

Math 5102
due Fri **March 13**, 2015

Homework 5

1. Show that if u is a solution of

$$-(p(x)u')' + (q(x) - \lambda w(x))u = 0 \quad (1)$$

then $v = pu'/u$ satisfies the Riccati equation

$$v' + p(x)^{-1}v^2 = q(x) - \lambda w(x)$$

(*Note:* Riccati equations can be studied by turning them into them to a linear second order equation.)

2. (*Sturm-Liouville normal form*) Show that the differential equation (1) transforms into one with $w = p = 1$: using the transformation

$$y(x) = \int_a^x \sqrt{\frac{w(t)}{p(t)}} dt, \quad v(y) = u(x(y)) [w(x(y)) p(x(y))]^{1/4}$$

then

$$-\frac{d^2v}{dy^2} + Q(y)v = \lambda v$$

where

$$Q = q - \frac{(pw)^{1/4}}{w} \left\{ p [(pw)^{-1/4}]' \right\}'$$

Moreover, the change of variable preserves the L^2 norm: show that

$$\int_a^b |u(x)|^2 w(x) dx = \int_0^c |v(y)|^2 dy \quad \text{where } c = \int_a^b \sqrt{\frac{w(t)}{p(t)}} dt$$

3. Given one solution $u(x)$ of (1), make a variation of constants ansatz $v(x) = c(x)u(x)$ and show that a second solution is given by

$$v(x) = u(x) \int \frac{1}{p(t)u(t)^2} dt$$

Are u and v linearly independent?