Hilbert spaces are defined by abstracting the structure needed for the properties above to hold.

21 Norms, seminorms and inner products

Definitions

Given a vector space V over a subfield $\mathbb{K}$ of the complex numbers, a **norm** on V is a nonnegative-valued scalar function $p : V \to [0, +\infty)$ with the following properties: for all $a \in \mathbb{K}$ and all $u, v \in V$,

1.

$$p(u + v) \leqslant p(u) + p(v)$$

(p is subadditive, or: p satisfies the triangle inequality).

2. $p(av) = |a|p(v)$ (p is absolutely homogeneous, or absolutely scalable).

3. If $p(v) = 0$ then $v = 0$ is the zero vector (p is positive definite).

A **seminorm** on V is a function $p : V \to \mathbb{R}$ with the properties 1 and 2 above.

Every vector space V with seminorm p induces a normed space V/W, called the quotient space, where W is the subspace of V consisting of all vectors v in V with $p(v) = 0$ (check that W is a subspace). The induced norm on V/W is defined by:

$$p(W + v) = p(v)$$

Two norms (or seminorms) p and q on a vector space V are equivalent if there exist two positive constants c_1 and c_2 such that $c_1 q(v) \leqslant p(v) \leqslant_2 q(v)$ for every vector v in V.

A topological vector space is called **normable** (seminormable) if the topology of the space can be induced by a norm (seminorm).

An inner product $\langle x, y \rangle$ over a vector space is a complex-valued function that satisfies the following properties:

1. The inner product of a pair of elements is equal to the complex conjugate of the inner product of the swapped elements:

$$\langle y, x \rangle = \overline{\langle x, y \rangle}\,. \tag{1}$$

2. The inner product is linear in its first argument. For all complex numbers a and b,

$$\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle\,. \tag{2}$$

3. The inner product of an element with itself is positive definite:

$$\langle x, x \rangle \geq 0 \tag{3}$$

where the case of equality holds precisely when $x = 0$. It follows from properties (1) and (2) that a complex inner product is antilinear in its second argument, meaning that

$$\langle x, ay_1 + by_2 \rangle = \bar{a}\langle x, y_1 \rangle + \bar{b}\langle x, y_2 \rangle\,. \tag{1+2}$$

It is easily checked that the quantity $\|x\| := \sqrt{\langle x, x \rangle}$ is a norm on $\mathcal{H}$.

22 Hilbert spaces

A Hilbert space $\mathcal{H}$ is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product (that is, the distance between $x, y \in \mathcal{H}$ is $\|x - y\|$).

Theorem 255 (Cauchy-Schwarz). *For any $x, y \in \mathcal{H}$ we have $|\langle x, y\rangle| \leqslant \|x\|\|y\|$.*

We have equality iff x, y are linearly dependent.

Proof. There is nothing to prove if $x = 0$ or $y = 0$, so we assume this is not the case. Note now that for any z, $\|z\| \geqslant 0$. In particular, for any $a \in \mathbb{C}$ we have

$$0 \leqslant \|x - ay\|^2 = \langle x, x\rangle + |a|^2\langle y, y\rangle - 2\Re(a\langle y, x\rangle) = f(a) \tag{68}$$

We write $\langle y, x\rangle = |\langle x, y\rangle|e^{i\alpha}$ (if $\langle x, y\rangle = 0$ any α works). For $t \in \mathbb{R}$,

$$f(te^{-i\alpha}) = \langle x, x\rangle + t^2\langle y, y\rangle - 2t|\langle x, y\rangle| \geqslant 0$$

is a nonnegative quadratic polynomial in t and thus it has nonpositive discriminant: $4|\langle x, y\rangle|^2 - 4\langle x, x\rangle\langle y, y\rangle \leqslant 0$, which is what we intended to prove. □

Proposition 256. *The function $x \to \|x\| = \sqrt{\langle x, x\rangle}$ is a norm.*

Proof. First of all, by the definition of the inner product and norm, $\|x\| = 0$ iff $x = 0$ and $\|\lambda x\| = |\lambda|\|x\|$. To prove the triangle inequality, we note that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re\langle x, y\rangle \leqslant \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

□

22.1 Example: ℓ^2

Definition 257. *Let*

$$\ell^2 = \{x : \mathbb{N} \to \mathbb{C} | \|x\|^2 = \sum_{i\in\mathbb{N}} |x(i)|^2 =: \sum_{i\in\mathbb{N}} |x_i|^2 < \infty\}$$

and define

$$\langle x, y\rangle = \sum_{i\in\mathbb{N}} x_i\overline{y_i}$$

which, by Cauchy-Schwarz is well-defined on ℓ^2.

Proposition 258. *ℓ^2 is complete thus it is a Hilbert space.*

Proof. If $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^2, then for every $i \in \mathbb{N}$ the number sequence of the ith component $\{(x_n)_i\}_{n\in\mathbb{N}}$ is Cauchy (indeed $|(x_n)_i - (x_m)_i|^2 \leqslant \|x_n - x_m\|^2$). Let $y_i = \lim_n (x_n)_i$.

We need to show that $y \in \ell^2$, and y is the limit of x_n. Let n_0 be s.t. $(\forall n, m \geqslant n_0), (\|x_n - x_m\| < 1)$. The triangle inequality implies that $\forall n \geqslant n_0, \|x_n\| \leqslant C$ where $C = 1 + \|x_{n_0}\|$. It follows that, for all n, $\sum_{i=1}^{n} |y_i|^2 = \lim_{k\to\infty} \sum_{i=1}^{n} |(x_k)_i|^2 \leqslant C$ and since $|y_i|$ are positive and the sums are bounded, the sum converges to $\|y\|^2 \leqslant C$, that is $y \in \ell^2$. Similarly, since $\lim_{k\to\infty} \sum_{i=0}^{n} |(x_k)_i - y_i|^2 = 0$ for any n, by the above we can use dominated convergence to show that $\|x_k - y\| \to 0$. □

Proposition 259. *The inner product is a continuous function from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$. In particular, if $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to (x, y) then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Proof. By Cauchy-Schwarz, as $\|h_1\|, \|h_2\| \to 0$ we have,

$$|\langle x + h_1, y + h_2\rangle - \langle x, y\rangle| = |\langle x, h_2\rangle + \langle h_1, y\rangle + \langle h_1, h_2\rangle| \leqslant \|x\|\|h_1\| + \|y\|h_2\| + \|h_1\|\|h_2\| \to 0$$

□

Proposition 260 (The parallelogram law)**.**

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \tag{P.L.}$$

Proof. A straightforward calculation, see (68) above. □

We see that a Hilbert space is a complete normed space where the norm comes from an inner product. A natural and important question arises: given a norm, can we always define an inner product that induces the norm? The answer is no and, remarkably, (P.L.) is the necessary and sufficient condition for the norm to come from an inner product.

Proposition 261. *Let $\mathcal{S}$ be a complete normed space, with norm $\|\cdot\|$. Then the norm comes from an inner product iff it satisfies the parallelogram law.*

Proof. We have already shown that an inner-product-induced norm satisfies the parallelogram law. In the opposite direction, a calculation assuming the existence of an inner product leads the following explicit formula for the inner product, called *the polarization identity*:

$$\langle x,\ y\rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right)\ \forall\, x, y \in \mathcal{H}$$

(for Hilbert spaces over $\mathbb{R}$ it has the form $\langle x,\ y\rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right)$).

It remains to check that *assuming the parallelogram law* the formula above defines an inner product (meaning: with properties (1)...(3) above). This is elementary, but by no means trivial! See original proof by P. Jordan & J. von Neumann, Annals, 1935. A geometric argument based on Euclid's three line theorem is N. Falkner, MAA 100,3, (1993). □

Corollary 262. *The inner product is continuous.*

22.2 Orthogonality

The notion of orthogonality, $x \perp y$ if by definition $\langle x, y\rangle = 0$ obviously extends to general Hilbert spaces. So does the following

Proposition 263 (Pythagorean equality)**.** *If $x_1, ..., x_n$ are pairwise orthogonal, then*

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Proof.

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \left\langle \sum_{i=1}^{n} x_i \middle| \sum_{i=1}^{n} x_i \right\rangle = \sum_{i,j\leqslant n} \langle x_i, x_j\rangle = \sum_{i=1}^{n}\langle x_i, x_i\rangle = \sum_{i=1}^{n} \|x_i\|^2$$

□

Definition 264. *The linear span (linear hull, or simply span) of a set of vectors $S = \{v_\alpha : \alpha \in A\}$ over the scalar field $\mathbb{K}$ is*

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \middle| k \in \mathbb{N}, v_i \in S, \lambda_i \in \mathbb{K} \right\}$$

22.3 The Gram-Schmidt process

Given a family $\{x_i\}_{i\in\mathbb{N}}$ of linearly independent vectors, we can construct, from them, an orthonormal family $\{e_i\}_{i\in\mathbb{N}}$, inductively: start with $v_1 = x_1$; let c be s.t. $v_2 = cv_1 + x_2 \perp v_1$ (which gives $c = -\langle x_2, v_1\rangle/\|v_1\|^2$). Having constructed $v_1, ..., v_n$ pairwise orthogonal, choose $c_{n1}, ..., c_{nn}$ s.t. $v_{n+1} = c_{n1}x_1 + ... + c_{nn}x_n + x_{n+1}$ is orthogonal on $v_1, ..., v_n$ (this is a linear system with nonzero determinant). Then $\{v_i\}_{i\in\mathbb{N}}$ is an orthogonal family with the property that span($\{x_1, ..., x_n\}$=span($\{v_1, ..., v_n\}$ for all n. All these v_i are nonzero vectors, and an orthonormal family is simply given by $e_i = v_i/\|v_i\|$.

Proposition 265. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a set of vectors in $\mathcal{H}$ and let $\mathcal{V}$ be the closure of span$(\{x_i\}_{i\in\mathbb{N}})$. We assume that $\mathcal{V}$ is infinite dimensional (the finite dimensional case is similar, and simpler). Then there exists an orthonormal set $\{e_i\}_{i\in\mathbb{N}}$ such that $\mathcal{V}$ is the closure of span$(\{e_i\}_{i\in\mathbb{N}})$.*

Proof. We can assume w.l.o.g. that $\{x_i\}_{i\in\mathbb{N}}$ are linearly independent, since we can inductively eliminate the dependent vectors without affecting the span. With $\pi_v x = \dfrac{\langle v, x\rangle}{\langle v, v\rangle} v$, the Gram-Schmidt procedure is:

$$\begin{aligned}
v_1 &= x_1, & e_1 &= \frac{v_1}{\|v_1\|}\\
v_2 &= x_2 - \pi_{v_1}x_2, & e_2 &= \frac{v_2}{\|v_2\|}\\
&\vdots & &\vdots\\
v_k &= x_k - \sum_{j=1}^{k-1}\pi_{v_j}x_k, & e_k &= \frac{v_k}{\|v_k\|}.\\
&\dots
\end{aligned}$$

Note that, for all $k \in \mathbb{N}$, we have span$\{x_1, ..., x_k\}$ =span$\{v_1, ..., v_k\}$, implying that span$\{x_i : i \in \mathbb{N}\}$ = span$\{v_i : i \in \mathbb{N}\}$ =span$\{e_i : i \in \mathbb{N}\}$, hence the closures of these spans also coincide. □

22.4 A very short proof of Cauchy-Schwarz

Proof. In case x, y are linearly dependent the inequality is an equality. Otherwise wlog, we may assume $\|x\| = \|y\| = 1$. Define $e_1 = x$ and let e_2 be obtained from e_1 and y by Gram-Schmidt.

Then $y = y_1 e_1 + y_2 e_2$ and

$$|\langle x, y\rangle|^2 = |y_1|^2 \leqslant |y_1^2| + |y_2^2| = 1$$

□

22.5 Orthogonal projections

In the following, $\mathcal{H}$ is a Hilbert space.

Definition 266 (The orthogonal complement of a space). *If $\mathcal{S}$ is a subspace of $\mathcal{H}$, then its orthogonal complement, $\mathcal{S}^\perp$ is the* **closed linear subspace** *(check these properties!) of $\mathcal{H}$ defined by*

$$\mathcal{S}^\perp = \{x \in \mathcal{H} : (\forall y \in \mathcal{S})(\langle x, y\rangle = 0)\}$$

The sum of two subspaces $\mathcal{V}_1, \mathcal{V}_2$ is defined by

$$\mathcal{V}_1 + \mathcal{V}_2 = \{v_1 + v_2 : v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\}$$

If $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$, then the sum is **direct**, *written $\mathcal{V}_1 \oplus \mathcal{V}_2$ and for any $x \in \mathcal{V}_1 \oplus \mathcal{V}_2$ there is a unique pair v_1, v_2, $v_i \in \mathcal{V}_i$ s.t. $x = v_1 + v_2$ (check!).*

Lemma 267 (Orthogonality and an extremal property). *If $\mathcal{M}$ is a* **closed** *subspace of $\mathcal{H}$, then*

1. *there is a unique $\mu \in \mathcal{M}$ s.t. $\forall m \in \mathcal{M}$, $m \neq \mu$, $\|x - m\| > \|x - \mu\|$.*

2. *If μ is as in 1., then $x - \mu \in \mathcal{M}^\perp$. Conversely, if $y \in \mathcal{M}$ is s.t $x - y \in \mathcal{M}^\perp$, then $y = \mu$.*

Proof. 1. Let $d = \inf_{y\in\mathcal{M}} \|x - y\|$. Since $0 \in \mathcal{M}$, $d \leqslant \|x\|$. Thus there is a sequence $\{y_m\}_{m\in\mathbb{N}}$ in $\mathcal{M}$ s.t. $d - \|x - y_m\| \to 0$. We show that this sequence is convergent to some $\mu \in \mathcal{M}$. Note that this proves both existence and uniqueness of a $\mu \in \mathcal{M}$ s.t. $\|x - \mu\|$ is minimal.

Since $\mathcal{M}$ is a closed subspace of the complete Hilbert space $\mathcal{H}$, it suffices to show that $\{y_m\}_{m\in\mathbb{N}}$ is Cauchy. Here we use the parallelogram law:

$$\|y_m - y_n\|^2 = \|(x - y_n) - (x - y_m)\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - \|2x - y_n - y_m\|^2$$
$$= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\|x - \tfrac{1}{2}(y_n + y_m)\|^2 \leqslant 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d^2 \to 0 \text{ as } n, m \to \infty$$

2. Next we show that $x - \mu \in \mathcal{M}^\perp$. Let $y \in \mathcal{M}$ be arbitrary and define $m = \mu - \alpha y$. Then

$$\|x - m\|^2 = \|x - \mu\|^2 + |\alpha|^2\|y\|^2 + 2\Re(\overline{\alpha}\langle x - \mu, y\rangle$$

Assume $\langle x - \mu, y\rangle \neq 0$, write $\langle x - \mu, y\rangle = |\langle x - \mu, y\rangle| e^{i\phi}$ and choose $\alpha = -|\alpha|e^{i\phi}$. We get

$$\|x - m\|^2 = \|x - \mu\|^2 + |\alpha|^2\|y\|^2 - 2|\alpha||\langle x - \mu, y\rangle| < d^2$$

if $|\alpha| < 2|\langle x - \mu, y\rangle|\|y\|^{-2}$, a contradiction.

Finally, if $y \in \mathcal{M}$ is s.t. $x - y \in \mathcal{M}^\perp$, then in particular $x - y \perp y - \mu$, hence

$$\|x - \mu\|^2 = \|x - y\|^2 + \|y - \mu\|^2 = d^2 + \|y - \mu\|^2 \Rightarrow y = \mu$$

□

Corollary 268. *If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then any $x \in \mathcal{H}$ can be uniquely written as $x = m + m^\perp$ with $m \in \mathcal{M}$ and $m^\perp \in \mathcal{M}^\perp$. Hence $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.*

Definition 269 (Orthogonal projections)**.** *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ and $\mathcal{M}^\perp$ its orthogonal complement. Let $x = m + m^\perp$ with $m \in \mathcal{M}$ and $m^\perp$ in $\mathcal{M}^\perp$and define*

$$\pi_{\mathcal{M}} x = m; \ \pi_{\mathcal{M}^\perp} x = m^\perp$$

The operator $\pi_{\mathcal{M}}$ is called the orthogonal projection on $\mathcal{M}$.

Proposition 270. *1. The operator $\pi_{\mathcal{M}}$ is the identity on $\mathcal{M}$, and is idempotent: $(\pi_{\mathcal{M}})^2 = \pi_{\mathcal{M}}$.*
2. Furthermore, $\pi_{\mathcal{M}^\perp}$ is the orthogonal projection on $\mathcal{M}^\perp$ and $(\pi_{\mathcal{M}^\perp})^2 = \pi_{\mathcal{M}^\perp}$.
3. We have $(\mathcal{M}^\perp)^\perp = \mathcal{M}$.

Proof. 1. If $t \in \mathcal{M}$, then the unique decomposition of t in $\mathcal{M} \oplus \mathcal{M}^\perp$ is $t = t + 0$ and thus $\pi_{\mathcal{M}} t = t$. Since, by definition, $\pi_{\mathcal{M}} x \in \mathcal{M}$ for any $x \in \mathcal{H}$, we have $(\pi_{\mathcal{M}})^2 = \pi_{\mathcal{M}}$.

2. The space $\mathcal{M}^\perp$ is also linear and closed, because of the continuity of the scalar product. Now, by the uniqueness of the decomposition $x = m + m^\perp$ and the fact that $m \perp \mathcal{M}^\perp$, Lemma 267 implies that $m^\perp = \pi_{\mathcal{M}^\perp} x$.

3. Clearly any vector in $\mathcal{M}$ is in $(\mathcal{M}^\perp)^\perp$. Conversely, $x \in (\mathcal{M}^\perp)^\perp \Rightarrow \pi_{\mathcal{M}^\perp} x = 0 \Rightarrow x = \pi_{\mathcal{M}} x \in \mathcal{M}$. □

Corollary 271. *1. The closure of a subspace $\mathcal{M} \subset \mathcal{H}$ is $\overline{\mathcal{M}} = (\mathcal{M}^\perp)^\perp$.*
If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then

$$\pi_{\mathcal{M}} + \pi_{\mathcal{M}^\perp} = I$$

where I is the identity on $\mathcal{H}$.

22.6 Bessel's inequality, Parseval's equality, orthonormal bases

Theorem 272 (Bessel's inequality)**.** *Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal sequence in $\mathcal{H}$. Then*

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leqslant \|x\|^2$$

Proof. Let $\mathcal{H}_n = \text{span}(\{e_1, ..., e_n\}) := \{c_1 e_1 + \cdots c_n e_n | c_i \in \mathbb{C}\}$. Clearly, $\mathcal{H}_n$ is a closed subspace of $\mathcal{H}$. We can then write

$$x = \pi_{\mathcal{H}_n} x + x^\perp = \sum_{i=1}^{n} \langle x, e_i \rangle e_i + x^\perp$$

and, by the Pythagorean equality,

$$\|x\|^2 = \|\pi_{\mathcal{H}_n} x\|^2 + \|x^\perp\|^2 \geqslant \|\pi_{\mathcal{H}_n} x\|^2 = \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$$

Since this holds for any n, taking $n \to \infty$, the result follows. □

Corollary 273. *Let $\{e_\alpha\}_{\alpha\in A}$ be an orthonormal family in $\mathcal{H}$. Then, for any $x \in \mathcal{H}$*

$$\sum_{\alpha\in A} |\langle x, e_\alpha\rangle|^2 := \sup_{\alpha_1,\dots,\alpha_n\in A, n\in\mathbb{N}} \sum_{i=1}^{n} |\langle x, e_{\alpha_n}\rangle|^2 \leqslant \|x\|^2$$

and the set $\{\alpha \in A : \langle x, e_{\alpha_n}\rangle \neq 0\}$ is countable.

Proof. Only countability needs to be shown. It is well known however that an uncountable sum of strictly positive numbers is infinity. □

Definition 274. *An orthonormal set $\{e_\alpha\}_{\alpha\in A}$ is called* **an orthonormal basis (Hilbert space basis)** *in the Hilbert space $\mathcal{H}$ if any $x \in \mathcal{H}$ can be written as a finite or countable infinite linear combination*

$$x = \sum_{k=1}^{\infty} c_k e_{\alpha_k}$$

Note 275.

1. *An orthonormal basis is* not *a vector space basis (unless $\mathcal{H}$ is finite-dimensional).*
2. *Using Bessel's inequality, Cauchy-Schwarz and dominated convergence, we see that $c_k = \langle x, e_k\rangle$, hence*
$$x = \sum_{k=1}^{\infty} \langle x, e_{\alpha_k}\rangle e_{\alpha_k} \tag{69}$$
3. *If $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis and $\langle x, e_\alpha\rangle = 0$ for all $\alpha \in A$, then $x = 0$.*

Proposition 276. *Any separable Hilbert space $\mathcal{H}$ has a countable orthonormal basis.*

Proof. Let $\{v_i\}_{i\in\mathbb{N}}$ be a countable dense set in $\mathcal{H}$. The closure of the span of $\{v_i\}_{i\in\mathbb{N}}$ is, of course, $\mathcal{H}$, and so is the span of $\{e_i\}_{i\in\mathbb{N}}$, constructed by Gram-Schmidt. Note that, by Bessel's inequality,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leqslant \|x\|^2 \Rightarrow \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k \in \mathcal{H}$$

The difference $x - \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$ is orthogonal to all the e_k, $k \in \mathbb{N}$, thus, by Note 275 3, is zero. □

Theorem 277. *If $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal set in a separable Hilbert space $\mathcal{H}$, then the following are equivalent:*

a. *(**Completeness**) If $\forall j$, $\langle x, e_j\rangle = 0$, then $x = 0$.*

b. *(**Parseval's identity**) $\forall x \in \mathcal{H}$, $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2$.*

c. *$\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$.*

Proof. (b.$\Rightarrow$ a.) is clear.

(a.$\Rightarrow$ c.) We see that $x - \sum_{k\in\mathbb{N}} \langle x, e_k\rangle e_k$ is orthogonal to all $e_j, j \in \mathbb{N}$, and thus it is zero.

(c.$\Rightarrow$ b.) This is simply the Pythagorean theorem plus the continuity of the norm. □

Exercise 58. *Let $\mathcal{H}$ be a Hilbert space, separable or not, and let $\{e_\alpha\}_{\alpha\in A}$ be an orthonormal set in $\mathcal{H}$. Then, the following statements are equivalent.*

1. (**Completeness condition**) *$\forall \alpha$, $\langle x, e_\alpha \rangle = 0$ holds iff $x = 0$*
2. (**Density condition**) *The span of $\{e_\alpha\}_{\alpha \in A}$ is dense in $\mathcal{H}$.*
3. (**Orthonormal basis condition**) *For any $x \in \mathcal{H}$, $\langle x, e_\alpha \rangle = 0$ except for a countable set $(e_{\alpha_k})_{k \in \mathbb{N}}$ and*
$$x = \sum_{k \in \mathbb{N}} \langle x, e_{\alpha_k} \rangle e_{\alpha_k}$$
4. (**Maximality condition**) *If $\{e'_\beta\}_{\beta \in B}$ is an orthonormal set in $\mathcal{H}$ which contains $\{e_\alpha\}_{\alpha \in A}$, then $\{e'_\beta\}_{\beta \in B} = \{e_\alpha\}_{\alpha \in A}$.*
5. (**Parseval's identity condition**) *$\forall x \in \mathcal{H}$, $\|x\|^2 = \sum_{\alpha \in A} |\langle x, e_\alpha \rangle|^2$.*

Exercise 59. Show that the $S = \{1, x, x^2, ...\}$ is a linearly independent set in $\mathcal{H} = L^2([-1,1])$ whose span is dense in $\mathcal{H}$. Thus Gram-Schmidt produces an orthonormal system of polynomials P_n out of S. ($\sqrt{\frac{2}{2n+1}} P_n$ are the Legendre polynomials.) Thus, any $f \in \mathcal{H}$ can be written as $f = \sum_{k \in \mathbb{N}} c_k P_k$. Show that, although as mentioned, the span of S is dense in $\mathcal{H}$, the set $\{f \in \mathcal{H} : f = \sum_{k \in \mathbb{N}} c_k x^k\}$ is a strict subspace of $\mathcal{H}$. Is it closed? Can you identify it?

Note 278. Nonseparable Hilbert spaces rarely occur in applications. A prototypical example is
$$\ell^2(A) := \left\{ f : A \to \mathbb{C} \,\middle|\, \sum_{\alpha \in A} |f(\alpha)|^2 < \infty \right\}$$
when A is not countable.

Also, Corollary 273 shows that *even in non-separable Hilbert spaces* we only need a countable family *at a time*.

Theorem 279. *In a Hilbert space $\mathcal{H}$, any orthonormal set S is contained in an orthonormal basis for $\mathcal{H}$.*

Proof. Let $\mathcal{E}$ be the family of all orthonormal sets containing S ordered by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{E}$, then it has a maximal element, namely the union of the sets in $\mathcal{C}$ as it is easily verified. Now, Zorn's Lemma implies that $\mathcal{E}$ has a maximal element, which by Exercise 58 4, is a basis for $\mathcal{H}$. □

An example of a Hilbert basis in ℓ^2 is the set $e_k = (0, .., 1, 0...)$, with 1 in the kth position.

Definition 280. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $U : \mathcal{H}_1 \to \mathcal{H}_2$ be linear and norm preserving, that is $\|Ux\|_2 = \|x\|_1$ for all $x \in \mathcal{H}_1$. Then U is called an isometry.*

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $U : \mathcal{H}_1 \to \mathcal{H}_2$ be linear, inner product preserving, $\langle Ux, Uy \rangle = \langle x, y \rangle$, and onto. Then U is called unitary.

Proposition 281. *U is unitary iff it is an isometry and onto.*

Note 282. *Unitary maps are isomorphisms, w.r.t the structure of a Hilbert space.*

Proof. If U is unitary, then $\|Ux\|^2 = \langle Ux, Ux \rangle = \langle x, x \rangle = \|x\|^2$. Conversely, the polarization identity shows that any isometry preserves the inner product. □

Proposition 283 (Any two separable Hilbert spaces are isomorphic). *Any separable Hilbert space $\mathcal{H}$ is isomorphic to ℓ^2.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$. Define $U : \mathcal{H} \to \ell^2$ by $U(x) = (\langle x, e_1\rangle, ..., \langle x, e_n\rangle, ...)$ and check that this is an isometry. □

For a nonseparable Hilbert space with a Hilbert basis $\{e_\alpha\}_{\alpha\in A}$, a similar statement holds, except $\ell^2 = \ell^2(\mathbb{N})$ is replaced by the more general $\ell^2(A)$, for an adequate set A.

23 Normed vector spaces

Definition 284. 1. *A vector space V endowed with a norm $\|\cdot\|$ is called a* **normed vector space***.*

2. *Two norms $\|\|_1, \|\|_2$ on the vector space V are* **equivalent** *if there exist two positive constants $c_1\, c_2$ s.t. $\forall v \in V$, $c_1\|v\|_1 \leqslant \|v\|_2 \leqslant c_2\|v\|_1$.*

3. *A* **Banach space** *is a normed space which is complete w.r.t the norm topology, that is the distance between x, y is $\|x - y\|$.*

4. *A series $\sum_{n\in\mathbb{N}} v_n$ of vectors in a normed space is* **absolutely convergent** *if $\sum_{n\in\mathbb{N}} \|v_n\|$ converges.*

Proposition 285. *An absolutely convergent series $\sum_{n\in\mathbb{N}} v_n$ is Cauchy. In the opposite direction, if $\sum_{n\in\mathbb{N}} v_n$ is Cauchy, then there exists a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ s.t. $n_1 = 1$ and s.t., with $w_i = \sum\limits_{n_i\leqslant j<n_{i+1}} v_j$, the series $\sum\limits_{i\in\mathbb{N}} w_i$ is absolutely convergent.*

Proof. Assume $\sum\limits_{n\in\mathbb{N}} v_n$ is absolutely convergent, and let $\epsilon > 0$. Then, the series of norms $\sum\limits_{n\in\mathbb{N}} \|v_n\|$ is Cauchy and there is an n_0 s.t. for all $m \geqslant n \geqslant n_0$ we have

$$\left\|\sum_{j=n}^{m} v_j\right\| \leqslant \sum_{j=n}^{m} \|v_j\| < \epsilon$$

Hence $\sum\limits_{n\in\mathbb{N}} v_n$ is Cauchy.

In the opposite direction, assume $\sum_{n\in\mathbb{N}} v_n$ is Cauchy. Choose $\epsilon_i = 2^{-i}, i \in \mathbb{N}$, let $n_1 = 1$ and, inductively for $i > 1$, define $n_i > n_{i-1}$ so that $\forall n \geqslant m \geqslant n_i$ we have $\left|\sum\limits_{m\leqslant j\leqslant n} v_i\right| \leqslant \epsilon_i$. Defining $w_i = \sum\limits_{n_i\leqslant j<n_{i+1}} v_j$, the result follows. □

Theorem 286. *A normed vector space V is complete iff every absolutely convergent series in V converges.*

Proof. Note first that, in a linear space, every Cauchy *sequence* converges iff every Cauchy *series* converges. Let $\sum_{n\in\mathbb{N}} v_n$ be Cauchy in V. With the construction of Proposition 285, the series $w_i = \sum\limits_{n_i\leqslant j<n_{i+1}} v_j$ is absolutely convergent, thus convergent, to some $v \in V$. Then, for any integer $m \in [n_i, n_{i+1})$, $\left\|v - \sum\limits_{j\geqslant m} v_j\right\| \leqslant \left\|\sum\limits_{m\leqslant j<n_i} v_j\right\| + \left\|v - \sum\limits_{k\geqslant i} w_i\right\|$ hence $\sum\limits_{n\in\mathbb{N}} v_n$ also converges to v. □

Proposition 287. *1. If V_1, V_2 are normed vector spaces, then the product space $V_1 \times V_2$ is a normed vector space under the product norm defined, for $v_i \in V_i$, by $\|(v_1, v_2)\| := \|v_1\|_1 + \|v_2\|_2$.*

2. If V' is a closed linear subspace of V, then the quotient space V/V' is a normed vector space under the quotient norm

$$\|v + V'\| := \inf_{v' \in V'} \{\|v + v'\|\} \tag{70}$$

Proof. This is easy to check. □

Note 288. *Recall that all norms in $\mathbb{C}^n$ are equivalent. Therefore, the product norm is equivalent to many other choices, s.a.* $\max\{\| \|_1, \| \|_2\}$.

23.1 Functionals and linear operators

Definition 289. *1. A **linear operator** (or map) between two vector spaces V_1, V_2 over the same scalar field is a function $L : V_1 \to V_2$ which satisfies $L(ax + by) = aLx + bLy$ for all $x, y \in V_1$ and all scalars a, b.*

*2. A linear operator having the scalar field as the target space is called **linear functional**.*

*3. An operator $L : V_1 \to V_2$ between two normed vector spaces V_1, V_2 is called **bounded** if there exists a constant $C \in [0, \infty)$ s.t., for all $v \in V_1$ we have*

$$\|Lv\|_2 \leqslant C\|v\|_1 \tag{71}$$

4. If V_1, V_2 are normed vector spaces, then $\mathcal{L}(V_1, V_2)$ denotes the space of linear bounded operators from V_1 to V_2.

*5. A **Banach algebra** is a Banach space which is an algebra for which the norm of the product is bounded by the product of the norms, that is, $\|xy\| \le \|x\| \, \|y\|$.*

*6. If X is a normed vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then the space of its bounded linear functionals $X^* := \mathcal{L}(X, \mathbb{K})$ is the very important **dual of** X.*

Note 290. *1. $L \in \mathcal{L}(V_1, V_2)$ iff L is linear from V_1 to V_2 and*

$$\|L\| := \sup_{\|v\|_1 = 1} \|Lv\|_2 < \infty \tag{72}$$

*The quantity $\|L\|$ is called the **norm of the linear map** L.*

2. $\mathcal{L}(V_1, V_2)$ is a normed space with the operator norm.

Proposition 291. *Let Y be a complete normed space and X a normed space. Then:*
a) $\mathcal{L}(X, Y)$ is a complete normed space, and
b) $\mathcal{L}(Y, Y)$ with the operator norm is a Banach algebra.

Proof. a) If $(T_n)_n$ is a Cauchy sequence in $\mathcal{L}(X, Y)$, then for any $x \in X$ $(T_n x)_n$ is Cauchy in Y, thus convergent. Now, $Tx = \lim_n T_n x$ defines a linear operator $T \in \mathcal{L}(X, Y)$, since it is easy to check that $\|T - T_n\| \to 0$ and $\|T_n\| \to \|T\|$ as $n \to \infty$.