Math 5102 **Homework 1**

due Fri Jan 25

1. The set of sequences $x = (x_k)_{k\in\mathbb{Z}_+} = (x_1, x_2, x_3, \ldots)$

$$\mathbb{R}^{\mathbb{Z}_+} = \left\{ x = (x_k)_{k\in\mathbb{Z}_+} \,|\, x_k \in \mathbb{R} \text{ for all } k \in \mathbb{Z}_+ \right\}$$

with the usual component-wise operations:

$$x + y = (x_k + y_k)_{k\in\mathbb{Z}_+}, \ \ \alpha x = (\alpha x_k)_{k\in\mathbb{Z}_+} \text{ for all } \alpha \in \mathbb{R}$$

is known to be a vector/linear space. (It is in fact very easy to check, just like for $\mathbb{R}^n$.)

Show that $\ell^2(\mathbb{R}) = \{x \in \mathbb{R}^{\mathbb{Z}_+} \mid \sum_{k=0}^{\infty} x_k^2 < \infty\}$ is a subspace of $\mathbb{R}^{\mathbb{Z}_+}$.

2. For which complex numbers a, b the sequences $x = (a\, b^k)_{k\in\mathbb{Z}_+}$ belong to $\ell^2(\mathbb{C})$? (Recall that if a series $\sum_{k=0}^{\infty} c_k$ converges, then necessarily $\lim_{k\to\infty} c_k = 0$.)

3. a) Show that the functions f_n, $n \in \mathbb{Z}$ defined as $f_n(x) = e^{inx}$ form an orthogonal set in the complex Hilbert space $L^2[0, 2\pi]$.

b) Normalize them to obtain an orthonormal set.

4. Let a, b be real numbers. Show that if $f \in L^2[a, b]$ then $\int_a^b |f(t)|\, dt < \infty$ (in other words, $f \in L^1[a, b]$). (Hint: Cauchy-Schwarz)

Note: this is not true on infinite intervals!

5. The converse of the above is not true: give an example of a function for with $\int_0^1 |f(t)|\, dt < \infty$ but with $f \notin L^2[0, 1]$.