Homework 1: Nets vs sequences

Ovidiu Costin

01/17/2018

If $\{X_i\}_{i\in I}$ are topological spaces then the product space is defined as $X = \prod_{i\in I} X_i = \left\{ f : I \to \bigcup_{i\in I} X_i \;\middle|\; (\forall i)(f(i) \in X_i) \right\}$; the product topology is defined to be the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections p_i ($p_i(x) = x_i$) are continuous. In this topology a net $\langle f_\alpha \rangle$ converges iff $\forall i \in I, f_\alpha(i)$ converges, that is, the topology is that of pointwise convergence of functions. The fact that X is nonempty for general X_i is equivalent to the axiom of choice, AC. The construction below however makes virtually *no use of AC*.

For $r \in [0,1]$ let $X_r = [0,1]$. Then $X = \prod_{r\in[0,1]} X_r$ is the space of all functions from $[0,1]$ into itself. [1]

Let $\mathcal{F}$ be the set of all finite subsets of $[0,1]$. That is, $F \in \mathcal{F}$ if for some $n \in \mathbb{N}$ $F = \{r_1, .., r_n\}, r_i \in [0,1]$. Check that $\mathcal{F}$ is a directed set under the following order relation: $F_1 \preccurlyeq F_2$ if, by definition, $F_1 \subset F_2$.

For $F \in \mathcal{F}$ define the function f_F by $f_F(x) = 0$ if $x \in F$ and $f_F(x) = 1$ otherwise. Consider the net $\tilde{f} = \langle f_F \rangle_{F\in\mathcal{F}}$.

1. Show $\lim_{F\in\mathcal{F}} f_F = 0$, the zero function.
2. Consider a strictly ascending infinite sequence $F_1 \prec F_2 \prec \cdots \prec F_n \cdots$. Find $\lim_{i\to\infty} f_{F_i}$.
3. (“*Sequences are too short for this topology.*”) Is there any choice of F_i as above such that $\lim_{i\to\infty} f_{F_i} = 0$?
4. Is X first countable?
5. Is there any sequence $f_1, f_2, \cdots$ with elements in the net $\tilde{f}$, as above, a *subnet* of $\tilde{f}$?

[1] By Tychonoff’s theorem, X is compact; this fact uses AC, but we don’t use compactenss.