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DISTRIBUTIONS

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Contents

1. Setting the stage

For a long time mathematicians, physicists, engineers worked, particularly in problems in differential equations, by differentiating functions which are not differentiable, integrating against "functions" which are zero everywhere except at one point, and obtaining a nonzero integral, integrating functions whose integral is not defined, and, in the end, obtaining valid results!

The ideas and techniques have been gathered and set on a rigorous mathematical ground by Sobolev (1930's), and, separately, by Laurent Schwartz (1940's). The main idea in working with generalized functions is that these are only needed and defined in integrals, that is, they are linear functionals.

Distributions are continuous linear functionals on spaces of *test functions.*

1.1. **Test functions.** These are defined on a domain of interest for a particular problem: on $\mathbb{R}$, or $\mathbb{R}^2$, $\mathbb{R}^n$ or a subset $(a,b)\subset\mathbb{R}$, $\Omega\subset\mathbb{R}^n$) and are very, very, nice functions, having all the properties one may wish for in tackling a problem at hand. Namely:
- they are infinitely many times differentiable (we say for short that they are class C^∞), and
- they vanish, together with all their derivatives, at the boundary of the domain (so that when we integrate by parts the boundary terms vanish).

Here are the most popular spaces of test functions.

$\mathcal{S}(\mathbb{R})$, the *rapidly decaying functions* (the Schwartz space) contains functions which decay, towards $\pm\infty$, more rapidly than any power, and so do all their derivatives. For example, $\phi(x)=e^{-x^2}\in\mathcal{S}(\mathbb{R})$.

It can be shown that the Fourier transform $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ is one-to-one and onto. This requires a proof, but here is the intuitive reason: f has decay $\leftrightarrow$ $\mathcal{F}f$ is smooth, and f is smooth $\leftrightarrow$ $\mathcal{F}f$ has decay, and functions in $\mathcal{S}$ are both smooth and have decay.

$C_0^\infty(\Omega)$, the *functions with compact support* on $\Omega\subset\mathbb{R}^n$ are functions which are non-zero only on a bounded set, and they are identically zero before the boundary of Ω. For example,

$$\phi(x)=\begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x|<1\\ 0 & \text{otherwise}\end{cases}\qquad\in C_0^\infty(\mathbb{R})$$

and in fact $\phi(x)\in C_0^\infty(-a,a)$ for any $a>1$.

Note: the *spaces of test functions are linear subspaces of* L^2 (and they are dense in L^2).

A sequence of test functions ϕ_n is said to *converge* to a test function ϕ (in the same space) if it converges in the strongest sense you may imagine (not only point-wise, but also all the derivatives $\phi_n^{(k)}(x) \to \phi^{(k)}(x)$, and, moreover, they converge uniformly on any compact subset... this ensures that one may pass to the limit freely all the operations needed: summation, differentiation, integration, without worry).

1.2. **Distributions.** Given a space of test functions, *distributions are continuous linear functionals*, that is, if u is a distribution

$$\text{we denote } u(\phi) := (\phi, u) \quad \text{for any test function } \phi$$

and

$$(c\phi + d\psi, u) = c(\phi, u) + d(\psi, u) \quad \text{for test functions } \phi, \psi \text{ and constants } c, d$$

$$\text{whenever } \phi_n \to \phi \text{ as test functions, then } (\phi_n, u) \to (\phi, u)$$

1.3. **Examples of distributions.** In what follows we assume our test functions are $C_0^\infty(\mathbb{R})$. For other spaces of test functions minimal extra care may be needed.

1.3.1. *Function type distributions.* Any function which is not too wild (i.e. it is integrable on compact sets) can be regarded as a distribution: say, $f(x) = 1$, or $f(x) = \sin x$, or $f(x) = e^{ix}$, or $f(x) = H(x)$, defines a distribution by the formula

$$(\phi, f) = \int_{-\infty}^{\infty} f(x)\phi(x)\, dx \tag{1}$$

The integral converges since any test function $\phi(x)$ vanishes outside some interval $[-M, M]$.[1] The functional defined by (1) is clearly linear; a bit of mathematical argumentation would also show continuity (which we do not pursue here).

Note the underlying idea: (1) is, essentially, the inner product of L^2. We want to use it for more general functions f, like the examples above, which are not in L^2; we do that, but the price we pay is that we can only do pairing with very rapidly decaying functions ϕ.

In the same spirit: when working with complex-valued functions, to use the full force of Hilbert space theory we need to define *distributions as conjugate-linear functionals*, that is, $u(\phi) := \langle \phi, u\rangle$ satisfies

$$\langle c\phi + d\psi, u\rangle = \overline{c}\langle \phi, u\rangle + \overline{d}\langle \psi, u\rangle$$

The constructions and definitions for distributions defined as linear functionals can be easily transcribed for distributions defined as conjugate-linear functionals (and we will use the inner product notation in this latter case).

[1] Of course, if the test functions are $\mathcal{S}$, then we should require f not to increase faster than some power.

1.4. **Dirac's delta function.** δ is defined as

$$(\phi, \delta) = \phi(0) \quad \text{for any test functions } \phi \tag{2}$$

and shifted delta, $\delta(x-a)$ is defined as

$$(\phi, \delta(x-a)) = \phi(a) \tag{3}$$

Note that (2),(3) are commonly written, in the spirit of (1), using an integral sign:

$$(\phi, \delta) = \int_{-\infty}^{\infty} \phi(x)\delta(x)\, dx = \phi(0)$$

and

$$(\phi, \delta(x-a)) = \int_{-\infty}^{\infty} \phi(x)\delta(x-a)\, dx = \int_{-\infty}^{\infty} \phi(t+a)\delta(t)\, dt = \phi(a)$$

These are not bonafide integrals, but they work like integrals.

2. Operations with distributions

2.1. **Linear combinations.** Any linear combination of distributions is a distribution (why?).

2.2. **Multiplication by C^∞ functions.** We can multiply distributions u with C^∞ functions $f(x)$ and we obtain another distribution fu, defined as

$$(\phi, fu) = (f\phi, u) \quad \text{for any test function } \phi$$

(the definition is, of course, inspired by what happens for function type distributions). Note that the definition makes sense since if $\phi \in C_0^\infty(\mathbb{R})$, and $f \in C^\infty(\mathbb{R})$, hence $\phi f \in C_0^\infty(\mathbb{R})$.

Note: When we work with distributions over the test functions $\mathcal{S}$ (these care called *tempered distributions*) we also need to require that the function f increases no faster than some power at $\pm\infty$, to ensure that for whatever $\phi \in \mathcal{S}$ then also $f\phi \in \mathcal{S}$.

2.3. **Any distribution is differentiable.** If our distribution is a nice, differentiable function $f(x)$, what should its derivative be? As a distribution, let us see how it acts on test functions:

$$(\phi, f') = \text{ [since } f' \text{ is a function] } = \int_{-\infty}^{\infty} \phi(x) f'(x) dx$$

and integrating by parts

$$= \phi(x)f(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi'(x)f(x)dx = \text{ [since } \phi(\pm\infty) = 0] \ = -(\phi', f)$$

This justifies

Definition 1. *The derivative u' of a distribution u is defined by the rule*

$$(\phi, u') = -(\phi', u) \quad \textit{for any test function } \phi$$

Theorem 2. *If u is a function-type distribution, and u is differentiable, then its derivative in the sense of distributions coincides with its derivative as a function.*

Why: denoting for the moment by Du the derivative of u as a function, and by u' its derivative as a distribution, we have

$$(\phi, u') = \text{ [by definition] } -(\phi', u) = \text{ [since u is a function] } -\int_{\mathbb{R}} \phi'(x)u(x)\,dx$$

$$\text{[by parts] } = \int_{\mathbb{R}} \phi(x)Du(x)\,dx = \text{ function-type distribution } Du \text{ applied to } \phi \ \square$$

Exercises.

1. Show that $|x|$ is differentiable as a distribution and

$$\frac{d}{dx}|x| = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

2. Show that the derivative of

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is δ.

3. Show that $(\phi, \delta') = -\phi'(0)$.

4. More generally, show that if $f(x)$ has continuous derivatives on $(-\infty, x_0]$ and on $[x_0, +\infty)$ and (possibly) has a jump discontinuity at $x = x_0$, then its derivative (in distribution sense) is $f_c'(x) + [f(x_0+) - f(x_0-)]\delta(x - x_0)$ where $f_c'(x)$ denotes the derivative of $f(x)$ as a function on $(-\infty, x_0) \cup (x_0, \infty)$.

2.4. **Convolution of distributions with test functions.** If ψ is a test function, then for each fixed x, the function $\psi(x - .)$ defined as $\psi(x - .)(t) = \psi(x - t)$ (as a function of t) is also a test function (it is a flip and a shift of ψ). It is natural to define $\psi * u$, for any distribution u, by

$$(\psi * u)(x) = \text{ [like] } \int_{\mathbb{R}} \psi(x - t)u(t)dt = \text{ [formal definition] } (\psi(x - .), u(.))$$

It is not hard to see (with some work) that $\psi * u$ is a function-type distribution, even $\psi * u \in C^\infty$! (The intuitive reason is that the dependence on x is only through ϕ, which is C^∞.) *Convolution is regularizing (smoothing).*

Very important example: $\psi * \delta = \psi$, or,

$$\int_{\mathbb{R}} \psi(x - t)\delta(t)dt = \psi(x) = \int_{\mathbb{R}} \psi(t)\delta(x - t)dt \tag{4}$$

Why: $(\psi * \delta)(x) = (\psi(x - .), \delta(.)) = \psi(x)$. $\square$

Equality (4) is the continuous analogue of the finite dimensional fact $\mathbf{x} = \sum_{k=1}^n x_k \mathbf{e}_k$ where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis.

And (4) can be intuitively interpreted as: a function $\psi(x)$ can be decomposed as a superposition of impulses located at t, of amplitude $\psi(t)$.

2.5. **The Fourier transform of distributions.** Let us see how its Fourier transform $\mathcal{F}f$ of a super-nice function (from the point of view of the Fourier transform), say, $f \in \mathcal{S}$ acts on test functions:

$$(\phi, \mathcal{F}f) = \int_{-\infty}^{\infty} (\mathcal{F}f)(\xi)\phi(\xi)d\xi = \int_{-\infty}^{\infty} d\xi \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-ix\xi} f(x)\phi(\xi)$$

$$= \int_{-\infty}^{\infty} dx \, f(x) \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi \, e^{-ix\xi}\phi(\xi) = \int_{-\infty}^{\infty} dx \, f(x) \, (\mathcal{F}\phi)(x) = (\mathcal{F}\phi, f)$$

It is natural then to define:

Definition 3. *For any distribution u on $\mathcal{S}$ (i.e. tempered distribution) define its Fourier transform $\mathcal{F}u$ by the rule*

$$(\phi, \mathcal{F}u) = (\mathcal{F}\phi, u) \quad \textit{for any test function } \phi$$

Examples.

1. The Fourier transform of δ is 1 (well, up to a normalization constant... you need to check the normalization you are using). For the one here

$$\mathcal{F}\delta = \frac{1}{\sqrt{2\pi}}$$

(check!) Also

$$\mathcal{F}\delta(x-a) = \frac{1}{\sqrt{2\pi}} e^{-ia\xi}$$

Exercise.

Show that $\delta(-x) = \delta(x)$. Find the Fourier transforms of 1, and of e^{ibx}, $\sin x$, $\cos x$.

3. Calculus with distributions

3.1. Limits of distributions.

Definition 4. *A sequence of distributions $\{u_n\}_n$ converges to the distribution u (all defined on the same space of test functions) if $(\phi, u_n) \to (\phi, u)$ for all test functions ϕ.*

The point is that distributions can be obtained as limits of functions.

3.2. Fundamental sequences for δ.

Intuitively, one may think of $\delta(x)$ as a "function" which is zero at all points except for $x = 0$, where "it is infinite", and such that $\int_{-\infty}^{\infty} \delta(x)dx = 1$. Of course, there are no such functions, but $\delta(x)$ is a limit (in distribution sense) of functions which are zero outside intervals around 0), of lengths shrinking to zero, while the total integral of these functions remains one. Of course, in this case the maximum of these functions must go to infinity.

There are many sequences of functions which converge, in the distribution sense, to Dirac's δ function. They are called fundamental sequences for δ.

3.3. A sequence of step functions.

Consider a narrower and narrower step, centered at 0, of total area 1:

$$\Delta_n(x) = \begin{cases} \frac{n}{2} & \text{if } x \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ \\ 0 & \text{otherwise} \end{cases}$$

Then $\lim_{n\to\infty} \Delta_n = \delta$. Indeed, for any test function ϕ,

$$(\phi, \Delta_n) = \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(x)\, dx = \phi(c_n) \quad \text{for some } c_n \in \left[-\frac{1}{n}, \frac{1}{n}\right]$$

(by the mean value theorem) which converges to $\phi(0) = (\phi, \delta)$.

3.4. Other sequences.

We could smooth out the corners of our step functions to make them even C^∞.

In fact, one can find many fundamental sequences for δ by rescaling, as follows. Pick a function f which is C^∞ on $\mathbb{R}$ which is zero outside some interval $[-M, M]$ and has total integral one: $\int_{-\infty}^{\infty} f(x)dx = 1$. For each $\epsilon > 0$ use rescaling to shrink $[-M, M]$ to $[-\epsilon M, \epsilon M]$ while dialting f so that the total area remains one: define

$$f_\epsilon(x) = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)$$

Then $f_\epsilon(x) = 0$ outside $[-\epsilon M, \epsilon M]$ and $\int_{-\infty}^{\infty} f_\epsilon(x)dx = 1$ (check!). We have $\lim_{\epsilon\to 0} f_\epsilon = \delta$ since

$$(\phi, f_\epsilon) = \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)dx = \int_{-\infty}^{\infty} f(y)\phi(\epsilon y)dy = \int_{-M}^{M} f(y)\phi(\epsilon y)dy$$
$$= \int_{-M}^{M} \{f(y)\,[\phi(\epsilon y) - \phi(0)] + f(y)\phi(0)\}\, dy$$

$$= \phi(0) \int_{-M}^{M} f(y)\, dy \, + \, \int_{-M}^{M} \epsilon y f(y)\, \frac{\phi(\epsilon y) - \phi(0)}{\epsilon y}\, dy$$

and using the mean value theorem for the function $\phi(\epsilon y)$

$$= \phi(0) \, + \, \epsilon \int_{-M}^{M} y f(y) \phi'(\epsilon c) \quad \text{for some } c \in (0, \epsilon M)$$

hence the last term in the sum goes to 0 as $\epsilon \to 0$ and the limit is $\phi(0)$.

4. Green's function

4.1. **Non-homogeneous conditions can be reduced to homogeneous conditions, but for a non-homogeneous equation.** If the initial and boundary conditions are not zero, the problem is first turned into one with homogeneous conditions (but with a non-homogeneous term) by subtracting from the unknown function a function witch has the prescribed initial and/or boundary conditions. This is easiest to see when illustrated on a few examples.

4.1.1. *First order equations.* Consider the simplest example

$$\frac{dy}{dx} + a(x) y = g(x), \quad y(x_0) = y_0, \quad \text{for } x \geq x_0 \tag{5}$$

We find a function $h(x)$ so that $h(x_0) = y_0$, then substitute $y = h + u$. For example, we can take $h(x) = y_0$. Problem (5) becomes

$$\frac{du}{dx} + a(x) u = g(x) - a(x) y_0, \quad u(x_0) = 0, \quad \text{for } x \geq x_0$$

which is the type (11) studied before.

4.1.2. *Second order equations.* An equation with *non-homogeneous boundary conditions*, say:

$$P(x) y + Q(x) y' + R(x) y = 0, \quad y(0) = A, \; y(1) = B$$

can be transformed into a second-order *non-homogeneous equation*, but with *homogeneous boundary conditions*: just find a function $g(x)$ so that $g(0) = A$ and $g(1) = B$, then substitute $y = g + u$. (For example, one could take $g(x) = (B - A)x + A$). The problem becomes

$$P(x) y'' + Q(x) y' + R(x) y = f(x), \quad y(0) = A, \; y(1) = B$$

where $f(x) = -(P(x) g'' + Q(x) g' + R(x) g)$.

So from now on, when dealing with non-homogeneous equations, we can restrict our considerations to homogeneous conditions.

4.2. **Superposition.** Suppose we have a linear, non-homogeneous differential equation, $Lu = f$ where $f(x)$ is a given function (the forcing term, or the non-homogeneity), and $L = L(x, \frac{d}{dx})$ is a linear differential operator, for example,

$$L = \tfrac{d}{dx} + \alpha \text{ for equation } \tfrac{du}{dx} + \alpha u = f(x) \tag{6}$$

$$L = \tfrac{d}{dx} + a(x) \text{ for equation } \tfrac{du}{dx} + a(x)u = f(x) \tag{7}$$

$$L = \tfrac{1}{w(x)}\left[-\tfrac{d}{dx}(p(x)\tfrac{d}{dx}) + q(x)\right] \text{ for equation } \tfrac{1}{w(x)}\left[-\tfrac{d}{dx}(p(x)\tfrac{du}{dx}) + q(x)u\right] = f(x) \tag{8}$$

$$L = \tfrac{\partial}{\partial t} - \tfrac{\partial^2}{\partial x^2} \text{ for equation } u_t - u_{xx} = f \tag{9}$$

and so on.

Then: if u_1 solves $Lu_1 = f_1(x)$, and u_2 solves $Lu_2 = f_2(x)$, then a linear combination $u = c_1u_1 + c_2u_2$ solves $Lu = c_1f_1 + c_2f_2$ (because L is linear). Similarly, we can superimpose any number of solutions: if u_k solves $Lu_k = f_k(x)$ for $k = 1, \dots, n$ then $u = \sum_k c_k u_k$ solves $Lu = \sum_k c_k f_k$.

The same is true if we also have some *homogeneous conditions*, initial (IC) or boundary (BC). A condition is called homogeneous when a linear functional of u is required to be zero. For example, we could consider the problem $Lu = f$ with
-L as in (7) and the (IC) $u(x_0) = 0$
-L as in (8) and the (IC) $u(x_0) = 0,\ u'(x_0) = 0$
-L as in (8) and the (BC) $u(x_0) = 0,\ u(x_1) - 3u'(x_1) = 0$
-L as in (9) and the (IC) $u(t_0, x) = 0$ and (BC) $u(x_0) = 0,\ u(x_1) = 0$
and so on.

For this non-homogeneous problems with homogeneous conditions, again, a superposition of forcing terms is solved by a superposition of solutions.

Why not consider a "continuous superposition"? After all, f can be written as a "superposition of impulses":

$$f(x) = \int_{-\infty}^{\infty} f(t)\delta(x - t)\, dt$$

Then solve, for each parameter t, the equation $LG = \delta(x - t)$, denote its solution by $G = G(x, t)$, and then superimpose:

$$u(x) = \int_{-\infty}^{\infty} f(t)G(x, t)\, dt \tag{10}$$

is expected to solve $Lu = f$. We can similarly solve homogeneous (IC) or (BC) problems.

$G(x, t)$ is called the *Green's function* of the problem.

The tasks ahead are then: to find ways of calculating the Green's function of given problems, to show that the superposition (10) is indeed a solution,

and to see what information about solutions can be read directly in the Green's function.

4.3. **First examples.**

4.3.1. *The simplest example, C^∞ case.* Consider L as in (7) and the homogeneous initial value problem

$$\frac{du}{dx} + a(x)u = f(x), \quad u(x_0) = 0, \quad \text{for } x \geq x_0 \tag{11}$$

Let us find its Green's function: solve

$$\frac{dG}{dx} + a(x)G = \delta(x-t), \quad G(x_0, t) = 0 \text{ for all } t \tag{12}$$

The integrating factor is $\exp(\int a(x)\,dx)$.

Assume $a \in C^\infty$. Then so is the integrating factor, and then we can multiply the equation by it (since δ can be multiplied by C^∞ functions) and we get

$$\frac{d}{dx}\left(e^{\int_{x_0}^x a(s)\,ds} G\right) = e^{\int_{x_0}^x a(s)\,ds}\delta(x-t)$$

hence

$$\frac{d}{dx}\left(e^{\int_{x_0}^x a(s)\,ds} G\right) = e^{\int_{x_0}^t a(s)\,ds}\delta(x-t) \tag{13}$$

Recalling that

$$\frac{dH}{dx} = \delta(x), \quad \text{and similarly } \frac{d}{dx}H(x-t) = \delta(x-t)$$

integrating (13)[2]

$$e^{\int_{x_0}^x a(s)\,ds} G(x,t) = e^{\int_{x_0}^t a(s)\,ds}\, H(x-t) + C(t)$$

therefore

$$G(x,t) = \underbrace{e^{-\int_t^x a(s)\,ds}\, H(x-t)}_{\text{particular sol.}} + \underbrace{e^{-\int_{x_0}^x a(s)\,ds} C(t)}_{\text{general sol. homog. eq.}}$$

Imposing the initial condition we obtain $C(t) = -e^{-\int_t^{x_0} a(s)\,ds}\, H(x_0 - t)$ hence

$$G(x,t) = e^{-\int_t^x a(s)\,ds}\, \chi_{[x_0,x]}(t) \tag{14}$$

where

$$\chi_{[x_0,x]}(t) = H(x-t) - H(x_0 - t) = \begin{cases} 1 \text{ if } t \in [x_0, x] \\ 0 \text{ otherwise} \end{cases}$$

is a step function.

The solution of (11) is

$$u(x) = \int_{x_0}^{\infty} f(t)G(x,t)\,dt = \int_{x_0}^{x} f(t)e^{-\int_t^x a}\,dt$$

[2]We need to use here the fact that the only distributions whose derivatives are 0 are the constants, which requires a proof (not included here).

4.3.2. *The simplest example, non-smooth coefficients case.* Consider again the problem (11), only now with $a(x)$ not C^∞ (for example, we could have $a(x) = |x|$). Then we do not know that we can multiply (meaningfully) the distribution δ by the integrating factor.

Then the equation is solved as follows: we solve for $x < t$, and separately for $x > t$ and we match the two pieces. There are two main facts here.

Fact 1. We can rely on the intuition that $\delta(x) = 0$ for $x < 0$ and also $\delta(x) = 0$ for $x > 0$. (This can be rigorously stated, but we will not do that.)

Therefore $\delta(x - t) = 0$ for $x < t$ and also for $x > t$.

Fact 2. Examining (12) we see that G (as a function of x) has a jump discontinuity of magnitude 1 at $x = t$. This condition will match the two solutions on the two intervals.

Now let us solve (12). For $x < t$ the equation is $\frac{dG}{dx} + a(x)G = 0$ with the general solution $G(x,t) = A(t)e^{-\int_{x_0}^{x} a}$. Imposing the initial condition we must have $A(t) = 0$, hence $G(x,t) = 0$ for $x < t$.

For $x > t$ we have again $\frac{dG}{dx} + a(x)G = 0$, with the general solution $G(x,t) = B(t)e^{-\int_{t}^{x} a}$. There are no other conditions on this interval, and $B(t)$ is determined from the jump condition at $x = t$: we must have

$$G(x,t)\big|_{x=t+} - G(x,t)\big|_{x=t-} = 1$$

hence $B(t) = 1$, giving for G the formula (14).

4.4. **Autonomous equations.** If the coefficients of $L = L(\frac{d}{dx})$ do not depend on x, the equation $Lu = f(x)$ is called autonomous. For example, (6), (9) are autonomous equations.

Autonomous equations are translation-invariant, and we want to take advantage of that. More precisely, if $u(x)$ solves $Lu = 0$ then $u(x-t)$ solves the same (homogeneous) equation for any t. And is $u(x)$ solves $Lu = f(x)$ then also $u(x-t)$ solves $L[u(x-t)] = f(x-t)$ for any t. So instead of searching for a Green's function $G(x,t)$ by solving $LG = \delta(x-t)$ with boundary conditions, we could first solve for $LF = \delta$, then add to $F(x-t)$ the general solution of the homogeneous equation, and then impose the boundary conditions, the result being the Green's function. The procedure is illustrated below. But first, a definition.

Definition 5. *Let $L = L(\frac{d}{dx})$ be an autonomous differential operator.*

A solution $F(x)$ of $LF = \delta$ is called a fundamental solution *of the differential operator L.*

Then $u_{part}(x) = (f * F)(x) = \int f(t)F(x-t)dt$ is *a particular solution* of $Lu = f$. The general solution of $Lu = f$ is $u_{\rm part} + u_{\rm homog}$ where $u_{\rm homog}$ is the general solution of $Lu = 0$.

4.4.1. *A simple example.* Consider the autonomous problem (6) (which, of course, is a particular case of (7), for $a(x) \equiv \alpha$). Its fundamental solution is found as in §4.3.1 (only calculating for $t = 0$): we find a solution of

$$\frac{dF}{dx} + \alpha F = \delta(x) \tag{15}$$

(no conditions imposed). Multiplying be the integrating factor $e^{\alpha x}$ and integrating we obtain

$$F(x) = e^{-\alpha x}\,[H(x) + C]$$

Any particular value for C gives a fundamental solution of L. For example, for $C = -\frac{1}{2}$ we obtain the symmetric fundamental solution $F(x) = \frac{1}{2}\,\mathrm{sign}(x)\,e^{-\alpha x}$.

To find the Green's function of L with condition $u(x_0) = 0$, for $x \geq x_0$ we look for it in the form $F(x-t) + C(t)e^{-\alpha x}$ (note that the free constant in the solution of the homogeneous equation is allowed to depend on t). We determine $C(t)$ by imposing the boundary condition: $G(x_0,t) = e^{-\alpha(x_0-t)}H(x-t) - C(t)e^{-\alpha(x_0-t)} = 0$ and solving for $C(t)$ we get

$$G(x,t) = e^{-\alpha(x-t)}\,(H(x-t) - H(x_0-t))$$

The solution of the initial value problem is then given by $u(x) = \int_{x_0}^{\infty} G(x,t)f(t)\,dt$ (which for $x > x_0$ equals the familiar $u(x) = e^{-\alpha x}\int_{x_0}^{x} e^{\alpha t}\,f(t)\,dt$).

Alternatively, the solution can be found as follows. A particular solution of $\frac{du}{dx} + \alpha u = f$ is then (for, say, $C = 0$)

$$u_{\rm part}(x) = (f * F)(x) = \int_{-\infty}^{\infty} f(t)e^{-\alpha(x-t)}H(x-t)\,dt$$

The general solution is obtained by adding the general solution of the homogeneous equation:

$$u_{\text{gen}}(x) = u_{\text{part}}(x) + Ae^{-\alpha x}$$

where the constant A is determined by imposing the initial conditions.

4.4.2. *A second order example.*

(i) Find the fundamental solutions of $L = \frac{d^2}{dx^2}$, and

(ii) the Green's function of $-L$ with boundary conditions $u(0) = 0,\ u(\pi) = 0$ for $x \in [0, \pi]$.

We need to solve: $F'' = \delta$.

(i) *Solution I.* By direct integration, recalling that $H' = \delta$ we obtain $F'(x) = H(x) + C$, and integrating again $F(x) = \int_{-\infty}^{x} H(t)dt + Cx + D = xH(x) + Cx + D$.

Solution II. We solve the equation for $x < 0$, giving $F(x) = ax + b$, and for $x > 0$, giving $F(x) = cx + d$, and we match the two solutions at $x = 0$ using the following.

1) $-F'$ has a jump discontinuity of magnitude 1 at $x = 0$ (the graph of F has an "angle" at $x = 0$): $F'(0+) - F'(0-) = -1$, which gives $c - a = -1$, and

2) F must be continuous at $x = 0$: $F(0+) = F'(0-)$, giving $b = d$.

Therefore $f(x) = cx + d$ for $x > 0$ and $F(x) = (c + 1)x + d$ for $x < 0$, which is, of course $F(x) = cx + d + xH(x)$, the same we had before.

(ii) A fundamental solution of $-L$ is $-xH(x)$ hence we look for $G(x, t)$ in the form $-(x-t)H(x-t)+a(t)x+b(t)$. Imposing the boundary conditions $G(0, t) = 0$, $G(\pi, t) = 0$ we obtain $G(x, t) = -(x - t)H(x - t) + (1 - \frac{t}{\pi})x$.

5. Green's function of self-adjoint Sturm-Liouville problems

5.1. **Liouville transformation.** Linear, second order, differential equations can be brought to the simple form $\frac{d^2y}{dz^2} - g(z)y = 0$ by a change of both dependent and independent variables. Moreover, the transformation preserves norms and eigenvalues. This is the Liouville transformation, explained in the present section.

Given an eigenvalue equation $L(x, \frac{d}{dx})u = \lambda u$ where

$$L(x, \tfrac{d}{dx}) = -P(x)\tfrac{d^2}{dx^2} - Q(x)\tfrac{d}{dx} - R(x)$$

we saw that if

$$p(x) = \exp\left(\int_{x_0}^{x} \frac{Q}{P}\right),\ w = \frac{1}{P}\,p,\ q = -\frac{R}{P}\,p$$

then substituting

$$P = \frac{p}{w},\ Q = \frac{p'}{w},\ R = -\frac{q}{w}$$

the operator L is presented in a self-adjoint form

$$L(x, \tfrac{d}{dx}) = \frac{1}{w(x)}\left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right] \tag{16}$$

which is formal self-adjoint in a domain in $L^2(I, w(x)dx)$ (where I is an interval and some homogeneous boundary conditions are assumed).

We can simplify further the equation/operator, bringing it to a form with $p = 1$ and $w = 1$, using the *Liouville transformation* as follows.

I. There is a change of the independent variable $z = z(x)$ so that in the new variable z the weight w and p become 1. To find this substitution, using the fact that $\frac{d}{dx} = z' \frac{d}{dz}$, and $\frac{d^2}{dx^2} = z'^2 \frac{d^2}{dz^2} + z'' \frac{d}{dz}$, L defined by (16)

$$L = \frac{1}{w}\left[-p\frac{d^2}{dx^2} - p'\frac{d}{dx} + q\right] u$$

becomes

$$\begin{aligned} L &= \frac{1}{w}\left[-pz'^2\frac{d^2}{dz^2} - (pz'' + p'z')\frac{d}{dz} + q\right] u \\ &= \frac{pz'^2}{w}\left[-\frac{d^2}{dz^2} - \frac{pz'' + p'z'}{pz'^2}\frac{d}{dz} + \frac{q}{pz'^2}\right] \end{aligned}$$

We want $pz'^2/w = 1$, therefore $z(x) = \int_a^x \sqrt{\frac{w(t)}{p(t)}}\, dt$ where a can be chosen at will. With this choice L is

$$L = -\frac{d^2}{dz^2} - \left(\frac{z''}{z'^2} + \frac{p'}{pz'}\right)\frac{d}{dz} + \frac{q}{w}$$

II. The first derivative term can be eliminated (for any linear second order equation) by a substitution of the dependent variable of the form $u = \phi y$ where the function ϕ is suitably chosen. In our case, this substitution in the eigenvalue equation $Lu = \lambda u$ gives

$$-\phi\frac{d^2y}{dz^2} - 2\frac{d\phi}{dz}\frac{dy}{dz} - y\frac{d^2\phi}{dz^2} - \left(\frac{z''}{z'^2} + \frac{p'}{pz'}\right)\phi\frac{dy}{dz} - \left(\frac{z''}{z'^2} + \frac{p'}{pz'}\right)y\frac{d\phi}{dz} + \frac{q}{w}\phi y = \lambda\phi y$$

hence, and dividing by ϕ

$$\frac{d^2y}{dz^2} - \left(2\frac{1}{\phi}\frac{d\phi}{dz} + \frac{z''}{z'^2} + \frac{p'}{pz'}\right)\frac{dy}{dz} + \left[-\frac{1}{\phi}\frac{d^2\phi}{dz^2} - \left(\frac{z''}{z'^2} + \frac{p'}{pz'}\right)\frac{1}{\phi}\frac{d\phi}{dz} + \frac{q}{w}\right] y = \lambda y \tag{17}$$

The condition that the coefficient of $\frac{dy}{dz}$ vanishes is that

$$2\frac{1}{\phi}\frac{d\phi}{dz} + \frac{z''}{z'^2} + \frac{p'}{pz'} = 0 \tag{18}$$

or

$$2\frac{\phi'}{\phi} + \frac{z''}{z'} + \frac{p'}{p} = 0 \;\text{ therefore } \phi^2 pz' = C \text{ and choose } \phi = (pw)^{-1/4}$$

Furthermore, this change of coordinates is unitary:

Theorem 6. Liouville transformation

Assume $p(x), w(x) > 0$ on $[a, b]$, and p, p', q, w continuous on $[a, b]$. Then the change of variables $(x, u) \mapsto (z, y)$ given by

$$z(x) = \int_a^x \sqrt{\frac{w(t)}{p(t)}}\, dt, \quad u = \phi y, \ \phi = (pw)^{-1/4} \tag{19}$$

transforms the eigenvalue equation $Lu = \lambda u$ with L given by (16) into

$$-\tfrac{d^2y}{dz^2} + g(z)y = \lambda y \tag{20}$$

where

$$g(z) = q - \phi \frac{d}{dz}\left(\frac{1}{\phi^2}\frac{d\phi}{dz}\right) = q - \frac{(pw)^{1/4}}{w}\left\{p\left[(pw)^{-1/4}\right]'\right\}' \tag{21}$$

Moreover, the change of variable preserves the L^2 norm: if $c = \int_a^b \sqrt{\frac{w}{p}}$ then

$$\int_a^b |u(x)|^2 w(x) dx = \int_0^c |y(z)|^2 dz$$

and it is an isomorphism between the Hilbert spaces $L^2([a, b], w(x)dx)$ and $L^2([0, c], dz)$.

Remark. A linear change of variables in x can be used to replace the interval $[0, c]$ by any other interval: the linear operator T defined as $(Tf)(t) = f(ct/\alpha)$ is an isomorphism between $L^2([0, c], dz)$ and $L^2([0, \alpha], \frac{c}{\alpha}dt)$. Replacing the constant weight $\frac{c}{\alpha}$ by 1 represents just a multiplication of all functions by the same factor, and does not modify the eigenvalues and eigenspaces of operators.

Proof of Theorem 6. The formula for $z(x)$ and ϕ have been found before. To deduce the formula for $g(z)$ we only need to simplify the coefficient of y in (17); using (18) this coefficient becomes

$$-\frac{1}{\phi}\frac{d^2\phi}{dz^2} + 2\left(\frac{1}{\phi}\frac{d\phi}{dz}\right)^2 + \frac{q}{w} = \phi\frac{d}{dz}\left(\frac{1}{\phi^2}\frac{d\phi}{dz}\right) + \frac{q}{w}$$

giving the first equality in (21); the second equality follows immediately using the formulas for z' and for ϕ.

To show that the Liouville transformation preserves the norm we simply change the variable of integration:

$$\int_a^b |u(x)|^2 w(x) dx = \int_0^c \phi^2 |y|^2 w \frac{1}{z'}\, dz = \int_0^c \frac{1}{(pw)^{1/2}} |y|^2 w \frac{\sqrt{p}}{\sqrt{w}}\, dz$$

□

As a consequence, all the theorems we prove for (20) can be transcribed for (16), and the first form is certainly much simpler (it is called a *normal*

form). In concrete problems, however, if we need specific formulas, we do not want to be bothered by sometimes involved changes of variables, or, we are interested in some $w(x)$ which vanishes at a or at b. For this reasons we will keep the form (16) for most times.

5.2. **Existence and uniqueness of the Greens's function for non-homogeneous problems are not guaranteed.** Not all non-homogeneous Sturm-Liouville problems have solutions. And when they exist, solutions may not be unique.

To see an example, solve

$$\frac{d^2u}{dx^2} = f(x), \quad u'(0) = 0, \ u'(1) = 0 \tag{22}$$

We attempt to find the Green's function of the problem. For this, we need to solve

$$G'' = \delta(x-t); \quad G'(0,t) = G'(1,t) = 0 \quad (*)$$

One integration of (*) gives

$$G'(x,t) = H(x-t) + C(t); \quad H(-t) + C(t) = H(1-t) + C(t) = 0$$

This is already impossible! No point in trying to integrate further.

What is happening?

If we had such a Green's function, then (22) would have a solution for any say continuous f, and the solution would be

$$u(t) = \int_{-\infty}^{\infty} G(x,t) f(t) dt$$

Note the similarity with the linear algebra problem $Ax = b \Rightarrow x = A^{-1}b$; $x_i = \sum A^{-1}_{ij} b_j$.

Let's try to solve one of the simplest instances of (22), when $f = 1$.

The solution of this inhomogeneous equation is, as we know, any particular solution u_0 plus the general solution of the homogeneous equation $u_h'' = 0$. The latter is $u_0 = ax + b$. For the former we can choose $u_0 = x^2/2$. Now we impose the boundary conditions:

$$(x^2/2 + ax + b)'|_{x=0} = a = 0; \quad (x^2/2 + ax + b)'|_{x=1} = 1 + a = 0$$

giving $-1 = 0$. What does this mean? Remember from linear algebra that $Ax = B$ where A is a matrix is guaranteed to have a solution if A is invertible. Likewise, an equation of the form $Lf = g$ where L is some differential operator is guaranteed to have a solution if L is invertible. The operator d^2/dx^2 defined on $\{f \in C^2[0,1] | f'(0) = f'(1) = 0\}$ is simply not invertible!

It is desirable to have a theorem which gives conditions guaranteeing existence and uniqueness of the Green's function. This is done in the following section.

6. Conditions for existence and uniqueness of the solution to real-valued, non-homogeneous Sturm-Liouville problems

In the present section we assume problems with real coefficients. Equations with complex coefficients can be split into the real and the imaginary parts.

6.1. **Existence of eigenvalues for real-valued Sturm-Liouville problems.** Recall the following facts:

Theorem 7. *Consider the linear second order differential operator in self-adjoint form* (16)*, with* $p(x), w(x) > 0$ *for* $x \in (a,b)$[3]*, and* p, p', w, q *continuous on* $[a,b]$ *and real-valued.*

Denote $\mathcal{H} = L^2((a,b), w(x)dx)$*. Let* $\alpha, \alpha' \in \mathbb{R}$*, not both zero, and* $\beta, \beta' \in \mathbb{R}$*, not both zero. Denote*

$$D(L) = \{u \in \mathcal{H} \,|\, u', u'' \in \mathcal{H},\ p(\alpha u + \alpha' u')\big|_{x=a} = 0,\ p(\beta u + \beta' u')\big|_{x=b} = 0\}$$

Then $D(L)$ *is dense in* $\mathcal{H}$*, and* L *is self-adjoint on the domain* $D(L)$*. Therefore:*

(i) all the eigenvalues of $L : D(L) \to \mathcal{H}$ *are real, and eigenfunctions are (can be chosen) real-valued,*

(ii) eigenfunctions corresponding to distinct eigenvalues are orthogonal in $\mathcal{H}$*,*

(iii) the eigenvalues are simple,

(iv) there is a lowest eigenvalue, and the eigenvalues form an increasing sequence $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$

(v) $\lim_{n\to\infty} \lambda_n = \infty$*, and the corresponding eigenfunctions form an orthogonal basis for* $\mathcal{H}$*.*

Notes.

1. If $p(a) = 0$, then no condition is needed at $x = a$ in $D(L)$, and similarly if $p(b) = 0$, no condition is needed at $x = b$.

2. The derivatives of functions in L^2 (but not necessarily differentiable as functions) are considered in the sense of distributions–when these derivatives are *functions*. For example $|x| \in L^2(-1,1)$ has its derivative $|x|' \in L^2(-1,1)$ (see Exercise 1. of §2.3).

Furthermore: For boundary conditions which are not regular, but mixed (including the periodic case):

$$\alpha u(a) + \alpha' u'(a) + \alpha_2 u(b) + \alpha_2' u'(b) = 0$$
$$\beta u(b) + \beta' u'(b) + \beta_2 u(a) + \beta_2' u'(a) = 0$$

results similar to those of Theorem 7 hold, *except* that the eigenvalues may not be simple (they are at most double, since this is a second order differential equations).

[3]Note that $p(x), w(x)$ may vanish at the endpoints $x = a,\ x = b$.

7. Green's function for self-adjoint Sturm-Liouville problems

7.1. **When the Green's function of Sturm-Liouville problems exists and is unique.** We will find conditions which ensure existence and uniqueness of the Green's function of given self-adjoint Sturm-Liouville problems and of the solution to non-nomogeneous problems. Theorems 8 and 9 embody the Fredholm alternative for self-adjoint Sturm-Liouville problems: either the homogeneous problem has a nontrivial solution, or any non-homogeneous problem has a unique solution. This theorem is true in more general contexts: for compact operators, as are, for example, the integral operators which give solutions to second order problems.

Remark. As before, L denotes a differential operator, and $D(L)$ its domain. The initial and boundary condition of the problem are specified by conditions on the functions in $D(L)$. We choose $D(L)$ to consist of square integrable functions so that we can take advantage of the powerful Hilbert spaces techniques and insights. On the other hand, the Green's function of a given problem is defined without reference to a precise space of functions: one only needs to specify a differential operator, the domain of the independent variable and boundary conditions; all calculations are done in distribution sense. For example, instead of spaces of square integrable functions, we could consider spaces of C^∞ functions (with the same boundary conditions), or L^p functions.

Recall that $\lambda = 0$ is an eigenvalue of $(L, D(L))$ if and only if the homogeneous problem has a nontrivial (i.e. nonzero) solution (i.e. there is $u \in D(L)$, $u \not\equiv 0$ so that $Lu = 0$).

Theorem 8 below states that the Green's function is guaranteed to exist, and be unique, if there are no nontrivial solutions of the homogeneous equation. Recall the example in §5.2 where the Green's function either did not exist, or was not unique, depending on a condition on the forcing term; it has a nontrivial solution of the homogeneous problem, namely $u(x) = 1$.

Furthermore, Theorem 8 also states that in this case, then there is a unique solution of the non-homogeneous problem, for any forcing term (in the appropriate L^2 space).

Theorem 8. *Let $(L, D(L))$ be a regular Sturm-Liouville problem, as in Theorem 7 with $p > 0$ on $[a, b]$.*

Denote by $\mathcal{H} = L^2((a, b), w(x)dx)$ and $\langle \cdot, \cdot \rangle_w$ its inner product.

Assume $\lambda = 0$ is not an eigenvalue. Then:

1. The problem has a Green's function.

2. The Green's function is unique.

3. Denoting by $u_1, u_2, \ldots, u_n, \ldots$ an orthonormal basis of $\mathcal{H}$ formed by eigenfunctions then

$$G(x,t) = w(t) \sum_{n=1}^{\infty} \frac{1}{\lambda_n} u_n(x)\, u_n(t) \tag{23}$$

where the series converges in $\mathcal{H}$ for each t.

4. For any $f \in \mathcal{H}$ the solution $u \in D(L)$ of $Lu = f$ is given by

$$u(x) = \int_a^b G(x,t)\, f(t)\, dt := (\mathcal{G}f)(x) \tag{24}$$

and its expansion in terms of eigenfunctions is

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle u_n, f\rangle_w \, u_n(x) \tag{25}$$

5. Furthermore,

$$w(t) \sum_{n=1}^{\infty} u_n(t)\, u_n(x) = \delta(x-t) \tag{26}$$

Remarks. Before looking at the proof of the Theorem, let us examine its implications.

1. The existence of a Green's function $G(x,t)$ so that solutions $u \in D(L)$ of $Lu = f$ are given by (24) can be stated as the existence of the inverse of the operator L (with given conditions); the inverse in an integral operator $\mathcal{G}$, with integral kernel $G(x,t)$, and whose domain needs to be specified. Theorem 8 gives a class of problems for which the inverse is defined on the whole Hilbert space: if 0 is not an eigenvalue, Theoren 8 point 4. states that the operator $\mathcal{G} : \mathcal{H} \to D(L)$ is the inverse of the operator $L : D(L) \to \mathcal{H}$. Otherwise, Theorem 9 establishes the domain of the inverse as the orthogonal complement of the eigenfunction corresponding to the 0 eigenvalue.

2. Moreover, the operator $\mathcal{G}$ is bounded: from (25) and by Theorem 7 (iv), it follows that $\|u\| \le M\|f\|$ where $M = \max 1/\lambda_n^2$.

3. We see from (23) that

$$w(x)G(x,t) = w(t)G(t,x) \tag{27}$$

Proof of Theorem 8. (To make it fully rigorous, we would need to be more careful with some steps...) 1. We construct the Green's function: Equation $Ly = \delta(x-t)$ for $x \in [a,b]$ can be written as

$$-G_{xx} - \tfrac{p'(x)}{p(x)}\, G_x + \tfrac{q(x)}{p(x)}\, G = \tfrac{w(x)}{p(x)}\, \delta(x-t)$$

hence

$$-G_{xx} - \tfrac{p'(x)}{p(x)}\, G_x + \tfrac{q(x)}{p(x)}\, G = \tfrac{w(t)}{p(t)}\, \delta(x-t)$$

Since the right hand side is zero for $x < t$, and also for $x > t$, solving on these two intervals with their respective boundary condition we get

$$G(x,t) = \begin{cases} C(t)\, y_a(x) & \text{for } x < t \\ D(t)\, y_b(x) & \text{for } x > t \end{cases}$$

where y_a, y_b are nonzero solutions of $Ly = 0$, satisfying $\alpha y_a(a) + \alpha' y_a'(a) = 0$, respectively $\beta y_b(b) + \beta' y_b'(b) = 0$ (since $p \neq 0$).

The two branches of $G(x,t)$ must be matched at $x=t$. As in §4.4.2, we must have $G(x,t)$ continuous in x at $x=t$, and its x-derivative must have a jump discontinuity of magnitude $\frac{w(t)}{p(t)}$ at $x=t$:

$$\begin{aligned} C(t)y_a(t) &= D(t)y_b(t) \\ D(t)y_b'(t) - C(t)y_a'(t) &= \tfrac{w(t)}{p(t)} \end{aligned} \tag{28}$$

If y_a and y_b are linearly dependent, then $y_b = \text{const}\, y_a$ then y_a satisfies the boundary condition at $x=b$ as well, and since $Ly_a = 0$, then y_a is an eigenfunction to the eigenvalue 0, which contradicts the assumption that 0 is not an eigenvalue.

Therefore y_a, y_b are linearly independent, hence their Wronskian $W := y_a y_b' - y_a' y_b \neq 0$. This means that the linear system (28) has a unique solution:

$$C(t) = \frac{w(t)y_b(t)}{W(t)p(t)}, \quad D(t) = \frac{w(t)y_a(t)}{W(t)p(t)}$$

On the other hand we can calculate the Wronskian $W(t)$ from the differential equation that $y_{a,b}$ satisfy, $-py'' - p'y' + qy = 0$, or, in matrix form

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac{q}{p} & -\frac{p'}{p} \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

and, recalling that $W = \text{const}\exp[\int \text{Tr}(Matrix)]$, it follows that $W = \text{const}/p$, or $W(x)p(x) = \text{const} \equiv \frac{1}{\kappa}$. Therefore $C(t) = \kappa w(t) y_b(t)$, $\quad D(t) = \kappa w(t) y_a(t)$, giving

$$G(x,t) = \begin{cases} \kappa\, w(t)\, y_b(t)\, y_a(x) & \text{for } x < t \\ \kappa\, w(t)\, y_a(t)\, y_b(x) & \text{for } x > t \end{cases}$$

2. Suppose G_1 and G_2 are two Green's function of the problem. Then $G_1 - G_2$ belongs to $D(L)$ and satisfies $L(G_1 - G_2) = 0$. Since 0 is not an eigenvalue, we must have $G_1 - G_2 = 0$ hence $G_1 = G_2$.

3. We know that the eigenfunctions of $(L, D(L))$ form an orthonormal basis of $\mathcal{H}$ (by Theorem 7). Since $G(\cdot, t) \in \mathcal{H}$ for each t (because it is continuous on $[a,b]$), then

$$G(x,t) = \sum_{n=1}^{\infty} c_n(t)\, u_n(x) \quad \text{for some } c_n(t) \text{ with } \{c_n(t)\}_{n=1,2,\ldots} \in \ell^2 \tag{29}$$

and where

$$c_n(t) = \langle u_n, G(\cdot,t)\rangle_w, \ n = 1, 2, \ldots$$

Now apply L to both sides of (29). We need to be careful: L is not bounded, i.e. it is not continuous, therefore we cannot say that $L(\lim_{N\to\infty} \sum_{n=1}^{N} *) = \lim_{N\to\infty} L(\sum_{n=1}^{N} *)$. It turns out to be true in distribution sense. Here is a justification.

We know that $Lu_n = \lambda_n u_n$ for all n. Therefore

$$\lambda_n c_n(t) = \lambda_n \langle u_n, G(\cdot,t)\rangle_w = \langle \lambda_n u_n, G(\cdot,t)\rangle_w = \langle Lu_n, G(\cdot,t)\rangle_w \tag{30}$$

Now,

$$\langle Lu_n, G(\cdot,t)\rangle_w = \int_a^b \left[(-pu_n')' + qu_n\right] G(x,t)\,dx = (u_n, wLG(\cdot,t))$$

(we define, in the sense of distributions, as usual, $(u, L\phi) = (Lu, \phi)$) hence

$$(u_n, wLG(\cdot,t)) = (u_n, \delta(x-t)\,w(t)) = u_n(t)w(t)$$

Question: Which steps above are not quite correctly justified?

Therefore

$$c_n(t) = \frac{u_n(t)w(t)}{\lambda_n}$$

which used in (29) gives (23).

4. Formula (24) for the solution of a second order linear equation follows from the classical theory of linear differential equations (or can be checked directly). Using (23) in (24) gives (25).

5. Besides the expansion (25), we can expand any $u \in \mathcal{H}$ as

$$u(x) = \sum_{n=1}^{\infty} \langle u_n, u\rangle_w u_n(x) = \sum_{n=1}^{\infty} \int_a^b u_n(t)u(t)w(t)dt\, u_n(x)$$

$$= \text{[exchange order in distribution sense]} \int_a^b \left[\sum_{n=1}^{\infty} u_n(t)u_n(x)\right] u(t)w(t)\,dt$$

therefore

$$w(t)\sum_{n=1}^{\infty} u_n(t)u_n(x) = \delta(x-t)$$

□

7.2. **When the Green's function for self-adjoint Sturm-Liouville problems exists and it is not unique.**

Note: If 0 is an eigenvalue of a self-adjoint Sturm-Liouville problem, the dimension of its eigenspace must be one (any eigenspace is at most two-dimensional).

Theorem 9 shows that if there are nontrivial solutions of the homogeneous problem (i.e. 0 is an eigenvalue), then a non-homogeneous equation is solvable only when the forcing term is orthogonal to the kernel of the homogeneous problem; the solution is not unique because we can add to it any solution of the homogeneous problem (i.e. in its kernel).

Theorem 9. *Let $(L, D(L))$ be as in Theorem 7 with $p > 0$ on $[a, b]$.*

If 0 is an eigenvalue of L, then an equation $Lu = f$ has solutions $u \in D(L)$ if and only if

$$\langle u_0, f\rangle_w = 0 \quad \text{for } u_0 \in D(L) \text{ with } Lu_0 = 0$$

When the solution exists, it is not unique, but there is a one-parameter family of solutions, namely

$$u(x) = Cu_0(x) + \sum_{n;\, \lambda_n \neq 0} \frac{1}{\lambda_n} \langle u_n, f\rangle_w \, u_n(x) \tag{31}$$

Proof.

We only need to retrace the proof of Theorem 8 and see what happens if 0 is an eigenvalue.

We first see that we cannot find a Green's function. Then a direct verification show that (31) satisfies $Lu = f$. Are there other solutions? Suppose that u_2 also satisfies $Lu_2 = f$. Then $u - u_2$ satisfies $L(u - u_2) = 0$, hence $u - u_2$ is a solution of the homogeneous equation, hence $u_2 = u +$ [a solution in $D(L)$ of the homogeneous equation].

If u_2 is linearly dependent of u_0, then u_2 has the same formula as u, only with a different constant C.

If u_2 and u_0 are linearly independent, then there are two independent solutions of the homogeneous equation satisfying the boundary conditions, hence any solutions satisfies the boundary conditions, but this is not possible unless $\alpha = \alpha' = \beta = \beta' = 0$, contradiction! □

7.3. **The resolvent.** We can use a line of reasoning similar to that of §7.1 and §7.2 to find the formula for the resolvent of L. Let $z \in \mathbb{C}$, not an eigenvalue of L. This means that 0 is not an eigenvalue of $L - zI$. Using Theorems 8 and 9 for the operator $L - z$ instead of L we find

Theorem 10. *Let $(L, D(L))$ be a regular Sturm-Liouville problem, as in Theorem 7 with $p > 0$ on $[a, b]$.*

(i) Let $z \in \mathbb{C}$ be not an eigenvalue of L.

Then Green's function $G(x,t,z)$ of $L-z$ has the expansion

$$G(x,t,z) = w(t)\sum_{n=1}^{\infty} \frac{1}{\lambda_n - z}\, u_n(x)\, u_n(t) \tag{32}$$

and the solution of $Lu - zu = f$, $u \in D(L)$ is

$$u(x) = (L-z)^{-1} f = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z}\, \langle u_n, f\rangle_w \; u_n(x) \tag{33}$$

(ii) If $z = \lambda_k$ is an eigenvalue of L then $Lu - zu = f$, $u \in D(L)$ has solutions if and only if $\langle u_k, f\rangle_w = 0$, and in this case there is a one-parameter family of solutions, namely

$$u(x) = Cu_k + \sum_{n=1,\, n\neq k}^{\infty} \frac{1}{\lambda_n - \lambda_k}\, \langle u_n, f\rangle_w \; u_n(x)$$

Theorem 10 can be formulated as an instance of the Fredholm's alternative:

Given $(L, D(L))$:
either $(L-z)u = f$ has a unique solution for every f,
or there is a nonzero solution of $(L-z)u = 0$.

Remark: formula (32) shows that the Green's function of the resolvent $(L-z)^{-1}$ of $(L, D(L))$ is analytic in the parameter z, except for z equal to the eigenvalues of $(L, D(L))$, where G has poles of order one. This is intuitively expected, by functional calculus. In infinite dimensions this happens only for special operators, though.

8. Continuous spectrum

Eigenvalue problems on infinite intervals, or when coefficients are not smooth enough, may have a continuum of "eigenvalues".

For example, consider the problem

$$u'' + \lambda u = 0, \quad \text{for } x \in (-\infty, \infty) \tag{34}$$

with the boundary condition

$$u \text{ is bounded at } \pm\infty \tag{35}$$

Then any $\lambda \geq 0$ is an "eigenvalue", corresponding to the "eigenfunctions" $\sin(\sqrt{\lambda}x)$, $\cos(\sqrt{\lambda}x)$. I used quotation makes because we can only use these words in a specified a Hilbert space, and we specified none. However, the problem (34), (35) has a valid physical significance, and we need a solid mathematical foundation for its treatment.

To have an intuitive picture on what is going on with the problem (34), (35), we should consider the even simpler operator

$$L = -i\tfrac{d}{dx} \tag{36}$$

with the boundary conditions (35), that is, on

$$D(L) = \{u \in L^2(\mathbb{R}) \,|\, u' \in L^2(\mathbb{R}), u \text{ is bounded at } \pm\infty\} \tag{37}$$

We have

$$-i\tfrac{d}{dx} e^{i\xi x} = \xi\, e^{i\xi x}$$

so $e^{i\xi x}$ behaves like an eigenfunction corresponding to the eigenvalue ξ, for any $\xi \in \mathbb{R}$. Since $e^{i\xi x}$ does not belong to the Hilbert space $L^2(\mathbb{R})$, we will call it a *generalized eigenfunction*, and ξ will be called a *generalized eigenvalue*.

Moreover, these generalized eigenfunctions are "complete" in $L^2(\mathbb{R})$ in the sense that any f in the Hilbert space can be expanded in terms of the generalized eigenfunctions—only not as a series, but *as an integral*: any $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \int_{-\infty}^{\infty} e^{ix\xi}\, g(\xi)\, d\xi$$

where, of course, $g(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f}(\xi) \in L^2(\mathbb{R})$ is the continuous analogue of the coefficients of the expansion of f.

The Fourier transform is the unitary transformation of $L^2(\mathbb{R})$ which diagonalizes the operator $\frac{d}{dx}$. Indeed, since

$$-i\tfrac{d}{dx} f(x) = -i\tfrac{d}{dx} \mathcal{F}^{-1} \hat{f}(\xi) = -i\tfrac{d}{dx} \int_{-\infty}^{\infty} e^{ix\xi} \tfrac{1}{\sqrt{2\pi}} \hat{f}(\xi)\, d\xi$$

$$= \int_{-\infty}^{\infty} e^{ix\xi} \tfrac{1}{\sqrt{2\pi}} \xi \hat{f}(\xi)\, d\xi = \mathcal{F}^{-1}[\xi \hat{f}(\xi)]$$

we see that

$$\mathcal{F}\left(-i\tfrac{d}{dx}\right)\mathcal{F}^{-1} = \xi$$

it is the operator of multiplication of $\hat{f}(\xi)$ by ξ. *These calculations are formal, more work would be needed to justify differentiation under the integral operator etc. But the result above, carefully stated is correct Given a function g, the multiplication operator M_g defined as $M_g f = gf$ is the continuum analogue of a diagonal operator.*

Notes: Recall that $\frac{d}{dx}$ is skew-symmetric in $L^2(\mathbb{R})$, hence $\pm i\frac{d}{dx}$ is symmetric, justifying the choice of the factor i in front of $\frac{d}{dx}$. We choose the minus sign so that the eigenvalue is ξ (rather than $-\xi$ for the other choice).

In general, in the case of continuous spectrum, we have generalized eigenfunctions, and, when they form a complete set, functions can be expanded as integrals, rather than series.

The formal definition of the spectrum of an operator is as follows.

Definition 11. *Consider an operator $L, D(L)$. The resolvent set of L consists of all the numbers $z \in \mathbb{C}$ for which $(L-z)^{-1}$ exists and it is bounded.*

The spectrum of L, $\sigma(L)$, consists of all complex numbers which are not in the resolvent set.

Note that $\lambda \in \sigma(L)$ is an eigenvalue means that $L - z$ is not one-to-one. But there are other reasons why some points belong to the spectrum of L: these are the generalized eigenvalues.

8.0.1. *A first order example: $L = -i\frac{d}{dx}$ on $\mathbb{R}$.* Consider the problem (36), (37), with spectrum $\sigma(L) = \mathbb{R}$, and which has no proper eigenfunctions (but plenty of generalized ones). Take z in its resolvent set, that is, $z \in \mathbb{C} \setminus \mathbb{R}$. This means can invert $L - z$ and $(L-z)^{-1}$ is a bounded operator. Let us find the resolvent $(L-z)^{-1}$. A simple way to solve $(L-z)u = f$ for $f \in L^2(\mathbb{R})$ (or any differential equations with constant coefficients defined on the whole line) is to take the Fourier transform in the equation, which gives

$$\xi\hat{u} - z\hat{u} = \hat{f} \quad \text{so that } \hat{u} = \frac{\hat{f}}{\xi - z}$$

therefore

$$u(x) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\hat{f}(\xi)}{\xi - z}\, d\xi \tag{38}$$

which is a bounded operator, since, using the Cauchy-Schwartz inequality, and then Parseval's identity

$$\|u\|^2 \le \int_{-\infty}^{\infty} \frac{d\xi}{|\xi - z|^2}\, \|\hat{f}\|^2 = \frac{\pi}{|\Im z|}\, \|f\|^2 \tag{39}$$

To exhibit the Greens's function of the problem, rewrite (38) as

$$\begin{aligned} u(x) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\hat{f}(\xi)}{\xi - z}\, d\xi &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{\xi - z} \int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} \frac{e^{i(x-y)\xi}}{\xi - z} d\xi \end{aligned} \tag{40}$$

hence

$$G(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi - z}\, d\xi \tag{41}$$

We can further simplify (41): for $\Im z > 0$ we have $G(x,t) = ie^{izt}\, H(t)$, which is seen by deformation of the path of integration upwards in the complex ξ plane, towards ∞ and collecting the residue at $\xi = z$ if $t > 0$, and downwards if $t < 0$, yielding zero. Similarly, for $\Im z < 0$, $G(x,t) = ie^{izt}\, H(-t)$.

We see that $u(x)$ in (38) is not defined if $z \in \mathbb{R}$. This alone does not imply that the equation $(L - z)u = f$ has no solution, for now it only implies that a particular formula for a possible solution does not apply.

Let's however try to solve

$$(L - z)u = f; \quad f \in L^2(\mathbb{R}), \quad z \in \mathbb{R} \tag{42}$$

We can proceed by variation of parameters. Let's take the particular example $z = 0$, as for general z we can do a similar analysis by variation of parameters. We get $f(x) = i\int_{-\infty}^{x} g(s)ds + C$. Take $g(x) = e^{-x^2} \in L^2$. Since we want $f \in L^2$ we must have $C = 0$. But then , since $\int_{-\infty}^{\infty} g(s)ds = \sqrt{\pi}$, $\lim_{x\to\infty} f(x) = i\sqrt{\pi}$ and f is not in L^2, thus L is not invertible in L^2, where L^{-1} was supposed to be defined.

Note that the generalized eigenvalues ($z \in \mathbb{R}$) are still singular points of the Green's function, only in this case they are points of jump discontinuities. But in its integral representation (41) they are poles of the integral kernel. Also, the norm of the resolvent goes to infinity when z approaches the spectrum of L, as seen in (39).

Of course, (38) coincides with the expression obtained by solving $(L - z)u = f$ by multiplying with an integrating factor: imposing the boundary condition (37) to the general solution $u(x) = e^{-izx}(C + i\int^x e^{izy} f(y)dy)$ we obtain $u(x) = i\int_0^{\infty} e^{izt} f(x-t)dt$ for $\Im z > 0$ and the symmetric formula for $\Im z < 0$.

8.0.2. *A second order example: $L = -\frac{d^2}{x^2}$ on $\mathbb{R}$.* Let

$$L = -\tfrac{d^2}{dx^2} \tag{43}$$

with the boundary conditions (35), that is, on

$$D(L) = \{u \in L^2(\mathbb{R}) \,|\, u', u'' \in L^2(\mathbb{R}), u \text{ is bounded at } \pm\infty\} \tag{44}$$

Since (43) is the square of (36) we expect these two operators to have a common set of generalized eigenvectors, and indeed, $-\frac{d^2}{dx^2}e^{\pm ix\xi} = \xi^2 e^{\pm ix\xi}$ so that the spectrum of (43) , (44) is $[0, +\infty)$.

8.1. **Example: Bessel operator.**

8.1.1. *Bessel functions.* Consider the Bessel equation

$$x^2u'' + xu' + (x^2 - n^2)u = 0 \tag{45}$$

For n real, we can assume $n \geq 0$.

Solutions of Bessel's equation are called *Bessel functions*. The point $x = 0$ is a singular point of the equation. It is not hard to see that there are two independent solutions of the form x^n, respectively x^{-n}, multiplying a convergent power series having an infinite radius of convergence. The first solution is the Bessel function J:

$$J_n(x) = x^n \frac{1}{2^n} \left(1 + O(x^2)\right)$$

This is the only solution of (45) which is bounded at $x = 0$ (if $n > 0$).

It can be shown that all solutions of (45) are bounded for $x \to \infty$ (they oscillate and decay as $x^{-1/2}$).

8.1.2. *An eigenvalue problem for the Bessel operator.* Consider the eigenvalue problem

$$\frac{1}{x}(xu')' + \left(\lambda - \tfrac{n^2}{x^2}\right) u = 0, \quad \text{for } x \in [0, \infty)$$

with boundary conditions

$$u \text{ is finite for } x \to 0 \text{ and for } x \to \infty$$

For any $\lambda > 0$ the Bessel functions $J_n(\sqrt{\lambda}x)$ are generalized eigenfunctions and it can be shown that any $f \in L^2([0, \infty), xdx)$ can be expanded in terms of the eigenfunctions: there exists a function F so that

$$f(x) = \int_0^\infty J_n(\sqrt{\lambda}x) F(\lambda)\, d\lambda = \int_0^\infty J_n(tx) F(t^2)\, 2t\, dt$$

A general representation theorem of this nature exists for self-adjoint operators.

8.2. **A cautionary tale: $-i\frac{d}{dx}$ on $L^2([0, \infty)$ Formally self-adjoint is not the same as self-adjoint!** Consider the operator $L = -i\frac{d}{dx}$ on $D(L) = \{f \in L^2([0, \infty), f' \in L^2([0, \infty), f(0) = 0\}$. Check first that the operator is symmetric.

Now let's look at $(L + i)$ and see whether it is invertible.

For that, we have to solve the equation

$$-iu' + iu = f; \quad f \in L^2([0, \infty)$$

Equivalently,

$$u' - u = if; \quad f \in L^2([0, \infty)$$

By variation of parameters, and imposing the condition $u(0) = 0$ we get

$$u(x) = e^x \int_0^x e^{-s} f(s) ds$$

Take any $f > 0$ in L^2. Check that $e^{-s} f \in L^2$ and $\int_0^\infty e^{-s} f(s) ds = C_f > 0$. It then follows that

$$u(x) e^{-x} \to C_f \quad \text{as } x \to \infty$$

and thus $u \notin L^2$.

In fact you can similarly check that for any $z = x + iy$ with $y < 0$ $(L - z)$ is not invertible. Thus the spectrum of L contains the whole lower half plane in z, far from being purely real! That's not what happens with self-adjoint matrices. In fact : $-i\frac{d}{dx}$ on $L^2([0,\infty)$ on $D(L) = \{f \in L^2([0,\infty), f' \in L^2([0,\infty), f(0) = 0\}$ is **not** self-adjoint!