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MORE ON THE FOURIER TRANSFORM

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Contents

0.1. **General properties.** We employ the usual definition

$$\mathcal{F}f = \hat{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$$

Proposition 1.

(1) $\widehat{f(x+h)} = \hat{f}(\xi)e^{i\xi h}$
(2) $\widehat{f(x)e^{-ixh}} = \hat{f}(\xi+h);\ h\in\mathbb{R}$
(3) $\widehat{f(ax)} = |a|^{-1}\hat{f}(a^{-1}\xi);\ a\neq 0$
(4) $\widehat{f'(x)} = i\xi\hat{f}(\xi)$
(5) $\widehat{xf} = i\frac{d}{d\xi}\hat{f}(\xi)$
(6) $\widehat{fg} = \frac{1}{\sqrt{2\pi}}\hat{f}*\hat{g} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(s)\hat{g}(\xi-s)ds$

Many of the proofs we have done them already. The rest are simple exercises, except perhaps for the last one, which we show by taking

$$(1)\quad \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\int_{-\infty}^{\infty}\hat{f}(s)\hat{g}(\xi-s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}\int_{-\infty}^{\infty}\hat{f}(s)\hat{g}(\xi-s)dsd\xi$$
$$= \int_{-\infty}^{\infty}e^{ixt}\frac{1}{\sqrt{2\pi}}\hat{f}(t)e^{ixu}\frac{1}{\sqrt{2\pi}}\hat{g}(u)dudt = fg$$

where we made the change of variables $s=t, \xi-s=u$

Let $\mathcal{S}(\mathbb{R})$ be the Schwarz space.

Proposition 2. *If $f\in\mathcal{S}(\mathbb{R})$ then $\hat{f}\in\mathcal{S}(\mathbb{R})$.*

We have shown this before as well.

An important invariance property is the following.

Theorem 3. *Let $f(x)=\exp(-x^2/2)$. Then $\hat{f}(\xi)=\exp(-\xi^2/2)$.*

In other words $\exp(-x^2/2)$ is an eigenfunction of $\mathcal{F}$ corresponding to the eigenvalue 1. What other eigenvalues are possible?

Proof. Let $f(x)=e^{-x^2/2}$. Then,

$$F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}e^{-ix\xi}dx$$

Then,

$$\sqrt{2\pi}F'(\xi) = \int_{-\infty}^{\infty}(-ix)e^{-x^2/2}e^{-ix\xi}dx = i\int_{-\infty}^{\infty}f'(x)e^{-ix\xi}dx$$

On the other hand,

$$F'(\xi) = \xi\hat{f}(\xi) = -\xi F(\xi)$$

(why?) It follows that

$$F(\xi) = Ce^{-x^2/2}$$

Now, $F(0) = 1$ (why?). Thus

$$F(\xi) = e^{-x^2/2}$$

□

Using Proposition 1 (3) we see that

$$\mathcal{F}(e^{-\beta x^2}) = \sqrt{\frac{\pi}{\beta}} e^{-\frac{\xi^2}{4\beta}}$$

1. Solving PDEs by Fourier transform

1.1. **The heat equation.**
Consider again the heat equation in one dimension

$$u_t = u_{xx}; \quad u(t = 0, x) = f(x) \in L^2$$

By taking the Fourier transform in x we get

$$\hat{u}_t = -\xi^2 \hat{u} \Rightarrow \hat{u}(t, \xi) = C(t) e^{-t\xi^2}$$

and imposing the boundary condition, we must have

$$\hat{u}(t, \xi) = \hat{f}(\xi) e^{-t\xi^2}$$

and by taking $\mathcal{F}^{-1}$ we get

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \mathcal{F}^{-1}(e^{-x^2/(4t)}) * f = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-u^2/(4t)} f(x - u) du$$

1.2. **The Laplace equation in the upper half plane.**
Consider the equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad u(x, y = 0) = f(x) \in L^2$$

Taking the Fourier transform in x we get

$$-\xi^2 \hat{u} + \hat{u}_{yy} = 0$$

with the only admissible solution (one which is not growing as $\xi \to \infty$ and imposing the boundary condition we get

$$\hat{u}(\xi) = \hat{f}(\xi) e^{-|\xi| y}$$

and it follows that

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{t^2 + y^2} f(x - t) dt$$

2. The Fourier transform in $\mathbb{R}^d$ (Based on [1])

2.1. **Notations.** Given $(x_1, ..., x_d) \in \mathbb{R}^d$ one writes

$$|x| = \sqrt{x_1^2 + \cdots + x_d^2}$$

and we often abbreviate $\langle x, y\rangle = x \cdot y$. Also, for $x \in \mathbb{R}^d, m \in \mathbb{Z}^d$ we write

$$x^m = x_1^{m_1} \cdots x_d^{m_d}$$

and also

$$\left(\frac{\partial}{\partial x}\right)^m = \left(\frac{\partial}{\partial x_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{m_d} = \frac{\partial^{|m|}}{\partial x_1^{m_1} \cdots \partial x_d^{m_d}}$$

where (there is some some ambiguity of notation) $|m| = m_1 + \cdots + m_d$.

Symmetries play an important role in the analysis of PDEs and in other problems as well. These symmetries are: translations, dilations, and rotations. The translation by h is simply $x \mapsto x + h$, dilations are $x \mapsto ax$ with $a > 0$ and rotations are linear orthogonal transformations, represented by matrices with real valued entries, s.t. $\langle Rx, Ry\rangle = \langle x, y\rangle$. As matrices, these are unitary matrices with real entries, and preservation of scalar product simply means $RR^* = R^*R = I$ where R^* is the adjoint of R, and since R is real-valued, $R^* = R^t$. We have $\det(R) = \pm 1$. In particular $-I$ is a rotation, but an *improper one*: $\det(I) = -1$. It represents a reflection (symmetry) about the origin. Rotations with $\det(R) = 1$ are called proper rotations. General rotations are then proper rotations composed with a symmetry w.r.t. 0.

In $\mathbb{R}^3$ the description of all possible rotations was provided by Euler. For any proper rotation, there is an axis of rotation d: $R(d) = d$; If P is the plane through $0 \perp$ to d, then $R(P) = P$ and on P, which is isomorphic to $\mathbb{R}^2$, R is a two-dimensional rotation matrix R_2:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

2.2. **Functions with rapid decrease in $\mathbb{R}^d$.** By definition, these are functions with the property

$$\sup_{x\in\mathbb{R}^d} |x^k||f(x) < \infty \ \forall k \in \mathbb{N}$$

Integrals over the whole of $\mathbb{R}^d$ are defined in particular on functions of rapid decrease. They are improper integrals, defined as

$$\int_{\mathbb{R}^d} f(x)dx = \lim_{R\to\infty} \int_{B_R} f(x)dx$$

where B_R is the ball of radius R. Instead of B_R we could take, with the same result, Q_R, the (hyper)cube of side R. In the latter interpretation, this is an iterated improper integral.

You can convince yourself that the limit exists if

$$\sup_{x\in\mathbb{R}^d} x^{d+\epsilon}|f(x)| < \infty \text{ for some } \epsilon > 0$$

Functions with moderate decrease are defined as above, with $\epsilon = 1$.

2.2.1. *Properties.*

(1)
$$\int_{\mathbb{R}^d} f(x+h)dx = \int_{\mathbb{R}^d} f(x)dx$$

(2)
$$a^d \int_{\mathbb{R}^d} f(ax)dx = \int_{\mathbb{R}^d} f(x)dx$$

(3) For any rotation R,
$$\int_{\mathbb{R}^d} f(Rx)dx = \int_{\mathbb{R}^d} f(x)dx$$

2.3. **(Hyper)Spherical coordinates.** Recall that polar coordinates in $\mathbb{R}^2$ are defined by (r,θ) where r is the distance to the origin and $\theta \in [0, 2\pi)$ is the angle with the x axis, and we have

$$\int_{\mathbb{R}^2} f(x)dx = \int_0^{2\pi} \int_0^{\infty} f(r\cos\theta, r\sin\theta)r dr d\theta$$

In $\mathbb{R}^3$ we similarly have

$$\begin{aligned} x_1 &= r\sin\theta\cos\phi \\ x_2 &= r\sin\theta\sin\phi \\ x_3 &= r\cos\theta \end{aligned}$$

and

$$\int_{\mathbb{R}^3} f(x)dx = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)r^2 \sin\theta d\theta d\phi dr$$

This is generalized as follows: We write a point on the hypersphere S^{d-1} of radius 1 as γ and write

$$\int_{\mathbb{R}^d} f(x)dx = \int_{S^{d-1}} \int_0^{\infty} f(r\gamma)r^{d-1} d\sigma(\gamma)$$

where $d\sigma(\gamma)$ is the surface element on S^{d-1}.

2.4. **The Schwarz space in $\mathbb{R}^d$.** The Schwarz space in $\mathbb{R}^d$ $\mathcal{S}(\mathbb{R}^d)$ consists of all indefinitely differentiable functions on $\mathbb{R}^d$ with the property

$$\sup_{x\in\mathbb{R}^d} \left| |x|^m \left(\frac{\partial}{\partial x}\right)^n f(x) \right| < \infty$$

for all multi-indices m, n.

2.5. **The Fourier transform on $\mathcal{S}(\mathbb{R}^d)$.** If $f \in \mathcal{S}(\mathbb{R}^d)$ we define, for $\xi \in \mathbb{R}^d$, in one convention

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx$$

and in a notation often used in PDEs,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} dx$$

From this point on, we will use the latter definition.

Some properties of the Fourier transform in $\mathbb{R}^d$ are listed below. We write $\mathcal{F}(f) = \hat{f}$ as $f(x) \mapsto \hat{f}(\xi)$.

Proposition 4. (1)

$$f(x+h) \mapsto \hat{f}(\xi) e^{2\pi i \xi h}; \quad h \in \mathbb{R}^d$$

(2)

$$f(x) e^{-2\pi i x\cdot h} \mapsto \hat{f}(\xi + h); \quad h \in \mathbb{R}^d$$

(3)

$$f(ax) \mapsto a^{-d} \hat{f}(a\xi); \quad a \in \mathbb{R}^+$$

(4)

$$\left(\frac{\partial}{\partial x}\right)^m f(x) \mapsto (2\pi i \xi)^m \hat{f}(\xi)$$

(5)

$$(-2\pi i x)^m f(x) \mapsto \left(\frac{\partial}{\partial \xi}\right)^m \hat{f}(\xi)$$

(6) *If R is a rotation, then*

$$f(Rx) \mapsto \hat{f}(Rx)$$

Proof. Only the last property requires a proof, as the proof of the others is similar to the one-dimensional case. For the last property, we make the change of variable $t = Rx$ and remember that $\langle R^{-1}x, R^{-1}\xi\rangle = \langle x, \xi\rangle$ and that $|\det(R)| = 1$.

□

Proposition 5. *The Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ into itself.*

Proof. The proof is similar to the one-dimensional one. □

Definition 6. *A function is radial if $f(x) = f_r(|x|)$ for some f_r.*

Proposition 7. *A function is radial if and only if it has radial symmetry, that is $f(Rx) = f(x)$ for all x.*

Proof. Indeed, if $f(x) = f_r(|x|)$, then $f(Rx) = f_r(|Rx|) = f_r(|x|) = f(x)$. In the opposite direction let x and x' be s.t. $x \neq x'$, $|x| = |x'|$ and let's for now prove the statement for $\mathbb{R}^3$. The general proof is not much more difficult. Taking the Π plane generated by x, x', there is a 2-d rotation R_2 s.t. $R_2 x' = x$. A 3-d rotation that does the same is R_2 about the normal to Π. Then $f(x) = f(Rx) = f(x')$ and thus f only depends on $|x|$.

How would you generalize this argument to $\mathbb{R}^d$? □

Corollary 8. *The Fourier transform of a radial function is radial.*

Proof. This follows from Proposition 4 (6), since $\hat{f}(R\xi) = \hat{f}(\xi)$ □

The $d-$dimensional Gaussian $f(x) = e^{-ar^2}, r = |x|$ is an example of a radial function.

Proposition 9 (The inversion formula)**.** *If* $f \in \mathcal{S}(\mathbb{R}^d)$ *and* $\hat{f} = \mathcal{F}(f)$, *then*

$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

Proposition 10 (Plancherel formula in $\mathbb{R}^d$)**.**

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Definition 11. *Convolution of two functions, say in* $\mathcal{S}(\mathbb{R}^d)$ *is defined in a way similar to convolution in* $\mathbb{R}$*:*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(t) g(x-t) dt$$

Proposition 12.

$$\widehat{f * g} = \hat{f}\hat{g}; \quad \widehat{fg} = \hat{f} * \hat{g}$$

Proof. The proofs can be obtained from the fact that the $\mathbb{R}^d$ Fourier transform in $\mathbb{R}^d$ is an iterated 1d Fourier transform. □

2.6. **The wave equation in** $\mathbb{R} \times \mathbb{R}^d$**.** The homogeneous wave equation with initial condition $u(t=0,x) = f(x)$, or the **Cauchy problem** for the wave equation is similar to the 1d one:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta u; \quad u(t=0,x) = f(x); \quad u_t(t=0,x) = g(x) \tag{2}$$

where

$$\Delta u := \sum_{k=1}^{d} \frac{\partial^2 u}{\partial {x_k}^2}$$

The strategy for solving this equation is similar to the one used in 1d initial value problems: We Fourier transform the problem w.r.t. the space variable, after which we end up with an *ODE*.

In (2) we remember that differentiation with respect to x_k is transformed into multiplication by $2\pi i\xi_k$, and the time derivative of the Fourier transform is the Fourier transform of the time derivative. Thus

$$\frac{1}{c^2}\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2\left(\sum_{k=1}^{n}\xi_k^2\right)\hat{u} = -4\pi^2|\xi|^2\hat{u} \tag{3}$$

This is indeed an ODE, with general solution

$$\hat{u}(t,\xi) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t) \tag{4}$$

We now note that on the one hand

$$\hat{u}(t=0,\xi) = \hat{f}(\xi); \quad \hat{u}_t(t=0,\xi) = \hat{g}(\xi); \tag{5}$$

and on the other hand

$$\hat{u}(t=0,\xi) = A(\xi); \quad \hat{u}_t(t=0,\xi) = 2\pi|\xi|B(\xi) \tag{6}$$

Combining (5) and (6) we get

Theorem 13. *The solution of the Cauchy problem for the d-dimensional wave equation is*

$$u(x,t) = \int_{\mathbb{R}^d}\left[\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right]e^{2\pi i x\cdot\xi}d\xi \tag{7}$$

Proof. This does require a proof since we only derived the solution formally, assuming it exists, assuming we can take the Fourier transform etc. Part of this proof is relatively straightforward: we should check that (7) is a solution of (2). The more difficult part is to show uniqueness of this solution, which is done by *energy arguments*, see [1] p. 187. □

What does this give in one dimension?

For this, we use Euler's formulas:

$$\cos(2\pi|\xi|) = \tfrac{1}{2}\left(e^{2\pi i|\xi|} + e^{-2\pi i|\xi|}\right); \sin(2\pi|\xi|) = \tfrac{1}{2i}\left(e^{2\pi i|\xi|} - e^{-2\pi i|\xi|}\right)$$

and get d'Alembert's formula,

$$u(x,t) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy$$

Check the formula above, both in terms of it solving the Cauchy problem, and also by deriving it from (7)*!*

2.7. The heat equation in $\mathbb{R}^d$.

This is the equation

$$\frac{\partial u}{\partial t} = \Delta u = \sum_{k=1}^{d}\frac{\partial u}{\partial x_k^2}; \quad u(t=0,x) = f(x) \in \mathcal{S}(\mathbb{R}^d)$$

Taking the Fourier transform in x, we get

$$\hat{u}_t = (2\pi i)^2\sum_{k=1}^{d}\xi_k^2 = -4\pi^2|\xi|^2$$

and thus

$$\hat{u} = C(\xi)e^{-4\pi^2|\xi|^2}$$

The initial condition implies that

$$\hat{u} = \hat{f}(\xi)e^{-4\pi^2|\xi|^2}$$

Now,

$$\int_{-\infty}^{\infty} e^{-4\pi^2\xi_k^2 t+2\pi i\xi_k x_k} = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x_k^2}{4t}}$$

and thus

$$\mathcal{F}^{-1}e^{-4\pi^2|\xi|^2} = \left(\frac{1}{\sqrt{4\pi t}}\right)^d e^{\frac{|x|^2}{4t}}$$

and therefore, by Proposition 12 we have

$$u(x,t) = (4\pi t)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}}f(y)dy \tag{8}$$

The condition that $f \in \mathcal{S}(\mathbb{R}^3)$ is not needed, provided (8) can be justified.

2.8. **The Poisson summation formula.** Let $f \in \mathcal{S}(\mathbb{R})$. Note first that

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

is convergent and *periodic with period one.*

Theorem 14 (Poisson summation formula)**.** *Under the assumptions above,*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi i n x} \tag{9}$$

and in particular we have the symmetric formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \tag{10}$$

Proof. On the left side of the identity we have, as mentioned, a smooth periodic function of period one. It suffices to check that the Fourier coefficients of both sides of the equation coincide. The series on the right side of (9) converges pointwise and rapidly so (why?).

The $k-$th coefficient on the right side of (9), calculated now with the definition

$$\hat{g}_k = \int_0^1 g(s)e^{-2\pi i k s}ds$$

is clearly $\hat{f}(k)$. For the left side we have

$$(11)\quad \int_0^1 \sum_{n=-\infty}^{\infty} f(s+n)e^{-2\pi iks}ds = \sum_{n=-\infty}^{\infty} \int_0^1 f(s+n)e^{-2\pi iks}ds$$
$$= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(t)e^{-2\pi ikt}dt = \int_{-\infty}^{\infty} f(t)e^{-2\pi ikt}dt = \hat{f}(k)$$

The formula extends to the case when f is smooth and decays fast enough, for instance

$$|f(x)| \le \frac{|C|}{1+x^2}$$

for some C. Recall that, for $a > 0$,

$$\int_{-\infty}^{\infty} e^{2\pi ix\xi}e^{-2\pi a|x|} = \frac{a}{\pi}\frac{1}{a^2+\xi^2}$$

□

Thus,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = \frac{\pi}{a}\coth(\pi a)$$

This identity is the *Mittag Leffler decomposition* of $\frac{\pi}{a}\coth(\pi a)$, a generalization of the decomposition by partial fractions to meromorphic functions (analytic except for poles).

By taking limits carefully, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(How?)

Exercise: Show that

$$\mathcal{F}^{-1}\left(\frac{\sqrt{\frac{\pi}{2}}\left(-2e^{-a|k|} - ie^{-ia|k|} + ie^{ia|k|}\right)}{4a^3}\right) = \frac{1}{n^4-a^4}$$

Assuming that $a^4 \notin \mathbb{Z}$ calculate

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^4-a^4}$$

and show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

2.9. **The Laplace transform.** Consider the differential equation

$$y'' - y = 0$$

. We can attempt to solve it by Fourier transform. What do we get?

$$(-\xi^2 - 1)\hat{y}(\xi) = 0$$

thus $y = 0$.

Why did we fail to obtain any interesting solution? Because neither of the two solutions of the equation, $e^{\pm x}$ is Fourier transformable.

There are other ways in which d/dx can be diagonalized.

Assume f is analytic in the upper half plane and decays faster than $1/|z|^{1+a}$, for some $a > 0$, as $z \to \infty$ in the upper half plane. Then, the integral

$$\hat{f}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp} f(x)dx = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{pt} f(-it)dt$$

is zero for all $p < 0$.

Indeed, the integral is the limit as $N \to \infty$ of

$$\frac{1}{2\pi}\int_{-N}^{N} e^{-ixp} f(x)dx$$

and it is also, for $p < 0$, the limit of

$$\frac{1}{2\pi}\int_{-N+ia}^{N+ia} e^{-ixp} f(x)dx$$

(Why)? It is easy to see that, if we take the limit $a \to \infty$, the limit is zero.

Thus, under these assumptions,

$$\hat{f}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp} f(x)dx = 0 \text{ for } p < 0$$

Then, the inverse transform is

$$\int_{-\infty}^{\infty} e^{ixp}\hat{f}(p)dp = \int_{0}^{\infty} e^{ixp}\hat{f}(p)dp \tag{12}$$

Note now that this integral makes sense for $x = x_1 + ix_2$ provided that $x_2 > 0$. In fact, the integral is analytic in x in the upper half plane: the value of the integral for $x = x_1 + ix_2$ is the analytic continuation of (12) to the upper half plane.

We can take $x_1 = 0$ and the inverse Fourier transform becomes (the analytic continuation of)

$$\int_{0}^{\infty} e^{-xp} F(p)dp \tag{13}$$

The formula

$$g(x) = \int_{0}^{\infty} e^{-xp} G(p)dp \tag{14}$$

is called the Laplace transform. By the calculations above, the inverse of the Laplace transform is

$$G(p) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px} g(x)dx$$

Note that this transform exists if $F(p)$ does not grow faster than $e^{\nu p}$ provided that $x > \nu$ (or $\Re(x) > \nu$ more generally). In this case, the inversion formula becomes

$$G(p) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} e^{px} g(x)dx$$

Note also that

(15)
$$\widehat{F'}(p) = \int_0^\infty e^{-xp}F'(p)dp = p = \int_0^\infty e^{-xp}F(p)dp - F(0) = \widehat{F}(p) - F(0)$$

Similarly,

$$\widehat{F''}(p) = \int_0^\infty e^{-xp}F'(p)dp = p = \int_0^\infty e^{-xp}F(p)dp - F'(0) - pF(0) \tag{16}$$
$$= \widehat{F}(p) - F'(0) - pF(0)$$

Now, the Laplace transform of our toy model $y'' - y = 0$ is

$$(p^2 - 1)\hat{y}(p) = py(0) + y'(0)$$

with solution

$$\hat{y}(p) = \frac{py(0) + y'(0}{(p^2 - 1)} \Rightarrow y(x) = y(0)\cosh(x) + y'(0)\sinh(x)$$

We can now easily solve the forced pendulum equation,

$$y'' + y = \cos(\omega x)$$

Indeed, the Laplace transformed equation is

$$p^2\hat{y} + \hat{y} - py(0) - y'(0) - \frac{p}{\omega^2 + p^2} = 0$$

with inverse Laplace transform

$$y(x) = y(0)\cos(x) + y'(0)\sin(x) + \frac{\cos(x) - \cos(\omega x)}{\omega^2 - 1}$$

or, similarly, the damped forced pendulum and so on.

2.10. **An application to Laplace transforms.** For now let $F \in L^1(\mathbb{R})$. The Laplace transform

$$\mathcal{L}F := \int_0^\infty e^{-px}F(p)dp \tag{17}$$

is well defined and continuous in x in the closed $\mathbb{H}^+$ and analytic in the open RHP (the open $\mathbb{H}^+$). (Obviously, we could allow $Fe^{-|\alpha|p} \in L^1$ and then $\mathcal{L}F$ would be defined for $\Re x > |\alpha|$.) F is uniquely defined by its Laplace transform, as seen below.

Lemma 15 (Uniqueness)**.** *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of x with an accumulation point in $\mathbb{H}^+$. Then $F = 0$ a.e.*

We will from now on write $F = 0$ on a set to mean $F = 0$ *a.e.* on that set.

Proof. By analyticity, $\mathcal{L}F = 0$ in the open RHP and by continuity, for $s \in \mathbb{R}$, $\mathcal{L}F(is) = 0 = \hat{\mathcal{F}}F$ where $\hat{\mathcal{F}}F$ is the Fourier transform of F (extended by zero for negative values of p). Since $F \in L^1$ and $0 = \hat{\mathcal{F}}F \in L^1$, by the known Fourier inversion formula, $F = 0$. ☐

More however can be said. We can draw interesting conclusions about F just from the rate of decay of $\mathcal{L}F$.

2.11. **A Laplace inversion formula.**

Theorem 16. *Assume $c \geq 0$, $f(z)$ is analytic in the closed half plane $\mathbb{H}_c := \{z : \Re z \geq c\}$. Assume further that $\sup_{c' \geq c} |f(c' + it)| \leq g(t)$ with $g(t) \in L^1(\mathbb{R})$. Let*

$$F(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} f(x)dx =: (\mathcal{L}^{-1}F)(p) \tag{18}$$

Then for any $x \in \{z : \Re z > c\}$ we have

$$\mathcal{L}F = \int_0^\infty e^{-px}F(p)dp = f(x) \tag{19}$$

Proof. Note that for any $x' = x_1' + iy_1' \in \{z : \Re z > c\}$

$$\int_0^\infty dp \int_{c-i\infty}^{c+i\infty} \left| e^{p(s-x')} f(s)\right| d|s| \;\leq\; \int_0^\infty dp e^{p(c-x_1')}\|g\|_1 \;\leq\; \frac{\|g\|_1}{x_1' - c} \tag{20}$$

and thus, by Fubini we can interchange the orders of integration:

$$\begin{aligned} U(x') &= \int_0^\infty e^{-px'} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} f(x)dx \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dx f(x) \int_0^\infty dp e^{-px'+px} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{f(x)}{x'-x}dx \end{aligned} \tag{21}$$

Since $g \in L^1$ there must exist subsequences $\tau_n, -\tau_n'$ tending to ∞ such that $|g(\tau_n)| \to 0$. Let $x' > \Re x = x_1$ and consider the box $B_n = \{z : \Re z \in [x_1, x'], \Im z \in [-\tau_n', \tau_n]\}$ with positive orientation. We have

$$\int_{B_n} \frac{f(s)}{x'-s}ds = -f(x') \tag{22}$$

while, by construction,

$$\lim_{n\to\infty}\int_{B_n} \frac{f(s)}{x'-s}ds = \int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s}ds - \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{x'-s}dx \tag{23}$$

On the other hand, by dominated convergence, we have

$$\int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s} ds \to 0 \quad \text{as} \quad x' \to \infty \tag{24}$$

□

3. Asymptotic series

A simple example of an asymptotic series is the Taylor series of a C^∞ function.

Remark 1. *The Taylor series of a function f at a point a converges to f in an open interval around a iff f is analytic at a. This is a pretty strong requirement!*

Example 2. *What can happen at a point of nonanalyticity? Take $f(x) = e^{-1/x^2}$ for $x \neq 0$ and zero at $x = 0$. You can check that all derivatives of f at zero exist, and they are zero. Thus the Taylor series of f at zero exists and it is the zero series. Which, of course, converges to the zero function, $\neq f(x)$.*

Based on the integral of e^{-1/x^2}, in the next example we define a function which, while still nonanalytic, has a nonzero Taylor series. Consider

$$\frac{df}{dx} = e^{-1/x^2} \tag{1}$$

To simplify the terms of the ODE, write $f(x) = e^{-1/x^2} g(x)$:

$$x^3 g' + 2g = x^3 \tag{2}$$

The fact that the coefficient of g' vanishes at third order is responsible for very singular behavior at zero. There are various ways to find a power series solution for this equation, which will turn out to be nothing else but the Taylor series of g. One is to substitute a power series with unkown coefficients into (2), form a recurrence relation for the coefficients and solve for them. We get

$$g(x) = \frac{x^3}{2} - \frac{3x^5}{4} + \frac{15x^7}{8} - \frac{105x^9}{16} + \frac{945x^{11}}{32} + \cdots \tag{3}$$

3.0.1. *The method of dominant balance.* We aim at understanding the small solutions g of our ODE, for small x. Generally, the various terms in an equation such as $x^3 g' + 2g - x^3 = 0$ do not go to zero at the same rate, and in a first approximation we want to identify the largest. Of course we cannot have a single largest term, or else equality to zero is impossible. In absence of a more systematic method, one takes all possible pairs of terms, assuming they are the largest, and then check for consistency.

If $x^3 g' \gg 2g$ and $x^3 \gg g$ we get $x^3 g' \approx x^3$, $g' \approx 1$ thus $g \approx C + x$, which means $2g \gg x^3$ again invalidating our assumption.

If the largest terms are x^3g' and $2g$, then $x^3g' \gg x^3$ and $2g \gg x^3$ and $x^3g' + 2g \approx 0$ or $g \approx Ce^{1/x^2}$. This is consistent, as in this case $x^3 \ll Ce^{1/x^2}$, and is a possible dominant balance. It does not give us small solutions though.

We are left with one case: $2g \gg x^3g'$, $x^3 \gg x^3g'$ $2g \approx x^3$ giving $g \approx x^3/2$. Now, $x^3g' \approx 3x^5/2$, and since $x^5 \ll \min(x^3, g)$, another *consistent balance.*

However, $g \approx x^3/2$ is just the leading approximation. To obtain more accuracy, we use the method of successive approximations:

$$g^{[0]} \approx \frac{x^3}{2}; \qquad g^{[1]} \approx \frac{x^3}{2} - \frac{1}{2}x^3(g^{[0]})' = \frac{x^3}{2} - \frac{3x^5}{4}$$

and in general

$$g^{[n]} \approx \frac{x^3}{2} - \frac{1}{2}x^3(g^{[n-1]})'$$

which is easy to automatize, and gives again (3). Examining the coefficients of this series, we see that they are of the form $c_n = a_n/b_n$ where $b_n = 2^n$ and $a_n = (-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots = (-1)^{n+1}(2n-1)!!$. The ratio test shows that the radius of convergence of this series is zero, and the series is *asymptotic.* What does this mean?

3.1. **The asymptotic series** (3)**, cont.**

Going back to the ODE and using the method of integrating factors, or equivalently of variation of parameters, we get

$$g(x) = e^{\frac{1}{x^2}} \int_0^x e^{-\frac{1}{s^2}} ds + Ce^{\frac{1}{x^2}} \tag{4}$$

Let us try to understand the behavior of the term $J = e^{\frac{1}{x^2}} \int_0^x e^{-\frac{1}{s^2}} ds$ for small x. Given all these fractions, we are better of substituting $x = 1/t$ where now $t \to \infty$:

$$J = e^{t^2} \int_t^\infty s^{-2} e^{-s^2} ds$$

3.2. **The method of integration by parts.**

In order to obtain the behavior of integrals such as J, which contain both exponentials and powers, a simple method is to integrate by parts, at each step aiming at *making the power smaller*, and this means differentiating the power, in our case:

$$J = e^{t^2} \int_t^\infty s^{-2}(2se^{-s^2})/(2s) ds = e^{t^2} \int_t^\infty s^{-2}(-e^{-s^2})'/(2s) ds \tag{5}$$
$$= \frac{1}{2t^3} - \frac{3}{2} e^{t^2} \int_t^\infty s^{-4} e^{-s^2} ds$$

Continuing integration by parts, we get

$$J = \frac{e^{x^2}}{2}\int_x^\infty \frac{e^{-s^2}}{s^2}ds \tag{6}$$
$$= \frac{1}{2t^3} - \frac{3}{2t^5} + \frac{15}{8t^7} + \cdots + (-1)^n \frac{(2n-1)!!}{2^{n-1}} e^{t^2}\int_t^\infty s^{-(2n-1)}e^{-s^2}ds$$

We note that the successive terms that we get in this way correspond exactly to (3), while this procedure gives us an identity at each order, and thus a way to control the error in the approximations. The error is given by

$$(-1)^n \frac{(2n-1)!!}{2^{n-1}} e^{t^2}\int_t^\infty s^{-(2n-1)}e^{-s^2}ds \tag{7}$$

and we see that the integrand gets smaller and smaller as n becomes large.

We also see that this error term is alternating in sign, and thus J itself always fits between two successive terms of the asymptotic series

$$J < \frac{1}{2t^3}; \quad J > \frac{1}{2t^3} - \frac{3}{2t^5}; \quad J < \frac{1}{2t^3} - \frac{3}{2t^5} + \frac{15}{8t^7}$$

etc. The error itself is of the same order as the next term of the asymptotic expansion. Indeed, by L'Hospital,

$$\frac{\displaystyle\int_t^\infty \frac{e^{-s^2}}{s^{2m}}ds}{\dfrac{e^{-t^2}}{t^{2m+1}}} \to \frac{1}{2} \text{ as } t \to \infty \tag{8}$$

How can we evaluate J when t is large? The series still does not converge. But it provides successive approximations of J.

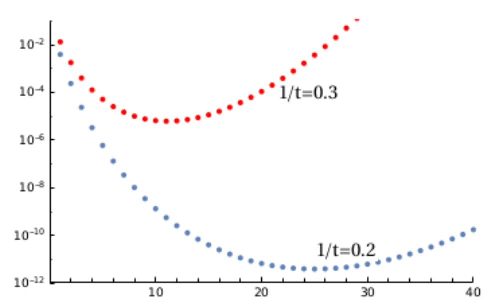

FIGURE 1. The size of the error as a function of n for two values of t.

3.3. **More general asymptotic series.** Classical asymptotic analysis typically deals with the qualitative and quantitative description of the behavior of a function close to a point, usually a singular point of the function. This description is provided in the form of an *asymptotic expansion.* The expansion certainly depends on the point studied and, as we have noted, often on the direction along which the point is approached (in the case of several variables, it also depends on the relation between the variables as the point is approached). If the direction matters, it is often convenient to change variables to place the special point at infinity.

Asymptotic expansions are formal series[1] of simpler functions f_k,

$$\tilde{f} = \sum_{k=0}^{\infty} f_k(t) \tag{9}$$

in which each successive term is much smaller than its predecessors (one variable is assumed for clarity). For instance if the limiting point is t_0 approached from above along the real line this requirement is written

$$f_{k+1}(t) = o(f_k(t)) \quad \text{or} \quad f_{k+1}(t) \ll f_k(t) \quad \text{as} \quad t \downarrow t_0 \tag{10}$$

denoting

$$\lim_{t \to t_0^+} f_{k+1}(t)/f_k(t) = 0 \tag{11}$$

We will often use the variable x when the limiting point is $+\infty$ and z when the limiting point is zero. Simple examples are the Taylor series, e.g.

$$\sin z + e^{-\frac{1}{z}} \sim z - \frac{z^3}{6} + \ldots \quad (z \to 0^+)$$

and the expansion in the Stirling formula

$$\ln \Gamma(x) \sim x \ln x - x - \frac{1}{2} \ln x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}, \ x \to +\infty$$

where B_k are the Bernoulli numbers.

(The asymptotic expansions in the examples above are the formal sums following the "$\sim$" sign, the meaning of which will be explained shortly.)

Examples of expansions that are *not* asymptotic expansions are

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (x \to +\infty)$$

[1]That is, there are no convergence requirements. More precisely, they are defined as sequences $\{f_k\}_{k \in \mathbb{N} \cup \{0\}}$, the operations being defined in the same way as if they represented convergent series; see also §3.4.

which converges to $\exp(x)$, but it is not an asymptotic series for large x since it fails (10); another example is the series

$$\sum_{k=0}^{\infty} \frac{x^{-k}}{k!} \ + e^{-x} \quad (x \to +\infty) \tag{12}$$

(because of the exponential terms, this is not an ordered *simple series* satisfying (10)). Note however expansion (12), *does* satisfies all requirements in the *left* half plane, if we write e^{-x} in the first position.

We also note that in this particular case the first series is convergent, and if we replace (12) by

$$e^{1/x} + e^{-x} \tag{13}$$

then (13) *is* a valid asymptotic expansion, of a very simple kind, with two nonzero terms. Since convergence is relative to a topology, this elementary remark will play a crucial role when we will speak of Borel summation.

Functions asymptotic to a series, in the sense of Poincaré. The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits:

Definition 3. *A function f is asymptotic to the formal series $\tilde{f}$ as $t \to t_0^+$ if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) =: f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{14}$$

We note that condition (14) can then be also written as

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{15}$$

where $g(t) = O(h(t))$ means $\limsup_{t\to t_0^+} |g(t)/h(t)| < \infty$. Indeed, this follows from (14) and the fact that $f(t) - \sum_{k=0}^{N+1} \tilde{f}_k(t) = o(\tilde{f}_{N+1}(t))$.

3.4. **Asymptotic power series.** In many instances the functions f_k are exponentials, powers and logarithms. This is not simply a matter of choice or an accident, but reflects some important fact about the relation between asymptotic expansions and functions which will be clarified later.

A special role is played by power series which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \ z \to 0^+ \tag{16}$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$) the series can be centered at t_0 (or $+\infty$, respectively).

Remark. If a c_k is zero then Definition 3 fails trivially in which case (16) is not an asymptotic series. This motivates the following definition.

Definition 4 (Asymptotic power series)**.** *A function possesses an asymptotic* power *series if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \tag{17}$$

We use the boldface notation $\boldsymbol{\sim}$ for the stronger asymptoticity condition in (14) when confusion is possible.

Example Check that the Taylor series of an analytic function at zero is its asymptotic series there.

In the sense of (17), the asymptotic power series at zero of e^{-1/x^2} is the zero series. It is however surely not the case that e^{-1/x^2} behaves like zero as $x \to 0$ on $\mathbb{R}$. Rather, in this case, the asymptotic *behavior* of e^{-1/x^2} is e^{-1/x^2} itself (only exponentials and powers involved).

Asymptotic power series form an algebra; addition of asymptotic power series is defined in the usual way:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} c_k' z^k = \sum_{k=0}^{\infty} (Ac_k + Bc_k')z^k$$

while multiplication is defined as in the convergent case

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} c_k' z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j c_{k-j}'\right) z^k$$

Remark 5. *If the series $\tilde{f}$ is convergent and f is its sum (note the ambiguity of the "sum" notation) $f = \sum_{k=0}^{\infty} c_k z^k$ then $f \sim \tilde{f}$.*

The proof of this remark follows directly from the definition of convergence.

Lemma 6. *(Uniqueness of the asymptotic series to a function) If $f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$ as $z \to 0$ then the $\tilde{f}_k$ are unique.*

Proof. Assume that we also have $f(z) \sim \tilde{F} = \sum_{k=0}^{\infty} \tilde{F}_k z^k$. We then have (cf. (14))

$$\tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = o(z^N)$$

which is impossible unless $g_N(z) = \tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = 0$, since g_N is a polynomial of degree N in z. □

Corollary 7. *The asymptotic series at the origin of an analytic function is its Taylor series at zero. More generally, if F has a Taylor series at* 0 *then that series is its asymptotic series as well.*

The proof of the following lemma is immediate:

Lemma 8. *(Algebraic properties of asymptoticity to a power series) If* $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ *and* $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ *then*
(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$
(ii) $fg \sim \tilde{f}\tilde{g}$

Sometimes it is convenient to check a formally weaker condition of asymptoticity:

Lemma 9. *Let* $\tilde{f} = \sum_{n=0}^{\infty} a_n z^n$. *If* f *is such that there exists a sequence* $p_n \to \infty$ *such that*

$$\Big(\forall n \exists p_n\Big) \text{ s.t. } \ f(z) - \tilde{f}^{[p_n]}(z) = o(z^n) \quad \text{as} \quad z \to 0$$

then $f \sim \tilde{f}$.

Proof. We let m be arbitrary and choose $n > m$ such that $p_n > m$. We have

$$f(z) - \tilde{f}^{[m]} = (f(z) - \tilde{f}^{[p_n]}) + (\tilde{f}^{[p_n]} - \tilde{f}^{[m]}) = o(z^m) \ (z \to 0)$$

by assumption and since $\tilde{f}^{[p_n]} - \tilde{f}^{[m]}$ is a polynomial for which the smallest power is z^{m+1} (we are dealing with truncates of the same series). ☐

3.5. **Integration and differentiation of asymptotic power series.** While asymptotic power series can be safely integrated term by term as the next proposition shows, differentiation is more delicate. In suitable spaces of functions and expansions, we will see the asymmetry largely disappears if we are dealing with analytic functions in suitable regions.

Anyway, for the moment note that the function $e^{-1/z}\sin(e^{1/z^2})$ is asymptotic to the zero power series as $z \to 0^+$ although the derivative is unbounded and thus not asymptotic to the zero series.

Proposition 10. *Assume* f *is integrable near* $z = 0$ *and that*

$$f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} \tilde{f}_k z^k$$

Then

$$\int_0^z f(s)ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{k+1} z^{k+1}$$

Proof. This follows from the fact that $\int_0^z o(s^n)ds = o(z^{n+1})$ as can be seen by immediate estimates. ☐

Asymptotic power series of analytic function, if they are valid in wide enough regions can be differentiated.

Asymptotics in a strip. Assume $f(x)$ is analytic in the strip $S_a = \{x : |x| > R, |\Im(x)| < a\}$. Let $\alpha < a$ and and $S_\alpha = \{x : |x| > R, |\Im(x)| < \alpha\}$ and assume that

$$f(x) \sim \tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k} \quad (|x| \to \infty, x \in S_\alpha) \tag{18}$$

It is assumed that that the limits implied in (18) hold uniformly in the given strip.

Proposition 11. *If (18) holds, then, for $\alpha' < \alpha$ we have*

$$f'(x) \sim \tilde{f}'(x) := \sum_{k=0}^{\infty} -\frac{kc_k}{x^{k+1}} \quad (|x| \to \infty, x \in S_{\alpha'})$$

Proof. We have $f(x) = \tilde{f}^{[N]}(x) + g_N(x)$ where clearly g is analytic in S_a and $|g_N(x)| \le Const.|x|^{-N-1}$ in S_α. But then, for $x \in S_{\alpha'}$ and $\delta = \frac{1}{2}(\alpha - \alpha')$ we get

$$|g_N'(x)| = \frac{1}{2\pi}\left|\oint_{|x-s|=\delta} \frac{g_N(s)ds}{(s-x)^2}\right| \le \frac{1}{\delta}\frac{Const.}{(|x| - |\delta|)^{N+1}}$$
$$= O(x^{-N-1}) \quad (|x| \to \infty,\ x \in S_{\alpha'})$$

By Lemma 9, the proof follows. □

3.6. **Watson's Lemma.** In many instances integral representations of functions are amenable to Laplace transforms

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-xp} F(p) dp \tag{19}$$

The behavior of $\mathcal{L}F$ for large x relates to the behavior for small p of F.

It is shown in the later parts of this book that solutions of generic analytic differential equations, under mild assumptions can be conveniently expressed in terms of Laplace transforms.

For the error function note that

$$\int_N^\infty e^{-s^2} ds = N \int_1^\infty e^{-N^2u^2} du = \frac{\sqrt{x}e^{-x}}{2} \int_0^\infty \frac{e^{-xp}}{\sqrt{p+1}} dp; \quad x = N^2$$

For the Gamma function, writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ in

$$n! = \int_0^\infty e^{-t} t^n dt = n^{n+1} \int_0^\infty e^{n(-s+\ln s)} ds \tag{20}$$

we can make the substitution $t - \ln t = p$ in each integral and obtain

$$n! = n^{n+1} e^{-n} \int_0^\infty e^{-np} G(p) dp$$

3.7. The Riemann-Lebesgue lemma.

Theorem 12. *If f is in $L^1(\mathbb{R}^d)$ that is to say, if the integral of $|f|$ is finite, then the Fourier transform of f satisfies*

$$\hat{f}(z) := \int_{\mathbb{R}^d} f(x)\exp(-iz\cdot x)\,dx \to 0 \text{ as } |z| \to \infty$$

Proof. We prove this in one dimension; the generalization is easy. Let first ϕ be a Schwarz function. Then, integrating by parts, we get

$$\int_{\mathbb{R}} \phi(x)\exp(-izx)\,dx = -\frac{1}{-iz}\int_{\mathbb{R}} \phi'(x)\exp(-izx) \to 0$$

as $z \to \infty$. □

Let now $f \in L^1$ and recall that $\mathcal{S}$ is dense in L^1. Take $\phi \in \mathcal{S}$ so that $\|f-\phi\|_1 < \epsilon$. Then,

$$\lim_{z\to\infty}\left|\int_{\mathbb{R}} f(x)\exp(-izx)\,dx\right| \le \lim_{z\to\infty}\left(\left|\int_{\mathbb{R}} f(x)-\phi(x)\exp(-izx)\,dx\right| + \left|\int_{\mathbb{R}}\phi(x)\exp(-izx)\,dx\right|\right) < \epsilon \tag{21}$$

and since ϵ is arbitrary, the result follows.

3.8. **The method of stationary phase.** Let f be a smooth function and Σ be the set of critical points of f (i.e. points where $\nabla f = 0$. Assume g is continuous and decays fast enough (e.g., exponentially or is compactly supported). Further, assume that all critical points of f are nondegenerate (i.e., the Hessian of f is nonzero at each point in Σ). Then,

$$\int_{\mathbb{R}^n} g(x)e^{ikf(x)} = \sum_{x_0\in\Sigma} e^{ikf(x_0)}|\det(\mathrm{Hess}(f))|^{-1/2}e^{\pi i/4\mathrm{sign}(\mathrm{Hess}(f))}(2\pi/k)^{n/2}g(x_0) + o(k^{-n/2})$$

For $n = 1$, this reduces to:

$$\int_{\mathbb{R}} g(x)e^{ikf(x)}dx = \sum_{x_0\in\Sigma} g(x_0)e^{ikf(x_0)+\mathrm{sign}(f''(x_0))i\pi/4}\left(\frac{2\pi}{k|f''(x_0)|}\right)^{1/2} + o(k^{-1/2})$$

Here is a sketch of the proof, in one dimension. We assume that g is smooth with compact support, say $[-a, a]$.

The main statement is local, i.e., we can assume that there is only one x_0, and without loss of generality we take $x_0 = 0$. We have $f'(0) = 0, f''(0) \neq 0$. Also without loss of generality we may assume $f''(0) > 0$. By symmetry, it is enough to show that the contribution to the integral of the interval $(0,\infty)$ is half of the stated result.

Claim Under these assumptions, there is an interval, say $(-\epsilon,\epsilon)$ s.t. f is decreasing on $(-\epsilon, 0)$ and on $(0,\epsilon)$ and, since $f' \neq 0$ on $[-a, 0)\cup(0, a]$, f is increasing on $(0,a]$. Furthermore, the change of variables $u = u(x)$ defined by $f(x) = f''(0)u^2/2$ is differentiable on $(0,a]$ and $dx/du = 1$ at $u = x = 0$.

Exercise 13. *Prove this claim.*

Now we can write

$$\int_0^\infty g(x)e^{ikf(x)}dx = \frac{\sqrt{2}}{\sqrt{f''(0)}}\int_0^\infty g(x(u))e^{iku^2}du \quad \text{(take now } u^2 = v\text{)} \tag{22}$$
$$= \frac{1}{\sqrt{2f''(0)}}\int_0^\infty g(x(\sqrt{v}))e^{ikv}v^{-1/2}dv$$

where $g(x(\sqrt{v})) = h(v)$ is smooth, with compact support $[-A,A]$ for some A. Now we integrate by parts:

$$\int_0^\infty g(x(\sqrt{v}))e^{ikv}v^{-1/2}dv \tag{23}$$
$$= \left(h(v)\int_\infty^v e^{iku}u^{-1/2}du\right)\Big|_0^\infty - \int_0^\infty \left(\int_\infty^v e^{iku}u^{-1/2}du\right)h'(v)dv$$
$$= \frac{1}{\sqrt{k}}\left(\sqrt{\pi}\sqrt{i}h(0) - \int_0^A \left(\int_\infty^{\sqrt{kt}} e^{ikt}t^{-1/2}du\right)h'(v)dv\right)$$

Since the last integral goes to zero, the result follows.

Proposition 14. *Assume $f \in C^{n-1}[a,b]$ and $f^{(n)} \in L^1([a,b])$. Then we have*

$$\int_a^b e^{ixt}f(t)dt = e^{ixa}\sum_{k=1}^n c_k x^{-k} + e^{ixb}\sum_{k=1}^n d_k x^{-k} + o(x^{-n}) \tag{24}$$
$$= e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b + o(x^{-n}),$$

where $c_k = -f^{(k-1)}(a)/i^k$ and $d_k = f^{(k-1)}(b)/i^k$.

Proof. This follows by integration by parts and the Riemann-Lebesgue lemma since

$$\int_a^b e^{ixt}f(t)dt = e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b \tag{25}$$
$$+ \frac{(-1)^n}{(ix)^n}\int_a^b f^{(n)}(t)e^{ixt}dt$$

□

Corollary 15. *(1) Assume $f \in C^\infty[0,2\pi]$ is periodic with period 2π. Then $\int_0^{2\pi} f(t)e^{int}dt = o(n^{-m})$ for any $m > 0$ as $n \to +\infty, n \in \mathbb{Z}$.*

(2) Assume $f \in C_0^\infty[a,b]$ vanishes at the endpoints together with all derivatives; then $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ for any $m > 0$ as $x \to \pm\infty$.

3.9. **The WKB ansatz.** This applies to linear equations (ODEs, PDEs etc) at irregular singularities, where exponential behavior is expected. An example is the Airy equation,

$$y'' = xy$$

when $x \to \infty$. You can convince yourselves that, as a power series in inverse powers behavior of x, only the zero series works, and that corresponds to the zero solution.

The WKB ansatz is: (1) look for exponential behavior, $y = e^{W(x)}$. This gives

$$W'' + W'^2 = x$$

(2) Find the dominant balance as $x \to \infty$. The rule of thumb is that $W'' \ll W'^2$. The equation above becomes

$$W' = \pm\sqrt{x - W''}$$

or

$$f = \pm\sqrt{x - f'}$$

where $f = W'$, to which we apply the method of successive approximations,

$$f^{[n+1]} = \pm\sqrt{x - f'^{[n]}}$$

This gives

$$W(x) \sim \pm\frac{2x^{3/2}}{3} - \frac{1}{4}\log(x) + C + \pm\frac{5}{48}\left(\frac{1}{x}\right)^{3/2} + \frac{5}{64x^3} + \cdots$$

Formally for now, this shows that there are two linearly independent solutions to the Airy equations (denoted Ai,Bi) whose behavior for large x is, up to a constant

$$x^{-1/4}e^{\pm\frac{2x^{3/2}}{3}}\left(1 \pm \frac{5}{48}\left(\frac{1}{x}\right)^{3/2} + \cdots\right)$$

3.10. **Watson's Lemma.** This important result states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 16. *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants β_i with $\Re(\beta_i) > 0$, $i = 1, 2$. Then*

$$\mathcal{L}F \sim \sum_{k=0}^\infty c_k \Gamma(k\beta_1 + \beta_2) x^{-k\beta_1-\beta_2}$$

along any ray ρ in the open right half plane H.

Proof. Induction, using the simpler version, Lemma 17, proved below. □

Lemma 17. *Let* $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ *and assume*

$$F(p) \sim p^\beta \quad \text{as } p \to 0^+$$

with $\Re(\beta) > -1$. *Then*

$$\int_0^\infty F(p) e^{-px} dp \sim \Gamma(\beta+1) x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $F_1(p) = p^{-\beta} F(p)$ we have $\lim_{p\to 0} F_1(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. For any $a > 0$ we have

(26)

$$\left| \int_a^\infty F(p) e^{-px} \mathrm{d}p \right| \le e^{-|x| a \cos\phi} \|F\|_1 = O(|x|^{\beta+1} e^{-|x| a \cos\phi}) = o(x^{-\beta-1})$$

Thus, we only need to show that

$$\int_0^a F(p) e^{-px} dp = \int_0^\infty p^\beta F_1(p) \chi_{[0,a]}(p) dp \sim \Gamma(\beta+1) x^{-\beta-1} \quad (\rho \to \infty)$$

where $\chi_{[0,a]}(p)$ is the characteristic function of the interval $[0,a]$. After the change of variable $s = p|x|$,

(27)

$$\begin{aligned} x^{\beta+1} \int_0^\infty p^\beta F_1(p) e^{-px} \chi_{[0,a]}(p) \mathrm{d}p &= e^{i\phi(\beta+1)} \int_0^\infty s^\beta F_1(s/|x|) \chi_{[0,a]}(s/|x|) e^{-se^{i\phi}} \mathrm{d}s \\ &\to e^{i\phi(\beta+1)} \int_0^\infty s^\beta e^{-se^{i\phi}} \mathrm{d}s = \Gamma(\beta+1) \end{aligned}$$

as $|x| \to \infty$, where we took the limit inside the integral which is justified by the exponential decay of the integrand.

4. Example 1: The Gamma function

One of the remarkable formulas for the Gamma function is

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \exp\left(\int_0^\infty \frac{(p/2)\coth(p/2) - 1}{p^2} e^{-xp} dp \right) \tag{28}$$

We first note that the integrand in (28) is analytic. Indeed, at $p = 0$,

$$(p/2)\coth(p/2) = 1 + \frac{p^2}{12} - \frac{p^4}{720} + \cdots$$

In fact,

$$\frac{(p/2)\coth(p/2) - 1}{p^2} e^{-xp} = \frac{1}{12} - \frac{p^2}{720} + \frac{p^4}{30240} - \frac{p^6}{1209600} + \cdots \tag{29}$$

where the series has nonzero radius of convergence.

Exercise 18. *What is the radius of convergence of the series above?*

Watson's Lemma gives

(30)
$$\int_0^\infty \frac{(p/2)\coth(p/2)-1}{p^2}e^{-xp}dp \sim \frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\cdots$$

In fact, essentially by definition, the coefficients of the series above are

$$\frac{B_{2n}}{2n(2n-1)}$$

where B_{2n} are the Bernoulli numbers. Hence,

(31)
$$\exp\left(\int_0^\infty \frac{(p/2)\coth(p/2)-1}{p^2}e^{-xp}dp\right) \sim 1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\cdots$$

Exercise 19. *Justify this exponentiation.*

(32)
$$\Gamma(x) \sim \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\cdots\right)$$

which is Stirling's formula. The pattern of signs of the terms in Stirling's formula is $++--++--\cdots$. Can you explain why?

4.0.1. *The Airy equation.* Let us look again at the Airy equation,

$$y''-xy=0 \tag{33}$$

Here, the behavior of solutions at infinity, that we have already obtained by WKB is

$$y \sim Cx^{-\frac{1}{4}}e^{-\frac{2}{3}x^{3/2}} \tag{34}$$

We use the transformation $y(x)=g(\frac{2}{3}x^{\frac{3}{2}})$ to achieve the **important normalization where the asymptotic exponential has linear exponent**, that is $(2/3)x^{3/2}=t$ and get

$$g''+\frac{1}{3t}g'-g=0 \tag{35}$$

In view of (34) we have

$$g(t) \sim Ct^{-\frac{1}{6}}e^{\pm t} \tag{36}$$

To eliminate the exponential behavior of one solution, say of the decaying one, we substitute $g=he^{-t}$, and get

$$h''-\left(2-\frac{1}{3t}\right)h'-\frac{1}{3t}h=0 \tag{37}$$

To obtain a second solution, we can resort to the substitution $g=he^t$. **Taking inverse Laplace transform** we get

$$p(2+p)H'+\frac{5}{3}(1+p)H=0 \tag{38}$$

with the solution

$$H = Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} \tag{39}$$

Exercise 20. *Use Watson's Lemma to find 6 terms in the asymptotic expansion of $h(t)$.*

and thus

$$h(t) = \mathcal{L}\left(Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}\right) \tag{40}$$

and, comparing the asymptotic expansion obtained from (40) with that of Airy functions we get

$$\mathrm{Ai}(x) = \frac{3^{-\frac{1}{6}}\exp(-\frac{2}{3}x^{3/2})}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{6})}\int_0^\infty e^{-\frac{2}{3}x^{\frac{3}{2}}p}p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}dp \tag{41}$$

Exercise 21. *Apply Watson's Lemma to $h(t)$ and find the asymptotic behavior of $Ai(x)$ as $x \to \infty$.*

4.1. **Laplace's method.**

Proposition 22. *Assume $f, g : [a, b] \to \mathbb{R}$ are smooth*

(1) *If f is strictly decreasing on $[a, b]$, then, as $x \to \infty$,*
$$\int_a^b g(s)e^{xf(s)}ds = e^{xf(a)}g(a)\frac{x^{-1}}{|f'(a)|}(1+o(1))$$

(2) *If f has a unique nondegenerate maximum at s_0, as $x \to \infty$,then*
$$\int_a^b g(s)e^{xf(s)}ds = e^{xf(s_0)}g(s_0)\frac{\sqrt{2\pi}x^{-1/2}}{\sqrt{|f''(s_0)|}}(1+o(1))$$

Proof. Both follow from Watson's lemma. Alternatively, for 1 change variable $f(s) = u$ and integrate by parts, and for 2 mimic the proof in the section with the stationary phase method. ☐

Exercise 23. *Prove the Proposition by the second method indicated.*

5. Reminder: Analytic functions

A continuous function defined in some open disk D in $\mathbb{C}$ is called analytic in D if for any $z_0 \in D$ $f'(z_0)$ exists (and then it follows that **all** derivatives exist) where f' is defined in the complex plane in the same way as you know it from calculus:

$$f'(z) = \lim_{|h|\to 0}\frac{f(z+h)-f(z)}{h}$$

or

$$f(z+h) - f(z) = f'(z)h + o(h)$$

A function which is analytic everywhere is called entire. Writing $f = u+iv$, $z = x+iy$, $f'(z) = a+ib$ and $h = dx+idy$ we have

$$df = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = (a+ib)(dx+idy)$$

and thus

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = adx - bdy \tag{42}$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = ady + bdx \tag{43}$$

giving

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ (= a); \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ (= -b) \tag{44}$$

These are the Cauchy-Riemann or CR equations.

Proposition 24. *The field* $\Big(u(x,y), v(x,y)\Big)$ *is conservative. Furthermore, u, v satisfy Laplace's equation*

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{45}$$

Proof. The CR equations imply $\nabla \times u = 0 = \nabla \times v$. Using the infinite differentiability of f and the CR equations, (45) merely states the equality of mixed partial derivatives.

Given a curve $\gamma = (x(t), y(t))$, the complex integral of the analytic function f along γ is defined as follows

$$\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx)$$

□

Note that **the integral only depends on the endpoints, or if a curve is closed, it is zero.** This follows from the CR equations, or, which is the same, from the fact that the field (u, v) is conservative.

6. Steepest descent (Saddle point) method

6.1. **Motivation: Fourier coefficients of a periodic analytic function.** Take $f(x) = \dfrac{1}{a + \cos(x)}$ where $a > 1$. Then f is periodic of period 2π and is analytic at all points whose imaginary part is not large (how large is too large?). For the sake of getting nice expressions for the Fourier coefficients f_k, take $a = \cosh(1)$. If we apply Proposition 14 (the method of integration by parts, really) we get that $f_k = o(k^{-n})$, for any $n \in \mathbb{N}$. Which means "very fast". But this is not very precise. On the other hand, we can calculate the Fourier coefficients in closed form, since we are dealing with trigonometric integrals, of the kind studied in Calc 1. Here is an even simpler method: write $\cos x = \frac{1}{2}(\zeta + 1/\zeta)$ $(\zeta = e^{ix})$ and $\cosh 1 = \frac{1}{2}(e + 1/e)$ and expand f by partial fractions in ζ:

$$f = \frac{2e^2}{(e^2-1)(\zeta+e)} - \frac{2e}{(e^2-1)(e\zeta+1)}$$

For the first term, we would expand in Taylor series in ζ, clearly convergent for $|\zeta| < e$ which is the case when $\zeta = e^{ix}$. Upon substituting $\zeta = e^{ix}$ the expansion becomes exactly the Fourier series of the first term (why?). We cannot do this for the second term, but we can rewrite it, and f, as

$$f = \frac{2e^2}{(e-1)(1+e)(\zeta+e)} - \frac{2}{\zeta}\frac{1}{(e-1)(1+e)\left(1+\frac{1}{e\zeta}\right)}$$

and $\left(1+\frac{1}{e\zeta}\right)$ can also be expanded in geometric series. All in all, we get

$$f_k = \frac{2}{e^{|k|-1} - e^{|k|+1}}$$

and we see that the coefficients decrease **exponentially** in k, $|f_k| = O(e^{-|k|}$.

*

Proposition 25. *Let f be periodic of period 2π, analytic in a strip $|\Im z| \le a$ and continuous up to the boundary. Then, for large $|k|$, $\hat{f}_k = O(e^{-a|k|})$.*

Proof. Consider the path

$$\gamma = [0, 2\pi] \cup [2\pi, 2\pi + ia] \cup [2\pi + ia, 0 + ia] \cup [0 + ia, 0]$$

Since f is analytic inside this path and continuous to the boundary, we have $\oint_\gamma f(z)e^{-ikz}dz = 0$. Note however that, by periodicity

$$\int_{[2\pi,2\pi+ia]} f(z)dz = \int_{[0,ia]} f(z)dz$$

and thus,

$$\int_{[2\pi,2\pi+ia]+[ia,0]} f(z)e^{-ikz}dz = 0$$

hence, for $k < 0$,

(46)

$$\int_{[0,2\pi]} f(z)e^{-ikz}dz = \int_{[ia,2\pi+ia]} f(z)e^{-ikz}dz = \int_{[0,2\pi+ia]} f(s+ia)e^{-ik(s+ia)}ds$$
$$= e^{ka}\int_{[0,2\pi+ia]} f(s+ia)e^{-iks}ds = O(e^{-|k|a})$$

since

$$\int_{[0,2\pi+ia]} |f(s+ia)e^{-iks}|ds \le 2\pi \max_{s\in[0,2\pi]} |f(s+ia)|$$

For positive k, take similarly a box in the lower half plane. □

We say that we deformed the contour of integration homotopically, from $[0, 2\pi]$ to $[ia, 2\pi + ia]$. For functions which are not periodic, or for more complicated cases, we need to understand how to deform the contour to get best estimates.

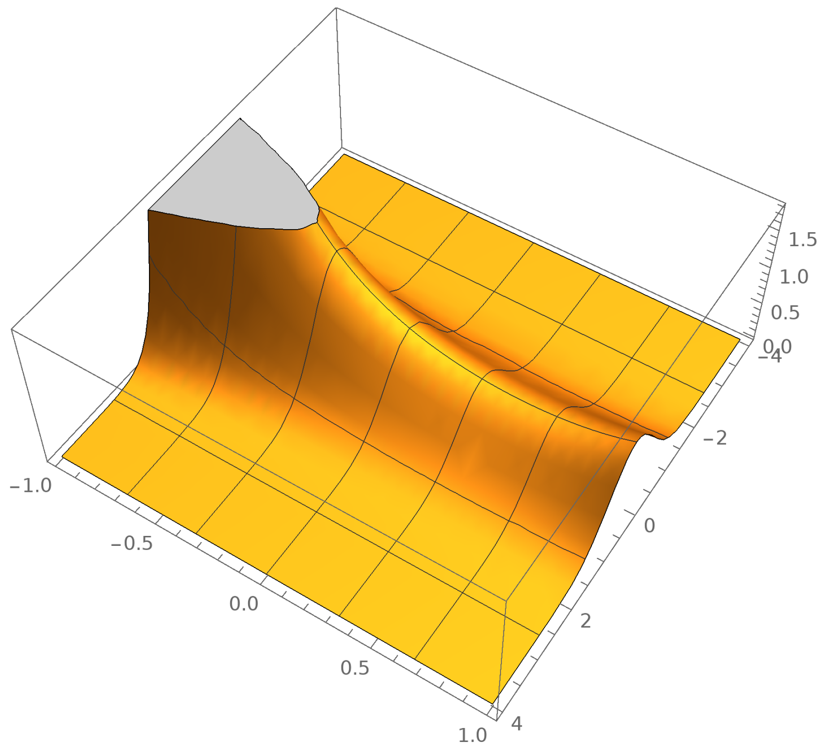

FIGURE 2. The function $|e^{-z^2+iz}|$

The next two examples reflect a common type of integral representations,

$$I(\nu) = \int_{-\infty}^{\infty} g(z) e^{\nu f(z)} dz \tag{47}$$

Take first a Fourier transform that we can calculate in closed form,

$$I = \int_{-\infty}^{\infty} e^{-z^2+i\nu z} dz = \sqrt{\pi} e^{-\frac{\nu^2}{4}}$$

and assume we did not know how to calculate it, but we still want to determine the large ν behavior. How should we change the contour? It is natural to push the various parts of the contour in such a way that the integrand becomes smaller in absolute value. Since the function is entire, it cannot have any minima or maxima, only saddle points. That does not mean that *along the path* we can't have maxima or minima! We push the contour to go over the saddle points along which, relative to the path, the absolute value of the function is minimal.

We imagine that the path is made of infinitely stretchable rubber, and we let the contour fall down under its own weight, until it stops, hanging over some saddles.

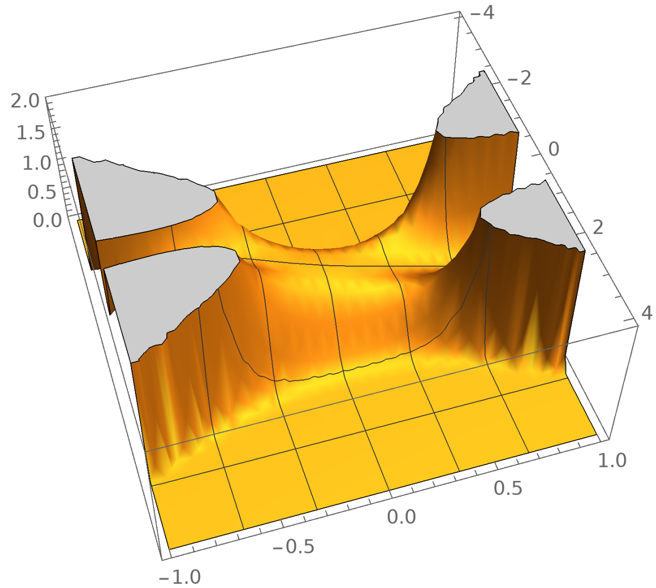

FIGURE 3. The function $|e^{-z^4+iz}|$

6.2. **Steepest descent method: an outline.** We seek to determine the asymptotic behavior of $I(\nu)$ in (47) as $\nu \to +\infty$ for f and g that are analytic in some some region of the complex plane[2], and $\mathcal{C}$ is some simple curve that may be finite or infinite. The problem is to determine the asymptotics of I as $\nu \to +\infty$. [3]

The idea of the steepest descent method is to use the analyticity of the integrand in (47) in z to deform $\mathcal{C}$ homotopically into one or more paths, each of which characterized by $v = \Im f = C$, a constant. Along such a path, $e^{i\nu v}$ is constant, it can be taken out of the integral, and inside the integral we are left with $e^{\nu u}$ which is real-valued, and now we can apply Laplace's method.

6.3. **An example.** Typically, $\mathcal{C}$ is homotopic to a finite number of finite or infinite piecewise smooth curves of constant imaginary part, each with finitely many non-differentiability points. As we will see in a moment, non-differentiable points of the steepest descent decomposition correspond to singularities of f and zeros[4]. We write

$$f(z) = u(x, y) + iv(x, y) \tag{48}$$

[2]The region of analyticity will be dictated by the need to deform C into one or more steepest descent paths and will depend on the specifics of the problem.

[3]More generally, if $\nu \to \infty$ along some complex ray $\arg \nu = \phi$, we can replace ν by $|\nu|$ and f by $e^{i\phi} f$ to obtain asymptotics along complex rays.

[4] It is understood that a zero of f is a point where f is analytic and $f' = 0$.

and note that $f' = 0$ implies that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$, and, since u and v are harmonic, such points are *saddle points*.

We define **special points** to be singularities of f, endpoints, saddle points and the point at infinity. If $f' \neq 0$, the path of constant imaginary part ($v = const$) is a smooth curve (since $\nabla v \neq 0$). Let $t \mapsto \gamma(t) = \alpha(t) + i\beta(t)$ be a parameterization one of these smooth pieces. We have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\alpha'(t) + \frac{\partial u}{\partial x}\beta'(t) \tag{49}$$

and also, since v is constant,

$$0 = v' = \frac{\partial v}{\partial x}\alpha'(t) + \frac{\partial v}{\partial y}\beta'(t) \tag{50}$$

At a point where, say $u_x := \partial u/\partial x \neq 0$ and $\alpha' \neq 0$ we solve for α' from (50), and use the Cauchy-Riemann equations to obtain

$$d\gamma = \frac{\alpha'}{u_x}\langle u_x, u_y\rangle\, dt \tag{51}$$

where $u_y = \partial u/\partial y$ and thus $d\gamma$ is tangent at every point to the steepest variation direction of u. If $\alpha'/u_x > 0$, it is a direction of steepest ascent of u, and of steepest descent otherwise. Between every two *special points* as defined above, we choose to traverse the curve in the steepest descent direction, reversing the sign of the integral if needed; hence the name "steepest descent" for the method. Note that the saddle points are of finite order since $(\forall n)(f^{(n)}(z_0) = 0)$ implies $f \equiv 0$.

For simplicity we assume for now that the homotopic deformation of $\mathcal{C}$ does not cross singularities of f. Between each two special points, the integral becomes

$$e^{i\nu C}\int_0^1 e^{\nu u(\alpha(t),\beta(t))} g(\gamma(t))\gamma'(t)dt \tag{52}$$

where C is the constant value of v $\langle\gamma_x', \gamma_y'\rangle = \gamma_x' + i\gamma_y'$ and similarly for g.

The integral (52) is one in which the exponent is monotonic and thus one-to-one, and all conditions of Laplace's method apply. In particular, we can take as a new variable $u(\alpha(t), \beta(t))$ and reduce the question to a Laplace transform of the type $\int_0^a e^{-u\nu}G(u)du$ for $a \in (0, \infty]$ to which Watson's lemma applies. Generally, multiple steepest descent paths, each with a different value of C, are involved in homotopic deformation of $\int_{\mathcal{C}}$; these paths may also join up at *sinks* where $\Re f \to -\infty$ such as ∞ or other singularities of f. Multiple descent paths will definitely be needed when $\Im f$ is different at the end points of $\mathcal{C}$, as in the example in §6.4. In such cases, the calculation of $I(\nu)$ generally requires adding up the contributions from each steepest descent path $\int_{\mathcal{C}_s}$ in the manner outlined in the last paragraph. Therefore, the only new element in the steepest descent method is to determine steepest curves which are homotopically equivalent to the original path $\mathcal{C}$. It should be further noted that without homotopic deformation into descent paths,

(47) will typically be an oscillatory integral; asymptotics obtained through the stationary phase method often leads to substantially weaker results. The stationary phase method, however, does not require analyticity of f and g.

Note 26. Also, it is *important to note* that Watson's lemma applies in a half plane, and the resulting asymptotic expansion depends only on the behavior of the integrand near zero. If the curve of steepest descent starting at some point z_0 is clumsy, it can be replaced with a segment of line in the same direction, or even in the same open half-plane centered on the direction of steepest descent at z_0

6.4. **Simple illustrative example.** Consider

$$I(\nu) = \int_0^1 \frac{e^{i\nu z^2}}{z+1} dz \text{ for } \nu \to +\infty \tag{53}$$

This first example is taken to be as simple as possible, to the point of being a bit oversimplified. In particular, the stationary phase method (most often suboptimal in $\mathbb{C}$) would apply with the same result, and in the deformation of contour process we do not cross singularities of the integrand, nor do saddle points interfere with the deformation. Indeed, the steepest descent line at the saddle $z = 0$ is vertical, and, since each point on the curve is moved along a steepest descent path $z = 0$ simply moves up too. However, for $\arg \nu \neq 0$ (more precisely, when $\Im \nu = 0$) this situation changes. In the notation of (47), $f(z) = iz^2$, $g(z) = \frac{1}{z+1}$. Steepest descent paths emanating at $z = 0$ are determined by

$$\Im f = \Im f(0) = 0 \text{ implying } \Re z^2 = 0,\ i.e.\ z = re^{\pm i\pi/4} \text{ for } r \in (-\infty, \infty) \tag{54}$$

However, since $\Re f \to -\infty$, along the ray $z = \{e^{i\pi/4} : r \in [0, \infty)\}$ as $r \to \infty$, it follows that $\infty e^{i\pi/4}$ is a sink that is connected to $z = 0$ along the steepest descent path $z = re^{i\pi/4}$. The steepest descent path from the other end point $z = 1$ in the integral (53) is found by setting

$$\Im f = \Im f(1) = 1 \text{ implying } \Re z^2 = 1,\ i.e.\ x^2 - y^2 = 1 \tag{55}$$

A simple way to determine the local descent direction at a point z_0 is to analyze the differential $df = f'(z_0)dz$ and determine the direction of dz for which $df \in \mathbb{R}^-$ (note that $df = du$ since $dv = 0$). In our example $df = 2izdz = 2idz$ and $df < 0$ if $dx = 0, dy > 0$. Since only one branch of the hyperbola passes through (1, 0) and it asymptotes to $y = x$, *i.e.* approaches the sink $\infty e^{i\pi/4}$, by simple estimates a homotopic deformation of the $\int_0^1$ may be made to coincide with descent paths $z = re^{i\pi/4}$, $0 \le r < \infty$ followed by integration along steepest descent path C that connects $\infty e^{i\pi/4}$ to 1 along the hyperbola[5] $x^2 - y^2 = 1$. Therefore,

$$I(\nu) = \int_0^{\infty e^{i\pi/4}} \frac{e^{i\nu z^2}}{1+z} dz + \int_C \frac{e^{i\nu z^2}}{1+z} dz \equiv I_1(\nu) + I_2(\nu) \tag{56}$$

[5]We do not have the option of going along $re^{-i\pi/4}$,$0 < r < \infty$ since $\Re f \to +\infty$ and so contribution at $\infty e^{-i\pi/4}$ cannot be ignored as it can be for a *sink*.

For $I_1(\nu)$, using $z = re^{i\pi/4}$ for $0 < r < \infty$, we obtain after change of variable and application of Watson's Lemma

$$(57)\quad I_1(\nu) = e^{i\pi/4}\int_0^\infty \frac{e^{-\nu r^2}}{1+re^{i\pi/4}}dr = e^{i\pi/4}\int_0^\infty \frac{e^{-\nu p}dp}{2p^{1/2}[1+p^{1/2}e^{i\pi/4}]}$$
$$\sim \frac{1}{2}e^{i\pi/4}\sum_{j=0}^\infty (-1)^j\Gamma\left(\frac{j+1}{2}\right)e^{ij\pi/4}\nu^{-(j+1)/2}$$

For $I_2(\nu)$, we know that $-p := f(z) - f(1) = iz^2 - i$ is real valued and monotonically decreasing on the parabolic path C from $z = 1$ to $z = \infty e^{i\pi/4}$, since $f' \neq 0$ on this path. Therefore, solving for z, inversion leads to

$$(58)\qquad z = Z(p) = (1+ip)^{1/2},$$

where we can readily check that for this branch of square-root, as $p \to +\infty$, $z \to \infty e^{i\pi/4}$ as required. Therefore,

$$(59)\qquad I_2(\nu) = -e^{i\nu}\int_0^\infty \frac{e^{-p\nu}}{1+Z(p)}Z'(p)dp.$$

Taylor expansion gives

$$(60)\qquad \frac{Z'(p)}{1+Z(p)} = \frac{i}{2}(1+ip)^{-1/2}\left[1+(1+ip)^{1/2}\right]^{-1} = \sum_{j=0}^\infty a_j p^j,$$

where the first few coefficents are: $a_0 = \frac{i}{4}$, $a_1 = \frac{3}{16}$, $a_2 = -\frac{5i}{32}$, $a_3 = -\frac{35}{256}$. Applying Watson's Lemma to (59), it follows

$$(61)\qquad I_2(\nu) \sim -e^{i\nu}\sum_{j=0}^\infty a_j\nu^{-j-1}\Gamma(j+1),$$

The full asymptotic expansion of $I(\nu) = I_1(\nu) + I_2(\nu)$ is then obvious from (57) and (61).

Note 27. (1) The Taylor expansion in (60) can be written explicitly, and in a simple way, in terms of the binomial series by multiplying the numerator and the denominator by $\left[1-(1+ip)^{1/2}\right]$ and expanding it out.

(2)If we replace the integrand $\frac{e^{i\nu z^2}}{z+1}$ in (53), by $\frac{e^{i\nu z^2}}{z-z_0}$, where z_0 is in the upper-half plane region between $e^{i\pi/4}\mathbb{R}^+$ and steepest descent contour C connecting $\infty e^{i\pi/4}$ to 1, for *e.g.* $z_0 = \frac{1+i}{2}$, then the singulariy at $z = z_0$ interferes with the homotopic deformation into steepest descent paths. Nonetheless, since this singularity is a pole, after collecting residue at $z = z_0$, we can use the same descent paths as in Example 6.4. Since $\Im z_0^2 > 0$, the residue contribution will be exponentially small in ν relative to (61) and (57). If this z_0 were a branch point instead, in addition to the steepest descent paths, the homotopically deformed path will include a contour that wraps around z_0. Nonetheless, as in the case of the pole, the contribution of the branch point is exponentially small in ν.

Note 28. The end result of this procedure, after changes of variables, is indeed a sum of Laplace transforms on $[0, a)$, $a \in [0, \infty]$ to which Watson's lemma applies.

References

[1] Elias M. Stein and Rami Shakarchi, Fourier Analysis, an introduction, Princeton University Press (2003).