Math 5102 **Homework 3**
due Wed March 6

1. A linear differential equation of the form $ax^2u'' + bxu' + cu = 0$ where a, b, c are constants is called an Euler equation.

a) Substituting $u(x) = x^r$ in the Euler equation we obtain a quadratic equation for r: $ar(r-1) + br + c = 0$. If this equation has two distinct roots $r_1 \neq r_2$, show that $u_1(x) = x^{r_1}$ and $u_2(x) = x^{r_2}$ are two independent solutions of the Euler equation.

b) Show that the substitution $x = e^s$ transforms this equation in a linear equation with constant coefficients: $a\frac{d^2u}{ds^2} + (b-a)\frac{du}{ds} + cu = 0$

c) Solve the equation at b) to show that, indeed, solutions of the Euler equation should be searched in the form $u(x) = x^r$ and find two independent solutions in the case $r_1 = r_2$.

2. Solve the Sturm-Liouville problem $(xu')' + \lambda\frac{u}{x} = 0,\ u(1) = 0,\ u(b) = 0$ (note that this is an Euler equation).

3. Solve the Sturm-Liouville problem $(x^3u')' + \lambda xu = 0,\ u(1) = 0,\ u(e) = 0$ (note that this is an Euler equation).

4. Bring the following to a Sturm-Liouville form $(pu')' + (-q + \lambda w)u = 0$:

a) Hermite equation: $u'' - 2xu' + \lambda u = 0$;

b) Laguerre equation: $xu'' + (1-x)u' + \lambda u = 0$;

c) Chebyshev equation: $(1-x^2)u'' - xu' + \alpha^2 u = 0$.