_1 = 0, \ \ldots, c_r = 0$$

And more generally:

Definition 11. *An (infinite) set of vectors $S \subset V$ is called* linearly independent *if all its finite subsets are linearly independent.*

Remark 12. *A linearly independent set cannot contain the zero vector.*

Indeed, consider a collection $\mathbf{v}_1 = \mathbf{0},\ \mathbf{v}_2, \ldots, \mathbf{v}_r \in V$. Then they are linearly dependent, since we have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r = 0$ for $c_1 = 1,\ c_2 = 0, \ldots, c_r = 0$. $\Box$

Examples.

4. The vectors $(1,0),\ (2,1) \in \mathbb{R}^2$ are linearly independent.
5. The vectors $(1,0,0), (0,1,0), (0,0,1) \in \mathbb{R}^3$ are linearly independent.
6. The vectors $(1,0)$, $(0,1)$, $(-2,3) \in \mathbb{R}^2$ are linearly dependent.
7. The polynomials $1,\ t,\ t^2 \in \mathcal{P}_2$ are linearly independent.
8. The polynomials $1,\ t,\ t^2, \ldots, t^n \ldots \in \mathcal{P}$ are linearly independent.
9. The polynomials $1 + t + t^2, t^2 - 1, 3t$ are linearly independent in $\mathcal{P}$.

Exercise. Prove the statements above.

Remark 13. *Consider a collection of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$, all $\mathbf{x}_j \neq \mathbf{0}$. Then $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$ are linearly dependent if and only if one of them belongs to the span of the others.*

The proof is left to the reader.

1.6. **Basis and Dimension.**

Definition 14. *A set of vectors $S \subset V$* is a basis of V *if:*
(i) it is linearly independent and
(ii) its span equals V.

In the Examples above, the vectors in Examples 1,2,4,5 above form bases for the stated vector spaces, but this is not true for Example 3.

Theorem 15. *Any vector space has a basis. Moreover, all the basis of V have the same number of elements, which is called the dimension of V.*

(The proof will not be discussed here; basically one chooses a maximal set of linearly independent vectors.)

Remark: if the dimension of V is infinite, we can also distinguish between different magnitudes of infinity...but this has to wait until next semester.

Examples

0. $\mathbb{R}^3$ is a vector space over $\mathbb{R}$ of dimension 3, with a basis consisting of the vectors $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$. Indeed, any $\mathbf{x} \in \mathbb{R}^3$ having coordinates $\mathbf{x} = (x_1, x_2, x_3)$ can be written as a linear combination

$$\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

and the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent.

1. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ of dimension n, with a basis consisting of the vectors

$$\mathbf{e}_1 = (1,0,\dots,0),\ \mathbf{e}_2 = (0,1,0\dots,0),\dots,\mathbf{e}_n = (0,0,\dots 0,1) \tag{6}$$

Indeed, any $\mathbf{x} \in \mathbb{R}^n$ having coordinates $\mathbf{x} = (x_1, \dots x_n)$ can be written as a linear combination

$$\mathbf{x} = (x_1, \dots x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$$

Also, the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are linearly independent (why?).

The basis (6) is called the *standard basis* of $\mathbb{R}^n$.

2. Similarly, $\mathbb{C}^n = \{(z_1, \dots z_n) \,|\, z_j \in \mathbb{C}\}$ is a vector space over $\mathbb{C}$ of dimension n, with a basis consisting of the vectors (6).

3. $\mathcal{P}_n(F)$, see (3), is a vector space over the field of scalars F, of dimension $n+1$, and a basis is $1,\, t\,, t^2, \dots\, t^n$.

4. The set of all polynomials with coefficients in F, see (4), (5) is a vector space over the field of scalars F, has infinite dimension.

Exercise. Justify the statements in the examples above.

Let V be a vector space over F and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis. Since $Sp(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = V$ then any $\mathbf{x} \in V$ belongs to $Sp(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$, hence has the form

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n \quad \text{for some scalars } c_1, c_2, \dots, c_n \tag{7}$$

which is called *the representation of the vector* $\mathbf{x}$ in the basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, and the scalars $c_1, \dots, c_n$ are called *the coordinates* of the vector in the given basis.

The representation of a vector in a given basis is unique. Indeed, suppose that $\mathbf{x}$ can be represented as (7), and also as

$$\mathbf{x} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n \quad \text{for some scalars } d_1, d_2, \dots, d_n \tag{8}$$

Then subtracting (7) and (8) we obtain

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n - (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_n\mathbf{v}_n) = \mathbf{0}$$

which (using the properties of the operations in a vector space) can be written as

$$(c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \dots + (c_n - d_n)\mathbf{v}_n = \mathbf{0}$$

and since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, then necessarily $(c_1 - d_1) = 0, \dots, (c_n - d_n) = 0$ therefore $c_1 = d_1, \dots, c_n = d_n$. $\Box$

Once a basis is specified, operations are done coordinate-wise:

Remark 16. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis of V. If $\mathbf{x}, \mathbf{y} \in V$ then $\mathbf{x} = \sum_{j=1}^n c_j \mathbf{v}_j$, $\mathbf{y} = \sum_{j=1}^n d_j \mathbf{v}_j$ for some $c_j, d_j \in F$. It follows that*

$$\mathbf{x} + \mathbf{y} = \sum_{j=1}^n (c_j + d_j)\mathbf{v}_j, \quad \alpha\mathbf{x} = \sum_{j=1}^n \alpha c_j \mathbf{v}_j, \text{ for any } \alpha \in F$$

Theorem 17. *If a vector space V has finite dimension n, then any collection consisting of $n+1$ vectors is linearly dependent.*

Why: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis for V. Assume, to arrive at a contradiction, that $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1} \in V$ are linearly independent.

The plan is to express $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in terms of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, and then the linear dependence of $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}$ will give a linear relation among $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

If one of $\mathbf{x}_j$ equals $\mathbf{0}$, then $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1} \in V$ are linearly dependent, which is a contradiction. So we may assume that all $\mathbf{x}_j \neq \mathbf{0}$.

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis then

$$\mathbf{x}_1 = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n \quad \text{for some } c_1, \ldots, c_n \in F$$

Since $\mathbf{x}_1 \neq \mathbf{0}$ then at least one scalar c_j is not zero, say $c_1 \neq 0$ (we can always renumber the $\mathbf{v}_j$) . We can solve for $\mathbf{v}_1$ in terms of $\mathbf{x}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\mathbf{v}_1 = \frac{1}{c_1}\mathbf{x}_1 - \frac{c_2}{c_1}\mathbf{v}_2 - \ldots - \frac{c_n}{c_1}\mathbf{v}_n \equiv \tilde{c}_1\mathbf{x}_1 + \tilde{c}_2\mathbf{v}_2 + \ldots + \tilde{c}_n\mathbf{v}_n \tag{9}$$

We repeat the argument for $\mathbf{v}_2$:

$$\mathbf{x}_2 = d_1\mathbf{v}_1 + \ldots + d_n\mathbf{v}_n \quad \text{for some } d_1, \ldots, d_n \in F$$

and replacing $\mathbf{v}_1$ from (9) it follows that

$$\mathbf{x}_2 = \tilde{d}_1\mathbf{x}_1 + \tilde{d}_2\mathbf{v}_2 + \ldots + \tilde{d}_n\mathbf{v}_n \tag{10}$$

Noting that not all $\tilde{d}_2, \ldots, \tilde{d}_n$ can be zero (otherwise $\mathbf{x}_2, \mathbf{x}_1$ would be linearly dependent) one of them, say $\tilde{d}_2$, is not zero, hence we can solve (10) for $\mathbf{v}_2$ in terms of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$:

$$\mathbf{v}_2 = \tilde{\tilde{d}}_1\mathbf{x}_1 + \tilde{\tilde{d}}_2\mathbf{x}_2 + \tilde{\tilde{d}}_3\mathbf{v}_3 + \ldots + \tilde{\tilde{d}}_n\mathbf{v}_n$$

Continuing the argument, in the end we obtain $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as linear combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

But $\mathbf{x}_{n+1}$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, hence as a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, which contradicts their assumed linear independence. □

Example. $Sp(1 + t + t^2, t^2 - 1, 3t) = \mathcal{P}_2$.
Indeed, by Example 9 of §1.5, the polynomials $1 + t + t^2$, $t^2 - 1$, $3t$ are linearly independent. Hence they span a 3-dimensional subspace in $\mathcal{P}_2$. Since $\dim \mathcal{P}_2$=3, then any polynomial in $\mathcal{P}_2$ belongs to $Sp(1+t+t^2, t^2-1, 3t)$, hence $Sp(1 + t + t^2, t^2 - 1, 3t) = \mathcal{P}_2$. □

Theorem 18. *Every linearly independent set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ in a finite dimensional vector space V can be completed to a basis of V.*

How: If $Sp(\mathbf{v}_1, \ldots, \mathbf{v}_r) = V$ then $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis and we are done.

Otherwise, there is some vector in V, call it $\mathbf{v}_{r+1}$, that cannot be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_r$. Then this means that $\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is a linearly independent set (convince yourselves!).

We then repeat the steps above with $\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$.

The procedure must end, since by Theorem 17 we can have at most $\dim V$ linearly independent vectors. $\square$

1.6.1. *More examples.*

1. Let $\mathbf{u} = (1, 2, -3) \in \mathbb{R}^3$. What is $Sp(\mathbf{u})$?

Solution: $Sp(\mathbf{u}) = \{c\mathbf{u} \,|\, c \in \mathbb{R}\}$ is the line through O in the direction of $\mathbf{u}$.

2. Let $\mathbf{u} = (1, 2, -3), \mathbf{v} = (2, 4, -6) \in \mathbb{R}^3$. What is $Sp(\mathbf{u}, \mathbf{v})$?

Solution: note that $\mathbf{v} = 2\mathbf{u}$ hence $Sp(\mathbf{u}, \mathbf{v}) = Sp(\mathbf{u}, 2\mathbf{u}) = Sp(\mathbf{u}) =$ as above.

3. Let $\mathbf{u} = (1, 2, -3), \mathbf{w} = (1, 1, 0) \in \mathbb{R}^3$. What is $Sp(\mathbf{u}, \mathbf{w})$?

Solution: (think geometrically) if $\mathbf{u}, \mathbf{w}$ are dependent, then they are multiple of each other (by Remark 9), and it is obvious by inspection that it is not case. Hence $Sp(\mathbf{u}, \mathbf{w}) =$ the smallest subspace containing two independent vectors = the plane (through O) containing them.

4. Let $\mathbf{u} = (1, 2, -3), \mathbf{v} = (2, 4, -6), \mathbf{w} = (1, 1, 0) \in \mathbb{R}^3$. What is $Sp(\mathbf{u}, \mathbf{v}, \mathbf{w})$? Find a basis for this subspace. What is its dimension?

Solution: since $\mathbf{v} = 2\mathbf{u}$ then $Sp(\mathbf{u}, \mathbf{v}, \mathbf{w}) = Sp(\mathbf{u}, 2\mathbf{u}, \mathbf{w}) = Sp(\mathbf{u}, \mathbf{w}) =$ the plane containing $\mathbf{u}, \mathbf{w}$.

Basis: clearly $\mathbf{u}, \mathbf{w}$ are independent and span $Sp(\mathbf{u}, \mathbf{v}, \mathbf{w})$ so they form a basis. Dimension 2.

What if we are not quite sure if $\mathbf{u}, \mathbf{w}$ are independent? Let's check: suppose that for some scalars c, d we have $c\mathbf{u} + d\mathbf{w} = 0$. But $c\mathbf{u} + d\mathbf{w} = c(1, 2, -3) + d(1, 1, 0) = (c+d, 2c+d, -3c) = (0, 0, 0)$ hence $c+d = 0,\ 2c+d = 0 - 3c = 0$ hence $c = d = 0$, independent!

5. Show that $\mathbf{x} = (1, 0, 0)$, $\mathbf{y} = (1, 1, 0)$, $\mathbf{z} = (1, 1, 1)$ form a basis for $\mathbb{R}^3$. Do they form a basis for $\mathbb{C}^3$ (as a complex vector space)?

6. Show that $\mathbb{C}$ is a real vector space, find a basis and its dimension. Same questions for $\mathbb{C}^2$ and for $\mathcal{P}_n(\mathbb{C})$.

1.7. **Direct sum of subspaces.** Let V be a vector space over the field F (which for us is either $\mathbb{R}$ or $\mathbb{C}$).

Recall that if U, W are two subspaces of V then their sum is defined as

$$U + W = \{\mathbf{u} + \mathbf{w} \,|\, \mathbf{u} \in U,\ \mathbf{w} \in W\}$$

and that $U + W$ *is also a subspace of* V.

Definition 19. *If $U \cap W = \{\mathbf{0}\}$ then their sum $U + W$ is called the* **direct sum** *of the subspaces U and W, denoted by $U \bigoplus W$.*

Examples. Let $V = \mathbb{R}^3$.
1) If U and W are two distinct planes through O then $U + W = \mathbb{R}^3$, and the sum is not direct (the intersection of two distinct planes is a line).
2) If U and W are two distinct lines through O then $U + W = U \bigoplus W =$ their plane.
3) If U is a line and W is a plane through O, then $U + W = U \bigoplus W =$ the whole space if $U \not\subset W$, and $U + W = W$ is $U \subset W$ and the sum is not direct.

Theorem 20. Existence of the complement space. *If U is a subspace of V, then there exists W a subspace of V so that $U \bigoplus W = V$.*

Proof.
By Theorem 15, U has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$. By Theorem 18, this can be completes to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ of V. Take $W = Sp(\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n)$. It only remains to show that $U \cap W = \{0\}$ and that $U + W = V$, which are left to the reader. $\square$

Examples.
1. Let U be the x_1-axis in $V = \mathbb{R}^2$ (the x_1x_2-plane). Any different line W (through O) is a complement of U. (Why?)
2. Let U be the x_1-axis in $V = \mathbb{R}^3$. Any plane (through O) not containing W is a complement of U. (Why?)

Theorem 21. *Let U, W be two subspaces of V with $U \cap W = \{\mathbf{0}\}$. Then if $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis of U, and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a basis of V then $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}_1, \ldots, \mathbf{v}_m$ is a basis of $U \bigoplus W$.*

In particular

$$\dim U \bigoplus W = \dim U + \dim W$$

The proof is left as an exercise.

Remark: more is true, namely

$$\dim(U + W) = \dim U + \dim W - \dim U \cap W$$

Remark 22. *If $V = U \bigoplus W$ then any $\mathbf{x} \in V$ can be uniquely decomposed as $\mathbf{x} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U, \mathbf{w} \in W$.*

The proof is left as an exercise.