Math 5102 **Homework 1**
due Fri Jan 30, 2015

Recall that $\mathbb{R}$ and $\mathbb{C}$ are complete: a sequence $s = (s_k)_{k\in\mathbb{N}}$ converges if and only if s is Cauchy, that is $\forall\epsilon\ \exists n_0$ s.t. for all $n_1, n_2 \geq n_0$, $|s_{n_2} - s_{n_1}| < \epsilon$ (Cauchy criterion). Applied to series,

$$S = \sum_{k=0}^{\infty} s_k, \quad s_k \in \mathbb{C}$$

we see that S converges if and only if $\forall\epsilon\ \exists n_0$ s.t. for all $n_1, n_2 \geq n_0$, $\left|\sum_{k=n_1}^{n_2} s_k\right| < \epsilon$. In particular this implies, by taking $n_2 = n_1 + 1$ that **if** a series is convergent, then

$$\lim_{n\to\infty} s_n = 0 \tag{1}$$

1. Show that taking $n_2 = n_1 + 1$ is *not* enough to ensure convergence. That is, there are sequences s.t. (1) holds but $\sum_{k=0}^{\infty} s_n$ does not exist. Thus (1) is a **necessary** but **not sufficient** condition for convergence. (Only the full Cauchy criterion is necessary **and** sufficient.)

2. (A) For which complex numbers a, b the sequences x, where for all $k \in \mathbb{N}$ $x_k = a\, b^k$, belong to $\ell^2(\mathbb{N}) = \left\{ (c_0, c_1, c_2, ...) : \sum_{k=0}^{\infty} |c_k|^2 < \infty \right\}$?

(B) For which complex numbers a, b the sequences x, where for all $k \in \mathbb{Z}$ $x_k = a\, b^k$, belong to $\ell^2(\mathbb{Z}) = \left\{ (\cdots c_{-2}, c_{-1}, c_0, c_1, c_2, ...) : \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty \right\}$?

3. a) Show that the functions f_n, $n \in \mathbb{Z}$ defined as $f_n(x) = e^{inx}$ form an orthogonal set in the complex Hilbert space $L^2[0, 2\pi]$.

b) Normalize them to obtain an orthonormal set.

4. Let a, b be real numbers. Show that if $f \in L^2[a, b]$ then $\int_a^b |f(t)|\, dt < \infty$ (in other words, $f \in L^1[a, b]$). (Hint: use Cauchy-Schwartz.)

Note: this is not true on infinite intervals!

5. The converse of the above is not true: give an example of a function for with $\int_0^1 |f(t)|\, dt < \infty$ but with $f \notin L^2[0, 1]$.