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SPECIAL FUNCTIONS

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1. Why more functions?

Many natural phenomena are described, quantitatively, as how the rates of change of different quantities are linked together. That is, are described by differential equations. Often times these equations are not linear, and their study is usually very difficult. For example, the Navier-Stokes equations are still not completely understood. Yet these equations are really important, since they describe the motion of fluids: movement of the air, much needed to predict the weather, or the movement of water, much needed to understand water currents for example.

Usually good linear approximations are sought, and can be found. Now linear differential equations are much more manageable, as we have already seen. Here is a nice, simple linear equation: $y'' = xy$. Airy arrived at this equation when studying the form of the intensity near an optical directional caustic, such as that of the rainbow, but this equation often arises as a good approximation to more complicated equations in certain instances. While is can be established mathematically that Airy's equation has very nice solutions, the problem is that these cannot be expressed in terms of the functions for which we have names (polynomials, trigonometric functions , exponential, etc.). So the solutions of this equation are considered new functions: the Airy functions.

Special functions are solutions of certain linear differential equations (there are other ways of generating new interesting functions).

2. The Helmholtz equation

In Sturm-Liouville theory the simplest, and most important equation is $y''+k^2y = 0$. Its generalization to two (or more) dimensions is the Helmholtz equation:

$$\nabla^2 u + k^2 u = 0 \tag{1}$$

where ∇^2 is the Laplacian. For functions of two variables, $u(x,y)$, the Laplacian is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

hence

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

2.1. **Vibrating membrane.** The two-dimensional analogue of the vibrating string is the vibrating membrane. When the edges clamped to be motionless, this is the drum! The Helmholtz equation was solved for many basic shapes in the 19th century, and it lead to many interesting linear differential equations defining new functions.

Let us consider the membrane in form of a circular disk of radius a. Then it is appropriate to introduce polar coordinates: if $x = r\cos\theta,\ y = r\sin\theta$ equation (1) takes the form:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + k^2 u = 0 \tag{2}$$

We impose the boundary condition that u vanishes on the boundary of the disc: $u(a,\theta) = 0$ for all θ. Let us try to find solutions using the method of separation of variables. That is, look for u in the form $u(r,\theta) = R(r)\Theta(\theta)$ where Θ must be periodic of period 2π. Then (2) becomes

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + k^2 R\Theta = 0$$

and, dividing by $R\Theta$ we obtain

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2 = 0$$

and to finally separate the variables, multiply be r^2:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} + k^2 r^2 = 0$$

hence

$$r^2\frac{R''}{R} + r\frac{R'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta}$$

The left side of the equation is a function of r, and the right side is a function of θ, therefore both must be constant: denoting this constant by n^2, we have

$$r^2\frac{R''}{R} + r\frac{R'}{R} + k^2 r^2 = n^2 \qquad \text{and} \qquad -\frac{\Theta''}{\Theta} = n^2$$

The equation for Θ:

$$\Theta'' + n^2\Theta = 0$$

has the general solution

$$\Theta = \alpha \cos n\theta + \beta \sin n\theta$$

Since periodic of period 2π, n must be an integer.

Now the equation for R is

$$r^2 R'' + rR' + (r^2 k^2 - n^2)R = 0$$

It is a linear equation whose solutions cannot be expressed in terms of already known functions. It can be simplified a bit if we substitute $R(r) = J_n(\rho)$ where $\rho = kr$, the equation becomes

$$\rho^2 J_n'' + \rho J_n' + (\rho^2 - n^2)J_n = 0 \tag{3}$$

which is called Bessel's equation, and its solutions, J_n, are the special functions called Bessel functions.

So the radial component R has the form

$$R(r) = \gamma J_n(\rho)$$

Now we are looking for solutions so that $R(a) = 0$. For this we need to know the zeroes of the Bessel functions...

We will see that J_n has infinitely many roots for each value of n, they are denoted by $\rho_{m,n}$. The boundary condition that R vanishes where $r = a$ will be satisfied if the corresponding wavenumbers k are given by

$$k_{m,n} = \frac{1}{a}\rho_{m,n}$$

The general solution u then takes the form of a doubly infinite sum of terms involving products of

$$\sin(n\theta) \text{ or } \cos(n\theta), \text{ and } J_n(k_{m,n}r)$$

These solutions are the modes of vibration of a circular drumhead.

3. The Bessel functions

The Bessel equation (3) is defined even if n is not an integer:

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{4}$$

where α is a complex parameter, called the *order* of the Bessel function.

The most important cases are when α is an integer or half-integer. Bessel functions for integer α are also known as cylinder functions or the cylindrical harmonics because they appear in the solution to Laplace's equation in cylindrical coordinates. Spherical Bessel functions with half-integer α are obtained when the Helmholtz equation is solved in spherical coordinates.

Because this is a second-order differential equation, there must be two linearly independent solutions. Depending upon the circumstances, however, various formulations of these solutions are convenient. We will present here the Bessel functions of the first kind, J_α and of the second kind, Y_α. Then any solution of (4) can be written asa linear combination: $y = c_1 J_\alpha + c_2 Y_\alpha$ for some constants c_1, c_2.

3.1. **Bessel functions of the first kind: J_α.** Let us look for solutions of (4) as a power series at $x = 0$. This is a singular point of the equation, since the coefficient of the higher derivative vanishes there. So it is not guaranteed that we can find an integer power series solution. However, this is a singularity of a benign type (it is called a *regular singularity*) and it is well established (called Frobenius Theory) that solutions will have expansions of the type: $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ for some complex number r. We may assume $a_0 = 1$, since if $a_0 = 0$ then we just increase r, and if a_0 is another nonzero number, then $y(x)/a_0$ is also a solution of the same equation, since the equation is linear.

To determine r, we substitute $y(x) = x^r + a_1 x^{r+1} + ...$ in the equation, and obtain:

$$x^2\left[r(r-1)x^{r-2} + a_1(r+1)rx^{r-1} + ...\right] + x\left[rx^{r-1} + a_1(r+1)x^r + ...\right] + x^2\left(x^r + a_1 x^{r+1} + ...\right) + \alpha^2\left(x^r + a_1 x^{r+1} + ...\right) = 0 \tag{5}$$

Expanding, we see that the smallest power of x is x^r, and its coefficient is $r(r-1) + r + \alpha^2$. A power series is zero when all its coefficients are zero, so we must have $r(r-1) + r + \alpha^2 = 0$ hence $r = \pm\alpha$.

By definition, J_α is the solution with $r = \alpha$. The other coefficients of the series of y can be calculated similarly, recursively. It turns out that

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

where $\Gamma(z)$ is the gamma function, a shifted generalization of the factorial function to non-integer values, defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

It can be shown that when $\alpha = n$ is an integer, $J_n(x)$ has an infinite number of zeroes for $x > 0$.

3.2. **Bessel functions of the second kind: Y_α.** For non-integer α

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}$$

and In the case of integer order n, the function is defined by taking the limit as a non-integer ? tends to n:

$$Y_n(x) = \lim_{\alpha \to n} Y_\alpha(x).$$