Homework 8, The Fourier transform

Problems: Folland pp. 262–263 **25,26,30,31** and

(*) Let $\theta(x)$ be the characteristic function of $[0, \infty)$ and $b > 0$. Show that $\mathcal{F}\left[\theta(x)e^{-x}x^{b-1}\right] = (1 + 2i\pi k)^{-b}\Gamma(b)$. For $\alpha > 1/2$ calculate

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^{\alpha}} dx$$

(Parseval, maybe?) Bonus: There are methods to evaluate this without Fourier analysis; try to find one.

(**) Find a function with exponential decay at infinity whose Fourier transform is not in L^1. (Hint: try to reverse-engineer the problem.)

1 Turn in only the problem below

Here we are seeking to establish a dictionary between regularity and decay of the Fourier transform:

(a) Assume f is Hölder continuous of order $\alpha \in (0, 1)$ and with L^1 decay. More precisely, assume that $\exists C > 0, \epsilon > 0$ s.t. for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \le \frac{C|x-y|^{\alpha}}{(1+|x|+|y|)^{1+\epsilon}}$$

Show that $|k|^{\alpha}\left[\mathcal{F}(f)\right](k)$ is a bounded functions from $\mathbb{R}$ to $\mathbb{R}$.

(b) Conversely [1], show that if for some $\alpha \in (0, 1)$ we have $(1 + |x|)^{\alpha} f(x) \in L^1$, then $\mathcal{F}f$ is Hölder continuous of order α.

[1] The converse is always slightly imperfect in Fourier analysis because the Dirichlet kernel is unbounded in L^1