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MORE ON THE FOURIER TRANSFORM

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Contents

0.1. **General properties.** We employ the usual definition

$$\mathcal{F}f = \hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx$$

Proposition 1.

(1) $\widehat{f(x+h)} = \hat{f}(\xi) e^{i\xi h}$
(2) $\widehat{f(x)e^{-ixh}} = \hat{f}(\xi + h);\ \ h \in \mathbb{R}$
(3) $\widehat{f(ax)} = |a|^{-1} \hat{f}(a^{-1}\xi);\ \ a \neq 0$
(4) $\widehat{f'(x)} = i\xi \hat{f}(\xi)$
(5) $\widehat{xf} = i\frac{d}{d\xi}\hat{f}(\xi)$
(6) $\widehat{fg} = \frac{1}{\sqrt{2\pi}} \hat{f} * \hat{g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\hat{g}(\xi - s) ds$

Many of the proofs we have done them already. The rest are simple exercise, except perhaps for the last one, which we show by taking

$$(1)\quad \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\int_{-\infty}^{\infty}\hat{f}(s)\hat{g}(\xi-s)ds=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}\int_{-\infty}^{\infty}\hat{f}(s)\hat{g}(\xi-s)dsd\xi$$
$$=\int_{-\infty}^{\infty}e^{ixt}\frac{1}{\sqrt{2\pi}}\hat{f}(t)e^{ixu}\frac{1}{\sqrt{2\pi}}\hat{g}(u)dudt=fg$$

where we made the change of variables $s=t,\xi-s=u$

Let $\mathcal{S}(\mathbb{R})$ be the Schwarz space.

Proposition 2. *If $f\in\mathcal{S}(\mathbb{R})$ then $\hat{f}\in\mathcal{S}(\mathbb{R})$.*

We have shown this before as well.

An important invariance property is the following.

Theorem 3. *Let $f(x)=\exp(-x^2/2)$. Then $\hat{f}(\xi)=\exp(-\xi^2/2)$.*

In other words $\exp(-x^2/2)$ is an eigenfunction of $\mathcal{F}$ corresponding to the eigenvalue 1. What other eigenvalues are possible?

Proof. Let $f(x)=e^{-x^2/2}$. Then,

$$F(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}e^{-ix\xi}dx$$

Then,

$$\sqrt{2\pi}F'(\xi)=\int_{-\infty}^{\infty}(-ix)e^{-x^2/2}e^{-ix\xi}dx=i\int_{-\infty}^{\infty}f'(x)e^{-ix\xi}dx$$

On the other hand,

$$F'(\xi)=\xi\hat{f}(\xi)=-\xi F(\xi)$$

(why?) It follows that

$$F(\xi)=Ce^{-x^2/2}$$

Now, $F(0)=1$ (why?). Thus

$$F(\xi)=e^{-x^2/2}$$

□

Using Proposition 1 (3) we see that

$$\mathcal{F}(e^{-\beta x^2})=\sqrt{\frac{\pi}{\beta}}e^{-\frac{\xi^2}{4\beta}}$$

1. Solving PDEs by Fourier transform

1.1. The heat equation. Consider again the heat equation in one dimension

$$u_t = u_{xx}; \quad u(t=0,x) = f(x) \in L^2$$

By taking the Fourier transform in x we get

$$\hat{u}_t = -\xi^2 \hat{u} \Rightarrow \hat{u}(t,\xi) = C(t)e^{-t\xi^2}$$

and imposing the boundary condition, we must have

$$\hat{u}(t,\xi) = \hat{f}(\xi)e^{-t\xi^2}$$

and by taking $\mathcal{F}^{-1}$ we get

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\mathcal{F}^{-1}(e^{-x^2/(4t)}) * f = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-u^2/(4t)} f(x-u)du$$

1.2. The Laplace equation in the upper half plane. Consider the equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad u(x,y=0) = f(x) \in L^2$$

Taking the Fourier transform in x we get

$$-\xi^2 \hat{u} + \hat{u}_{yy} = 0$$

with the only admissible solution (one which is not growing as $\xi \to \infty$ and imposing the boundary condition we get

$$\hat{u}(\xi) = \hat{f}(\xi)e^{-|\xi| y}$$

and it follows that

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{t^2+y^2} f(x-t)dt$$

2. The Fourier transform in $\mathbb{R}^d$ (Based on [1])

2.1. Notations. Given $(x_1, ..., x_d) \in \mathbb{R}^d$ one writes

$$|x| = \sqrt{x_1^2 + \cdots + x_d^2}$$

and we often abbreviate $\langle x, y\rangle = x \cdot y$. Also, for $x \in \mathbb{R}^d, m \in \mathbb{Z}^d$ we write

$$x^m = x_1^{m_1} \cdots x_d^{m_d}$$

and also

$$\left(\frac{\partial}{\partial x}\right)^m = \left(\frac{\partial}{\partial x_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{m_d} = \frac{\partial^{|m|}}{\partial x_1^{m_1} \cdots \partial x_d^{m_d}}$$

where (there is some some ambiguity of notation) $|m| = m_1 + \cdots + m_d$.

Symmetries play an important role in the analysis of PDEs and in other problems as well. These symmetries are: translations, dilations, and rotations. The translation by h is simply $x \mapsto x+h$, dilations are $x \mapsto ax$ with $a > 0$ and rotations are linear orthogonal transformations, represented by matrices with real valued entries, s.t. $\langle Rx, Ry\rangle = \langle x, y\rangle$. As matrices, these are unitary matrices with real entries, and preservation of scalar product simply means $RR^* = R^*R = I$ where R^* is the adjoint of R, and since R is real-valued, $R^* = R^t$. We have $\det(R) = \pm 1$. In particular $-I$ is a rotation, but an *improper one*: $\det(I) = -1$. It represents a reflection (symmetry) about the origin. Rotations with $\det(R) = 1$ are called proper rotations. General rotations are then proper rotations composed with a symmetry w.r.t. 0.

In $\mathbb{R}^3$ the description of all possible rotations was provided by Euler. For any proper rotation, there is an axis of rotation d: $R(d) = d$; If P is the plane through $0 \perp$ to d, then $R(P) = P$ and on P, which is isomorphic to $\mathbb{R}^2$, R is a two-dimensional rotation matrix R_2:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2.2. **Functions with rapid decrease in $\mathbb{R}^d$.** By definition, these are functions with the property

$$\sup_{x\in\mathbb{R}^d} |x^k||f(x) < \infty \ \forall k \in \mathbb{N}$$

Integrals over the whole of $\mathbb{R}^d$ are defined in particular on functions of rapid decrease. They are improper integrals, defined as

$$\int_{\mathbb{R}^d} f(x)dx = \lim_{R\to\infty} \int_{B_R} f(x)dx$$

where B_R is the ball of radius R. Instead of B_R we could take, with the same result, Q_R, the (hyper)cube of side R. In the latter interpretation, this is an iterated improper integral.

You can convince yourself that the limit exists if

$$\sup_{x\in\mathbb{R}^d} x^{d+\epsilon}|f(x) < \infty \text{ for some } \epsilon > 0$$

Functions with moderate decrease are defined as above, with $\epsilon = 1$.

2.2.1. *Properties.*

(1)
$$\int_{\mathbb{R}^d} f(x+h)dx = \int_{\mathbb{R}^d} f(x)dx$$

(2)
$$a^d \int_{\mathbb{R}^d} f(ax)dx = \int_{\mathbb{R}^d} f(x)dx$$

(3) For any rotation R,
$$\int_{\mathbb{R}^d} f(Rx)dx = \int_{\mathbb{R}^d} f(x)dx$$

2.3. **(Hyper)Spherical coordinates.** We remind that polar coordinates in $\mathbb{R}^2$ are defined by (r, θ) where r is the distance to the origin and $\theta \in [0, 2\pi)$ is the angle with the x axis, and we have

$$\int_{\mathbb{R}^2} f(x)dx = \int_0^{2\pi} \int_0^{\infty} f(r\cos\theta, r\sin\theta) r dr d\theta$$

In $\mathbb{R}^3$ we similarly have

$$x_1 = r\sin\theta\cos\phi$$
$$x_2 = r\sin\theta\sin\phi$$
$$x_3 = r\cos\theta$$

and

$$\int_{\mathbb{R}^3} f(x)dx = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) r^2 \sin\theta d\theta d\phi dr$$

This is generalized as follows: We write a point on the hypersphere S^{d-1} of radius 1 as γ and write

$$\int_{\mathbb{R}^d} f(x)dx = \int_{S^{d-1}} \int_0^{\infty} f(r\gamma) r^{d-1} d\sigma(\gamma)$$

where $d\sigma(\gamma)$ is the surface element on S^{d-1}.

2.4. **The Schwarz space in $\mathbb{R}^d$.** The Schwarz space in $\mathbb{R}^d$ $\mathcal{S}(\mathbb{R}^d)$ consists of all indefinitely differentiable functions on $\mathbb{R}^d$ with the property

$$\sup_{x\in\mathbb{R}^d} \left| x^m \left(\frac{\partial}{\partial x}\right)^n f(x)\right| < \infty$$

for all multi-indices m, n.

2.5. **The Fourier transform on $\mathcal{S}(\mathbb{R}^d)$.** If $f \in \mathcal{S}(\mathbb{R}^d)$ we define, for $\xi \in \mathbb{R}^d$, in one convention

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-x\cdot\xi} dx$$

and in a more common notation in PDEs,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} dx$$

From this point on, we will use the latter definition.

Some properties of the Fourier transform in $\mathbb{R}^d$ are listed below. We write $\mathcal{F}(f) = \hat{f}$ as $f(x) \mapsto \hat{f}(\xi)$.

Proposition 4. (1)

$$f(x+h) \mapsto \hat{f}(\xi) e^{2\pi i \xi h}; \quad h \in \mathbb{R}^d$$

(2)

$$f(x) e^{-2\pi i x\cdot h} \mapsto \hat{f}(\xi + h); \quad h \in \mathbb{R}^d$$

(3)
$$f(ax) \mapsto a^{-d}\hat{f}(a\xi); \quad a \in \mathbb{R}^+$$
(4)
$$\left(\frac{\partial}{\partial x}\right)^m f(x) \mapsto (2\pi i\xi)^m \hat{f}(\xi)$$
(5)
$$(-2\pi i x)^m f(x) \mapsto \left(\frac{\partial}{\partial \xi}\right)^m \hat{f}(\xi)$$
(6) *If R is a rotation, then*
$$f(Rx) \mapsto \hat{f}(Rx)$$

Proof. Only the last property requires a proof, as the proof of the others is similar to the one-dimensional case. For the last property, we make the change of variable $t = Rx$ and remember that $\langle R^{-1}x, R^{-1}\xi\rangle = \langle x, \xi\rangle$ and that $|\det(R)| = 1$.

□

Proposition 5. *The Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ into itself.*

Proof. The proof is similar to the one-dimensional one. □

Definition 6. *A function is radial if $f(x) = f_r(|x|)$ for some f_r.*

Proposition 7. *A function is radial if and only if it has radial symmetry, that is $f(Rx) = f(x)$ for all x.*

Proof. Indeed, in one direction, $f(Rx) = f_r(|Rx|) = f_r(|x|) = f(x)$. In the opposite direction let x and x' be s.t. $x \neq x'$, $|x| = |x'|$ and let's for now prove the statement for $\mathbb{R}^3$. The general proof is not much more difficult. Taking the plane generated Π by x, x' there is a 2-d rotation s.t. $R_2 x' = x$. A 3-d rotation that does the same is R_2 about the normal to Π. Then $f(x) = f(Rx) = f(x')$ and thus f only depends on $|x|$.

How would you generalize this argument to $\mathbb{R}^d$? □

Corollary 8. *The Fourier transform of a radial function is radial.*

Proof. This follows from Proposition 4 (6), since $\hat{f}(R\xi) = \hat{f}(\xi)$ □

The $d-$dimensional Gaussian $f(x) = e^{-ar^2}, r = |x|$ is an example of a radial function.

Proposition 9 (The inversion formula)**.** *If $f \in \mathcal{S}(\mathbb{R}^d)$ and $\hat{f} = \mathcal{F}(f)$, then*
$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

Proposition 10 (Plancherel formula in $\mathbb{R}^d$)**.**
$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Definition 11. *Convolution of two functions, say in $\mathcal{S}(\mathbb{R}^d)$ is defined in a way similar to convolution in $\mathbb{R}$:*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(t)g(x-t)dt$$

Proposition 12.

$$\widehat{f * g} = \hat{f}\hat{g}; \quad \widehat{fg} = \hat{f} * \hat{g}$$

Proof. The proofs can be obtained from the fact that the $\mathbb{R}^d$ Fourier transform in $\mathbb{R}^d$ is an iterated 1d Fourier transform. □

2.6. **The wave equation in $\mathbb{R} \times \mathbb{R}^d$.** The homogeneous wave equation with initial condition $u(t=0,x) = f(x)$, or the **Cauchy problem** for the wave equation is similar to the 1d one:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta u; \quad u(t=0,x) = f(x); \quad u_t(t=0,x) = g(x) \tag{2}$$

where

$$\Delta u := \sum_{k=1}^{d} \frac{\partial^2 u}{\partial {x_k}^2}$$

The strategy for solving this equation is similar to the one used in 1d initial value problems: We Fourier transform the problem w.r.t. the space variable, after which we end up with an ODE.

In (2) we remember that differentiation with respect to x_k is transformed into multiplication by $2\pi i \xi_k$, and the time derivative of the Fourier transform is the Fourier transform of the time derivative. Thus

$$\frac{1}{c^2}\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2 \left(\sum_{k=1}^{n} \xi_k^2\right) \hat{u} = -4\pi^2 |\xi|^2 \hat{u} \tag{3}$$

This is indeed an ODE, with general solution

$$\hat{u}(t,\xi) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t) \tag{4}$$

We now note that on the one hand

$$\hat{u}(t=0,\xi) = \hat{f}(\xi); \quad \hat{u}_t(t=0,\xi) = \hat{g}(\xi); \tag{5}$$

and on the other hand

$$\hat{u}(t=0,\xi) = A(\xi); \quad \hat{u}_t(t=0,\xi) = 2\pi|\xi|B(\xi) \tag{6}$$

Combining (5) and (6) we get

Theorem 13. *The solution of the Cauchy problem for the d-dimensional wave equation is*

$$u(x,t) = \int_{\mathbb{R}^d} \left[\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right] e^{2\pi i x\cdot\xi} d\xi \tag{7}$$

Proof. This does require a proof since we only derived the solution formally, assuming it exists, assuming we can take the Fourier transform etc. Part of this proof is relatively straightforward: we should check that (7) is a solution of (2). The more difficult part is to show uniqueness of this solution, which is done by *energy arguments*, see [1] p. 187. □

What does this give in one dimension?

For this, we use Euler's formulas:

$$\cos(2\pi|\xi|) = \tfrac{1}{2}\left(e^{2\pi i|\xi|} + e^{-2\pi i|\xi|}\right); \sin(2\pi|\xi|) = \tfrac{1}{2i}\left(e^{2\pi i|\xi|} - e^{-2\pi i|\xi|}\right)$$

and get d'Alembert's formula,

$$u(x,t) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy$$

Check the formula above, both in terms of it solving the Cauchy problem, and also by deriving it from (7)*!*

2.7. **The heat equation in $\mathbb{R}^d$.** This is the equation

$$\frac{\partial u}{\partial t} = \Delta u = \sum_{k=1}^{d} \frac{\partial u}{\partial x_k^2}; \quad u(t=0,x) = f(x) \in \mathcal{S}(\mathbb{R}^d)$$

Taking the Fourier transform in x, we get

$$\hat{u}_t = (2\pi i)^2 \sum_{k=1}^{d} \xi_k^2 = -4\pi^2|\xi|^2$$

and thus

$$\hat{u} = C(\xi)e^{-4\pi^2|\xi|^2}$$

The initial condition implies that

$$\hat{u} = \hat{f}(\xi)e^{-4\pi^2|\xi|^2}$$

Now,

$$\int_{-\infty}^{\infty} e^{-4\pi^2\xi_k^2 t + 2\pi i\xi_k x_k} = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x_k^2}{4t}}$$

and thus

$$\mathcal{F}^{-1}e^{-4\pi^2|\xi|^2} = \left(\frac{1}{\sqrt{4\pi t}}\right)^d e^{\frac{|x|^2}{4t}}$$

and therefore, by Proposition 12 we have

$$u(x,t) = (4\pi t)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)dy \tag{8}$$

The condition that $f \in \mathcal{S}(\mathbb{R}^3)$ is not needed, provided (8) can be justified.

2.8. **The Poisson summation formula.** Let $f \in \mathcal{S}(\mathbb{R})$. Note first that

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

is convergent and *periodic with period one.*

Theorem 14 (Poisson summation formula)**.** *Under the assumptions above,*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x} \tag{9}$$

and in particular we have the symmetric formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \tag{10}$$

Proof. On the left side of the identity we have, as mentioned, a smooth periodic function of period one. It suffices to check that the Fourier coefficients of both sides of the equation coincide. The series on the right side of (9) converges pointwise and rapidly so (why?).

The $k-$th coefficient on the right side of (9), calculated now with the definition

$$\hat{g}_k = \int_0^1 g(s) e^{-2\pi i k s} ds$$

is clearly $\hat{f}(k)$. For the left side we have

$$\int_0^1 \sum_{n=-\infty}^{\infty} f(s+n) e^{-2\pi i k s} ds = \sum_{n=-\infty}^{\infty} \int_0^1 f(s+n) e^{-2\pi i k s} ds \tag{11}$$

$$= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(t) e^{-2\pi i k t} dt = \int_{-\infty}^{\infty} f(t) e^{-2\pi i k t} dt = \hat{f}(k)$$

The formula extends to the case when f is smooth and decays fast enough, for instance

$$|f(x)| \le \frac{|C|}{1+x^2}$$

for some C. Recall that, for $a > 0$,

$$\int_{-\infty}^{\infty} e^{2\pi i x \xi} e^{-2\pi a |x|} = \frac{a}{\pi} \frac{1}{a^2+\xi^2}$$

□

Thus,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \sum_{n=-\infty}^{\infty} e^{-2\pi a |n|}$$

and by taking limits carefully, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(How?)

References

[1] Elias M. Stein and Rami Shakarchi, Fourier Analysis, an introduction, Princeton University Press (2003).