Homework 3, Elementary Fourier series

Problems: **26 (p. 124) 49, 55 (p. 134-135) 67 (p. 178)**

a. Check: a1. the sequence $\{e_k\}_{k\in\mathbb{Z}} = \{\frac{e^{ikx}}{\sqrt{2\pi}}\}_{k\in\mathbb{Z}}$ is an orthonormal system in $\mathcal{H} = L^2(-\pi,\pi)$. a2. the series

$$1 + \sum_{k\in\mathbb{N}} \left(\frac{e^{ikx}}{ik} + \frac{e^{-ikx}}{-ik} \right)$$

is (conditionally) convergent for any $0 \neq x \in (-\pi,\pi)$. Also, note that

$$\sum_{k=-N}^{N} e^{ikx} = \frac{\sin(Nx + \frac{1}{2}x)}{\sin\frac{x}{2}}$$

b. Let $f \in C^1(-\pi,\pi)$ and take $[a,b] \subset (-\pi,\pi)$. Show (e.g. by integration by parts) that

$$\lim_{n\to\infty} \int_a^b e^{ins} f(s)ds = 0$$

c. Show (e.g. using a. and b.) that for any $(a,b) \subset (-\pi,\pi)$ s.t. $0 \notin (a,b)$,

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \int_a^b e^{-iks} ds = \lim_{N\to\infty} \int_a^b \frac{\sin(Ns + \frac{1}{2}s)}{\sin\frac{s}{2}} ds = 0 \qquad (*)$$

Show that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \int_{-\pi}^{\pi} e^{iks} ds = 2\pi \text{ and, by (*), } \forall\delta \in (0,\pi) \lim_{N\to\infty} \sum_{k=-N}^{N} \int_{-\delta}^{\delta} e^{iks} ds = 2\pi$$

d. With $f \in \mathcal{H}$ s.t. $f_a(x) = f(x+a) \in \mathcal{H}$ check that

$$\sum_{k\in\mathbb{Z}} \langle f_a, e_k\rangle e_k = \sum_{k\in\mathbb{Z}} e^{-ika} \langle f, e_k\rangle e_k \qquad (**)$$

For $\epsilon \in (0,\pi)$ and $\chi_{(-\epsilon,\epsilon)}$ the characteristic function of $(-\epsilon,\epsilon)$ calculate the series

$$S(\chi_{(-\epsilon,\epsilon)}) = \sum_{k\in\mathbb{Z}} \langle \chi_{(-\epsilon,\epsilon)}, e_k\rangle e_k$$

Show (e.g. by Parseval, using c. above) that $S(\chi_{(-\epsilon,\epsilon)}) = \chi_{(-\epsilon,\epsilon)}$. Now use (**) to show $S(\chi_{(a,b)}) = \chi_{(a,b)}$ for every interval in $[-\pi,\pi]$. Use density of step functions and linearity to show that $\{e_k\}_{k\in\mathbb{Z}}$ form a basis in $\mathcal{H}$.