Math 5102 **Homework 4**
due Wed, March **20**, 2019

1. Show that if u is a solution of

$$-(p(x)u')' + (q(x) - \lambda w(x))u = 0 \qquad (1)$$

then $v = pu'/u$ satisfies the Riccati equation

$$v' + p(x)^{-1}v^2 = q(x) - \lambda w(x)$$

(*Note:* Riccati equations can be studied by turning them into them to a linear second order equation.)

2. *(Sturm-Liouville normal form)* Show that the differential equation (1) transforms into one with $w = p = 1$: using the transformation

$$y(x) = \int_a^x \sqrt{\frac{w(t)}{p(t)}}\, dt, \quad v(y) = u(x(y))\left[w(x(y))\, p(x(y))\right]^{1/4}$$

then

$$-\frac{d^2v}{dy^2} + Q(y)v = \lambda v$$

where

$$Q = q - \frac{(pw)^{1/4}}{w}\left\{p\left[(pw)^{-1/4}\right]'\right\}'$$

Moreover, the change of variable preserves the L^2 norm: show that

$$\int_a^b |u(x)|^2 w(x)dx = \int_0^c |v(y)|^2 dy \qquad \text{where } c = \int_a^b \sqrt{\frac{w(t)}{p(t)}}dt$$

3. *Given one solution* $u(x)$ of (1), make a variation of constants ansatz $v(x) = c(x)u(x)$ and show that a second solution is given by

$$v(x) = u(x)\int^x \frac{1}{p(t)u(t)^2}\, dt$$

Are u and v linearly independent?