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1 Differential equations

A general scalar differential equation can be presented in explicit form,

$$y^{(n)} = F(t, y, y', ..., y^{(n-1)}) \tag{1}$$

or implicitly,

$$G(t, y, y', ..., y^{(n)}) = 0 \tag{2}$$

Here y ($\mathbf{y}$) is the dependent variable and t is the independent one.

An initial condition for equations such as (1) and (2) above are assignments of the form $y(t_0) = y_{0,0}, ..., y^{(n-1)}(t_0) = y_{0,n-1}$ and a problem of the form (1) or (2) with an initial condition is called an initial value problem (IVP). There are also other forms of selecting solutions, such as boundary value problems, in which the values of the function and a sufficient number of its derivatives are prescribed at two points. Such are eigenvalue problems, e.g. finding the λs for which $x'' = \lambda x,\ x(0) = 0, x(1) = 0$ has nonzero solutions, and the solutions corresponding to those special λs.

An explicit nth order system of equations has the generic form

$$\mathbf{y}^{(n)} = \mathbf{F}\left(t, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \cdots \mathbf{y}^{(n-1)}\right) \tag{3}$$

One can similarly define implicit systems, higher order systems and so on.

Most equations are not solvable in closed form. Even very simple equations, such as the Abel equation $y' = y^3 + x$ are not solvable in closed form. A closed form solution would be one given in terms of simple operations involving elementary or special functions. In fact, what is "special" about special functions? One feature is that there behavior is opposite to being chaotic. There are other important qualities they have. And what is chaos? We'll discuss these later in the course.

A system that received substantial attention is the Lorenz system

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= xy - \beta z. \end{aligned} \tag{4}$$

where "dot" is differentiation. The system (4) exhibits chaotic behavior (opposite to "integrable behavior") and the trajectories are attracted toward a "strange attractor", see Fig. 42. We will analyze this system in more detail later. Given how well-behaved elementary and special functions are, it is "clear" that the solutions of (4) cannot be given by some explicit closed-form expression. Proving this is another matter. What is remarkable about this system is of course its apparent simplicity, and also the fact that it arises naturally in many applications, including the Rayleigh-Bénard convection in fluid dynamics.

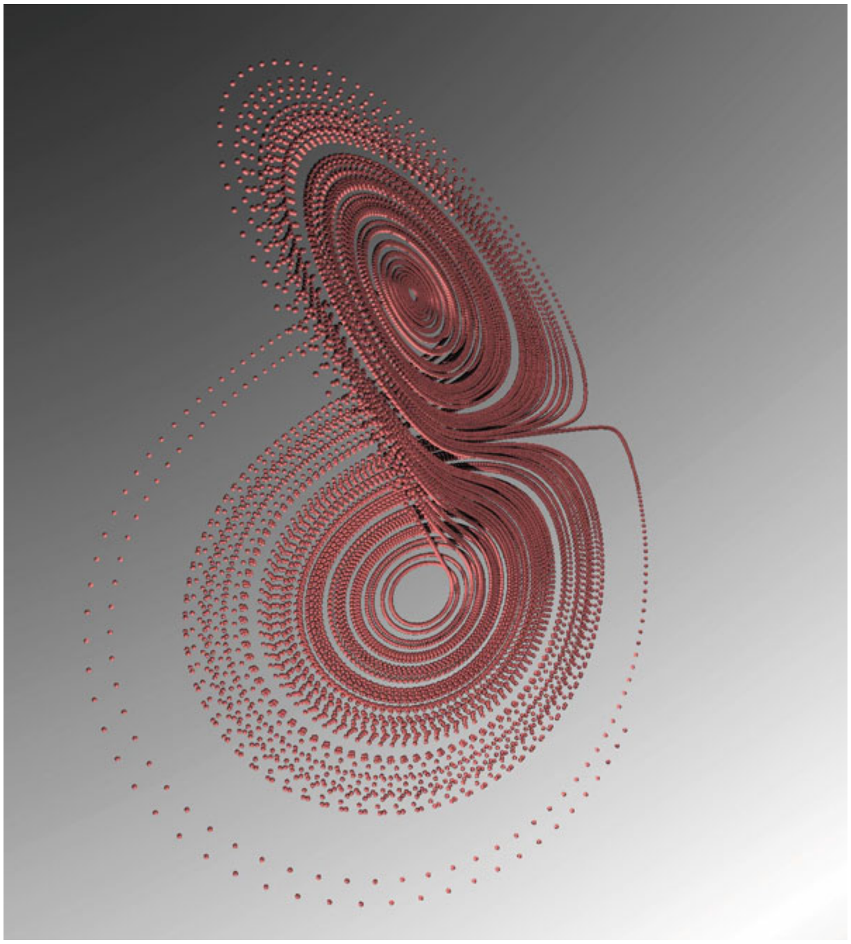

Figure 1: The Lorenz attractor

We will learn how to handle these systems, obtain as much information as possible *using other tools than simple explicit integration*. In fact, often even when explicit formulas exist, they may be quite cumbersome, and the questions we need might be answered in better ways than relying on the exact solution.

General questions of interest pertain to existence of solutions and uniqueness of solutions of an IVP, the behavior of the solutions near the initial point, domain of existence, either in $\mathbb{R}$ or in $\mathbb{C}$, behavior for large values of x (the *asymptotic behavior*) if the solution does exist for large x or near other special values of x. It is this kind of questions that we will mostly address. The question of explicit solvability is also one that can be addressed answered to a good extent, and worth pursuing since, while nongeneric, integrable equations play special role in physics and other applications.

1.1 Phase portraits

These are especially useful for 2-systems of autonomous equations, equations of the form (3) where $n = 1$ for two dimensional vectors, and where there is no explicit dependence on x. Two dimensional systems are important, often originating in problems in mechanics. They have properties that do not generalize to higher order, and are simpler in a number of ways. They deserve a special place in our course.

For notational simplicity we take $y_1 = y, y_2 = x$. Thus we will look at systems of the form

$$x' = F(x,y); \quad y' = G(x,y) \tag{5}$$

The vector field associated to (5) is the set of all vectors which equal $(F(x_0, y_0), G(x_0, y_0)$ (or a common nonzero multiple thereof) at all points $(x_0, y_0) \in \mathbb{R}^2$. Points where $(F(x_0, y_0), G(x_0, y_0) = 0$ are called *critical points of the field*, and they play an important role in the study of systems of ODEs.

Trajectories are defined as the curves (or points!) $\{x(t), y(t)) : t \in [0, a), x(0) = x_0, y(0) = y_0\}$ where a, depending on the solution is usually taken as the maximal existence time and may depend on x_0, y_0; it may be that $a = \infty$.

Of course, when a solution can be written explicitly, such as in the system

$$x' = x, y' = 2y \Rightarrow x(t) = x(0)e^t, \; y(t) = y(0)e^{2t} \tag{6}$$

Here we see that $y = Ax^2, A = y(0)/x(0)$ (assuming $x(0) \neq 0$) and it is easy to draw the phase portrait: just a bunch of parabolas, see Fig. 2, except for degenerate one when $A \in \{0, \infty\}$ when the trajectories are straight lines, and the important exception $(0, 0)$ *which is a solution in itself*, a *critical point of the system*.

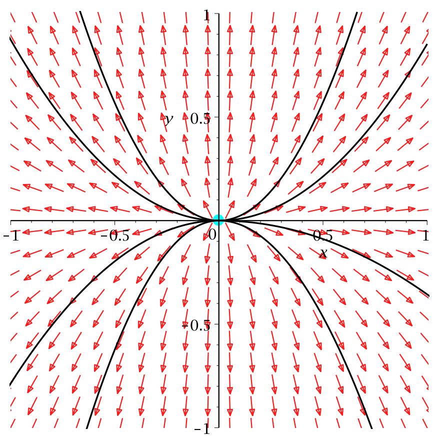

Figure 2: The system (6): vector field and some trajectories.

The arrows represent the vector field associated to (9): at the point (x, y) we have a vector proportional to $(x, 2y)$.

If we take now

$$x' = ye^y, y' = xe^x \tag{7}$$

we can still solve the equation by means of nonelementary integrals of nonelementary functions.

An implicit solution is obtained by separation of variables,

$$(y - 1)e^y = (x - 1)e^x + C \tag{8}$$

It is not immediately clear how these curves should look like as a function of C, and we are better off by constructing the phase portrait in a more qualitative fashion. Trajectories do not intersect except at critical points or singular points of the functions.

It is clear that a trajectory is tangent to this vector field. In principle, we can draw the phase portrait by painstakingly drawing many vectors and guessing the tangent lines, but there are better methods.

For systems with no singularities other than critical points and infinity. The phase portrait is then topologically determined by the type of critical points, and the behavior of the solutions for large values of x and y. Special solutions also help.

If the critical points and singularities are not too many, or too complicated, it is usually not so hard to draw the phase portrait based on the information above.

Here, $C = 0, x = y$ is a trajectory. $x = y = 0$ is a critical point, and there are no other critical points or fixed points. Near $(0, 0)$ we have **approximately** $x' = y, y' = x$, that is $x^2 - y^2 \approx C$ that is the curves are hyperbolas. $(0, 0)$ is a *saddle point*. The phase portrait is symmetric w.r.t. reflection by th line $y = x$. Check this. For $x < 0$ large and y, we have $|y'|$ small, thus the lines are nearly horizontal. Try to determine what happens in the third quadrant, looking at the cases $x/y \to -\infty$, etc.

The phase portrait you obtain should resemble that in Fig. 3 If we take now

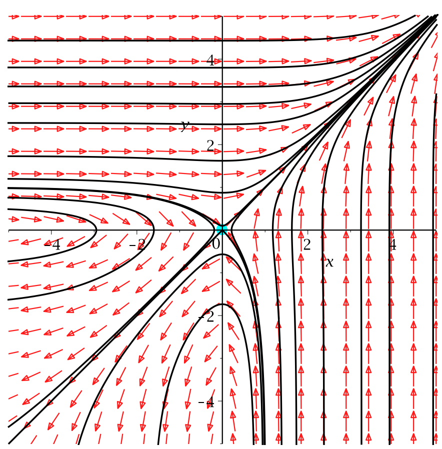

Figure 3: The system (7)

$$x' = y \sin(\pi y), y' = x \sin(\pi x) \tag{9}$$

we get the implicit representation

$$\sin(\pi y) - \pi y \cos(\pi y) = \sin(\pi x) - \pi x \cos(\pi x) + C \tag{10}$$

Plotting these curves does not seem very easy. To analyze what happens near zero is not too hard, and we leave it as an exercise. But the global behavior

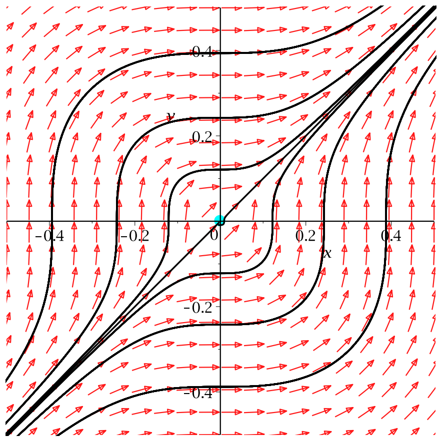

Figure 4: The system (9), small scale

is another question, and we would need better tools to do a good job for more complex systems (this example is still simple enough to be analyzed by "elementary phase portrait" methods). Not to speak that most autonomous ODEs cannot be solved in closed form, implicit or explicit!

Brush up your knowledge on phase portraits.

Exercise 1. Analyze the phase portrait of (9), and try to explain the patterns observed in Fig. 1.1. (Had the points been chosen more carefully the structure would become more obvious!)

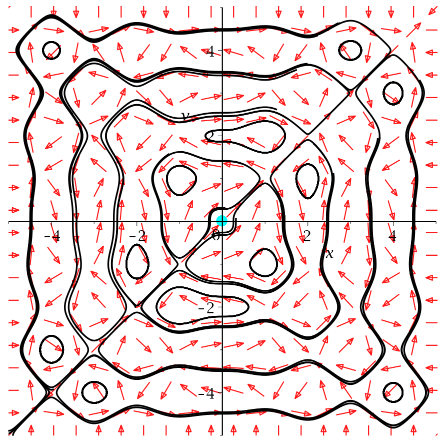

Figure 5: The system (9), larger scale. The trajectories are gotten by picking "random" initial conditions, observing the symmetry of the system.

2 Banach spaces and the contractive mapping principle

In rigorously proving local or asymptotic results about *solutions* of various problems, where a closed form solution does not exist or is awkward, the contractive mapping principle is a handy tool. Some general guidelines on how to construct this operator are discussed in §5.1.

In §2.0.1 we discuss, for completeness, a few basic facts about Banach spaces. There is of course a vast literature on the subject; see e.g. [80].

2.0.1 A brief review of Banach spaces

Familiar examples of Banach spaces are the n-dimensional Euclidian vector spaces $\mathbb{R}^n$. A norm exists in a Banach space, which has the essential properties of a length: scaling, positivity except for the zero vector which has length zero and the triangle inequality (the sum of the lengths of the sides of a triangle is no less than the length of the third one). Once we have a norm, we can define limits, by reducing the notion to that in $\mathbb{R}$: $x_n \to x$ iff $\|x - x_n\| \to 0$. A normed vector space $\mathcal{B}$ is a Banach space if it is complete, that is every sequence with the property $\|x_n - x_m\| \to 0$ uniformly in n, m (a Cauchy sequence) has a limit in $\mathcal{B}$.A standard example is the space of complex valued bounded measurable functions on a set S with the norm $\|f\| = \sup_S |f|$; this space is denoted by $L^\infty(S)$. Note that $\mathbb{R}^n$ is a special case ; it is $L^\infty(\{1, 2, ..., n\})$.

A function L between two Banach spaces which is linear, $L(x+y) = Lx+Ly$, is bounded (or continuous) if $\|L\| := \sup_{\|x\|=1} \|Lx\| < \infty$.

Definition 1. *Let $\mathcal{A}, \mathcal{B}$ be Banach spaces. The norm $\|.\|$ of a linear operator*

$L : \mathcal{A} \to \mathcal{B}$ *is simply defined as*

$$\|L\| = \sup_{\|x\|=1} \|Lx\| \overset{check!}{=} \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|}$$

Assume $\mathcal{B}$ is a Banach space and that S is a closed subspace of $\mathcal{B}$. In the *induced topology* (i.e., in the same norm), S is a complete normed space.

2.0.2 The contractive mapping theorem

Assume $\mathcal{N} : S \mapsto \mathcal{B}$ is a (linear or nonlinear) operator with the property that for any $x, y \in S$ we have

$$\|\mathcal{N}(y) - \mathcal{N}(x)\| \leqslant \lambda \|y - x\| \tag{11}$$

with $\lambda < 1$. Such an operators is called **contractive on** S. Note that if $\mathcal{N}$ is linear, this just means that the norm of $\mathcal{N}$ is less than one.

Exercise 1. *Check that contractivity implies continuity: if x_n, x are in S and $x_n \to x$, then $\mathcal{N}(x_n) \to \mathcal{N}(x)$.*

Theorem 1. *Assume $\mathcal{N} : S \mapsto S$, where S is a closed subset of $\mathcal{B}$ is a contractive mapping. Then the equation*

$$x = \mathcal{N}(x) \tag{12}$$

has a unique solution in S.

Proof. Consider the sequence $\{x_j\}_j \in \mathbb{N}$ defined recursively by

$$\begin{aligned} x_0 &= x_0 \in S \\ x_1 &= \mathcal{N}(x_0) \\ &\cdots \\ x_{j+1} &= \mathcal{N}(x_j) \\ &\cdots \end{aligned} \tag{13}$$

We see that

$$\|x_{j+2} - x_{j+1}\| = \|\mathcal{N}(x_{j+1}) - \mathcal{N}(x_j)\| \leqslant \lambda \|x_{j+1} - x_j\| \leqslant \cdots \leqslant \lambda^j \|x_1 - x_0\| \tag{14}$$

Thus,

$$\|x_{j+p+2} - x_{j+2}\| \leqslant \left(\lambda^{j+p} + \cdots \lambda^j\right) \|x_1 - x_0\| \leqslant \frac{\lambda^j}{1-\lambda} \|x_1 - x_0\| \tag{15}$$

and x_j is a Cauchy sequence, and it thus converges, say to x. Since by (11) $\mathcal{N}$ is continuous, passing the equation for x_{j+1} in (13) to the limit $j \to \infty$ we get

$$x = \mathcal{N}(x) \tag{16}$$

that is existence of a solution of (12). For uniqueness, note that if x and x' are two solutions of (12), by subtracting their equations we get

$$\|x - x'\| = \|\mathcal{N}(x) - \mathcal{N}(x')\| \leqslant \lambda \|x - x'\| \tag{17}$$

implying $\|x - x'\| = 0$, since $\lambda < 1$. □

Note 1. As we will see, contractivity and therefore existence of a solution of a fixed point problem depends on the norm. This is the case even for equivalent norms. [(1)] An adapted norm needs to be chosen for this approach to give results.

Exercise 2. Show that if L is a linear operator from the Banach space $\mathcal{B}$ into itself and $\|L\| < 1$ then $I - L$ is invertible, that is $x - Lx = y$ has always a unique solution $x \in \mathcal{B}$. "Conversely," assuming that $I - L$ is not invertible, then in whatever norm $\|\cdot\|_*$ we choose to make the same $\mathcal{B}$ a Banach space, we must have $\|L\|_* \geqslant 1$ (why?).

2.1 The abstract implicit function theorem

The *Fréchet derivative* of a possibly nonlinear operator A in a Banach space, if it exists, is defined to be the linear operator $D_x A$ with the property

$$\|A(x+h) - A(x) - (D_x A)h\| = o(\|h\|) \text{ as } \|h\| \to 0 \tag{18}$$

Exercise 3 (Implicit function theorem in Banach spaces)**.** Prove the following:

"Let X, Y, Z be Banach spaces. Assume that the mapping $f : X \times Y \to Z$ be continuously Fréchet differentiable. If $(a, b) \in X \times Y$ is s.t. $f(a, b) = 0$ and $Df(a, b)(0, y)$ is a Banach space isomorphism from Y onto Z, then there exist neighborhoods A of a, B of b and a Fréchet differentiable function $g : A \to B$ s.t. $f(x, g(x)) = 0$ and $f(x, y) = 0$ if and only if $y = g(x)$, for all $(x, y) \in A \times B$."

2.2 Some guidelines

For writing an ODE, PDE, etc in a contractive form, if indeed there is one, we have to make sure that the operators we are dealing with can be contractive. This means in particular, for instance that the operator cannot be one of differentiation-that operator is unbounded. We then transform first the equation in an equivalent integral form. Then the norm is chosen so that it is reflective of the regularity, rate of growth of the actual solutions (the norm should not *underestimate* the rate of growth).

3 Examples

3.1 Linear differential equations in Banach spaces

Consider the equation

$$Y'(t) = L(t)Y(t); \quad Y(0) = Y_0 \tag{19}$$

[(1)] This simply means that convergence in one norm implies convergence in the other (and vice-versa...).

in a Banach space X, where $L(t) : X \to X$ is linear, norm continuous in t and uniformly bounded,

$$\sup_{t\in[0,\infty)} \|L(t)\| < L \tag{20}$$

Then the problem (19) has a global solution on $[0,\infty)$, and

$$\|Y(t)\|_X \leqslant \frac{L}{a}\|Y_0\|_X e^{(L+a)t} \text{ for any } a > 0 \tag{21}$$

Proof. We already have a norm on X, $\|\cdot\|_X$ which we simply denote by $\|\cdot\|$. Now we have to deal with the time dependence and we need a norm on the *space of functions of t with values in X.*

By comparison with the case when $X = \mathbb{R}$, the natural growth is indeed Ce^{Lt}, so we rewrite (19) as an integral equation, in a space where the norm reflects this possible growth. Consider the space of continuous functions $Y : [0,\infty) \mapsto X$ in the norm

$$\|Y\|_{\infty,L} = \sup_{t\in[0,\infty)} e^{-Lt/\lambda}\|Y(t)\| \tag{22}$$

with $\lambda < 1$ and the auxiliary equation

$$Y(t) = Y_0 + \int_0^t L(s)Y(s)ds =: Y_0 + JY := \mathcal{N}[Y](t) \tag{23}$$

which is well defined on X and is contractive there since

$$e^{-Lt/\lambda}\left\|\int_0^t L(s)Y(s)ds\right\| \leqslant Le^{-Lt/\lambda}\int_0^t e^{Ls/\lambda}\|Y\|_{\infty,L}ds$$
$$= \lambda(1-e^{-Lt/\lambda})\|Y\|_{\infty,L} \leqslant \lambda\|Y\|_{\infty,L}, \tag{24}$$

and therefore the operator is contractive in the whole of X. Thus it has a unique solution which satisfies $\|Y(t)\|_X \leqslant const e^{Lt}$.

Note 2. Had we chosen a different norm, say the one above but with $\lambda > 1$, then contractivity would fail. Check! This is true even on a compact set in $t, t \in [0,T]$ if λ is too large. But on such a compact set you can check (do so!) that all norms as above with $\lambda > 0$ are equivalent!

Since the integral operator J vis linear, we could have proceeded as follows. Once we have shown that $\|J\| \leqslant \lambda < 1$ we can write

$$(1-J)Y = Y_0 \Rightarrow Y = (1-J)^{-1}Y_0 \tag{25}$$

(1 is the identity operator). The inverse is simply defined by the geometric-like series (called Neumann series)

$$(1-J)^{-1} = \sum_{k=0}^{\infty} J^k \tag{26}$$

which converges in norm (since $\|J\| \leqslant \lambda < 1$, we noted this already). This expression also shows that

$$\|(1-J)^{-1}\| \leqslant \frac{1}{1-\lambda} \tag{27}$$

□

4 Local existence and uniqueness of solutions of nonlinear systems

Consider the system of equations (or one vector equation if you prefer)

$$y' = F(x,y); \quad y(x_0) = y_0 \tag{28}$$

where $y \in \mathbb{R}^n$, $x \in \mathbb{R}$. The second condition, the initial value, makes (32) an *initial value problem, IVP*. You see that by taking $y = \tilde{y} + y_0$, $x = \tilde{x} + x_0$ and $\tilde{F}(\tilde{x},\tilde{y}) = F(\tilde{x}+x_0, \tilde{y}+y_0)$, we can assume, without loss of generality that our IVP is

$$y' = F(x,y); \quad y(0) = 0 \tag{29}$$

For existence of solutions *some* condition is needed on F, for the simple equation $y' = f(x)$ has no solution if f is, say, not Lebesgue measurable (why?) For mere existence of not necessarily unique solutions, continuity of F suffices; see [31] pp. 2–7.

Existence and uniqueness requires stronger properties. Indeed, the equation

$$y' = 2y^{1/2}, \quad y(0) = 0$$

has as solutions $y = 0, y = x^2$ (and many more; can you find them?)

Let $\mathcal{D} = \mathbb{D}_\varepsilon = \{z : |z| < \varepsilon\}$. We will assume that $F : \mathbb{D}_\delta \times \mathbb{D}_\varepsilon^n \mapsto \mathbb{R}^n$ is L^1 in x and Lipschitz continuous in y (for some C and all $(x,y) \in \mathcal{D}$ we have $|F(x,y_1) - F(x,y_2)| < C|y_1 - y_2|$).

By taking smaller ε, δ if needed, we can assume that F is continuous up to the boundary, that is continuous in $\overline{\mathbb{D}_\varepsilon} \times \overline{\mathbb{D}_\delta^n}$.

We consider the space of continuous functions y on $\mathbb{D}_\varepsilon$ with the sup norm, $\|y\|_\infty = \sup_{x \in \overline{\mathbb{D}_\delta}} |y|$ form a Banach space; call this Banach space $\mathcal{B}$.

We now consider a closed subspace of $\mathcal{B}$, the closed ball $B = \{y \in \mathcal{B} : \|y\| \leqslant \delta\}$.

Exercise. Check that the IVP (33) is equivalent to

$$y = \int_0^x F(s, y(s))ds \tag{30}$$

Let ε be small enough. How small that is, we'll calculate in a moment. We now consider the *nonlinear* operator $\mathcal{N}$ be defined on B with values in B, given by

$$\mathcal{N}(y) = \int_0^x F(s, y(s))ds \tag{31}$$

For $|\mathcal{N}(y)|$ to be bounded by δ, we need that $\varepsilon \max_{\overline{\mathbb{D}_\varepsilon} \times \overline{\mathbb{D}_\delta^n}} |F(s, y(s)| < \delta$. Check that this can be arranged by taking ε small enough.

For $\mathcal{N}(y)$ to be contractive, check that it suffices to have

$$C\varepsilon < \alpha < 1$$

This ensures contractivity and therefore existence and uniqueness of solutions of the IVP.

4.1 Fixed points and vector valued analytic functions

A theory of analytic functions with values in a Banach space can be constructed by almost exactly following the usual construction of analytic functions. For the construction to work, we need the usual vector space operations and a topology in which these operations are continuous. A typical setting is that of a Banach algebra, see §6. A detailed presentation is found in [62] and [72], but the basic facts are simple enough for the reader to redo the necessary proofs.

5 Local existence and uniqueness of analytic solutions: contractive mapping approach

The study of analytic systems mirrors the analysis of §4.

Consider the system of equations (or one vector equation if you prefer)

$$y' = F(x, y); \quad y(x_0) = y_0 \tag{32}$$

where $y \in \mathbb{C}^n$, $x \in \mathbb{C}$. The second condition, the initial value, makes (32) an *initial value problem, IVP*. You see that by taking $y = \tilde{y} + y_0$, $x = \tilde{x} + x_0$ and $\tilde{F}(\tilde{x}, \tilde{y}) = F(\tilde{x} + x_0, \tilde{y} + y_0)$, we can assume, without loss of generality that our IVP is

$$y' = F(x, y); \quad y(0) = 0 \tag{33}$$

We must specify the properties of F. Let where $\mathbb{D}_\varepsilon = \{z : |z| < \varepsilon\}$. We will assume that $F : \mathbb{D}_\varepsilon \times \mathbb{D}_\delta^n \mapsto \mathbb{C}^n$ is analytic in $\mathbb{D}_\varepsilon \times \mathbb{D}_\delta^n$ for some $\delta > 0, \varepsilon > 0$. This means that F has a convergent Taylor series in $(x, y_1, ..., y_n)$ in $\mathbb{D}_\varepsilon \times \mathbb{D}_\delta^n$.

It is known (by Hartog's theorem: google it!) that if F is separately analytic in each variable (thinking therefore of the others as being "frozen"), then it is analytic in the stronger sense above.

By taking a slightly smaller ε if needed, we can assume that F is continuous up to the boundary, that is continuous in $\overline{\mathbb{D}_\delta} \times \overline{\mathbb{D}_\varepsilon^n}$.

Check that the functions y which are analytic in $\mathbb{D}_\varepsilon$ and continuous in $\overline{\mathbb{D}_\varepsilon}$ endowed with the sup norm, $\|y\|_\infty = \sup_{x \in \overline{\mathbb{D}_\varepsilon}} |y|$ form a Banach space; call this Banach space $\mathcal{B}$.

We now consider a closed subspace of $\mathcal{B}$, the closed ball $B = \{y \in \mathcal{B} : \|y\| \leqslant \delta\}$.

Exercise. Check that the IVP (33) is equivalent to

$$y = \int_0^x F(s, y(s))ds \tag{34}$$

Let ε be small enough. How small that is, we'll calculate in a moment. We now consider the *nonlinear* operator $\mathcal{N}$ be defined on B with values in B, given by

$$\mathcal{N}(y) = \int_0^x F(s, y(s))ds \tag{35}$$

For $|\mathcal{N}(y)|$ to be bounded by ε, we need that $\delta \max_{\overline{\mathbb{D}_\varepsilon} \times \overline{\mathbb{D}_\delta^n}} |F(s, y(s)| < \varepsilon$. Check that this can be arranged by taking δ small enough.

For $\mathcal{N}(y)$ to be contractive, check that it suffices to have

$$\varepsilon \sup_{|x|<\delta; |y|<\varepsilon} \left\| \frac{\partial F_i}{\partial y_j} \right\|_{\mathbb{R}^n \to \mathbb{R}^n} < \alpha < 1$$

where the norm of the Jacobian is the usual matrix norm. This ensures contractivity and therefore existence and uniqueness of solutions of the IVP.

Exercise 1. Formulate and prove an analytic analog of the result in §3.1 in the case A is analytic in a connected, open and bounded domain in $\mathbb{C}$.

5.1 Choice of the contractive map

An equation can be rewritten in a number of equivalent ways. In solving an asymptotic problem, as a general guideline we mention:

1. The operator $\mathcal{N}$ appearing in the final form of the equation, which we want to be contractive, should not contain derivatives of highest order, divided differences with small denominators, or other operations poorly behaved with respect to estimates and it should only depend on the sought-for solution in a formally small way. The latter condition should be, in a first stage, checked for consistency: those discarded terms which contain the dependent variable, calculated using the first order approximation, should indeed turn out to be small.

   For instance, in Eq. (34) near zero, the approximation

   $$F(s, y(s)) \approx F(s, 0) + \left(\frac{\partial F_i}{\partial y_j} \right) (s, 0) y$$

   where $\left(\frac{\partial F_i}{\partial y_j} \right) (s, 0)$ is the Jacobian matrix, leads to the approximate equation

   $$y(x) \approx \int_0^x F(s, 0)ds + \left[\int_0^x \left(\frac{\partial F_i}{\partial y_j} \right) (s, 0) \right] y$$

   where we see that these criteria are satisfied. Of course, in simple problems, we don't necessarily have to write down approximate equations first.

To obtain an equation where the discarded part is manifestly small one may need to write the sought-for solution as the sum of the first few terms of the approximation, plus an exact remainder, say δ (if it were needed in our example, we could have Taylor-expanded to higher order in y; the equations we dealt with so far are simple enough so that we don't have to worry about this). See the last item in this list though.

The equation for δ is usually more contractive. It also becomes, up to smaller corrections, linear.

2. The norms should reflect as well as possible the expected growth/decay tendency of the solution itself and the spaces chosen should be spaces where this solution lives. We have already seen this in §5.

3. All freedom in the solution has been accounted for, that is, we should make sure the final equation cannot have more than one solution.

   For instance the Painlevé equation P1,

$$y'' = 6y^2 + x \tag{36}$$

   has, as we will see later in this course, meromorphic solutions of the form

$$y = \frac{1}{(x - x_0)^2} + \sum_{k=2}^{\infty} c_k (x - x_0)^k \tag{37}$$

   Trying to write $y = 1/(x - x_0)^2$ and the equation for h in integral form, we get

$$h = a(x - x_0) + b + \int_{x_0}^{x} \int_{x_0}^{t} \left(6h(s)^2 + \frac{12h(s)}{(s - x_0)^2} + s \right) ds dt$$

   the equation will *fail to be contractive in any norm.* This is because there is a hidden degree of freedom left. Calculating the c_k we get $c_0 = c_1 = 0, c_3 = -1/6, c_4 = C$ where C is undetermined! C does not enter our attempted contractive equation, and (37) will also have multiple solutions whereas we know that contractive maps have unique solutions.

   Going further in the expansion you can convince yourselves that there are no other free constants than x_0 and C. A contractive mapping approach would work if we write instead $y = 12/(x - x_0)^2 - 1/6(x - x_0)^2 - \frac{1}{10}(x - x_0)^3 + C(x - x_0)^4 + \delta$ would succeed instead. There are however slightly softer ways to do the analysis, and we postpone this for later.

Note 3. At the stage where the problem has been brought to a contractive mapping setting, it usually can be recast without conceptual problems, but perhaps complicating the algebra, to a form where the implicit function theorem applies (and vice versa). The contraction mapping principle is often more natural, especially when the topology, suggested by the problem itself, is not one of the common ones. But an implicit function reformulation might bring in more global information.

6 The exponential and the log of a matrix

A natural setting in which functions of a matrices and more generally of (say bounded) operators are analyzed is that of a Banach algebra.

This is a Banach space endowed with multiplication which is distributive, associative and continuous in the Banach norm.

Continuity of the addition and multiplication are spelled out as

$$\|x+y\| \leqslant \|x\| + \|y\|; \quad \|xy\| \leqslant \|x\|\|y\|, \quad \forall x, y \tag{38}$$

Note that $n-$dimensional matrices form a Banach space w.r.t. the usual norm, $\|A\| = \max_{\|x\|=1} \|Ax\|$.

We can consider the sum

$$e^M = \sum_{k=0}^{\infty} M^k/k! \tag{39}$$

Since $\|M^k\| \leqslant \|M\|^k$ and the series

$$\sum_{k=0}^{\infty} \|M\|^k/k! \tag{40}$$

converges (to $e^{\|M\|}$), it follows that e^M is correctly defined, by a norm-convergent series. You can check the usual properties of the exponential. Careful though: $AB \neq BA$ in general, and in general

$$e^{A+B} \neq e^A e^B; \text{ however, } \textbf{if } \ AB = BA \text{ then } e^{A+B} = e^A e^B$$

Assume first that M is diagonalizable, $M = ADA^{-1}$ where D is diagonal. If 0 is not on the diagonal of D, then define $\log M$ to be $A[\log D]A^{-1}$. Here, A is the diagonalization matrix, D is diagonal, and so is, by definition $\log D$, consisting of the logs of the diagonal elements of D. If the eigenvalues are positive, then the log is the usual one, real on the real line. If an eigenvalue d_i is negative, the log is complex and not uniquely defined, but for some purposes, this is fine.

Exercise: If M has a nontrivial Jordan normal form with no zero eigenvalue, it is enough to define the log block by block. Each block is of the form $\lambda I - N$, $\lambda \in \mathbb{C}$, I the identity matrix and N a nilpotent. Then, the sum

$$(\log \lambda)I - \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{N}{\lambda}\right)^k = I \log \lambda - \sum_{k=1}^{m} \frac{1}{k} \left(\frac{N}{\lambda}\right)^k \tag{41}$$

since $N^{m+j} = 0 \ \forall j \in \mathbb{N}$ for some $m \leqslant \dim(M)$ since N is nilpotent.

One can define, in a similar way, more general functions of matrices.

6.1 Writing an n-th order scalar ODE as a system of first order ODEs

These are equations of the form

$$y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_n(z)y = 0; \quad a_j \text{ analytic in } \mathbb{D} \setminus \{0\} \tag{42}$$

Let $v_0 = y, ..., v_k = y^{(k)}, ...$ and note that (42) is equivalent to

$$\begin{pmatrix} v_0 \\ v_1 \\ \cdots \\ v_{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ -a_n(z) & -a_{n-1}(z) & -a_{n-2}(z) & \cdots & -a_1(z) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \cdots \\ v_{n-1} \end{pmatrix} \tag{43}$$

Note 4. There is **more than one way** to transform a scalar higher order ODE into a first order system. When studying regular singularities, we will use a different representation, in "better correspondence" with the scalar equation.

Studying first order systems is, at least algebraically, simpler.

7 The fundamental solution of a linear system

Consider a linear system of differential equations of the form

$$w' = A(z)w; \; w(x_0) = w_0 \tag{44}$$

Which we consider in two cases: A merely L^1 in an open interval J around z_0 and A analytic in some *simply connected* domain $\mathcal{D}$ containing x_0. As you will see, the proofs are very similar in the two cases. As before can be chosen to be zero; w_0 cannot be chosen to be zero, since the transformation $w(z) = w_0 + \tilde{w}$ would transform (44) to a linear-nonhomogeneous form, $\tilde{w}' = A(z)\tilde{w} + A(z)w_0$.

Proposition 2. *The IVP* (44) *has a unique solution in a neighborhood of* z_0. *If* A *is analytic at* z_0 *then so is* w.

Proof. $y \mapsto \int_0^x A(z)y(z)dz$ is well defined in $\mathcal{D}$ since the integral does not depend on the contour. By the results in §4 and §5, all that needs to be checked is that $y \mapsto \int_0^x A(z)y(z)dz$ is Lipschitz in y, and which is an easy exercise. □

For linear equations we can find the domain of existence of solutions: J in the L^1 case and $\mathcal{D}$ in the analytic case. As mentioned, we consider one of the two cases, because the approach is the same.

Proposition 3. *The IVP* (44) *has a unique solution in* $\mathcal{D}$ *and is analytic there.*

Proof. Since $\mathcal{D}$ is connected in $\mathbb{C}$, it is pathwise connected. Choose $L > 0, \varepsilon > 0$ and $K_{L,\varepsilon} \subset \mathcal{D}$ be the set of points z such that $d(z, \partial D) > \varepsilon$ that can be reached from zero by a curve of length $< L$; this is a simply connected open set (check) with a compact closure and $\cup_{L,\varepsilon} K_{L,\varepsilon} = \mathcal{D}$. It is enough to prove the result for each $K = K_{L,\varepsilon}$.

Consider $\mathcal{B}$, the Banach space of functions

$$\{f : K \to \mathbb{C} | f \text{ analytic in } K, \|f\|_\nu < \infty\}; \quad \|f\|_\nu = \sup_{z \in K} |f(z) e^{\nu |z|} \tag{45}$$

where $\nu > 0$ will be chosen later. Note that this norm is equivalent to the sup norm since $\overline{K}$ is compact. We have, with $\|\cdot\|$ without the subscript ν standing for the usual sup norm, and with $A_K = \|A\|$,

$$\left\| \int_0^z A(s) y(s) ds \right\|_\nu = \sup_{z \in K} e^{-\nu|z|} \left| \int_0^z A(s) y(s) ds \right|$$
$$\leqslant \sup_{z \in K} e^{-\nu|z|} \int_0^z \|A(s)\| e^{\nu|s|} \|y(s) e^{-\nu|s|}\| \, d|s| \leqslant \sup_{z \in K} e^{-\nu|z|} \int_0^z \|A(s)\| \|y\|_\nu e^{\nu|s|} \, d|s|$$
$$\leqslant \frac{A_K L}{\nu} \|y\|_\nu < \|y\|_\nu \tag{46}$$

if ν is large enough. The equation is linear, thus the whole of $\mathcal{B}$ is an invariant closed set in $\mathcal{B}$ if the norm of the operator is finite. Thus we only needed to show contractivity; the result follows. □

In a neighborhood of a regular point (i.e., a point where A analytic) there exist n linearly independent vector solutions, $\{w_j : j = 1, ..., n\}$ of (44) by the existence and uniqueness theorems we have proven. We will also see shortly that $w_i(t)$ are linearly independent at *all* t.

Furthermore, you can choose initial conditions so that $w_j(z = 0) = e_j$, the unit vector in the direction j. If you construct a matrix M having as the $j - th$ column the vector w_j, you can check immediately that

$$M' = A(z) M; \quad M(0) = I \tag{47}$$

Inverse matrix. Let's now see what equation M^{-1} woud satisfy, assuming momentarily it has one (we will show this is the case). Since $MN = I$ we have $M'N + MN' = 0 \Rightarrow MN' = -AMN = -A$ and thus *we expect*

$$\boxed{N' = -NA}; \; N(0) = I \tag{48}$$

thus N gives the backward evolution $z \to -z$.

Note 5. You can check that the matrix equation (47) is equivalent to the set of equations $w_i' = Aw_i, \; w_i(0) = e_i$, while (48) corresponds to the system $u_i' = -A^t u_i, \; u_i(0) = e_i$ where A^t is the transpose of A.

Proposition 4. *Consider the IVPs* (47) *and* (48)*, with $A \in L^1$ in open interval J containing the origin, (A analytic in $\mathcal{D}$, resp.). We have $MN = NM = I$ on J in ($\mathcal{D}$, resp.), where N is differentiable on J (analytic in $\mathcal{D}$, resp.) and satisfies* (48).

Proof. We consider the analytic case. MN should satisfy the equation

$$(MN)' = M'N + MN' = AMN - MNA; \quad (MN)(0) = I \tag{49}$$

Let X be a matrix s.t.

$$X' = AX - XA; \quad X(0) = I \tag{50}$$

Note that this is a *linear equation* for the matrix X ($X \mapsto AX - XA$ is a linear, bounded operator on matrices). Thus the IVP has a unique solution. But I is a solution, thus $X = I$ for all z.

Since (48) is also linear and analytic, it has a solution in $\mathcal{D}$ and MN satisfies (50). The result follows easily.

□

The fact that the determinant of M is nonzero at all t also follows from the following proposition, important in its own right.

Proposition 5. *If M is a solution of* (47) *then*

$$\det M(t) = \det M(0) \exp\left(\int_0^t \mathrm{Tr}A(s)\right) ds$$

Proof. Let $d(t) = \det(M(t))$. As long as M is invertible, we have

$$M(t+\varepsilon) = M(t)+(AM)(t)\varepsilon+o(\varepsilon) = (I+A\varepsilon)M(t) \Rightarrow d(t+\varepsilon) = \det(I+\varepsilon A)d(t) \tag{51}$$

and now the result follows from the following lemma, which is easily proved by induction on the size of the matrix. □

Lemma 6. *For any matrix A,* $\det(I + \varepsilon A) = 1 + \varepsilon \mathrm{Tr}A + o(\varepsilon)$.

Now either Proposition 4 or 5 shows that the vectors $w(t)$ are linearly independent at any t.

Take any z-independent vector w_0, and let $w = Mw_0$. We have

$$(Mw_0)' = M'w_0 = AMw_0 \Rightarrow w' = Aw; \quad w(0) = Iw_0 = w_0 \tag{52}$$

Thus, we see that the solution of the initial value problem $w' = Aw, w(0) = w_0$ is simply Mw_0. We will often work with the fundamental matrix solution M, as it often simplifies the calculations.

Note 6. There is a generalization in some sense of this result in the case of PDE evolution equations. The Schrödinger equation $i\phi_t = H\psi = -\Delta\psi + V(x)\psi$; $\psi(0) = \psi_0$ where Δ is the Laplacian has the solution

$$\psi(t) = U(x,t)\psi(0) \tag{53}$$

where $U(x,t)$ is a unitary family of operators, and in fact $U = e^{-iHt}$, and functional analysis arguments give a precise meaning to e^{-iHt} (as a unitary operator) whenever H is a self-adjoint operator. This form of U does *not* hold if V depends on t as well.

Lemma 7. *The matrix differential equation*

$$W' = AW \tag{54}$$

has the general solution is $W = MC$ where M is the fundamental solution and C is any matrix of constants.

Proof. Indeed, since M is invertible, we can define $Q = M^{-1}W$, which we write in the form $W = MQ$. We then have

$$M'Q + MQ' = AMQ \Leftrightarrow MQ' = 0 \Leftrightarrow Q' = 0 \tag{55}$$

(since $M' = AM$) which indeed means that Q is a constant matrix. □

The exercise below addresses a fairly common fallacy of " ODE beginners".

Exercise 1. Show that the following claim is **wrong** for general $n \times n$ analytic matrices **and** find the fallacy in its "proof".

"If $M' = A(z)M$ then $M = \exp\left(\int_0^z A(s)ds\right) M(0)$."

"proof".

$$M(z)' = \left[\exp\left(\int_0^z A(s)ds\right) M(0)\right]' = A(z)\exp\left(\int_0^z A(s)ds\right) M(0) = A(z)M(z)$$

□

Justify carefully your answer.

7.1 Domain of existence of regular solutions

As discussed, the differential system (44) is equivalent to the matrix differential equation (678) which, by Exercise 1, has an analytic solution throughout the domain of analyticity of A. Solutions of linear systems can only be singular only at singularities of the coefficients, in this case the singularities of A.

8 Isolated singularities of linear systems

Consider the system

$$w' = A(z)w, \ w \in \mathbb{C}^n \tag{56}$$

where A is a matrix valued analytic function, but now with *an isolated singularity at z_0*. Clearly, by translating z we can take $z_0 = 0$, and by rescaling z, we can assume that A is analytic in $\mathcal{D} = \mathbb{D} \setminus \{0\}$ where $\mathbb{D}$ is the open unit disk. Though the equation is single-valued in $\mathcal{D}$, since $\mathcal{D}$ is not simply connected, the solutions may not be, as seen by solving the equation $y' = ay/z$ with $a \notin \mathbb{Z}$. We can take $z = e^{\zeta}$ and $\mathcal{D}$ becomes $\{\zeta : \text{Re}\zeta \in \mathbb{R}^-\}$, a half plane. By the standard existence and uniqueness theorems, we find that there is a unique solution of the system, rewritten in ζ, and thus there is a fundamental solution of (56), in the form $M(\ln z)$, which shows once more that, in principle at least, the solution of (56) may not be single-valued.

9 Some general facts about solutions near isolated singularities

In the generality of the singular systems in §8 all we can say now, without a lot more theory, is the way the solution itself can be ramified. Once more, we consider that we rescaled everything so that $z = 0$ is the isolated singularity, and $\mathcal{D} = \mathbb{D} \setminus \{0\}$ is the domain of analyticity of A.

Theorem 2. *The general solution of* (56) *is of the form*

$$M(z) = S(z)z^P \quad (z^P := e^{\ln z\, P}) \tag{57}$$

where P is a constant matrix, and $S(z)$ is analytic in $\mathcal{D}$. With the price of changing the matrix M to MT, with T a constant matrix, we can write

$$MT = S_1 x^J \tag{58}$$

where J is the Jordan normal form of P.

Note 7. This implies in particular (and is implied by, as we will see) the relation

$$M(ze^{2\pi i}) = M(z)C \tag{59}$$

for some invertible constant matrix C. By (59), a rotation by 2π generates another solution (since MC is indeed a solution). The map $M \mapsto MC$ is a group, and it is the *the monodromy group* at the singularity (0). One also says that (59) is the monodromy at zero.

What this theorem says is that the solution itself is single-valued up to multiplication by z^P with P constant. Of course, there is no reason to expect that S is analytic at zero-just that 0 is an isolated singularity. For the proof we need the following result.

Lemma 8. *Assume M is any matrix analytic on the universal covering of $\mathcal{D}$ (that is, $M(z) = F(\ln(z))$ whereF is analytic in the left half plane) which satisfies*

$$M(ze^{2\pi i}) = MC \quad \textit{where } C \textit{ is a constant invertible matrix.} \tag{60}$$

Then

$$M(z) = S(z)z^P \tag{61}$$

where P is a constant matrix and $S(z)$ is analytic in $\mathcal{D}$. At the price of altering M by a constant matrix, P can be taken to be in Jordan normal form.

Proof of the lemma. Since C is invertible, we can define P (up to $2\mathbb{Z}\pi i I$) by $C = e^{2\pi i P}$. Let

$$S = Mz^{-P} \tag{62}$$

$$S(ze^{2\pi i}) = Me^{2\pi i P}e^{-P\ln z - 2\pi i P} = Me^{-P\ln z} = S(z) \tag{63}$$

since e^{aP} and e^{bP} commute, if a and b are scalars. Let now T be the change of basis that brings P to its Jordan normal form, that is $T^{-1}PT = -J$ where J is a Jordan matrix. We then have

$$MT = STT^{-1}z^PT = STz^J \tag{64}$$

where ST is also ingle valued, as required. □

Proof of the theorem. We only need to show that the assumptions of the lemma above hold. Take $N(z) = M(ze^{2\pi i})$. That is, we use the fact that M exists on the universal covering of $\mathcal{D}$, and look at its value on the second Riemann sheet. We have

$$N(z)' = e^{2\pi i}M'(ze^{2\pi i}) = A(ze^{2\pi i})M(ze^{2\pi i}) = A(z)M(ze^{2\pi i}) = A(z)N \tag{65}$$

where we used the fact that M is already a solution, and A is single-valued. Thus, by Remark 7, we must have $N = MC$ where C is a constant matrix. □

Remark 8. *If S happens to be analytic, note also the emerging noninteger powers of z and $\ln z^j$ through the term z^J.*

Indeed, if J_1 is an elementary Jordan block in J, we have

$$z^J = z^{\lambda I+N} = z^\lambda e^{N\ln z} = z^\lambda(1 + N\ln z + \cdots \ln z^l N^l/l!) \tag{66}$$

where $N^{l+1} = 0$, and thus $l < n$, the degree of the system.

10 Regular singular points of differential equations, nondegenerate case

10.1 Example

Consider the hypergeometric equation

$$x(x-1)y'' + y = 0 \tag{67}$$

around $x = 0$. The indicial equation is $r(r-1) = 0$ (a *resonant case*: the roots differ by an integer). Substituting $y_0 = \sum_{k=0}^\infty c_k x^k$ in the equation and identifying the powers of x yields the recurrence

$$c_{k+1} = \frac{k^2 - k + 1}{k(k+1)}c_k \tag{68}$$

with $c_0 = 0$ and c_1 arbitrary. By linearity we may take $c_1 = 1$ and by induction we see that $0 < c_k < 1$. Thus the power series has radius of convergence at least 1. The radius of convergence is in fact exactly one as it can be seen applying the ratio test and using (68); the series converges exactly up to the nearest singularity of (67).

Exercise 1. *What is the asymptotic behavior of c_k as $k \to \infty$?*

We let $y_0 = y_0 \int g(s)ds$ and get, after some calculations, the equation

$$g' + 2\frac{y_0'}{y_0} g = 0 \tag{69}$$

and, by the previous discussion, $2y_0'/y_0 = 2/x + A(x)$ with $A(x)$ is analytic. The point $x = 0$ is a regular singular point of (688) and in fact we can check that $g(x) = C_1 x^{-2} B(x)$ with C_1 an arbitrary constant and $B(x)$ analytic at $x = 0$. Thus $\int g(s)ds = C_1(a/x + b\ln(x) + A_1(x)) + C_2$ where $A_1(x)$ is analytic at $x = 0$. Undoing the substitutions we see that we have a fundamental set of solutions in the form $\{y_0(x), B_1(x) + \ln(x)y_0(x)\}\}$, or, as a series description, $xs_0(x), s_1(x) + x\ln(x)s_0(x)\}$ where s_0 and s_1 are analytic, $s_0(0) = s_0(1) = 1$.

10.1.1 Vector/matrix valued analytic functions

The definition of an analytic vector function $z \mapsto \mathbf{f}(z)$ is "the same" as the one in complex analysis. Namely f is analytic in the domain $\Omega \subset \mathbb{C}$ (connected, open set) in $\mathbb{C}$ iff $\mathbf{f}'$ exists in Ω where $(\mathbf{f}')_i = f_i'$. Same with matrices, they can be seen as vectors in some $\mathbb{C}^m$. Everything in complex analysis that does not involve division clearly goes through in this more general context.

10.2 Singularities of the first kind; regular singularities

For the system of equations (56) a singularity, say 0 , is of the first kind if A has a first order pole at zero and is analytic in a punctured disk, say $\mathcal{D}$. Regular versus irregular singularities are classified according to the type of *solutions* the system admits. If in (61) $S(z)$ is meromorphic, then the singularity is *regular*, whereas if 0 is an essential singularity of S, the singularity is *irregular*. Sometimes even in the irregular singular case we may get formal power series for S, but they usually have zero radius of convergence. In the regular singular case therfore, the components of the fundamental solution are *finite sums of analytic functions, each multiplied by terms of the form* $z^\lambda(\ln z)^m$ *where* $\lambda \notin \mathbb{N} \cup \{0\}$ *and* $m \in \mathbb{Z}^+$.

We now show that singularities of the first kind imply that the singularity is *regular* and S is meromorphic. The converse is not true, but something along these lines holds if the system is written as an nth order equation, as we shall see.

We then let

$$A(z) = z^{-1} A_0(z)$$

where A_0 is analytic in the full disk $\mathbb{D}$, and for the problem to be interesting, $A_0(0) \neq 0$. Since A is singular only at 0, we can find a fundamental matrix M (invertible, of course) and analytic in any simply connected domain in $\mathcal{D}$. We want to show that $M(z)$ *grows at most polynomially as* $z \to 0$.

For this we want to control the norm of M and one way would be to estimate $\|M\|'$.

The problem is that, with the usual matrix norm $\sup_{\|u\|=1}\|Mu\|$, the function $\|M+tT\|$ may not be differentiable in t at $z=0\in\mathbb{R}$! Why?

We introduce a different norm, the Euclidian norm on components:

$$\|M\|_e := \sqrt{\sum_{i,j}|m_{ij}|^2} \tag{70}$$

We are not aiming at optimality in the estimates, because this will be otained later from Frobenius theory. All norms in finite dimensional spaces are equivalent (there are positive constants bounding any one of them in terms of any other one). In our case, by Cauchy-Schwarz,

$$\|MN\|_e^2 = \sum_{i,k}\left(\sum_j M_{ij}N_{jk}\right)^2 \le \sum_i\sum_k\left(\sum_j M_{ij}^2\right)\left(\sum_{j'} N_{j'k}^2\right)$$
$$= \left(\sum_{i,j} M_{ij}^2\right)\left(\sum_{j,k} N_{jk}^2\right) = \|M\|_e^2\|N\|_e^2 \tag{71}$$

(2)

Lemma 9. *If $M\neq 0$, then the function $\|M(z)\|_e := \|M+zT\|_e\|$ is differentiable at $z=0$, $z\in\mathbb{R}$ and*

$$\|M(z)\|_e'(0) \leqslant \|T\|_e$$

Proof. This follows from

$$\|M+zT\|_e^2 = \|M\|_e^2 + 2z\sum_{i,j}\mathrm{Re}(M_{ij}\overline{T_{ij}}) + O(z^2) \tag{72}$$

while, again by Cauchy Schwarz,

$$\left|\sum_{i,j}\mathrm{Re}(M_{ij}\overline{T_{ij}})\right| \leqslant \sum_{i,j}|M_{ij}||T_{ij}| \leqslant \|M\|_e\|T\|_e \tag{73}$$

as above, and since $\|M\|_e\neq 0$, we have $2\|M\|_e\|M\|_e' \leqslant 2\|M\|_e\|T\|_e$. □

We choose a direction $\gamma = e^{i\phi}$ and evolve towards the origin, taking $z=t\gamma$, $t\in\mathbb{R}$. Let $\varepsilon>0$ be small. Note that $M(t\gamma-\varepsilon\gamma) = M(t\gamma)-\varepsilon\gamma AM(t\gamma)+O(\varepsilon^2)$. Using the results above, the triangle inequality and the fact that ε is small, we see that

$$\frac{d}{dt}\|M\|_e \geqslant -\kappa t^{-1}\|M\|_e\,;\ \kappa = \max_{\mathbb{D}}\|A(z)\|_e \tag{74}$$

and thus

$$\frac{\|M\|_e'}{\|M\|_e} \geqslant -\frac{\kappa}{t} \Rightarrow \|M\|_e \leqslant Ct^{-\kappa} \tag{75}$$

(2) Slight simplification of a calculation by Irfan Glogic.

Note 9. We found the bounds only when approaching the origin along straight lines. This suffices for our purpose. Why? Is this always sufficient to show a singularity is removable, even with no underlying ODE?

Recalling that $M = S(z)z^B$ for some matrix B, we see that $\|S\| \leqslant Cz^{\|B\|+\kappa}$. Thus, if $m = \lfloor |\kappa| + \|B\|_e \rfloor + 1$, then $z^m S$ is analytic in $\mathbb{D} \setminus \{0\}$ and it is bounded at zero; as we know this means that $z^m S$ has a removable singularity at zero thus S has (at most) a pole at 0.

This also means that we have

$$M = \tilde{S} z^{\tilde{B}} \tag{76}$$

where $\tilde{S}$ is analytic at zero and $\tilde{B}$ is a constant matrix.

These estimates are not optimal, but getting the right growth power will follow easily from Frobenius' theory.

We also see that, if A has a higher order pole, say double pole, then the best estimate that we can get by the method above is $\|M\|_e \leqslant \exp(\kappa t^{-1})$. While this does not show that this *must* be the growth, it is not hard to construct simple examples in which the growth rate is indeed exponential (e.g., the scalar equation $f' = z^{-2} f$).

10.2.1 Singularities of first kind and power series solutions

Consider the following differential equations:

$$f' + (1+z)f = 1 \tag{77}$$

$$z(1-z)f'(z) + (1+z)^2 f(z) = 1 \tag{78}$$

$$z^2 f' - f = z \tag{79}$$

where the analysis is done near $z = 0$.

Note 10. (i) Of course, these first order equations that we use for illustration in this section can be solved in closed form, but this is not the point.

(ii) These equations can be made homogeneous (second order) by differentiating them.

The first equation has no singularities and the solution is thus entire. Let's see how this is reflected at the level of the recurrence relation for the coefficients. Plugging in $f = \sum_{k=0}^{\infty} c_k z^k$ we get

$$c_{k+1} = -\frac{1}{m+1}\left(c_k + c_{k-1}\right) \tag{80}$$

It is not hard to show inductively that $c_k \leqslant AB^k (k/2)!$ for some A and B (check). For (78) we get

$$c_k = -\frac{k-3}{k+1} c_{k-1} + \frac{1}{k+1} c_{k-2} \tag{81}$$

which, as in the hypergeometric example can be shown to lead to a series with radius of convergence 1.

In the third example however, we get

$$c_k = (k-1)c_{k-1} \Rightarrow c_k = A(k-1)! \tag{82}$$

At the level of the series, a negative power of x shifts the coefficient c_k to c_{k+1} while differentiation essentially results in multiplying c_k by k. Depending on the strength of the singularity relative to the order of the equation, this results in a balance of the type $c_{k+1} \sim c_k/k$, $c_{k+1} \sim c_k$ or in case the pole is of higher order than the order of the equation, $c_{k+1} \sim kc_k$. The behavior of solutions depends critically on this balance.

10.3 Detailed analysis of singularities of the first kind

Consider the system

$$w' = \frac{1}{z}Bw + A_1(z)w; \quad \text{or, in matrix form,} \quad M' = \frac{1}{z}BM + A_1(z)M \tag{83}$$

where B is a constant matrix and A_1 is analytic at zero. Let J be the Jordan normal form of B and $T^{-1}BT = J$. Then, we see that

$$T^{-1}MT = \frac{1}{z}T^{-1}BTT^{-1}MT + T^{-1}A_1(z)TT^{-1}MT \tag{84}$$

with $\tilde{M} = T^{-1}MT$ we see that

$$\tilde{M}' = \frac{1}{z}J\tilde{M} + A_2(z)\tilde{M} \tag{85}$$

where clearly A_2 is also analytic. In other words,

Remark 11. *In* (83) *we can assume, without loss of generality that B is in its Jordan normal form, J. We will thus study equations of the form*

$$y' = \frac{1}{z}Jy + A(z)y \tag{86}$$

where $A(z)$ is analytic.

There are three cases leading to somewhat different analytic properties of M.

1. All eigenvalues are distinct and do not differ by integers. In this case, the elements of the fundamental matrix have the analytic structure $\sum z^{\lambda_k} A_k(z)$ where λ_k are the eigenvalues of J and A_k are analytic.

2. Some eigenvalues can be repeated, but no two eigenvalues differ by *nonzero* integers. Then, the elements of the fundamental matrix are of the form $\sum_{k,l\leqslant n} z^{\lambda_k} \ln z^l A_{kl}(z)$.

3. Some eigenvalues differ by *nonzero* integers. Then the powers of z may differ from the eigenvalues.

10.4 Nondegenerate case

Assumption. No two eigenvalues of B differ by a nonzero integer.

Theorem 3. *Under the assumption above, (86) has a fundamental matrix solution in the form* $M(z) = Y(z)z^J$*, where* $Y(z)$ *is a matrix analytic in* $\mathcal{D}$.

Exercise 2. *Check that, if we had not arranged for* B *to be in its Jordan normal form, the solution of (83) would be* $M(z) = Z(z)z^B$*, where* $Z(z)$ *is a matrix analytic at zero.*

Proof. Clearly, it is enough to prove the theorem for (86). The solution of (86) must be of the form $M = Yz^C$ for some C and $Y = \sum_{k=0}^{\infty} Y_k z^k$ where the sum is convergent and Y_k are constant matrices. We try $Y_0 = I$:

$$Y(z) = I + zY_1 + z^2Y_2 + \cdots \tag{87}$$

we get

$$Y'z^C + \frac{1}{z}YCz^C = \frac{1}{z}JYz^C + AYz^C \tag{88}$$

Multiplying by z^{-C} we obtain

$$Y' + \frac{1}{z}YC = \frac{1}{z}JY + AY \tag{89}$$

or

$$Y' = \frac{1}{z}\Big(JY - YJ\Big) + AY \tag{90}$$

Using (88) we get, to leading order,

$$z^{-1}J = z^{-1}C \Rightarrow C = J \tag{91}$$

$$\begin{aligned} Y_1 + 2zY_2 + 3z^2Y_3 + \cdots = &\Big[(JY_1 - Y_1J) + z(JY_2 - Y_2J) + \cdots\Big] \\ &+ A_0J + zA_1J + \cdots + zA_0Y_1 + z^2(A_0Y_2 + A_1Y_1) + \cdots \end{aligned} \tag{92}$$

The associated system of equations, after collecting the powers of z is

$$kY_k = (JY_k - Y_kJ) + A_{k-1}J + \sum_{j=1}^{k-1} Y_jA_{k-j-1}; \quad k \in \mathbb{N} \tag{93}$$

or

$$V_kY_k = T_k(\{Y_j\}_{j<k}, z) \tag{94}$$

$$T_k(\{Y_j\}_{j<k}, z) := A_{k-1}J + \sum_{j=1}^{k-1} Y_jA_{k-j-1}; \quad k \in \mathbb{N} \tag{95}$$

where

$$V_kM := kM - (JM - MJ) \tag{96}$$

is a linear operator on matrices $M \in \mathbb{R}^{n^2}$. As a linear operator on a finite dimensional space, $V_k X = Y$ has a unique solution for every Y iff $\det V_k \neq 0$ or, which is the same, $kX - JX + XJ = 0$ implies $X = 0$ for *all* $k \geqslant 1$.

We show that this is the case, by showing that $Xv = 0$ for all generalized eigenvectors of J.

Let $v_\lambda := v$ be one of the eigenvectors of J. If $V_k X = 0$ we obtain, since $Jv = \lambda v$,

$$k(Xv) - J(Xv) + X\lambda v = 0 \tag{97}$$

or

$$J(Xv) = (\lambda + k)(Xv) \tag{98}$$

Here we use our assumption: $\lambda + k$ is not an eigenvalue of J. This forces

$$Xv = 0 \tag{99}$$

We let $v_0 = v$ and take the next generalized eigenvector, v_1, in the same Jordan block as v, if any.

We remind that we have the following relations between these generalized eigenvectors:

$$Jv_i = \lambda v_i + v_{i-1} \tag{100}$$

where $v_0 = v_lambda$ is the eigenvector and $1 \leqslant i \leqslant m - 1$ where m is the dimension of the Jordan block. With $i = 1$ we get

$$k(Xv_1) - J(Xv_1) + X(\lambda v_1 + v_\lambda) = 0 \tag{101}$$

and, using (99) (i.e., $Xv_\lambda = \lambda$), we get the same equation (102), now for Xv_1:

$$J(Xv_1) = (\lambda + k)(Xv_1) \tag{102}$$

and thus $Xv_1 = 0$. Inductively, we see that $Xv = 0$ for any generalized eigenvector of J, and thus $X = 0$.

This shows that (96) has a unique solution for each k, so the power series expansion exists order by order and we are left with proving its convergence, and for each Jordan block.

Now, we claim that $V_k^{-1} \leqslant Ck^{-1}$ for some C. We let $\mathcal{C}$ be the commutator operator, $\mathcal{C}X = JX - XJ$ Now $\|JX - XJ\| \leqslant 2\|J\|\|X\|$ and thus

$$V_k^{-1} = k^{-1}\left(I - k^{-1}\mathcal{C}\right)^{-1} = k^{-1}(1 + o(1)); \quad (k \to \infty) \tag{103}$$

Therefore, the function kV_k is bounded for $k \in \mathbb{R}^+$.

We rewrite the system (93) in the form

$$Y_k = V_k^{-1} A_{k-1} J + V_k^{-1} \sum_{j=1}^{k-1} Y_j A_{k-j-1}; \quad k \in \mathbb{N} \tag{104}$$

or, in abstract form, with $\mathsf{Y} = \{Y_j\}_{j\in\mathbb{N}}$, $(\mathsf{LY})_k := V_k^{-1} \sum_{l=0}^{k-2} A_l Y_{k-1-l}$, where we regard Y as a function defined on $\mathbb{N}$ with matrix values, with the norm

$$\|\mathsf{Y}\| = \sup_{n\in\mathbb{N}} \|\mu^{-n}\mathsf{Y}(n)\|; \quad \mu > 1 \tag{105}$$

we have

$$\mathsf{Y} = \mathsf{Y}_0 + \mathsf{L}\mathsf{Y} \tag{106}$$

Exercise 3. *Show that (636) is contractive for μ sufficiently large, in an appropriate ball that you will find.*

The solution of this exercise is given in the appendix. □

Note 12. *We have* **not** *seriously used the fact that J is a Jordan matrix in this proof. It follows that if J is replaced by any B with eigenvalues not differing by integers, we have $M = Y(z)z^B$.*

Exercise 4. *We proved the convergence of the series using the contractive mapping principle. But did we need to prove convergence? Justify your answer carefully!*

11 General case. Lowering an eigenvalue

Clearly the above procedure *fails* if $\lambda + k$ is an eigenvalue for some $k \in \mathbb{Z}^+$. We know that there must exist a solution of the form $W = Y(z)z^C$ for some C, where Y is analytic and Y_0 is invertible. So what fails in the previous calculation?

If, by suitable transformations we can bring an equation for which roots differ by nonzero integers to one in which they don't, the problem is solved in general. We show that there exists a transformation that brings (83) to an equation of the same form, but for which all the eigenvalues except one, say λ_j are the same and $\lambda_j \mapsto \lambda_j - 1$. Then repeated applications of such transformations would bring (83) to a form in which there exist repeating eigenvalues, but none differing by integers. (Analyze this statement!) To solve the general case, in which eigenvalues *may* differ by positive integers, we find transformations which decrease one eigenvalue by one, leaving all others the same and without changing the structure of the ODE.

Write J in the form

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \tag{107}$$

where J_1 is the Jordan block we care about, $\dim(J_1) = m \geqslant 1$, while J_2 is a Jordan matrix, consisting of the remaining blocks. The transformation we are looking for would change J into $J - I_1$ where

$$I_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{108}$$

where I is the identity matrix. That, in turn, would change the fundamental solution to

$$Yz^{J-I_1} \tag{109}$$

This suggests we try this change of variables in our equation. In matrix form,

$$M' = z^{-1}JM + AM \tag{110}$$

where we take $M = M_1 z^{I_1}$.

Without loss of generality we assume that the eigenvalue we want to lower is λ_1. Let the space $V(= \mathbb{R}^n$ or $\mathbb{C}^n)$ be decomposed as $V = V_1 \oplus V_2$ where V_1 is spanned by all eigenvectors and generalized eigenvectors corresponding to λ_1 and V_2 its orthogonal complement, I_1 the identity matrix on V_1 and I_2 the identity matrix on V_2.

Let

$$U = \begin{pmatrix} zI_1 & 0 \\ 0 & I_2 \end{pmatrix} \tag{111}$$

U is clearly invertible, with inverse

$$U^{-1} = \begin{pmatrix} z^{-1}I_1 & 0 \\ 0 & I_2 \end{pmatrix} \tag{112}$$

and commutes with J. We have

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \tag{113}$$

where by construction J_1 is a Jordan form matrix with all eigenvalues equal to λ_1 .

Clearly

$$UJ = JU \Rightarrow U^{-1}JU = J \tag{114}$$

Let $W = UQ$ (or rather, define $Q = WU^{-1}$). We have

$$\begin{aligned} U'Q + UQ' = z^{-1}JUQ + AUQ \Rightarrow \\ Q' = z^{-1}U^{-1}JUQ - U^{-1}U'Q + U^{-1}AUQ \\ = z^{-1}JQ - U^{-1}U'Q + U^{-1}AUQ \end{aligned} \tag{115}$$

If the blocks of A are

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{116}$$

direct calculation shows that

$$Q' = z^{-1}BQ + \tilde{A}Q \tag{117}$$

where $\tilde{A}$ is analytic and

$$B = \begin{pmatrix} J_1 - I_1 & A_{12} \\ 0 & J_2 \end{pmatrix} \tag{118}$$

Writing down the eigenvalue problem for B we get

$$(J_1 - I_1 - \lambda)v_1 + A_{12}v_2 = 0 \tag{119}$$

$$J_2 v_2 = \lambda v_2 \tag{120}$$

Thus, either λ is one of the eigenvalues of J_2, by definition different from λ_1 or else $v_2 = 0$ and the eigenvalue is obviously $\lambda = \lambda_1 - 1$ which is what we wanted to show.

12 Scalar $n-$th order linear equations

These are equations of the form

$$y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_n(z)y = 0; \quad a_j \text{ analytic in } \mathbb{D} \setminus \{0\} \tag{121}$$

Definition 10. An equation of the form (121) has a singularity of the first kind at zero if

$$a_i(z) = b_i(z)/z^i \tag{122}$$

where b_i are analytic at zero. We will see shortly the reason for this terminology.

12.1 Connection between systems of equations and higher order scalar ones

There is an obvious way in which an n-th order equation can be transformed to an n-dimensional first order system. Take for simplicity a second order equation

$$y'' + a(z)y' + b(z)y = 0 \tag{123}$$

and write the equivalent system

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -b(z) & -a(z) \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} \Leftrightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a(z) & -b(z) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{124}$$

A natural attempt at a definition of a singularity of the first kind for n-th order equations would be that that the associated system (43) has a first order singularity. But as mentioned, there is more than one way to arrive at a system, an this is not right.

For instance, the Euler equation

$$y'' = 6z^{-2}y \tag{125}$$

has as general solution

$$y = C_1 z^{-2} + C_2 z^3 \tag{126}$$

and we would expect it to correspond to a system with a singularity of the first kind, since the vector solution is bounded by a power of z near the origin. However, the system associated via (43) (or, here (231)) is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \left[z^{-2} \begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{127}$$

We can see the nature of the problem in the following way: in (125), if $y \sim z^m$ then $y' \sim mz^{m-1}$, $y/y' = u_1/u_2 = O(z)$. This is quite general, certainly it is the case for equations admitting convergent series as solutions, as these can be differentiated term by term). In a system, no component should play a special role, but here we ended up with $u_2 \gg u_1$ if solutions behave like powers. It is instead natural to take $u_1 = y/z$ and $u_2 = y'$, or, equivalently, $u_1 = y, u_2 = zy'$. We then get $u_2' = y' + zy'' = y' + 6z^{-1}y$ and thus the system

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \frac{1}{z}\begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{128}$$

which is indeed a system with a singularity of first kind.

More generally, the natural substitution is

$$u_k = z^{k-1}y^{(k-1)}, \; k = 1, 2, ..., n \tag{129}$$

We then have

$$u_k' = (k-1)z^{k-2}y^{(k-1)} + z^{k-1}y^{(k)} \Rightarrow zu_k' = (k-1)u_k + u^{(k+1)} \tag{130}$$

while, as usual, u_n is special, since $y^{(n)}$ can be written in terms of lower order derivatives, using (121):

$$\begin{aligned} zu_n' = (n-1)u_n + z^n y^{(n)} = (n-1)u_n \\ - z^n(b_n z^{-n} y + b_{n-1}z^{-n+1}y' + ... + b_1 z^{-1} y^{(n-1)}) = (n-1)u_n \\ - b_n y - b_{n-1}zy' - ... - b_1 z^{n-1}y^{(n-1)} = (n-1)u_n - b_n u_1 - b_{n-1}u_2 - ... - b_1 u_n \end{aligned} \tag{131}$$

In matrix form, the end result is the system

$$\boldsymbol{u}' = z^{-1}B\boldsymbol{u} \tag{132}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ & & \cdots & & & \\ -b_n(z) & -b_{n-1}(z) & -b_{n-2}(z) & -b_{n-3}(z) & \cdots & (n-1) - b_1(z) \end{pmatrix} \tag{133}$$

or, recalling (122),

$$\boldsymbol{u}' = z^{-1}B(0)\boldsymbol{u} + A(z)\boldsymbol{u} \tag{134}$$

where A is analytic at zero.

Corollary 11. *If an equation of the form (121) has a singularity of the first kind, then the singularity is regular (the general solution is a convergent combination of powers and logs).*

We will see that the converse is true also.

12.2 Possible solutions to the n-th order scalar equation

Assume that there is a set of n linearly independent of solutions of (122) in the form

$$y_j = \sum_{k,m=1}^{n'} H_{kmj}(z) z^{p_{k;j}} \ln^{l_{m;j}} z, \quad j = 1, ..., n \tag{135}$$

where the H_{kmj} are analytic at zero and $l_{m;j}$ are integers. Since they are linearly independent, their Wronskian does not vanish at some point. From the general theory of systems we know then that it cannot vanish anywhere for $z \neq 0$. Thus the system of equations for the coefficients a_j.

$$\{y_j^{(n)} + a_1(z) y_j^{(n-1)} + \cdots + a_n(z) y_j = 0, j = 1, ..., n\} \tag{136}$$

allows for expressing the coefficients a_k as rational functions of $y_j^{(l)}$. Thus a_k grow at most algebraically at zero. Since they are single-valued in $\mathbb{D} \setminus \{0\}$, it follows that the coefficients are meromorphic.

Note also that if y is a solution of (121), then $u(z) = y(ze^{i\beta})$ is a solution of the equation

$$u^{(n)} e^{-in\beta} + a_1(z) u^{(n-1)} e^{-i(n-1)\beta} + \cdots + a_n(z) u = 0 \tag{137}$$

As we proceeded in §9, we can define $\tilde{y}_j$ to be $y(ze^{2\pi i})$ (a notation for the analytic continuation of y on a circle around the origin). From (137) we get that $\tilde{y}_j$ are also solutions of (121) (which, of course, have to be linear combinations of y_j). In this way, we can eliminate the logs in the expressions (135), if there are any, see below.

Note 13. (i) For one term $f = H_{km}(z) z^{p_k} \ln^{l_m} z$ the difference between f and and $e^{2\pi i p_k}$ times its analytic continuation along a circle around the origin is

$$\hat{f} = H_{km}(z) z^{p_k} e^{2\pi i p_k} (\ln z + 2\pi i)^{l_m} - H_{km}(z) e^{2\pi i p_k} z^{p_k} (\ln z)^{l_m} \tag{138}$$

which is a sum of the form (135) with the powers of the logs multiplying z^{p_k} reduced by one. So we can transform a solution with $\max l_m = M$ to one in which $\max l_m = M - 1$. Thus, applying M operations of the type (138) we get a solution of (121) with no logs. That is, there is at least one solution of the form

$$\hat{y}_0 = \sum_{k=1}^{n} H_k(z) z^{p_k} \tag{139}$$

Note also that if $\alpha = e^{2\pi i p_1}$ and $\beta = e^{2\pi i p_2}$ are different, then we can eliminate, say, the term $H_1(z) z^{p_1}$ by replacing $\hat{y}_0$ by $\hat{y}_0(z) - e^{-2\pi i p_1} \hat{y}_0(ze^{2\pi i})$. Proceeding in this manner, we see that there are solutions of the form

$$y_0 = H(z) z^a \tag{140}$$

Exercise 1. Show that if a sequence of $\lambda' s$ differ by integers, there is always a solution of the form (140) for the one with the *most positive integer*.

for *some* a, since in the process of eliminating a log we might also eliminate other $\lambda's$ iff they differ from a by integers. We can always assume $H(0) = 1$ since any other starting power than zero can be absorbed into a, and multiplicative constants don't matter since the equation is linear.

Exercise 2. (a) Substitute $y = y_0 g$ in (121) and show that g' satisfies an equation of type (121) of order lower by one.

(b)Show that the condition (122) is preserved by changes of dependent variable as in (a).

Note that if a first order scalar equation

$$y' + z^{-n}A(z)y = 0 \tag{141}$$

with A meromorphic admits a solution of the form

$$y(z) = z^a H(z)$$

with H analytic then

$$H' + az^{-1}H + AH = 0 \Rightarrow A = -az^{-1} - H'/H \tag{142}$$

and thus –since H'/H is has a pole of order at most one whenever H is meromorphic– A has a pole of order at most one.

(c) Use (a) and (b) to show that (121) has a complete set of solutions of the form (135), then (122) is satisfied.

12.3 Frobenius' theorem

Systematizing what we have obtained so far we have the following.

Definition 12. An equation of the form (121) has a regular singularity at zero if there exists a fundamental set of solutions in the form of finite combinations of functions of the form

$$y_i = z^{\lambda_i}(\ln z)^{m_i} f_i(z); \quad \text{(by convention, } f_i(0) \neq 0) \tag{143}$$

where f_i are analytic, $m_i \in \mathbb{N} \cup \{0\}$.

Theorem 4 (Frobenius)**.** *An equation of the form* (121) *has a regular singularity at zero iff the singular point is of the first kind, that is iff* (122) *holds.*

We see one advantage of the scalar formulation of a differential system: we have the Frobenius theorem as an "iff" statement (recall (127)).

Note 14. We could have allowed l_m to be noninteger, since, by converting the equation into a system and noting that the coefficients of the system are single-valued, Lemma 8 implies that l_m are necessarily $\in \mathbb{N} \cup \{0\}$.

12.4 The indicial equation

We saw in Note 13 that Frobenius type solutions can be found in the form $z^\lambda(1+o(1))$. We insert this into the differential equation and note that $y^{(j)} = \lambda(\lambda-1)\cdots(\lambda-j+1)z^{\lambda-j}(1+o(1))$ and also that $a_j z^{\lambda-n+j}(1+o(1)) = b_j(0)z^{\lambda-n})(1+o(1))$. Thus, the equation for the leading power of z is

$$\lambda(\lambda-1)\cdots(\lambda-n+1)+\lambda(\lambda-1)\cdots(\lambda-n+1)b_1(0)+\cdots+b_n(0)=0 \quad (144)$$

This is the **indicial equation** and it determines all possible lowest powers("p_1") in a Frobenius-type solution–of the form (139).

12.4.1 Eigenvalues of $B(0)$ in (134)

The eigenvalue equation, $(B-\lambda I)x=0$, is easy to solve. If we expand this out as a system, using the explicit form (176), we get

$$(0-\lambda)x_0+x_1=0 \quad (145)$$
$$(1-\lambda)x_1+x_2=0 \quad (146)$$
$$(2-\lambda)x_2+x_3=0 \quad (147)$$
$$... \quad (148)$$
$$-b_n(0)x_0-b_{n-1}(0)x_1-\cdots-(b_1-[n-1-\lambda])x_{n-1}=0 \quad (149)$$

Without loss of generality we can take $x_0=1$. Then

$$x_1=\lambda,\ x_2=\lambda(\lambda-1),\ \cdots, x_{n-1}=\lambda(\lambda-1)\cdots(\lambda-(n-2))$$

and thus (524) is equivalent to

$$-b_n(0)-\lambda b_{n-1}(0)-\cdots-(b_1-[(n-1)-\lambda])\lambda(\lambda-1)\cdots(\lambda-(n-2))=0 \quad (150)$$

which is precisely (144). We have shown

Proposition 13. *The eigenvalues of $B(0)$ are precisely the roots of the indicial equation.*

12.5 Equations of the form (121) with regular singularities; analysis using the system formulation

Note first that if we write (121) as a system, we get a fundamental matrix solution of the form (64). Singling out a Jordan block B , we see that we get

$$z^B = e^{\lambda z \ln B} = z^\lambda\left(1+N\lambda\ln z+...+\frac{\lambda^{m-1}(\ln z)^m}{(m-1)!}N^{m-1}\right)$$

which is a matrix of the form

$$z^\lambda\begin{pmatrix} 1 & \ln z & \frac{\ln^2 z}{2!} & \cdots & \frac{\ln^m z}{(m-1)!} \\ 0 & 1 & \ln z & \cdots & \frac{\ln^{m-1} z}{(m-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (151)$$

and the solution matrix is

$$z^{\lambda}\begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1m} \\ S_{21} & S_{22} & \cdots & S_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ S_{m1} & S_{m2} & \cdots & S_{mm} \end{pmatrix}\begin{pmatrix} 1 & \ln z & \frac{\ln^2 z}{2!} & \cdots & \frac{\ln^m z}{(m-1)!} \\ 0 & 1 & \ln z & \cdots & \frac{\ln^{m-1} z}{(m-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{152}$$

By the substitution (129), the element 11 of the matrix product above is a solution y. Thus there is a solution of (121) of the form $z^{\lambda}S_{11}(z)$ where S_{11} is single-valued. On the other hand, this also needs to be a linear combination of the form (135). By the assumption on the nature of the solutions, S_{11} has a convergent series thus

$$y =: y_1 = z^{\lambda-(n-1)}F(z), \; n \in \mathbb{N} \tag{153}$$

where F is analytic. From this point on, we can proceed as in the previous section to conclude that the singular point is regular. Combining Proposition 13 with Theorem 3 and (375), we obtain the following result.

Proposition 14. *If the roots of the indicial equation do not differ by nonzero integers, then for a root λ of multiplicity m, there are m linearly independent solutions of* (121) *in the form*

$$y_{1,\lambda} = z^{\lambda}f_1(x), \; y_{2,\lambda} = z^{\lambda}f_1(x)\ln x + f_{11}(x), \; ...,$$
$$y_{m,\lambda} = z^{\lambda}f_1(x)\ln^m x + f_{m1}(x)\ln^{m-1} x + ... + f_{mm}(x) \tag{154}$$

12.5.1 Writing a system of equations as one higher order equation

Take first a simple case,

$$u' = au + bv, \quad v' = cu + dv; \quad (a, b, c, d \;\text{ analytic in some domain } \mathcal{D}) \tag{155}$$

and assume for simplicity that b does not vanish in $\mathcal{D}$. Then we write $v = b^{-1}(u' - au)$ and get

$$[b^{-1}(u' - au)]' = cu + db^{-1}(u' - au) \tag{156}$$

which expanded and normalized is an equation of the type (121), equivalent to (155). The same is true if $d \neq 0$. What if however this $b = d = 0$?

Let's look at the very simple system

$$u' = u; \quad v' = v; \quad \text{or, in matrix form,} \quad M' = M \tag{157}$$

Of course we can integrate it in closed form and the general solution is $u = C_1e^z, v = C_2e^z$. We would be tempted to say that there is no scalar equation equivalent to this. Indeed, a genuinely second order linear ODE should have two linear independent solution, since the existence of a solution to the IVP must exist and be unique. But we remember that in passing from an n-th order equation to a system we had a number of choice, far from equivalent. Here we can try something similar to (129). If we take $v = z^{-1}w$ in the second equation we get

$$u' = u; \quad w' = zv' + v = zv + v = w + z^{-1}w \tag{158}$$

and now we can find a second order system.

Exercise 3. Show that if $f_1, ..., f_n$ are analytic and the Wronskian does not vanish,

$$\begin{vmatrix} f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \\ f_1^{(n-2)} & f_2^{(n-2)} & \cdots & f_n^{(n-2)} \\ \cdots & \cdots & \cdots & \cdots \\ f_1 & f_2 & \cdots & f_n \end{vmatrix} \neq 0 \tag{159}$$

then there is an $n - th$ order equation of the form (121) having $f_1, f_2, ..., f_n$ as solutions. One way is to take a general equation of the form (121) and see if its coefficients can be determined from the fact that $f_1, f_2, ..., f_n$ are solutions.

Exercise 4. Show that by transformations of the type that led to (158) if necessary, we can find an n-th order scalar equation equivalent to a given system.

Exercise 5. Show that if a scalar equation of the form (121) has a solution of the form $z^\lambda(\log z)^p A_1(z) + o(z^\lambda(\log z)^p)$, then λ satisfies the indicial equation (144).

Exercise 6. Show that there are directions $\phi = \arg(z)$ so that, for some λ, $|z^{\lambda'}(\ln z)^p| = o(|z^\lambda|)$ as $z \to 0$; $\arg z = \phi$.

12.6 Examples of resonant and nonresonant second order equations

In view of what we know already about n-th order equations, it is clear that we can always arrange that one root of the indicial equation is zero, by a substitution of the form $y = z^\lambda y_1$. We look for simplicity at second order equations.

We assume that the second root, λ is nonzero real part, since a double root falls in the nondegenerate case that we looked at already.

We only need to look at the case $\mathrm{Im}\lambda \neq 0$ or if $\mathrm{Im}\lambda = 0$, then $\mathrm{Re}(n) > 0$.

Consider then the equation

$$z^2y'' + (1-n)zy' + zy = 0 \tag{160}$$

The equation can be solved in terms of Bessel functions,

$$y = C_1 z^{n/2} J_n(2\sqrt{z}) + C_2 z^{n/2} Y_n(2\sqrt{z}) \tag{161}$$

The exact solution is not what we are looking for here; we aim at the more modest goal of finding the behavior of solutions near singular points.

We see that zero is the only singular point. We use it as to illustrate the way the nature of the Frobenius solutions depend on the roots of the indicial equation,

$$\lambda(\lambda - n) = 0 \tag{162}$$

We look first for a power series solution starting with z^0. The recurrence relation for the coefficients is

$$c_1 = (n-1)c_0;\ m(m-n)c_m + c_{m-1} = 0, \quad m > 1 \tag{163}$$

The coefficient c_0 is arbitrary; we can take it for definiteness to be one. It is clear that if $n \notin \mathbb{N}$, the recurrence (163) has a solution and the solution is entire. There is a second solution, starting with z^n which, after division by z^n, is also entire.

If $n \in \mathbb{N}$ we see that the equation for c_n is $0 \cdot c_n = c_{n-1}$, that is, no power series solution with $c_0 = 1$ exists in this case (for exceptional equations, we may have $c_{n-1} = 0$, allowing for analytic solutions).

We know that there are solutions of the form $z^n Y(z)$ with $Y(z)$ analytic (in fact, entire). This is at the origin of this phenomenon: for such a solution we have $c_{n-1} = 0$ and c_n is undetermined, as it should. This also suggests what we should try. The solution y_0 starting with $c_0 = 1$ is nonanalytic, and thus its monodromy is nontrivial. On the other hand by linearity and the usual arguments, $y_0(ze^{2\pi i}) - y_0(z)$ must be a solution of (664) which is $o(x^{n-1})$. Indeed, the solution must be a combination of powers and logs, the allowed powers are z^0 and z^n, and all coefficients up to c_{n-1} are determined uniquely. We expect then $y_0(ze^{2\pi i}) - y_0(z) = Cz^n Y(z)$ for some C which in turn suggests that $y_0(z) = c_0 + c_1 z + ... + c_{n-1}z^{n-1} + Cz^n \ln z + ...$

To keep things relatively simple, we take $n = 3$. If we substitute $y_0 = c_0 + c_1 z + c_2 z^2 + Cz^3 \ln z + c_4 z^4 + d_4 z^4 \ln z + ...$ in (664) we get

$$2c_1 = c_0,\ 2c_2 = c_1,\ 3C = -c_2,\ 4d_1 + 1 = 0, 4c_4 + 5Cd_1 + c_3 = 0$$

and so on, c_3 is *free* and the rest of the coefficients can now can be determined uniquely.

The general solution has thus the form

$$y_0 = C_1[A(z) + z^n \ln z Y(z)] + C_2 z^n Y(z)$$

where A is analytic and Y is the second solution (entire).

Exercise 7. *Is $A(z)$ entire?*

12.7 Example

Let's consider again the equation

$$x(x-1)y'' + y = 0 \tag{164}$$

We want to use the theory we have developed this far, to find the shape of the generic solution at $0, 1, \infty$ (the only singular points of the equation).

As usual, we write $u_1 = y, u_2 = xy'$.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ (x-1)^{-1} & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{165}$$

or

$$\mathbf{u} = x^{-1}B\mathbf{u} + A\mathbf{u} \tag{166}$$

where B has eigenvalues $0, 1$ (Tr$= 1$, det$= 0$), is resonant.

If instead we had tried the naive transformation $u_1 = y, u_2 = y'$ we get

$$\mathbf{u}' = x^{-1}B\mathbf{u} + A\mathbf{u}; \quad B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad A := \begin{pmatrix} 0 & 1 \\ (1-x)^{-1} & 0 \end{pmatrix} \tag{167}$$

which is now nonresonant! This shows that resonance is not invariant under changes of variables, and that we may be able to reduce a resonant case to a nonresonant one by suitable transformations. The matrix B is brought to the Jordan normal form by the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad T^{-1}BT = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{168}$$

$$B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad A := \begin{pmatrix} 0 & 1 \\ 1/(1-x) & 0 \end{pmatrix} \tag{169}$$

It follows that the fundamental solution of this equation is

$$M = Y(x)x^B \tag{170}$$

where $Y(x)$ is analytic near zero (in this case, analytic in the unit disk, since $x = 1$ is the singular point closest to the origin (other than the origin itself).

Thus,

$$\begin{aligned} M = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\left(I + \ln x \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) &= \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{pmatrix} 1 & \ln x \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} y_{11} & y_{11}\ln x + y_{12} \\ y_{21} & y_{21}\ln x + y_{22} \end{pmatrix} \end{aligned} \tag{171}$$

and thus, by applying M to some initial condition (a, b) we get that the general solution of (164) in a neighborhood of 0 is

$$y = Axy_{11} + B(xy_{11}\ln x + y_{12}) \tag{172}$$

12.8 Example:Bessel functions

The equation

$$f'' + \frac{3}{2x} f' + f = 0 \tag{173}$$

has the general solution

$$C_1 x^{-1/4} J_{1/4}(x) + C_2 x^{-1/4} Y_{1/4}(x) \tag{174}$$

where J and Y are Bessel functions.

The indicial equation is obtained by substituting $f(x) = x^\lambda$ in (173), and reads

$$a^2 + a/2 = 0 \Rightarrow a \in \{0, -\tfrac{1}{2}\} \tag{175}$$

The roots are nonresonant, and thus there

In matrix form, as in the example before, we have

$$\mathbf{f}' = M\mathbf{f} \tag{176}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ -1 & -\dfrac{3}{2x} \end{pmatrix} = \frac{1}{x} \begin{pmatrix} 0 & x \\ -x & -\dfrac{3}{2} \end{pmatrix} = \frac{1}{x} B + A \tag{177}$$

where

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{3}{2} \end{pmatrix}; \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{178}$$

and thus the fundamental matrix is

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} x^{\begin{pmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}} = \begin{pmatrix} f_{11} & x^{-3/2} f_{12} \\ f_{21} & x^{-3/2} f_{22} \end{pmatrix} \tag{179}$$

This means that the general solution of the equation is of the form

$$f = C_1 f_1(x) + C_2 x^{-3/2} f_2(x) \tag{180}$$

with f_1 and f_2 analytic.

Exercise 8. *It also follows from the analysis above that $f_2(0) = 0$. Why?*

13 Some special functions and their regular singular points

Here is a good and up to date online source of information about special functions: http://dlmf.nist.gov/.

13.1 Hypergeometric functions

The general solution of the equation

$$x(x-1)y'' + [(a+b+1)x - c]y' + aby = 0 \tag{181}$$

is

$$y = {}_2F_1(a,b;c;x)A + Bx^{1-c}\ {}_2F_1(a-c+1, b-c+1; 2-c; x), \tag{182}$$

With the substitution

$$y = u, y' = v/x$$

we get

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \frac{1}{x} B \begin{pmatrix} u \\ v \end{pmatrix} + A(x) \begin{pmatrix} u \\ v \end{pmatrix} \tag{183}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 1-c \end{pmatrix} \tag{184}$$

and

$$A = \begin{pmatrix} 0 & 0 \\ -ab/(x-1) & -(a+1+b-c)/(x-1) \end{pmatrix} \tag{185}$$

Note that B is already in Jordan normal form. The eigenvalues of B are clearly 0 and $1-c$. Note that they are resonant when $c \in \mathbb{Z} \setminus \{1\}$.

Exercise 1. (a) Find the behavior near the origin of the general solution in the nonresonant case.

(b) In the resonant case, show that there is always a solution of the form $x^{1-c}A(x)$ if $1-c > 0$ and $A(x)$ otherwise, where A is analytic. Find the behavior of the second solution.

13.2 The exponential integral

This is defined for $\mathrm{Re} z > 0$ by

$$\mathrm{Ei}_1(z) = e^{-z} \int_0^\infty \frac{e^{-zt}}{1+t} dt \tag{186}$$

The function is defined by analytic continuation on the Riemann surface of the log at zero (or simply only on $\mathbb{C} \setminus (-\infty, 0]$). What is the behavior of $\mathrm{Ei}_1(z)$ as $z \to 0^+$? There are many ways to answer this question, but one of the simplest is to write a differential equation for $\mathrm{Ei}_1(z)$.

A straightforward calculation shows that

$$\mathrm{Ei}_1(z)' = -\frac{e^{-z}}{z} \tag{187}$$

With the substitution $\mathrm{Ei}_1(z) = g(z)e^{-z}$ we get

$$zg' - zg + 1 = 0 \tag{188}$$

We transform this into a second order homogeneous equation by differentiating once more in z:

$$g'' + (1/z - 1)g' - g/z = 0 \tag{189}$$

Clearly, zero is the only singular point of this equation. We write as before $g = u, g' = v/z$ and we get

$$u' = g' = v/z; \tag{190}$$

We get

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \frac{1}{x} B \begin{pmatrix} u \\ v \end{pmatrix} + A(x) \begin{pmatrix} u \\ v \end{pmatrix} \tag{191}$$

where A is analytic at zero and

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{192}$$

where clearly the eigenvalues of B are $0, 0$, nonresonant. Note that

$$z^B = \begin{pmatrix} 1 & \ln z \\ 0 & 1 \end{pmatrix} \tag{193}$$

so near zero we have(check)!

$$g = A_1(z) \ln z + A_2(z) \tag{194}$$

Here, it is easy enough to find the behavior of $\mathrm{Ei}_1(z)$ directly from the integral expression. The behavior is

$$\mathrm{Ei}_1(z) = -\gamma - \ln z + z A_3(z) \tag{195}$$

where γ is the Euler constant and A_3 is entire. Can you show this?

13.3 Bessel functions

The Bessel functions of the first kind satisfy the equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \tag{196}$$

or, in normal form,

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0 \tag{197}$$

The general solution of this equation is

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x) \tag{198}$$

In this case, the system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \frac{1}{x} B \begin{pmatrix} u \\ v \end{pmatrix} + A(x) \begin{pmatrix} u \\ v \end{pmatrix} \tag{199}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix} \tag{200}$$

with eigenvalues $\nu, -\nu$. In the nonresonant case there are two solutions which behave near zero like

$$x^{\pm\nu} A_{\pm}(x) \tag{201}$$

with $A_{\pm}$ analytic.

The algorithm is clear. I attach a Maple file with the general procedure, in which, instead of $e1$ you would insert any second order ODE. Finally, let's make a simple connection with equilibria. If we have a system of the form

$$x' = ax + by \tag{202}$$

$$y' = cx + dy \tag{203}$$

and the associated matrix is diagonalizable, then we can bring it to the form

$$u' = \lambda_1 u; \quad v' = \lambda_2 v \tag{204}$$

Of course, this can be easily solved in closed form. But we also note that we can write

$$\frac{dv}{du} = b\frac{v}{u}; \quad b = \frac{\lambda_2}{\lambda_1} \tag{205}$$

which perhaps the simplest case we can think of within Frobenius theory. Suppose first that $b \in \mathbb{R}$, then based on Frobenius theory, it is very easy to draw the phase portrait. Discuss also the case when b is complex, and the case when the Jordan form of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is nontrivial.

13.4 Reduction of order

Let λ_1 be a characteristic root such that $\lambda_1 + n$ is not a characteristic root. Then, there is a solution of (121) of the form $y_1 = z^{\lambda_1}\varphi(z)$, where $\varphi(z)$ is analytic and we can take $\varphi(0) = 1$.

We can assume without loss of generality that $\lambda_1 = 0$. Indeed, otherwise we first make the substitution $y = z^{\lambda_1} w$ and divide the equation by z^{λ_1}.

The general term of the new equation is of the form

$$z^{-\lambda_1} b_l z^{-l} (z^{\lambda_1} w)^{n-l} = z^{-\lambda_1} b_l z^{-l} \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} (z^{\lambda_1})^{(n-l-j)}$$

$$= z^{-\lambda_1} b_l z^{-l} \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} z^{\lambda_1 - n + l + j} = b_l \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} z^{-(n-j)} \tag{206}$$

```
> with(linalg):
> e1:=z^2*diff(y(z),z,z)+z*diff(y(z),z)+(z^2-nu^2)*y(z) = 0;
```

$$e1 := z^2\left(\frac{d^2}{dz^2}y(z)\right) + z\left(\frac{d}{dz}y(z)\right) + \left(z^2 - \nu^2\right)y(z) = 0 \quad (1)$$

```
> d2:=diff(y(z), z, z)=solve(e1,diff(y(z), z, z));
```

$$d2 := \frac{d^2}{dz^2}y(z) = -\frac{z\left(\frac{d}{dz}y(z)\right) + y(z)\,z^2 - y(z)\,\nu^2}{z^2} \quad (2)$$

```
> s1:=y(z)=u(z);
```

$$s1 := y(z) = u(z) \quad (3)$$

```
> s2:=diff(y(z),z)=v(z)/z;
```

$$s2 := \frac{d}{dz}y(z) = \frac{v(z)}{z} \quad (4)$$

```
> diff(s2,z);
```

$$\frac{d^2}{dz^2}y(z) = \frac{\frac{d}{dz}v(z)}{z} - \frac{v(z)}{z^2} \quad (5)$$

```
> subs(d2,%);
```

$$-\frac{z\left(\frac{d}{dz}y(z)\right) + y(z)\,z^2 - y(z)\,\nu^2}{z^2} = \frac{\frac{d}{dz}v(z)}{z} - \frac{v(z)}{z^2} \quad (6)$$

```
> subs(s2,%);
```

$$-\frac{v(z) + y(z)\,z^2 - y(z)\,\nu^2}{z^2} = \frac{\frac{d}{dz}v(z)}{z} - \frac{v(z)}{z^2} \quad (7)$$

```
> subs(s1,%);
```

$$-\frac{v(z) + u(z)\,z^2 - u(z)\,\nu^2}{z^2} = \frac{\frac{d}{dz}v(z)}{z} - \frac{v(z)}{z^2} \quad (8)$$

```
> solve(%,diff(v(z),z));
```

$$\frac{u(z)\left(-z^2 + \nu^2\right)}{z} \quad (9)$$

```
> expand(%);
```

$$-u(z)\,z + \frac{u(z)\,\nu^2}{z} \quad (10)$$

```
> mm:=matrix([[0,1],[nu^2,0]]);
```

$$mm := \begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix} \quad (11)$$

```
> jordan(%);
```

Figure 6:

which is of the same type as (121).

Thus we assume $\lambda_1 = 0$ and take $y = \varphi w$. As discussed, we can assume

$\varphi(0) = 1$. The equation for w is

$$\sum_{l=0}^{n} z^{-l} b_l \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} \varphi^{(n-l-j)} = 0 \tag{207}$$

or

$$\sum_{j=0}^{n} w^{(j)} \sum_{l=0}^{n-j} z^{-l} b_l \binom{n-l}{j} \varphi^{(n-l-j)} = 0 \tag{208}$$

or also

$$\sum_{j=0}^{n} w^{(n-j)} \sum_{l=0}^{j} z^{-l} b_l \binom{n-l}{j} \varphi^{(j-l)} = 0 \tag{209}$$

We note that this equation, after division by φ (recall that $1/\varphi$ is analytic) is of the same form as (121). However, now the coefficient of w is

$$\sum_{l=0}^{n} z^{-l} b_l \binom{n-l}{0} \varphi^{(n-l)} = \sum_{l=0}^{n} z^{-l} b_l \varphi^{(n-l)} = 0 \tag{210}$$

since this is indeed the equation φ is solving.

We divide the equation by φ (once more, remember $\varphi(0) = 1$), and we get

$$\sum_{j=0}^{n-1} w^{(1+(n-1-j))} \tilde{b}_j = 0 \tag{211}$$

where

$$\tilde{b}_j = \sum_{l=0}^{j} z^{-l} b_l \binom{n-l}{j} \frac{\varphi^{(j-l)}}{\varphi} \tag{212}$$

has a pole of order at most j, or

$$\sum_{j=0}^{n-1} g^{(n-1-j)} \tilde{b}_j = 0 \tag{213}$$

with $w' = g$. This is an $(n-1)$th order equation for g, and solving the equation for w reduced to solving a lower order equation, and one integration, $w = \int g$.

Thus, by knowing, or assuming to know, one solution of the nth order equation, we can reduce the order of the equation by one. Clearly, the characteristic roots for the g equation are $\lambda_i - \lambda_1 - 1$, $i \neq 1$. We can repeat this procedure until the equation for g becomes of first order, which can be explicitly solved. This shows what to do in the degenerate case, other than, working in a similar (in some sense) way with the equivalent nth order system.

14 Nonlinear systems

A point, say $z = 0$ is a singular point of the first kind of a nonlinear system if the system can be written in the form

$$y' = z^{-1}h(z, y) = z^{-1}(L(z)y + f(z, y)) \tag{214}$$

where h is analytic in z, y in a neighborhood of $(0, 0)$. We will not analyze these systems in detail, but much is known about them, [6] [2]. The problem, in general, is nontrivial and the most general analysis to date for one singular point is in [6], and utilizes techniques beyond the scope of our course now. We present, without proofs, some results in [2], which are more accessible. They apply to several singular points, but we will restrict our attention to just one, in the setting of (214). In the nonlinear case, a "nonlinear nonresonance" condition is needed, namely: if λ_i are the eigenvalues of $L(0)$, we need a *diophantine condition*: for some $\nu > 0$ we have

$$\inf\left\{(|\mathbf{m}|+k)^{\nu}|k+\mathbf{m}\cdot\boldsymbol{\lambda}-\lambda_i| \;\middle|\; \mathbf{m}\in\mathbb{N}^n, |\mathbf{m}|>1, k\in\mathbb{N}\cup\{0\}; i\leqslant n\right\} > 0 \tag{215}$$

Furthermore, $L(0)$ is assumed to be diagonalizable. (In [6] a weaker nonresonance condition is imposed, known as the Brjuno condition, which is known to be optimal.)

Proposition 15. Under these assumptions, There is a change of coordinates $y = \Phi(z)u(z)$ where Φ is analytic with analytic inverse, so that the system becomes

$$u' = z^{-1}h(z, u) = z^{-1}Bu + \tilde{f}(z, u)) \tag{216}$$

where B is a constant matrix and $\tilde{f}$ is analytic.

Proposition 16. The system (216) is analytically equivalent *in a neighborhood of* $(0, 0)$, that is for small u as well as small z, to its linear part, namely to the system

$$w' = z^{-1}Bw \tag{217}$$

In terms of solutions, it means that the general *small* solution of (214) can be written as

$$y = H(z, \Phi(z)z^B C) \tag{218}$$

where $H(u, v)$ is analytic as a function of two variables, C is an arbitrary constant vector. The diophantine, and more generally, Brjuno condition is generically satisfied. If the Brjuno condition fails, equivalence is still possible, but unlikely. The structure of y in an equation of the form (218) is

$$y_j(z) = \sum_{m,k} c_{k,\mathbf{m}} z^k z^{\mathbf{m}\cdot\boldsymbol{\lambda}} \tag{219}$$

15 Variation of parameters

As we discussed, a linear nonhomogeneous equation can be brought to a linear homogeneous one, of higher order. While this is useful in a theoretical quest, in practice, it is easier to solve the associated homogeneous system and obtain the solution to the nonhomogeneous one by integration. Indeed, if the matrix equation

$$Y' = B(z)Y \tag{220}$$

has the solution $Y = M(z)$, then in the equation

$$Y' = B(z)Y + C(z) \tag{221}$$

we seek solutions of the form $Y = M(z)W(z)$. We get

$$M'W + MW' = B(z)MW + C(z) \quad \text{or} \quad M(z)W' = C(z) \tag{222}$$

giving

$$Y = M(z)\int_a^z M^{-1}(s)C(s)ds \tag{223}$$

16 Equilibria

We start with the simple example of the physical pendulum. It is helpful in a number of ways, since we have a good intuitive understanding of the system. Yet, the ideal (frictionless) pendulum has nongeneric features.

We can use conservation of energy to write

$$\frac{1}{2}mv^2 + mgl(1-\cos\theta) = const \tag{224}$$

where θ is the angle and $v = ld\theta/dt = l\omega$, so with $l = m = 1$ we get

$$\theta'' = -\sin\theta \tag{225}$$

16.1 Exact solutions

This equation can be solved exactly, in terms of Weierstrass elliptic functions. Integration could be based on (226), and also by multiplication by θ' and integration, which leads to the same.

$$\frac{1}{2}\theta'^2 - \cos\theta = C \tag{226}$$

$$\int_0^\theta \frac{ds}{\sqrt{C+2\cos s}} = t + t_0 \tag{227}$$

With the substitution $\tan(\theta/2) = u$ we get

$$\int_0^{\tan(\theta/2)} \frac{du}{\sqrt{1+u^2}\sqrt{C+1+(C-1)u^2}} = t + t_0 \tag{228}$$

Whenever a differential system can be reduced to mere integrations as above, we say that the system is integrable by quadratures. On the other hand, by definition the elliptic integral of the first kind, $F(z,k)$ is defined as

$$F(z,k) = \int_0^z \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{229}$$

and we get, with $K = \sqrt{2}/\sqrt{1+C}$,

$$iKF\left(\cos(z/2), K\right)\Big|_0^\theta = t + t_0 \tag{230}$$

Inverting, we see that θ is a Weierstrass elliptic function of t. At this point, we should study elliptic functions to proceed. They are in fact very interesting and worthwhile studying, but we'll leave that for later. For now, it is easier to gain insight on the system from the equation than from the properties of elliptic functions.

16.2 Discussion and qualitative analysis

Written as a system, we have

$$\begin{aligned} \theta' &= \omega \\ \omega' &= -\sin\theta \end{aligned} \tag{231}$$

The point $(0,0)$ is an equilibrium, and $\theta = 0, \omega = 0$ is a solution. So are the points $\theta = n\pi, \omega = 0$, $n \in \mathbb{N}$. In general, a point θ_0 is an equilibrium of the equation $\dot{\theta} = \mathbf{F}(\theta)$ if $\mathbf{F}(\theta_0) = 0$, implying that θ_0 is a solution. The physical interpretation here is clear, if $\theta = 0, \omega = 0$ the system is in equilibrium.

Note that (231) is a *Hamiltonian system*, i.e., it is of the form

$$\begin{aligned} \theta' &= \frac{\partial H(\theta,\omega)}{\partial \omega} \\ \omega' &= -\frac{\partial H(\theta,\omega)}{\partial \theta} \end{aligned} \tag{232}$$

where $H(\theta,\omega) = \frac{1}{2}\omega^2 + 1 - \cos\theta$. For Hamiltonian systems, we see that H is a *conserved quantity*, that is $H(\theta(t),\omega(t)) = const$ along a given *trajectory* $\{(\theta(t),\omega(t)) : t \in \mathbb{R}\}$ (just calculate $\frac{d}{dt}H$). The trajectories are thus the level lines of H, that is

$$H(\theta,\omega) = \frac{1}{2}\omega^2 + 1 - \cos\theta = E \tag{233}$$

(we artificially added 1 to make $H \geqslant 0$, since H is defined by the differential system up to an additive constant). Of course, not every 2-dimensional system $y' = F(y)$ is Hamiltonian; one obvious condition is $\nabla F = 0$. We will return to this later.

Drawing the phase portrait of the system (say two-dimensional) means plotting the vector field F, its special points, trajectories of interest and so on. A trajectory is a set $\{(\theta(t), \omega(t)) : t \in A \subset \mathbb{R};\ \theta(t_0) = \theta_0, \omega(t_0) = \omega_0\}$ where A is typically taken to be the domain of existence of the solution.

In our example each trajectory is associated with a given value of $E = H$ and the shape depends on E.

The trajectories are level sets of H; at all points where $\nabla H \neq 0$, that is where the right side of (578) is nonzero, by the implicit function theorem, the trajectories are analytic (more generally if H is smooth, so will be the trajectories); otherwise they are typically singular.

Let first $0 < E < 2$. We have $1 - \cos\theta = 2\sin^2(\theta/2) = E - \omega^2/2 < 2$ and thus $\theta \in (-\alpha, \alpha)$, $\alpha < \pi/2)$. Also, $\omega^2/2 = E - 2\sin^2(\theta/2) < 2$ and thus $\{(\theta(t), \omega(t)) : t \in \mathbb{R}\}$ are compact sets; in fact the curves are closed (why?) and non-intersecting (since $\nabla H \neq 0$), smooth boundaries of the domains $H \leqslant E$. Since $\nabla H = 0$ only at $(0,0)$ and $H > 0$ otherwise, the maximum of H occurs on the boundary of $\{(\theta, \omega) : H(\theta, \omega) \leqslant E\}$.

Physically, for initial conditions close to zero, the pendulum would periodically swing around the origin, with amplitude limited by the total energy.

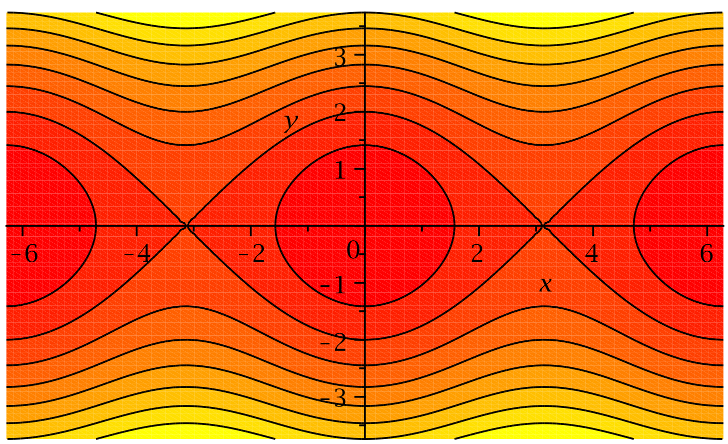

Figure 7: Contour plot of $\omega^2/2 - \cos\theta$

Fig. 21 represents a numerical contour plot of $\omega^2/2 - \cos\theta$. If we zoom in, we see that the program had difficulties at the critical points $\pm\pi$, showing once more that there is something singular there.

16.3 Linearization of the phase portrait

Let's first analyze the system approximately for small E. Along any such trajectory, θ is small too and we have $E = H(\theta, \omega) \approx \frac{1}{2}\omega^2 + \frac{1}{2}\theta^2$. The trajectories are approximately circles (this needs a more serious discussion, coming shortly). The flow for this approximate Hamiltonian is $\theta' = \omega, \omega' = -\theta$ or $\theta'' = -\theta$, the harmonic oscillator.

This suggests changing variables to u, ω with $u^2/2 = (1 - \cos\theta)$:

$$(1 - u^2/2) = \cos\theta; \quad u \in [-2, 2] \ \text{ or } u^2 = 4\sin(\theta/2)^2 \tag{234}$$

which defines two holomorphic changes of coordinates

$$u = \pm 2\sin(\theta/2) \tag{235}$$

These are indeed biholomorphic changes of variables until $\sin(\theta/2)' = 0$ that is, $\theta = \pm\pi$. With either of these changes of coordinates we get from (231)

$$u' = \pm\cos(\theta/2)\omega \Rightarrow \pm\sin(\theta/2)u' = uu' = \frac{1}{2}(2\sin(\theta/2)\cos(\theta/2)\omega$$

or

$$uu' = \omega\sin\theta \tag{236}$$

$$\omega' = -\sin\theta \tag{237}$$

Since the trajectories are graphs of curves $F(u, \omega)$ determined by $du/d\omega$ or $d\omega/du$ whichever of the two exists, this system would give the same trajectories family as

$$u' = \omega \tag{238}$$

$$\omega' = -u \tag{239}$$

for which the exact solution, $A\sin t, A\cos t$ gives rise to circles. This is consistent of course with what we would get by making the same substitution, (242) in (576). We note again that in (242) we have $u^2 \in [0, 4]$, so the equivalence does not hold beyond $u = \pm 2$. The level sets $H \leqslant E < 2$ are analytically conjugated to circles.

What about the other equilibria, $\theta = (2k + 1)\pi$? It is clear, by periodicity and symmetry that it suffices to look at $\theta = \pi$. If we make the change of variable $\theta = \pi - s$ we get

$$s' \qquad = -\omega \tag{240}$$

$$\omega' = -\sin(\pi - s) \quad = -\sin s \tag{241}$$

In this case, the same change of variable, $u = 2\sin(s/2)$ gives a set of orbits equivalent to

$$u' = \omega \tag{242}$$

$$\omega' = u \tag{243}$$

implying $\omega^2 - u^2 = E$ as long as the change of variable is meaningful, that is, for $u < 2$, or $|s| < \pi$. So the curves associated to (240) are analytically conjugated to the hyperbolas $\omega^2 - u^2 = E$. The equilibrium is unstable, points starting nearby necessarily moving far away. The point $\pi, 0$ is a saddle point.

The trajectories starting at π are *heteroclinic*: they link different saddles of the system. Only "exceptional" systems have heteroclinic trajectories (or homoclinic ones, connecting a fixed point to itself.

In our case, heteroclinic trajectories correspond to $E = 2$ and this gives

$$\omega^2 = 2(1 + \cos(\theta)) \tag{244}$$

or

$$\omega^2 = 4\cos(\theta/2)^2 \tag{245}$$

that is, the trajectories are given explicitly by

$$\omega = \pm 2\cos(\theta/2) \tag{246}$$

This is a case where the elliptic function solution reduces to elementary functions: The equation

$$\frac{d\theta}{dt} = 2\cos(\theta/2) \tag{247}$$

has the solution

$$\theta = 2\arctan(\sinh(t + C)) \tag{248}$$

We see that the time needed to move from one saddle point to the next one is infinite.

Note that we can fully describe the trajectories –in terms of elementary functions. In the process however, the time dependence, which was the parametrization, is lost. It is a different matter to solve (231).

16.4 Connection to regular singularities

Note that at the equilibrium point $(\pi, 0)$ the system of equations is analytically equivalent, insofar as trajectories go, to the system

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \tag{249}$$

The eigenvalues of the matrix are ± 1 with (unnormalized) eigenvectors $(1, 1)$ and $(-1, 1)$. Thus, the change of variables to bring the system to a diagonal form is $x = \xi + \eta, v = \xi - \eta$. We get

$$\xi' + \eta' = \xi - \eta \tag{250}$$

$$\xi' - \eta' = \xi + \eta \tag{251}$$

By adding and subtracting these equations we get the diagonal form

$$\xi' = \xi \tag{252}$$

$$\eta' = -\eta \tag{253}$$

or

$$\frac{d\xi}{d\eta} = -\frac{\xi}{\eta};\ or\ \xi_\eta + \frac{1}{\eta}\xi = 0 \tag{254}$$

a standard equation with a regular singularity. Clearly the solutions of (254) are $\xi = C/\eta$ with $C \in (-\infty, \infty)$, and insofar as the phase portrait goes, we could have written $\eta_\xi + \frac{1}{\xi}\eta = 0$, which means that the trajectories are the curves $\xi = C/\eta$ with $C \in [-\infty, \infty]$, hyperbolas and the coordinate axes. In the original variables, the whole picture is rotated by 45°.

16.5 Completing the phase portrait

We see that, for $E > 2$ we have

$$\omega = \pm\sqrt{2E + 2\cos\theta} \tag{255}$$

With one choice of branch of the square root (the solutions are analytic, after all), we see that $|\omega|$ is bounded, and it is an open curve, defined on the whole of $\mathbb{R}$. Note that the explicit form of the trajectories, given by (576) does not, in general, mean that we can solve the second order differential equation. The way the pendulum position depends on time, or the way the point moves along these trajectories, is still transcendental.

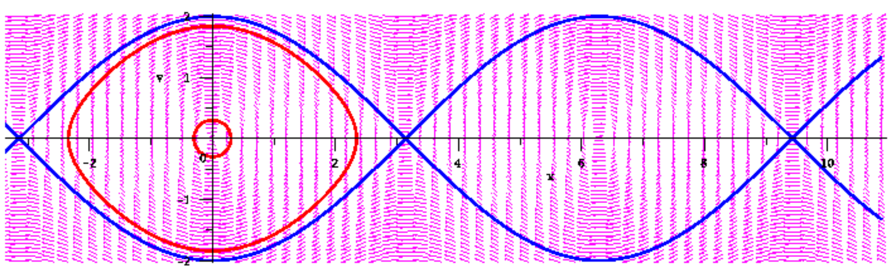

Figure 8: Contour plot of $y^2/2 - \cos x$

16.6 Local and asymptotic analysis

Near the origin, for $E = \varepsilon^2$ small, we have

$$\theta' = \omega \tag{256}$$
$$\omega' = -\theta + \theta^3/6 + ... \tag{257}$$

implying

$$\theta' = \omega \tag{258}$$
$$\omega' \approx -\theta \tag{259}$$

which means

$$\theta \approx \varepsilon \sin t \tag{260}$$

$$\omega \approx \varepsilon \cos t \tag{261}$$

For E very large, we have

$$\frac{d\theta}{\sqrt{E+\cos\theta}} = d\theta(E+\cos\theta)^{-1/2} = d\theta E^{-1/2}(1+\cos\theta/E)^{-1/2}$$

$$= d\theta(E^{-1/2} - \frac{1}{2}\cos\theta/E^{-3/2} + \cdots) \tag{262}$$

which means

$$E^{-1/2}\theta + \frac{1}{2}\sin\theta/E^{-3/2} + \cdots = t + t_0 \tag{263}$$

or

$$\theta = E^{1/2}(t+t_0) - \frac{1}{2}\sin(E^{1/2}t)/E^{-3/2} + \cdots = \tag{264}$$

The solutions near the critical point $(\pi, 0)$ can be analyzed similarly.

Local and asymptotic analysis often give sufficient qualitative, and sometimes quantitative information about all solutions of the equation. Of course we will justify these approximations.

17 Equilibria

In [7], Chapter 1.3 about nonlinear systems starts with the words:

"We must start by admitting that almost nothing beyond general statements can be made about most nonlinear systems. In the remainder of this book we will meet some of the delights and horrors about such systems, but the reader must bear in mind that the line of attack we develop in this text is only one and that any other tool in the workshop of applied mathematics, including numerical integration, perturbation methods, and asymptotic analysis, can and should be brought to bear on a specific problem". Since the book was written, there has been substantial progress, especially in using tools of asymptotic analysis, to find the behavior of nonlinear systems. We will see about these later but for now, we start with classical results and tools.

17.1 Flows

Consider the system

$$\frac{dx}{dt} = F(x) \tag{265}$$

where F is smooth enough. Such equations can be considered in $\mathbb{R}^n$ or, more generally, in Banach spaces.

As we know by now, if x_0 is a regular point for F, then there exists a unique local solution of (265) with $x(0) = x_0$.

Remark 15. (a) Note that equilibria, defined as points where $F(x_e) = 0$ are singular points of the *field*. *Trajectories* can intersect there. But this does not mean that *flows* are singular there. Indeed, if we write

$$x(t) = x_0 + \int_0^t F(x(s))ds \tag{266}$$

and the field is smooth at x_0, the map above is contractive for $|t| < \varepsilon$ for small enough $\varepsilon > 0$, and thus there is a unique solution for $|t| < \varepsilon$. In particular, if $x_0 = x_e$ then $x(t) \equiv x_e$. See also Remark 18 below.

The initial condition x_0 is mapped by the solution of the differential equation (265) into $x(t)$ where $t \in (-a, b)$ for some $a > 0, b > 0$ depending on the domain of existence of a solution to (266) and one or both a, b may be infinity.

Definition 17. The map $x(0) \to x(t)$ written as $f^t(x_0)$ is the flow associated to F.

Note 16 (Semigroup property)**.** We note the (commutative) semigroup property $f^0 = I, f^{s+t} = f^s f^t$ as long as the flow makes sense. This follows from uniqueness of solutions, giving $x(t+s; x_0) = x(s; x(t; x_0))$.

17.1.1 Fixed points, hyperbolic fixed points in $\mathbb{R}^n$

Example. If $F(x) = Bx$ where B **does not depend on** x, then the general solution is

$$x = e^{Bt} x_0 \tag{267}$$

where x_0 is the initial condition at $t = 0$. (Note again that a simple exponential formula does not exist, in general, if B depended on t.) Note that $(D_x f)(0) = B$ and the flow f is given by the linear map

$$f^t(x_0) = e^{DF(0)t} x_0 \tag{268}$$

Note 17. Remember that the eigenvalues of $e^{B\alpha}$ are $e^{\lambda_i \alpha}$ where λ_i are the eigenvalues of B.

Definition 18. *The point x_0 is a fixed point of f if $f^t(x_0) = x_0$ for all t.*

Proposition 19. Let f be the flow associated to the smooth field F. Then x_0 is a fixed point of f *iff* $F(x_0) = 0$.

Proof. Since x is differentiable in any domain where F is smooth, we have $x(t+\Delta t) = x(t) + F(x_0)\Delta t + o(\Delta t)$ for small Δt. Then $x(t+\Delta t) = x(t)$ implies $x(t) = x_0 = x(t+\Delta t)$. We get $F(x_0) = F(x_0)\Delta t + o(\Delta t)$ as $\Delta t \to 0$. In the limit $\Delta t \to 0$ we get $F(x_0) = 0$. Conversely, it is obvious that $F(x_0) = 0$ implies that $x(t) = x_0$ is a solution of (265), and this solution is unique, by Remark 15. $\square$

Remark 18. This also shows that if closures of trajectories intersect (as sets) at an equilibrium, then along any nontrivial trajectory ending at x_0 (that is, a trajectory other that $x(t) = x_0$ we must have $x(t) \neq x_0$ for all $t \in \mathbb{R}$. Indeed, we showed that there is a $\varepsilon > 0$ s.t. solutions of (266) exist and are unique. If $x(t) = x_0$ then $x(t \pm \varepsilon/2) = 0$ and it follows that $x(t) = x_0$ for all t. Thus, the only possibility for the closure of a nontrivial trajectory to be x_0 is $x_0 = \lim_{t\to\infty} x(t)$ (possibly along a subsequence).

Assume 0 is a fixed point of f, $F(0) = 0$. The flow f depends on two variables, x_0 and t. Since $x(t; x_0) = f^t(x_0)$, we clearly have

$$\frac{\partial f}{\partial t} = F(x(t; x_0)) = F(f^t t(x_0)) \tag{269}$$

This shows that f^t acts as a nonlinear analog of the fundamental matrix of a linear system.

To see what $\dfrac{\partial f}{\partial x_0}$ is near 0, we write

$$x' = F(x) = F(0) + (DF)(0)x + o(x) = (DF)(0)x + o(x) \tag{270}$$

We thus expect to leading order

$$x' = (DF)(0)x \Rightarrow x = e^{t(DF)(0)}x_0 \Rightarrow \frac{\partial f}{\partial x_0} = e^{t(DF)(0)} \tag{271}$$

This is indeed the case, and it is shown below.

Proposition 20. *If f is associated to the C^1 field F and x_0 is a fixed point of f, then $D_x f^t|_{x=x_0} = e^{DF(x_0)t}$.*

That is, the flow is tangent to the linear flow.

Proof. Without loss of generality we take $x_0 = 0$. Take the initial condition $x(t = 0) = x_0$ small enough. Let $DF(0) = B$. We have $F(x) = Bx + g(x)$ where $g(x) = o(x)$ for small x. Note that we also have for x_1, x_2 close to 0 and close to each other

$$F(x_2) - F(x_1) = DF(x_2)(x_2 - x_1) + o(x_2 - x_1) \tag{272}$$

and thus

$$F(x_2) - F(x_1) - B(x_2 - x_1) = g(x_2) - g(x_1) = o(1)(x_2 - x_1) \tag{273}$$

We have $x' = Bx + g(x)$. Taking $x = e^{Bt}u$ we get

$$e^{Bt}u' + Be^{Bt}u = Be^{Bt}u + g(e^{Bt}u) \tag{274}$$

Thus

$$u = x_0 + \int_0^t e^{-Bs} g(e^{Bs}u(s))ds \tag{275}$$

or

$$x = e^{Bt}x_0 + \int_0^t e^{B(t-s)} g(x(s))ds \tag{276}$$

Take initial the initial condition in $\mathcal{O} = \{x : \|x\|_\infty < \delta\}$. Let $\mathcal{B}$ be the Banach space of functions defined on $[0, T]$ with the sup norm, and the ball B_0 of radius $2e^{\|B\|T}|x_0|$.

We claim that (276) is contractive in B_0, if δ is small. Indeed, you can easily check that B_0 is preserved since $g(x) = o(x)$.

To show contractivity, we note that

$$x_1(t) - x_2(t) = \int_0^t e^{B(t-s)}[g(x_1(s)) - g(x_2(s))]ds \tag{277}$$

where we know that

$$\|g(x_1(s)) - g(x_2(s))\| = o(1)\|x_1(s) - x_2(s)\| \tag{278}$$

The rest of the contractivity proof is straightforward. We have by the definition of B_0

$$\left|\int_0^t e^{B(t-s)} g(x)dx\right| \leqslant |x_0|o(1)e^{\|B\|T}\|B\|^{-1} = o(x_0) \tag{279}$$

and thus

$$x = e^{Bt}x_0 + o(x_0)$$

proving the statement. □

17.2 Linearizations. The Hartman-Grobman theorem

Definition 21. • *(For a flow f) The fixed point $x = 0$ is hyperbolic if the matrix $D_x f|_{x=0}$ has no eigenvalue on the unit circle.*

• *(for a field F) Equivalently, if f is associated with F, the fixed point 0 is hyperbolic if the matrix $DF(0)$ has no purely imaginary eigenvalues.*

The following result generalizes to Banach space settings.

Let U and V be open subsets of $\mathbb{R}^n$. Let f be a diffeomorphism between U and V with a **hyperbolic** fixed point, that is there is $x_0 \in U \cap V$ so that $f(x_0) = x_0$ and $Df(x_0)$ has no spectrum on the unit circle. Without loss of generality, we may assume that $x = 0$.

The following result shows that there is a continuous change of coordinates which transforms f into its linear part.

Theorem 5 (Hartman-Grobman for maps)**.** *Under these assumptions, f and $Df(0)$ are topologically conjugate, that is, there are neighborhoods U_1, V_1 of zero, and a homeomorphism h from U_1 to V_1 such that*

$$h^{-1} \circ f \circ h = Df(0) \ \Leftrightarrow f(h(x)) = h(Df(0)x) \tag{280}$$

Note that in the process, $D(f)(0)$ can be brought to Jordan normal form, but any transformation of the form of the form (280) preserves the eigenvalues of $D(f)(0)$. The theorem states that, in new coordinates, the flow becomes linear.

Finally, note the *loss of regularity*. We start with C^1 and end up in C^0.

The proof is long but not very difficult; it is preferable to leave it for later; we however illustrate it on some simple cases.

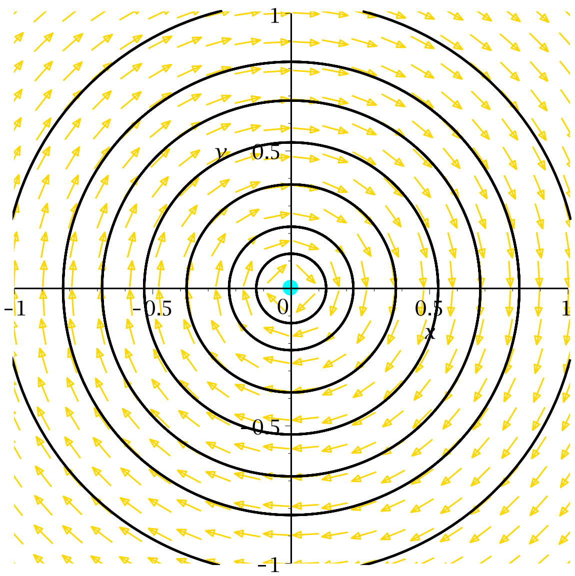

Figure 9: Phase portrait of $u' = v, v' = -u$

17.3 Conjugation of maps, a simple case: one-dimensional analytic maps

We note that the higher dimensional version of the analytic linearization result below is **not** a simple extension of the one-dimensional case.

A quadratic map, $f(x) = \alpha x + \beta x^2$ contains the core of the problem, while being algebraically easier to handle. Note that by the substitution $f(x) = a^{-1}\tilde{f}(ax)$ we get

$$\tilde{f}(t) = \alpha t + a^{-1}\beta t^2 = \alpha t + t^2 \quad \text{if } a = \beta \tag{281}$$

and we can assume without loss of generality that $\beta = 1$. We are looking for conjugation map, "tangent to the identity" ($h'(0) = 1$, to ensure that the existence of h means that the maps are close to each-other) with the property

$$h^{-1} \circ f \circ h = \alpha I, \quad \text{where } I \text{ is the identity} \tag{282}$$

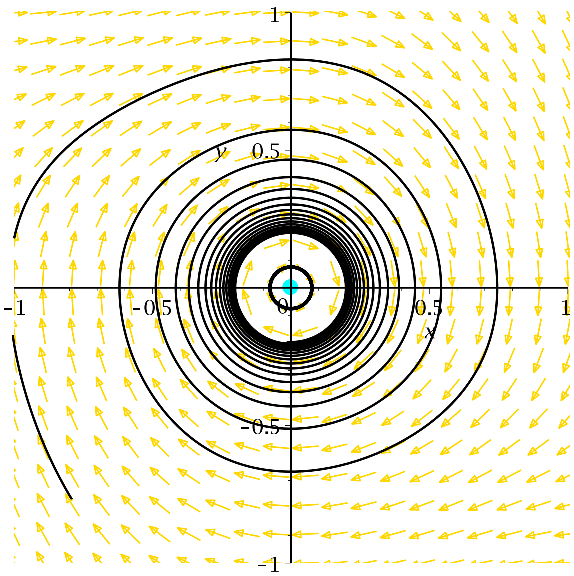

Figure 10: Phase portrait of $u' = v, v' = -u - yx^2$

In our example, it means

$$h(\alpha y) = \alpha h(y) + h(y)^2 \tag{283}$$

Let's first see why hyperbolicity is important. First of all, obviously with $\alpha = 1$ implies $h = 0$ which is not a valid solution. Neither can we find such a map if $\alpha = -1$. Indeed, in this case

$$\begin{aligned} &h(-y) = -h(y) + h(y)^2 \;\Rightarrow h(y) = -h(-y) + h(-y)^2 \\ &= -[-h(y) + h(y)^2] + [-h(y) + h(y)^2]^2 = h(-y) - 2h(-y)^3 + h(-y)^4 \end{aligned} \tag{284}$$

and then $h(y) = 0$ or $h(y) = 2$, both unacceptable. You can show that if ω is an n-th root of unity, we also get a contradiction. Now, for one-dimensional maps, it is not necessary that f be hyperbolic for h to exist. The Brjuno condition measuring how far an irrational angle is from the rationals is in fact necessary and sufficient for the existence of an analytic h.

Let's now take $|\alpha| \neq 1$. We are looking for $h(x) = x + O(x^2)$ and then it is natural to substitute $h(x) = x + x^2\delta(x)$ in the conjugation equation (284).

We get

$$\alpha x + \alpha^2 x^2 \delta(\alpha x) = \alpha x + \alpha x^2 \delta(x) + x^2 + 2x^3\delta(x) + x^4\delta(x)^2 \tag{285}$$

If $|\alpha| > 1$ we isolate $\delta(\alpha x)$,

$$\delta(\alpha x) = \alpha^{-2} + \alpha^{-1}\delta(x) + 2\alpha^{-2}x\delta(x) + x^2\alpha^{-2}\delta(x)^2 \tag{286}$$

One reason to attempt to place $\delta(x)$ on the left side is that, if there is any chance at contractivity, there must be an invariant domain in x. If, for calculating $\delta(x)$ we needed $\delta(\alpha x)$ with $|\alpha| > 1$, only $\mathbb{C}$ would be invariant, and then the conjugation map would need to be entire, which is a priori quite unlikely. We can rewrite(287) as

$$\delta(x) = \alpha^{-2} + \alpha^{-1}\delta(x/\alpha) + 2\alpha^{-3}x\delta(x/\alpha) + x^2\alpha^{-4}\delta(x/\alpha)^2 = \mathcal{N}(\delta) \tag{287}$$

Let's see if $\mathcal{N}$ is contractive as presented. In choosing an invariant ball of radius R in the sup norm for δ, R, as we know already, must be at least as large as the δ-independent term, $|\alpha|^{-2}$, meaning

$$\frac{1}{\alpha^2} + \frac{R}{\alpha} < R \Rightarrow R > \frac{1}{\alpha(\alpha-1)} \tag{288}$$

which, if a is close to 1 would force x to be very large, which we don't want either, because of the same reason: we expect δ to be well behaved for small x and not necessarily for any x.

We can attempt to pull out more terms, as usual. Unfortunately, $1/a$ is not a small parameter, and this would be hard work. If we write $\delta(x) = a^{-2} + \delta_1(x)$ we get

$$\delta_1(x) = \alpha^{-3} + \delta_1\left(\frac{x}{\alpha}\right)\alpha^{-1} + x^2\alpha^{-8} + 2\,x^2\delta_1\left(\frac{x}{\alpha}\right)\alpha^{-6} \\ + x^2\left(\delta_1\left(\frac{x}{\alpha}\right)\right)^2\alpha^{-4} + 2\,\frac{x}{\alpha^5} + 2\,x\delta_1\left(\frac{x}{\alpha}\right)\alpha^{-3} \tag{289}$$

For any $a > 1$, we will eventually succeed with this procedure, because for some large k a^{-k} will be sufficiently small.

Since however a^{-1} is not small enough, there is no compelling reason to "discard" in a first approximation, $\alpha^{-1}\delta(x/a)$ (this is what it would amount to, if we expect contractivity).

So a better way would be to keep it on the left, if we can still invert the resulting operator:

$$\delta(x) - \alpha^{-1}\delta(x/\alpha) = \alpha^{-2} + 2\alpha^{-3}x\delta(x/\alpha) + x^2\alpha^{-4}\delta(x/\alpha)^2 = \mathcal{N}(\delta) \tag{290}$$

Now every term on the right side containing δ is "guarded" by a power of x and we can hope to show that it will be $\ll a^{-2}$.

On the left side, we denote $(L(\delta))(x) = a^{-1}\delta(x/a)$ and note that L is a linear operator of norm < 1. We then have

$$\delta(x) = (I-L)^{-1}\left(\alpha^{-2} + 2\alpha^{-3}x\delta(x/\alpha) + x^2\alpha^{-4}\delta(x/\alpha)^2\right) \tag{291}$$

From here on, contractivity in a ball of radius $\|\delta\|_\infty \|(I-L)^{-1}a^{-2}\|(1+\varepsilon) \leqslant |\alpha^{-2}(1-\alpha^{-1})|$, if x is small enough, is immediate. For $\alpha < 1$ we can similarly write

$$\delta(x) - \alpha\delta(\alpha x) = -\alpha^{-1} - \alpha^{-1}(x^2\delta(x)^2 + 2x\delta(x)) \tag{292}$$

and we now define $(L(\delta))(x) = \alpha\delta(ax)$, again an operator of norm < 1 and we proceed as before.

We can work in a space of analytic functions δ; then we also obtain analyticity of h, in this case.

Let's see, in terms of a power series, what we would get. With $h = x + \sum_{k=2}^{\infty} c_k x^k$ we get

$$c_{m+1} = \frac{1}{a(a^m - 1)}(2c_m + \text{ terms of lower index}) \tag{293}$$

It is apparent that any root of the unity, or any a of the form $e^{2k\pi i/m}$ would prevent the series from existing, unless some "miraculous cancellation" occurs. It is also clear that we would have some difficulty proving analytic equivalence for any $a = e^{2\pi i\phi}$, since we will get small denominators every time k/m is close to ϕ.

17.4 General case

As noted we can assume without loss of generality, that $D(0) = J$ is a Jordan matrix.

We have to solve

$$f(x + \delta(x)) = Jx + \delta(Jx) \tag{294}$$

An expansion at zero, noting that $f(0) = 0$, gives

$$\begin{aligned} &Jx + J\delta(x) + o(Jx + \delta(x)) = Jx + \delta(Jx) \\ &\qquad\qquad \Leftrightarrow J\delta(x) - \delta(Jx) = o(Jx + \delta(x)) \end{aligned} \tag{295}$$

assuming both x and $\delta(x)$ are small. We also have, since f is C^1

$$o(\delta_1(x) - \delta_2(x)) = o(1)(\delta_1(x) - \delta_2(x))$$

The one-dimensional example shows us what to do. We split the space as $X_e \oplus X_c$, where in X_e the Jordan blocks J_e have eigenvalues $|\lambda| > 1$ and J_c with $|\lambda| < 1$, resp.; we also write $J = J_e \oplus J_c$.

We define an operator $L(\delta)(x) = \delta(Jx)$. We write $x = x_e \oplus x_c$ and write the system as

$$\begin{aligned} J_e\delta_e(x_e \oplus x_c) - \delta_e(J_e x_e \oplus J_c x_c) &= o_e(Jx + \delta(x)) \\ J_c\delta_c(x_e \oplus x_c) - \delta_c(J_e x_e \oplus J_c x_c) &= o_c(Jx + \delta(x)) \end{aligned} \tag{296}$$

and it remains to show that this can be arranged as a contractive mapping, which we leave as a (not so easy) exercise!

To apply this to flows, we would linearize the flow. A toy model is to take a discrete evolution

$$x(t+1) = \alpha x(t) + x(t)^2 \tag{297}$$

and substitute $x = h(v)$:

$$h(v(t+1)) = \alpha h(v(t)) + h(v(t))^2 = h(\alpha v(t)) \Rightarrow v(t+1) = \alpha v(t) \tag{298}$$

Consider

$$x' = F(x) \tag{299}$$

over a Banach space, where F is a C^1 vector field defined in a neighborhood of the origin 0 and $F(0) = 0$. As before,

$$DF(0) = B$$

Remember that, by Lemma 20, the flow f^t associated with F satisfies

$$D_x f^t|_0 = e^{Bt} \tag{300}$$

Note that the flow f^t is hyperbolic *iff* $\sigma(B) \cap i\mathbb{R} = \emptyset$. In this case we naturally say that F is hyperbolic.

Theorem 6 (Hartman-Grobman for flows, [10])**.** *Suppose that* 0 *is a hyperbolic fixed point of the flow described by F in* (299)*. Then there is a homeomorphism between the flows of F and $DF(0)$, that is a homeomorphism between a neighborhood of zero into itself so that*

$$f^t = h \circ e^{Bt} \circ h^{-1}; \quad B = DF(0) \tag{301}$$

In fact, Theorem 6 follows from Theorem 5. Indeed, then we take first $f(x) := f^1(x)$, the flow at time 1 which by the given assumptions has a hyperbolic fixed point at zero. Then, recalling (271), there is an h s.t.

$$f^1(h(x)) = h(e^B x) \tag{302}$$

We claim that this conjugation extends for all t, that is

$$\boxed{f^t(h(x)) = h(e^{tB} x)} \tag{303}$$

as long as $e^{Bt}x$ is in the domain of h.

For the proof note that, assuming (303) we would have

$$h(x) = f^t(h(e^{-tB}x)) \tag{304}$$

and this is what we will check. Fix $t = T$ and let

$$\hat{h}(x) = f^T[h(e^{-BT}x)]$$

and the goal is to show $\hat{h} = h$. Using (302) we get

$$\begin{aligned} f^1[\hat{h}(x)] &= f^1[f^T[h(e^{-BT}x)]] = f^T[f^1[h(e^{-BT}x)]] \\ &= f^T[h(e^B e^{-BT}x)] = f^T[h(e^{-BT}e^B x)] = \hat{h}(e^B x) \end{aligned} \tag{305}$$

That is (302) holds with $\hat{h}$ instead of h. By uniqueness $\hat{h}(x) = h(x)$ and thus (303) follows. See also [8].

Smoother linearizations The more regularity is needed, the more conditions are required. Let us now consider a two by two system,

$$\dot{u} = \lambda_1 u + A_0 v^2 + A_1 uv + A_2 u^2 + \cdots \tag{306}$$

$$\dot{v} = \lambda_2 u + B_0 u^2 + B_1 uv + B_2 v^2 + \cdots \tag{307}$$

We assumed without loss of generality that the linear part is diagonal (more generally, we should take a Jordan normal form), since this can be arranged by linear changes of variables.

We try, by a change of variables, to eliminate the quadratic correction (at the expense of course of introducing higher order terms).

$$u = U + a_0 V^2 + a_1 VU + a_2 U^2 \tag{308}$$

$$v = V + b_0 U^2 + b_1 VU + b_2 V^2 \tag{309}$$

Substituting (308) in (306) we get

$$\dot{U} = \lambda_1 U + (-\lambda_1 a_2 + A_2)U^2 + (-a_1\lambda_2 + A_1)UV + (\lambda_1 a_0 - 2\,a_0\lambda_2 + A_0)V^2 + \cdots \tag{310}$$

$$\dot{V} = \lambda_2 V + (\lambda_2 b_0 + B_0 - 2\,\lambda_1 b_0)U^2 + (-b_1\lambda_1 + B_1)UV + (-\lambda_2 b_2 + B_2)V^2 + \cdots \tag{311}$$

and require that the quadratic monomials in U, V vanish; we get a system of equations which we solve for a_i, b_i. The result is

$$a_0 = -\frac{A_0}{\lambda_1 - 2\lambda_2}, a_1 = \frac{A_1}{\lambda_2}, a_2 = \frac{A_2}{\lambda_1}, b_0 = \frac{B_0}{-\lambda_2 + 2\lambda_1}, b_1 = \frac{B_1}{\lambda_1}, b_2 = \frac{B_2}{\lambda_2} \tag{312}$$

we see that for more regularity, we need *nonresonance conditions*: so far, we need

$$\lambda_i \neq 0; \quad \lambda_1 \neq 2\lambda_2; \quad \lambda_2 \neq 2\lambda_1$$

Let $\mu_i = e^{\lambda_i}$ and $\boldsymbol{\mu}^{\mathbf{k}} = \mu_1^{k_1} \cdot \mu_2^{k_2} \cdots \mu_n^{k_n}$

Theorem 7 (Sternberg-Siegel, see [10])**.** *Assume F is differentiable, with a hyperbolic fixed point at zero, and DF is Hölder continuous near zero. Assume further that $A = DF(0)$ satisfies*

$$\lambda_{i,j,k} \in \sigma(A) \Rightarrow \mathrm{Re}\lambda_i \neq \mathrm{Re}\lambda_j + \mathrm{Re}\lambda_k \tag{313}$$

when $\mathrm{Re}\lambda_j < 0 < \mathrm{Re}\lambda_k$ (for maps $\mu_i \neq \mu_j\mu_k$ if $|\mu_j| < 1 < |\mu_k|$). Then the functions h in Theorems 5 and 6 can be taken to be diffeomorphisms.

Note 19. *In one or two dimensions, there is obviously no further restriction besides $|\lambda_i| \neq 1$.*

Smooth linearizations

Theorem 8 (Sternberg-Siegel, see [10])**.** *Assume $F \in C^\infty$ and the eigenvalues of $DF(0)$ are nonresonant, that is*

$$\lambda_i - \mathbf{k}\boldsymbol{\lambda} \neq 0 \tag{314}$$

for any $\mathbf{k} \in (\mathbb{N} \cup \{0\})^n$ with $|\mathbf{k}| > 1$. (For maps, $\mu_i \neq \boldsymbol{\mu}^{\mathbf{k}}$.) Then the functions h in Theorems 5 and 6 can be taken to be C^∞ diffeomorphisms.

17.4.1 Linearization proofs

Consider a nonlinear system in a neighborhood of 0 taken to be a fixed point:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{F}(\mathbf{x}),\ \ \mathbf{F}(\mathbf{x}) = O(\mathbf{x}^2) \tag{315}$$

Assume first that A is diagonalizable, and let $S^{-1}AS = \Lambda$; with $\mathbf{x} = S\mathbf{y}$ we get

$$S\dot{\mathbf{y}} = AS\mathbf{y} + \mathbf{F}(S\mathbf{y}) \Rightarrow \dot{\mathbf{y}} = \Lambda\mathbf{y} + \mathbf{F}(S\mathbf{y});\ \ \mathbf{F}(S\mathbf{y}) = O(\mathbf{y}^2) \tag{316}$$

(similarly for a more general Jordan form). Thus, without loss of generality we can assume $A = \Lambda$. Then, the problem is equivalent to

$$\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{F}(\mathbf{x});\quad \text{where } \mathbf{F} \text{ is at least quadratic} \tag{317}$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^d$, Λ is diagonal (or more generally a Jordan matrix; we will not yet deal with this extra layer of complication.) We want to find an $\mathbf{h}$ which conjugates (317) with the linear equation

$$\dot{\mathbf{w}} = \Lambda\mathbf{w} \tag{318}$$

We also $\mathbf{h}(0) = 0$ to preserve the fixed point, and that $\mathbf{h}$ be locally invertible. That means $D\mathbf{h}(0)$ is invertible. As we'll see first, without loss of generality we can assume that $B = D\mathbf{h}(0) = I$. We have

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{h}}{\partial \mathbf{w}}\dot{\mathbf{w}} = \frac{\partial \mathbf{h}}{\partial \mathbf{w}}\Lambda\mathbf{w} \tag{319}$$

(because we want (318) to hold), and on the other hand,

$$\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{F}(\mathbf{x}) = \Lambda\mathbf{h}(\mathbf{w}) + \mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w})) \tag{320}$$

and finally, we get the nonlinear PDE

$$\frac{\partial \mathbf{h}}{\partial \mathbf{w}}\Lambda\mathbf{w} = \Lambda\mathbf{h}(\mathbf{w}) + \mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w})) \tag{321}$$

Or, equivalently

$$L\mathbf{h} = \mathbf{F}(\mathbf{w} + \mathbf{h}) \tag{322}$$

where

$$L\mathbf{h} := \frac{\partial \mathbf{h}}{\partial \mathbf{w}}\Lambda\mathbf{w} - \Lambda\mathbf{h} \tag{323}$$

An equation of the form

$$\frac{\partial \mathbf{h}}{\partial \mathbf{w}}\Lambda\mathbf{w} - \Lambda\mathbf{h}(\mathbf{w}) = \mathbf{v} \tag{324}$$

is called a *homological equation*. On the other hand

$$\mathbf{h}(\mathbf{w}) = B\mathbf{w} + \mathbf{H}_2(\mathbf{w}) \tag{325}$$

where $\mathbf{H}_2$ is at least quadratic. Substituting (325) into (324), we get

$$(B + \mathbf{H}_2(\mathbf{w}))\Lambda\mathbf{w} = \Lambda(B + \frac{\partial \mathbf{H}_2}{\partial \mathbf{w}})\mathbf{w} + \mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w}) \tag{326}$$

in the limit $\mathbf{w} \to 0$ we have

$$B\Lambda = \Lambda B + o(1) \tag{327}$$

which means B (which by choice is invertible) is in the commutative C^*-algebra generated by Λ. Let $w = Bw_1$ We have

$$\frac{\partial \mathbf{h}}{\partial \mathbf{w_1}}\frac{\partial \mathbf{w}}{\partial \mathbf{w_1}}\Lambda B^{-1}\mathbf{w}_1 - \Lambda\mathbf{h}(B^{-1}\mathbf{w}_1) = \mathbf{F}(B^{-1}\mathbf{w}_1) = \tilde{\mathbf{F}}(\mathbf{w}_1) \tag{328}$$

or

$$\frac{\partial \tilde{\mathbf{h}}}{\partial \mathbf{w}_1}\Lambda\mathbf{w}_1 - \Lambda\tilde{\mathbf{h}}(\mathbf{w}_1) = \tilde{\mathbf{F}}(\mathbf{w} + \tilde{\mathbf{h}}(\mathbf{w})) \tag{329}$$

an equation of the same form, but where $\tilde{\mathbf{h}}(\mathbf{w}_1) = \mathbf{w}_1 + o(\mathbf{w}_1)$. So we can assume without loss of generality that $\mathbf{h}(\mathbf{w})$ was already f this form.

Now we Taylor expand $\mathbf{h}$ for small $\mathbf{w}$. The key to avoid very complicated algebra is to decompose the equations by homogeneous polynomials. A homogeneous polynomial $\mathbf{h}_n$ of degree n is one in which any monomial in $\mathbf{h}_n$ is of the form $\mathbf{w}^{\mathbf{k}}$ where $|\mathbf{k}| = k_1 + ... + k_d = n$:

$$\mathbf{h}(\mathbf{w}) = \mathbf{w} + \sum_{n=2}^{\infty}\sum_{|\mathbf{k}|=n}\mathbf{h}_{\mathbf{k}}\mathbf{w}^{\mathbf{k}} \tag{330}$$

Using homogeneous polynomial decompositions, an equation of the form (324) splits into an infinite set of *algebraic equations*.

Let us, for simplicity, first look at a two-dimensional model. The generalization will be quite straightforward. It is simple to generalize this if we see it in a 2 by 2 case, $\mathbf{h} = M_n\mathbf{e}_1 = w_1^{k_1}w_2^{k_2}\mathbf{e}_1$, $k_1 + k_2 = n$.

$$\nabla_{\mathbf{w}} M_n = \begin{pmatrix} \frac{d}{dw_1}M_n & \frac{d}{dw_2}M_n \end{pmatrix} \Rightarrow \frac{dM_n\mathbf{e}_1}{\partial \mathbf{w}} = \begin{pmatrix} n_1 w_1^{-1} & n_2 w_2^{-1} \\ 0 & 0 \end{pmatrix} M_n \tag{331}$$

$$\frac{\partial M_n}{\partial \mathbf{w}}\Lambda \mathbf{w} = M_n \begin{pmatrix} n_1 \dfrac{1}{w_1} & n_2 \dfrac{1}{w_1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 w_1 & 0 \\ 0 & \lambda_2 w_2 \end{pmatrix} = M_n \begin{pmatrix} \lambda_1 n_1 + \lambda_2 n_2 \\ 0 \end{pmatrix} \tag{332}$$

Likewise also,

$$\Lambda M_n \mathbf{e}_1 = \lambda_1 M_n \mathbf{e}_1 \tag{333}$$

17.4.2 Preservation of monomials

Let L be as defined in (323). Note that the monomials $M_{\mathbf{k}}\mathbf{e}_j = \mathbf{w}^{\mathbf{k}}\mathbf{e}_j$ where $\mathbf{e}_j$ is the j-th unit vector form a basis in the space of all vector polynomials of degree $\leqslant n$.

Lemma 22.

$$L\mathbf{w}^{\mathbf{k}}\mathbf{e}_j = (\mathbf{k}\cdot\boldsymbol{\lambda} - \lambda_j)\,\mathbf{w}^{\mathbf{k}}\mathbf{e}_j \tag{334}$$

Proof. We note that $\frac{\partial \mathbf{w}^{\mathbf{k}}\mathbf{e}_j}{\partial \mathbf{w}}$ is a matrix with only one nonzero row, the jth, and $\Lambda\mathbf{w}^{\mathbf{k}}\mathbf{e}_j = \mathbf{w}^{\mathbf{k}}\lambda_j\mathbf{e}_j$, and thus the jth component of $\frac{\partial \mathbf{w}^{\mathbf{k}}\mathbf{e}_j}{\partial \mathbf{w}}\Lambda\mathbf{w}^{\mathbf{k}}\mathbf{e}_j$ is nonzero, and it equals $\mathbf{w}^{\mathbf{k}}\sum_{i=1}^{d}\lambda_i k_i$ and the result follows. □

17.4.3 Homogeneous polynomial decomposition

We write

$$\mathbf{h}(\mathbf{w}) = \mathbf{w} + \sum_{n\geqslant 2}\sum_{|\mathbf{k}|=n}\mathbf{h}_{\mathbf{k}}\mathbf{w}^{\mathbf{k}} = \sum_{n\geqslant 2}\mathbf{h}^{[n]}(\mathbf{w}) \tag{335}$$

where $\mathbf{h}^{[n]}$ are homogeneous polynomials of degree n. Similarly, in the equation for x we write

$$\mathbf{F}(\mathbf{w}) = \sum_{n\geqslant 2}\sum_{|\mathbf{k}|=n}\mathbf{F}_{\mathbf{k}}\mathbf{w}^{\mathbf{k}} = \sum_{n\geqslant 2}\mathbf{F}^{[n]}(\mathbf{w}) \tag{336}$$

with $\mathbf{F}^{[n]}(\mathbf{w})$ homogeneous polynomials of degree n.

Definition 23 (Nonresonance)**.** *the eigenvalues $\lambda_1, ..., \lambda_d$ are nonoresonant if $\lambda_j \neq (\mathbf{k}\cdot\boldsymbol{\lambda})$ for $j = 1, ..., d$ and $\mathbf{k}$ with $|\mathbf{k}| \geqslant 2$.*

We assume from now on that $\lambda_1, ..., \lambda_d$ are nonresonant.

Exercise 1. Show that $\lambda_1 = i, \lambda_2 = -i$ are resonant.

Lemma 24. *The equation*

$$L\mathbf{P}^{[n]} = \mathbf{F}^{[n]};\ \mathbf{F}^{[n]}(\mathbf{x}) = \sum_{j=1}^{d}\sum_{\mathbf{k},|\mathbf{k}|=n} f_{j,\mathbf{k}}\mathbf{x}^{\mathbf{k}}\mathbf{e}_j$$

where $\mathbf{P}^{[n]}$ is a homogeneous polynomial, has the solution

$$\mathbf{P}^{[n]}(\mathbf{x}) = \sum_{j=1}^{d}\sum_{\mathbf{k},|\mathbf{k}|=n}\frac{f_{j,\mathbf{k}}}{\mathbf{k}\cdot\boldsymbol{\lambda} - \lambda_j}\mathbf{x}^{\mathbf{k}}\mathbf{e}_j \tag{337}$$

Proof. This follows from (334). □

Now we start by eliminating the quadratic terms in $\mathbf{F}$.

We substitute $\mathbf{x} = \mathbf{w} + \mathbf{h}^{[2]}(\mathbf{w}) + \boldsymbol{\delta}(\mathbf{w})$ in (319), where $\mathbf{h}^{[2]}(\mathbf{w})$, is a homogeneous polynomial of degree 2, where we want $\boldsymbol{\delta}(\mathbf{w}) = o(\mathbf{x}^2$. We write $\mathbf{F}_2(\mathbf{x}) = \mathbf{F}^{[2]}(\mathbf{x}) + o(\mathbf{x}^2)$. We get

$$\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{F}_2(\mathbf{x}) = \Lambda\mathbf{x} + \mathbf{F}^{[2]}(\mathbf{x}) + o(\mathbf{x}^2)$$
$$= \Lambda\mathbf{w}+\Lambda\mathbf{h}^{[2]}(\mathbf{w})+\mathbf{F}^{[2]}(\mathbf{w}+\Lambda\mathbf{h}^{[2]}(\mathbf{w}))+o(\mathbf{w}^2) = \Lambda\mathbf{w}+\Lambda\mathbf{h}^{[2]}(\mathbf{w})+\mathbf{F}^{[2]}(\mathbf{w})+o(\mathbf{w}^2) \tag{338}$$

and on the other hand, combining the substitution with (338) we get

$$\dot{\mathbf{x}} = \dot{\mathbf{w}}+\frac{\partial\mathbf{h}^{[2]}}{\partial\mathbf{w}}\dot{\mathbf{w}} = \Lambda\mathbf{w}+\frac{\partial\mathbf{h}^{[2]}}{\partial\mathbf{w}}\Lambda\mathbf{w}+o(\mathbf{w}^2) = \Lambda\mathbf{w}+\Lambda\mathbf{h}^{[2]}(\mathbf{w}))+\mathbf{F}^{[2]}(\mathbf{w})+o(\mathbf{w}^2) \tag{339}$$

and thus

$$\frac{\partial\mathbf{h}^{[2]}}{\partial\mathbf{w}}\Lambda - \Lambda\mathbf{h}^{[2]}(\mathbf{w}) = L\mathbf{h}^{[2]} = \mathbf{F}^{[2]}(\mathbf{w}) + o(\mathbf{w}^2) \tag{340}$$

The equation

$$\frac{\partial\mathbf{h}^{[2]}}{\partial\mathbf{w}}\Lambda - \Lambda\mathbf{h}^{[2]}(\mathbf{w}) = L\mathbf{h}^{[2]} = \mathbf{F}^{[2]}(\mathbf{w}) \tag{341}$$

(under the nonresonance condition) has a solution, c.f. (337).

Theorem 9 (Poincaré)**.** *If the eigenvalues are nonresonant, then there is a formal series transformation linearizing the system.*

Proof. We let $\mathbf{x} = \mathbf{w}+\mathbf{h}^{[2]}(w)+\mathbf{h}^{[3]}(\mathbf{w})+\boldsymbol{\delta}(\mathbf{w})$. After eliminating the quadratic terms the equation becomes

$$\frac{\partial\mathbf{h}^{[3]}(\mathbf{w})}{\partial\mathbf{w}}\Lambda - \mathbf{w}\Lambda\mathbf{h}^{[3]}(\mathbf{w}) = L\mathbf{h}^{[3]}(\mathbf{w}) = \tilde{\mathbf{F}}^{[3]} + o(\mathbf{w}^3) \tag{342}$$

Now, the equation

$$\frac{\partial\mathbf{h}^{[3]}(\mathbf{w})}{\partial\mathbf{w}}\Lambda\mathbf{w} - \Lambda\mathbf{h}^{[3]}(\mathbf{w}) = L\mathbf{h}^{[3]}(\mathbf{w}) = \tilde{\mathbf{F}}^{[3]} \tag{343}$$

has a unique solution by Lemma 24. Inductively we get at step n the equation

$$\frac{\partial\mathbf{h}^{[n]}(\mathbf{w})}{\partial\mathbf{w}}\Lambda\mathbf{w} - \Lambda\mathbf{h}^{[n]}(\mathbf{w}) = L\mathbf{h}^{[n]}(\mathbf{w}) = \tilde{\mathbf{F}}^{[n]}(\mathbf{w}) + o(\mathbf{w}^n) \tag{344}$$

Definition 25. A sequence of formal power series

$$\sum_{\mathbf{k};|\mathbf{k}\geqslant 0|} \mathbf{c}_{\mathbf{k}}^{[n]}\mathbf{w}^{\mathbf{k}}$$

converges to zero in the topology of formal series *if for any* $\mathbf{k}$ *there is an* n_0 *s.t. for all* $n > n_0$ $\mathbf{c}_{\mathbf{k}}^{[n]} = 0$*. In general* $S^{[n]} \to S$ *if* $S^{[n]} - S \to 0$*.*

Theorem 10 (Poincaré-Dulac)**.** *An equation of the form*

$$\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{F}_2(\mathbf{x})$$

is formally equivalent to

$$\dot{\mathbf{w}} = \Lambda\mathbf{w} + \mathbf{R}_2(\mathbf{x})$$

where $\mathbf{R}_2(\mathbf{x})$ *is a series (possibly finite) containing the resonant monomials in* $\mathbf{F}_2$ *and only those.*

Proof. If there are resonances, we follow the same procedure as in the nonresonant case, while keeping the monomials that cannot be eliminated because of resonannce. We leave the details as an exercise. □

□

Note 20 ([14])**.** (i) Even without having convergence, the nonlinear terms can be made arbitrarily small in a neighborhood of the fixed point by performing finitely many transformations as above. The equation becomes

$$\dot{\mathbf{w}} = \Lambda\mathbf{w} + o(\mathbf{w}^n)$$

(ii) If the right side of the homological equation has degree r, then by letting $\mathbf{w} = \mathbf{w}_1 + \mathbf{h}$ and solving the homological equation $L\mathbf{h} = \mathbf{v}^{[n]}$, the new right side will have degree $2r - 1$. This is very important to control the convergence of the expansion; see next subsection.

(iii) Multiple eigenvalues are allowed, provided they do not lead to resonance.

17.4.4 Improved convergence rate

Also, in the proof of convergence, a much more rapid scheme can be used, namely, instead of writing $\mathbf{x} = \mathbf{w} + \mathbf{h}^{[n]}(\mathbf{w})$ (where as the notation indicates $\mathbf{h}^{[n]}$ is a homogeneous polynomial of degree n, one takes a general $\mathbf{h}_n$ of degree at least n and solve for all the monomials present in the nonlinearity up to a higher order nonlinearity, namely by solving

$$\frac{\partial \mathbf{h}^{[n]}}{\partial \mathbf{w}}\Lambda\mathbf{w} - \Lambda\mathbf{h}^{[n]} = L\mathbf{h}^{[n]} = \mathbf{F}_n(\mathbf{w}) \tag{345}$$

where $\mathbf{F}_n$ contains all monomials of degree $\in [n, 2n-1]$. Then, the error terms generated come from taking $\mathbf{F}_n(\mathbf{w})$ instead of $\mathbf{F}(\mathbf{w}+\mathbf{h}_n)$ which only introduces terms of order at least $2n - 1$.

By following this procedure we have at step j a least power satisfying $p_j = 2p_{j-1} - 1$, $p_1 = 2$, that is $p_n = 2^{n-1} + 1$

Note 21. Alternatively, if we do not aim for computational simplicity but rather for the simplicity of the result, we can solve (345) to all orders, to avoid introducing additional error terms at the next stage.

*

17.5 Types of resonances

The resonances are classified according to their degree. This relates to the number of "partial linearizations" $\mathbf{h}_n$ can be performed. Clearly, if some λ is zero, this corresponds to $\lambda_j = 2\lambda_j$; it is also a resonance at higher orders, $\lambda_j = n\lambda_j$, for any $j \geqslant 2$. If λ_1, λ_2 are s.t.

$$n_1\lambda_1 = -n_2\lambda_2; \quad n_1, n_2 \in \mathbb{N}$$

then this constitutes a resonance since it implies

$$\lambda_2 = n_1\lambda_1 + (n_2 + 1)\lambda_2$$

Taking the opposite sign,

$$\lambda_1 = m\lambda_2$$

with $m \geqslant 2$ is a resonance since it is the same as

$$\lambda_1 = 0\lambda_1 + m\lambda_2$$

but it entails no higher order resonance since

$$\lambda_1 = n_1\lambda_1 + n_2\lambda_2 \Rightarrow m(1 - n_1)\lambda_2 = n_2\lambda_2 \Rightarrow (n_1 = 0 \text{ and } n_2 = m)$$

Furthermore, if p_1 and p_2 are relatively prime, then

$$p_1\lambda_1 = p_2\lambda_2$$

is not a resonance since, together with $\lambda_1 = n_1\lambda_1 + n_2\lambda_2$ it implies

$$p_1\lambda_1 = p_1n_1\lambda_1 + p_1n_2\lambda_2 \Rightarrow p_2\lambda_2 - p_2n_1\lambda_2 = p_1n_2\lambda_2 \tag{346}$$

and thus $n_1 = 0$ implying $p_2 = p_1n_2 \Rightarrow n_1 = 1 \Rightarrow p_1 = p_2$, contradiction.

If $N_1\lambda_1 = N_2\lambda_2$, $M, N \in \mathbb{Z}$, a more systematic way to look for the degree of resonance, their number, etc is, once we saw what happens in the case of 0 eigenvalues, is of course to write the systems

$$\begin{aligned} \lambda_1(1 - n_1) - \lambda_2 n_2 = 0 \\ N_1\lambda_1 - N_2\lambda_2 = 0 \end{aligned} \quad \Rightarrow N_1n_2 = N_2(1 - n_1) \tag{347}$$

and similarly

$$N_1(n_2 - 1) = n_1N_2 \tag{348}$$

17.5.1 Poincaré domains and Siegel domains

Definition 26. The eigenvalues $\lambda_1, ..., \lambda_n$ belong to the *Poincaré domain* (a subset of $\mathbb{C}^n$) if the convex hull of $\lambda_1, ..., \lambda_n$ in $\mathbb{C}$ does not contain zero inside. Otherwise it is said that they belong to the *Siegel domain*.

Theorem 11 (The convex separation theorem)**.** *A convex domain not containing the origin is contained in some half-plane.*

This is a special Hahn-Banach separation theorem. A direct proof goes as follows. The convex hull C is a convex set which contains, by definition, all the λ and by assumption $0 \notin C$. Then $\mathrm{dist}(0, C) = \inf_{x \in C} \mathrm{dist}(x, C) = a > 0$. Since dist is a continuous function and C is compact, there is a point in C (it must be one of the lambdas) s.t. $\mathrm{dist}(\lambda, 0) = a$. Any line ℓ perpendicular to the segment $[0, \lambda]$ passing through its interior does not cross C. For if $x \in \ell \cap C$, then the segment $[x, \lambda]$ would be in C and elementary geometry shows that there would be a point y on this segment s.t. $\mathrm{dist}(y, 0) < a$.

Theorem 12. *If the eigenvalues are in a Poincaré domain, then there are only finitely many (possibly zero) resonances.*

Proof. Indeed, without loss of generality, we can assume that $\lambda_1, ..., \lambda_n$ are all in the open right half plane. Then $\mathrm{Re}\lambda_i > 0$ for all i. This easily implies

$$\lim_{|\mathbf{k}| \to \infty} \mathrm{Re}\,(\mathbf{k} \cdot \boldsymbol{\lambda}) = \infty \tag{349}$$

□

Lemma 27. *For general $DF(0)$, there is a basis in which L is triangular, with the same eigenvalues as in the diagonal case.*

Theorem 13. *If $\boldsymbol{\lambda}$ is in the Siegel domain then either*
(i) There are infinitely many resonances
(ii) There exist sequences s.t.

$$\lambda_j - \mathbf{k} \cdot \boldsymbol{\lambda} \ \to 0 \ \textit{with } k \in \mathbb{N} \ \textit{as } |\mathbf{k}| \to \infty \tag{350}$$

Proof. The convex hull C is evidently a polygon. In case 0 is on side of the polygon $[\lambda_i, \lambda_{i+1}]$ then there exist $\alpha_{1,2} > 0, \alpha_1 + \alpha_2 = 1$ s.t. $\alpha_1\lambda_1 + \alpha_2\lambda_2 = 0$ (why?). Otherwise, any triangle with vertices $\lambda_i, \lambda_{i+1}, 0$ is contained in C. Then, there is a triangle with vertices $\lambda, \lambda', \lambda''$ containing 0 inside (why?) By relabeling the lambda's, for some $\alpha_{1,2,3} \in \mathbb{R}^+$. with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and

$$\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 = 0 \tag{351}$$

(why?) Evidently, both conditions (i) and (ii) are invariant under linear changes of coordinates. We have in mind linear invertible transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$, not only conformal ones.

There are two cases: (i) there are two linearly independent λs (over $\mathbb{R}$); (ii) the three λs are collinear.

If the points are collinear, then 0 is between, say λ_1 and λ_2. By a rotation we make both λ's real; one is positive and the other one is negative. If λ_1 and λ_2 are linearly dependent over $\mathbb{Q}$ then clearly there are infinitely many resonances. If not, it is known that the set of $k_1|\lambda_1| + k_2|\lambda_2|$ with $(k_1, k_2) \in \mathbb{Z}$ is dense in $\mathbb{R}$

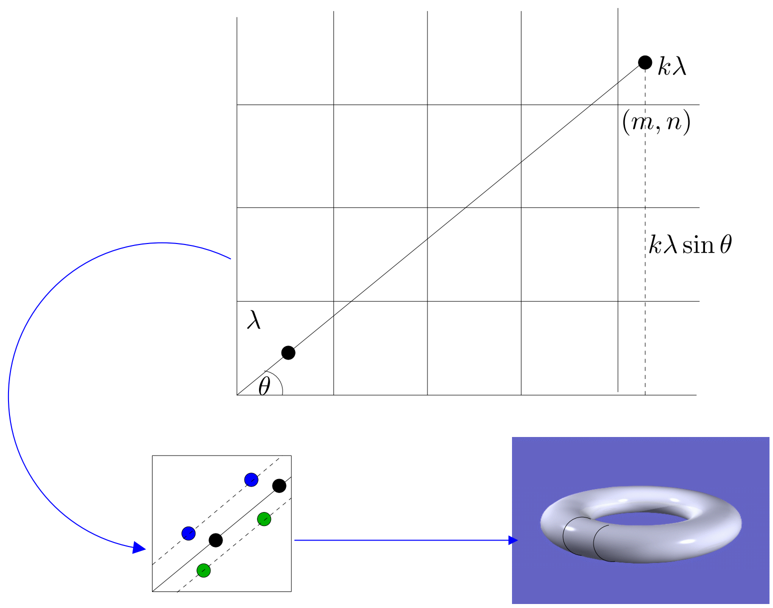

Figure 11: “Small denominators”. Black is for $k = 1, 2$, blue for $k = 3, 4$, green for $5, 6$.

from which the property follows trivially. We could use the Poincaré recurrence theorem, or simple results about continued fractions but here is a direct simple proof. We can assume without loss of generality, that $|\lambda_1/\lambda_2| = r$ is an irrational number in $[0, 1]$. Take all rational numbers m/n with $p < q$ both positive. There are n^2 of them. The rest is an exercise.

Case (ii). Through a linear invertible matrix we can make $\lambda_1 = 1$ and $\lambda_2 = i$. By assumption, 0 is inside the triangle $\Delta(1, i, \lambda)$. There is a convex combination satisfying (351), or

$$-\alpha_3\lambda_3 = \alpha_1 + \alpha_2 i \tag{352}$$

or

$$-\lambda_3 = \beta_1 + \beta_2 i \tag{353}$$

where β_1, β_2 are positive, or, $\lambda = -\lambda_3$ is in the first quadrant. Now we are looking at approximations

$$\lambda = k + mi - n\lambda \Leftrightarrow (n-1)\lambda = k + mi \tag{354}$$

Then, considering all linear combinations of $-n_3\lambda_3$, n_1 and $n_2 i$ is the same as looking at $-n_3\lambda_3$ for all $n_3 \in \mathbb{N}$ on a square lattice in the first quadrant, and determining whether $-n_3\lambda_3$ goes through, or passes near, a node. This in turn is equivalent to the question of the evolution $X \mapsto X - \lambda_3$ mod the unit square, a discrete rotation on the torus $\mathbb{T}$. Thus, either the rotation is rational in which case there are infinitely many resonances, or else the trajectory is dense. This follows from the Poincaré recurrence theorem.

Exercise 2. Find an elementary proof based on Fig. 11, counting the number of loops on the torus etc. The whole proof illustrates 3 strategies: using the invariance of the question under linear transformation, and using this freedom to convert it into a square lattice one; modding out irrelevant features and lifting the dimension (to 2, that of a torus) to decouple a system.

If the rotation is irrational (in both directions on the torus, that is the point never returns to the same horizontal or vertical line in the square) what is the *minimum* n, for large n s.t. there is a point on the trajectory of the point (shown as a colored ball) within ε of the center of the patch?

□

17.6 Analytic equivalence

17.6.1 The Poincaré-Dulac theorem

Theorem 14 (Poincaré). *Assume the eigenvalues of Λ are in the Poincaré domain and are nonresonant and in the system*

$$\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{F}(\mathbf{x}); \quad \mathbf{F} = O(\mathbf{x}^2) \tag{355}$$

the function $\mathbf{F}$ is analytic in a polydisk of radius R containing the origin. Then the system (355) *is analytically equivalent to*

$$\dot{\mathbf{w}} = \Lambda\mathbf{w} \tag{356}$$

for small enough $\mathbf{w}$.

Theorem 15 (Poincaré-Dulac). *Under the same assumptions as in Theorem 14 except nonresonance, the system* (355) *is analytically equivalent to*

$$\dot{\mathbf{w}} = \Lambda\mathbf{w} + \mathbf{R}(\mathbf{w}) \tag{357}$$

where $\mathbf{R}$ contains only resonant monomials.

Note 22. *The proof of Theorem 15 is very similar to that of Theorem 14 so we focus on Theorem 14. The C^∞ and C^r cases are similar and in fact simpler.*

17.7 Proof of Theorem 14

Proof. We look for an equivalence map

$$\mathbf{x} = \mathbf{w} + \mathbf{h}(\mathbf{w}); \quad \mathbf{h} = O(\mathbf{w}^2) \tag{358}$$

where $\mathbf{h}$ is analytic, reducing (355) to (356). The equation that $\mathbf{h}$ satisfies is then, see (329),

$$\frac{\partial \mathbf{h}}{\partial \mathbf{w}}\Lambda\mathbf{w} - \Lambda\mathbf{h} = \mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w})) \tag{359}$$

17.7.1 Spaces of vector valued analytic functions

Note 23 (Reminder: Hartog's theorem)**.** If $f : \mathbb{C}^n \to \mathbb{C}$ is an analytic function in each variable $w_i, 1 \leqslant i \leqslant n$, while the other variables are held constant, then f is analytic function in the $n-$variable sense (i.e. that locally it has a Taylor expansion).

Corollary 28 (Cauchy's formula in n variables)**.** *Let $\mathbb{D} = \prod_{i=1}^n \mathbb{D}_i$ be a polydisk (each $\mathbb{D}_i$ is a disk in $\mathbb{C}$). Then,*

$$f(\mathbf{w}) = \frac{1}{(2\pi i)^n} \int \cdots \iint_{\partial\mathbb{D}_1 \times \cdots \times \partial\mathbb{D}_n} \frac{f(z_1, \dots, z_n)}{(z_1 - w_1) \dots (z_n - w_n)} dz_1 \dots dz_n \tag{360}$$

Proof. This follows from Fubini, by starting with all variables but the first one frozen and applying the usual Cauchy formula, then moving to the second variable, etc. □

For the **proof of the theorem**, we consider the space of vector valued functions $\mathbf{f}$,

$$\mathbf{f}(\mathbf{w}) = \sum_{j=0}^{d} \sum_{\mathbf{k} \geqslant 0} f_{\mathbf{k},j} \mathbf{w}^j \mathbf{e}_j \tag{361}$$

analytic in a *polydisk* $\mathbb{D}_R = \{\mathbf{w} : |w_i| < R, i = 1, ..., d\}$ and continuous on $\overline{\mathbb{D}}_R$ with the norm

$$|||\mathbf{f}|||_R = |||\mathbf{f}||| = \sum_{j,\mathbf{k}} |f_{\mathbf{k},j}| \ R^{|\mathbf{k}|} \tag{362}$$

Definition 29. *We denote by $C^\omega(\mathbb{D}_R)$ the space of functions with finite norm* (362)*.*

Note 24. Associate to a function $\mathbf{f} \in C^\omega(\mathbb{D}_R)$ the function $\mathbf{f}_{\text{abs}}$ by

$$\mathbf{f} = \sum_{\mathbf{k}} f_{\mathbf{k}} \mathbf{w}^{\mathbf{k}} \Rightarrow \mathbf{f}_{\text{abs}} = f_{\text{abs}} = \sum_{\mathbf{k},j} |f_{\mathbf{k},j}| \mathbf{w}^{\mathbf{k}} \tag{363}$$

($\mathbf{f}_{\text{abs}}$ is in fact a scalar). The operator $\mathbf{f} \to \mathbf{f}_{\text{abs}}$ is unbounded in the usual sup norm. Note also that

$$|||\mathbf{f}||| = \|\mathbf{f}_{\text{abs}}\|_\infty \tag{364}$$

where the sup norm is taken over $\overline{\mathbb{D}}$. This is clear because the coefficients of $\mathbf{f}_{\text{abs}}$ are all positive.

Proposition 30. *(i) If $\mathbf{f}_{abs}$ is analytic in $\mathbb{D}_R$ and $\sup_{\mathbb{D}_R} |\mathbf{f}_{abs}| < \infty$, then $\mathbf{f}$ is analytic in $\mathbb{D}_R$ and continuous in $\overline{\mathbb{D}}_R$.*

(ii) If $\mathbf{f}$ is analytic $\mathbb{D}_R$ and $R' < R$, then the sup norms of $|\mathbf{f}_{abs}^{(n)}|$ over $\mathbb{D}_{R'}$ are finite for all n. (Note that $\mathbf{f}_{abs}^{(n)} = (\mathbf{f}^{(n)})_{abs}$.)

Proof. (i) Both analyticity and continuity follow by dominated convergence.

(ii) Cauchy's formula implies immediately that

$$\|\mathbf{f}_{\rm abs}\|_{\infty,R'} \leqslant \frac{\|\mathbf{f}_{\rm abs}\|_{\infty,R}}{(1-R'/R)^d} \tag{365}$$

One can differentiate under the integral sign in Cauchy's formula, by dominated convergence. Thus, similar bounds hold for the derivative (estimate them!). □

Proposition 31. *$C^\omega(\mathbb{D}_R)$ is a Banach space of analytic functions in $\mathbb{D}_R$ continuous in $\overline{\mathbb{D}}_R$. If $\mathbf{f} \in C^\omega(\mathbb{D}_R)$, then $|||\mathbf{f}||| = \sup_{\mathbf{w}\in\mathbb{D}_R} |\mathbf{f}_{abs}|$.*

Proof. Analyticity is clear. Continuity follows from the uniform convergence of the power series in $\mathbb{D}_R$. The fact that this is a Banach space can be seen either directly or as follows: the space $C^\omega(\mathbb{D}_R)$ is isomorphic to a weighted ℓ^1 space (with weight $R^{|\mathbf{k}|}$). The last part is clear since $f_{\rm abs}$ is an analytic function with positive coefficients whose sup, if finite, is reached at $w_i = R, i = 1, ..., d$. □

Proposition 32. *In the scalar case, $d = 1$, the space $C^\omega(\mathbb{D}_R)$ is a Banach algebra.*

Note 25 (Reminder)**.** *A Banach algebra is a normed algebra in which the product "$*$" is continuous: $\|f * g\| \leqslant \|f\|\|g\|$.*

Proof. It is straightforward to check that

$$|||fg||| = \sup|(fg)_{\rm abs}| \leqslant \sup(|f_{\rm abs}|\ |g_{\rm abs}|) \leqslant \sup|f_{\rm abs}|\ \sup|g_{\rm abs}| = |||f|||\ |||g|||$$

□

Definition 33. *If $\mathbf{H} = (H_1, H_2, ..., H_d)$, we naturally write*

$$\mathbf{H}^{\mathbf{k}}(\mathbf{w}) = H_1^{k_1}(\mathbf{w})H_2^{k_2}(\mathbf{w})\cdots H_d^{k_d}(\mathbf{w}) \tag{366}$$

Corollary 34.

$$|||\mathbf{H}^{\mathbf{k}}||| \leqslant |||H_1|||^{k_1}\cdots|||H_d|||^{k_d}$$

Proposition 35. *If $\mathbf{f}, \mathbf{H}$ are in $C^\omega(\mathbb{D}_R)$ and if $\mathbf{H}(0) = 0$ then for R' small enough $\mathbf{f}(\mathbf{H}) \in C^\omega(\mathbb{D}_{R'})$ and we have*

$$\mathbf{f}(\mathbf{H}(\mathbf{w})) = \sum_{\mathbf{k},j} f_{\mathbf{k},j}\mathbf{H}^{\mathbf{k}}\mathbf{e}_j \tag{367}$$

and

$$|||\mathbf{f}(\mathbf{H})||| \leqslant \sup_{|\mathbf{w}|<R'} |\mathbf{f}_{abs}(\mathbf{H}_{abs})| \tag{368}$$

Proof. This follows immediately from Corollary 34. □

17.8 The analytic equation

Recall that the eigenvalues of Λ are assumed nonresonant and that this implies $|\boldsymbol{\lambda}\mathbf{k} - \lambda_j| \to \infty$ as $|\mathbf{k}| \to \infty$. Then there is a lower bound

$$|\boldsymbol{\lambda} \cdot \mathbf{k} - \lambda_j| > (1/a) > 0 \Rightarrow \frac{1}{|\boldsymbol{\lambda}\mathbf{k} - \lambda_j|} < a;\ \forall \mathbf{k}, j \tag{369}$$

Proposition 36. *If $\boldsymbol{\lambda}$ is nonresonant and a is as in* (369), *then the operator*

$$L := \mathbf{H} \mapsto \frac{\partial \mathbf{H}}{\partial \mathbf{w}} \Lambda \mathbf{w} - \Lambda \mathbf{H} \tag{370}$$

is invertible from $C^\omega(\mathbb{D}_R)$ into and $C^\omega(\mathbb{D}_R)$ and the norm of the inverse is $|||L^{-1}||| \leqslant a$.

Proof. The inverse of L (say densely defined on a dense set, on polynomials) is, see Lemma 22,

$$L^{-1}\mathbf{f} = \sum_{j,|\mathbf{k}|\geqslant 2} \frac{f_{\mathbf{k},j}}{\boldsymbol{\lambda} \cdot \mathbf{k} - \lambda_j} \mathbf{w}^{\mathbf{k}} \mathbf{e}_j \tag{371}$$

and by (369) the norm is less than a. L^{-1} extends thus by continuity to the whole of C^ω with the same norm (check the details: polynomials are dense in $C^\omega(\mathbb{D}_R)$, the equation LL^{-1} holds on polynomials, etc.) ☐

We write (359) in the equivalent form

$$\mathbf{h}(\mathbf{w}) - L^{-1}\mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w})) = 0 \tag{372}$$

Proposition 37. *For small enough ε there exists a unique solution $\mathbf{h}$ of* (372) *in $C^\omega(\mathbb{D}_\varepsilon)$.*

Proof 1. It is easily checked that $\mathbf{h} \mapsto = L^{-1}\mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w}))$ is well defined in a small ball $|||\mathbf{h}||| < \varepsilon$. The Fréchet derivative at $\mathbf{h} = \mathbf{w} = 0$ is

$$D_{\mathbf{h}}\Big[\mathbf{h}(\mathbf{w}) - L^{-1}\mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w}))\Big]\Big|_{(0,0)} = I - D_{\mathbf{h}}[L^{-1}\mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w}))]\Big|_{(0,0)}$$
$$= I - L^{-1}\mathbf{F}_{\mathbf{w}}(0)I = I \tag{373}$$

and the implicit function theorem applies. ☐

One could have alternatively applied the contractive mapping principle, but in this case the fixed point theorem leads to a shorter proof. *However, note that if we have a concrete equation, the implicit function theorem, as applied above, does not give us any explicit ε.* ☐

17.8.1 The case when the Jordan form of A is not diagonal

Note: the result is stated incorrectly in Arnold [14]. It is however stated accurately and proved in [9]. The proofs are similar to those in the diagonalizable case. See [9] for details.

Theorem 16 (Formal Poincaré-Dulac theorem)**.** A formal vector field is formally equivalent to a vector field with the linear part in the Jordan normal form and only resonant monomials in the nonlinear part.

Theorem 17. A holomorphic vector field polynomial with the linear part of Poincaré type is holomorphically equivalent to its PoincaréDulac formal normal form.

In particular, if the field is non-resonant, then it can be linearized by a holomorphic transformation.

17.9 The Poincaré domain resonant case and the extended system

In this case the Poincaré Dulac theorem shows that we are generically left with resonant monomials, cf. (357). Ideas going back to Dulac [3] and developed by Kazhdan, Kostant and Sternberg [5], Walcher [11] and Gaeta [4], show that the system can be extended so that it becomes linear. The idea is essentially to take each resonant monomial as a new dependent variable. We illustrate this on a number of examples, following relatively closely [4] (modulo notation and small typos in [4]).

We recall that, by (334)

$$L\mathbf{w}^{\mathbf{k}}\mathbf{e}_j = (\mathbf{k}\cdot\boldsymbol{\lambda} - \lambda_j)\,\mathbf{w}^{\mathbf{k}}\mathbf{e}_j$$

and thus a resonant monomial corresponding to λ_j can only appear in the jth equation.

Take $d = 2$, $\mathbf{w} = (x, y)$ and

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

with k a positive integer; here $\boldsymbol{\lambda} = (1, k)$ is in the Poincaré domain. There is only one resonance $\mathbf{k} = (k, 0)$ (with $j = 2, n = k$ (that is, $\lambda_2 = k\lambda_1 + 0\lambda_2$, $k_1 + k_2 = n = 2$ and the only resonant monomial is $\mathbf{w}^{\mathbf{k}} = x^k$, possibly appearing in the equation for y. The Poincaré-Dulac normal form is

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= ky + \beta x^k \end{aligned}$$

with $\beta \in \mathbb{C}$. If $\beta = 0$ of course there are no resonant monomials left. Otherwise, let $q = x^k$, $\dot{q} = k(x^{k-1})\dot{x} = kq$. The extended system is

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= ky + \beta q \\ \dot{q} &= kq \end{aligned} \tag{374}$$

The matrix of the system has a nontrivial block corresponding to the resonant k:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & \beta \\ 0 & 0 & k \end{pmatrix} \tag{375}$$

This can be of course brought to its jordan normal form simply by taking

$$y = \beta y_1$$

but there is no need for this small step, as we can solve (374) as presented. The solution is

$$x(t) = x_0 e^t \ , \ y(t) = (y_0 + y_1 t)e^{kt} \ , \ q(t) = q_0 e^{kt}; \ y_1 = \beta k q_0$$

and of course, at the end we don't need $q(t)$. Thus,

$$x(t) = x_0 e^t; \quad y(t) = (y_0 + t x_0^k) e^{kt}$$

Example 2.

Take $d = 3$, $\mathbf{w} = (x, y, z)$,

$$\Lambda \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

and $\boldsymbol{\lambda} = (1, 2, 5)$ in the Poincaré domain. There are four resonances: $[\mathbf{k}_1 = (2,0,0), j = 2, n = 2]$, $[\mathbf{k}_2 = (1,2,0), j = 3, n = 3]$, $[\mathbf{k}_3 = (3,1,0), j = 3, n = 4]$ and $[\mathbf{k}_4 = (5,0,0), j = 3, n = 5]$. and correspondingly we have, with $q_i = \mathbf{w}^{\mathbf{k}_i}$,

$$q_1 = x^2 \ , \ q_2 = xy^2 \ , \ q_3 = x^3 y \ , \ q_4 = x^5$$

The normal form is

$$\begin{aligned} \dot{x} &= \ x \\ \dot{y} &= \ 2y + c_1 x^2 \\ \dot{z} &= \ 5z + c_2 xy^2 + c_3 x^3 y + c_4 x^5 \end{aligned}$$

with c_i arbitrary real coefficients.

$$\dot{q_1} = 2x\dot{x} = 2x^2 = 2q_1, \tag{376}$$

$$\dot{q_2} = \dot{x}y^2 + 2xy\dot{y} = xy^2 + 2xy(2y + c_1x^2) = 5q_2 + 2c_1q_3; \tag{377}$$

$$\dot{q_3} = 3x^2y\dot{x} + x^3\dot{y} = 3x^3y + x^3(2y + c_1x^2) = 5q_3 + c_1q_4 \tag{378}$$

$$\dot{q_4} = 5x^5 = 5q_4 \tag{379}$$

We take $q_1 = x^2, q_2 = xy^2, q_3 = x^3y, q_4 = x^5$ and the system extends to a linear system in $\mathbb{R}^7$,

$$\begin{pmatrix} x \\ y \\ z \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}^{\cdot} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & c_2 & c_3 & c_4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 2c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \tag{380}$$

with Jordan normal form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} = D + N \tag{381}$$

where N is a nilpotent, $N^4 = 0$. The general solution to the system (retaining only x, y, z) is

$$x(t) = x_0 e^t, y(t) = (y_0 + ty_1)e^{2t}, z(t) = (z_0 + z_1 t + z_2 t^2 + z_3 t^3)e^{5t} \tag{382}$$

(**Note.** There seem to be typos in the solution in [4].)

17.10 Connection with regular singular points and Frobenius theory

A system of the form

$$\dot{\boldsymbol{u}} = B(0)\boldsymbol{u} + zA(z)\boldsymbol{u}; \ \dot{z} = z; \ (\dot{f} := \frac{df}{dt}) \tag{383}$$

implying

$$\boldsymbol{u}' = \frac{d\boldsymbol{u}}{dz} = \frac{d\boldsymbol{u}}{dt}\frac{dt}{dz} = \frac{1}{z}\left(B(0)\boldsymbol{u} + zA(z)\boldsymbol{u}\right) \tag{384}$$

which is the same as (134). In the form (383) the system is nonlinear (in z only) but free of "explicit" singularities while (134) is linear but singular.

What resonances can we have? Note that we don't have monomials of total degree larger than 1 in $\mathbf{u}$, only possibly in z; "left-over" monomials can only be of the form $u_j^1 z^p \mathbf{e}_j$. Thus, the resonant ones that cannot be eliminated are of the form (383)

$$\lambda_i = \lambda_j + p \tag{385}$$

which is exactly the Frobenius resonance condition, of which we can only have finitely many (regardless of whether we are in a Poincaré or Siegel domain)!

Of course, if there are only finitely many resonances, then the proof of the Poincaré-Dulac theorem also applies to a Siegel case, as the surviving denominators are bounded below. The space of functions whose all Taylor monomials are of the type $u_i z^p \mathbf{e}_j$ forms a Banach space preserved by the right side of (383) (which is linear in u_i). Thus the Poincaré-Dulac proof applies with minor changes, and Frobenius theory is subsumed by Poincaré-Dulac linearization theory. Note also that after linearization and possibly extension of the system, Poincaré-Dulac reduces the study of a regular singularity, analytically, to a system with **constant coefficients**, analyzed at $t = \infty$, or to an Euler system by taking $t = \ln s$ to analyze it at $s = 0$!

For simplicity let's assume B is diagonalizable, let's place $\dot{z} = z$ in first place, that is $\lambda_1 = 1$ and assume there is only one Frobenius resonance which we place last, $\lambda_n = \lambda_1 + p$, and we assume to get something nontrivial, that the coefficient of the resonance is $\beta \neq 0$ which, as in the previous section can be chosen to be one by rescaling w_n. Then the system is analytically equivalent

$$\dot{z} = z \tag{386}$$

$$... \tag{387}$$

$$\dot{w_n} = \lambda_n w_n + w_n z^p \tag{388}$$

We take $q = w_n z^p$ and get

$$\dot{z} = z \tag{389}$$

$$... \tag{390}$$

$$\dot{w_n} = \lambda_n w_n + q \qquad\qquad \dot{q} = \lambda_n q \tag{391}$$

and the matrix of the extended system has one just one nontrivial Jordan block,

$$\begin{pmatrix} \lambda_n & 1 \\ 0 & \lambda_n \end{pmatrix}$$

which yields

$$z = C_1 e^t, ..., w_j = C_j e^{\lambda_j t}, ..., \dot{w}_n = w_{n,0}(1 + w_{n_0} z_0^p t) e^{\lambda_n t}, q = C_n C_1^p e^{\lambda_n t} \tag{392}$$

which we solve as before. There is only one log in this case, when we write the solution in terms of z.

Note also that the imaginary line plays no role. However, if the singularity is irregular, say $\dot{z} = z^2$, then we do have infinitely many resonances. But this is a case which cannot be analytically linearized in general, as we shall see.

18 Newton's method

Consider a simple contractive mapping setting, a linear one,

$$X = X_0 + LX \tag{393}$$

where $\|L\| = \lambda < 1$. If we iterate $X_{n+1} = X_0 + LX_n$ we have

$$\|X_{n+1} - X_n\| \leqslant \lambda \|X_n - X_{n-1}\| \Rightarrow \|X_{n+1} - X_n\| \leqslant C\lambda^n \tag{394}$$

and this is all that is guaranteed in full generality; indeed we can look at the one-dimensional case $x_{n+1} = a + \lambda x_n$, $|\lambda| < 1$ to see that this is optimal.

The actual convergence rate, depending on L may however may be faster. An example is the equation for the exponential $f' = f$; $f(0) = 1$ written in integral form, and iterated,

$$f(x) = 1 + \int_0^x f(s)ds; \quad f_{n+1} = 1 + \int_0^x f_n(s)ds; \quad f_0 = 0 \tag{395}$$

A simple induction argument shows that

$$f_n = \sum_{j=0}^{n-1} \frac{x^n}{n!} \tag{396}$$

and since $f_{n+1} - f_n = O(x^n/n!)$ and f_n approaches f factorially rather than geometrically.

Note 26. *If we are in a Hilbert space setting and the contractive operator is self-adjoint (or normal), then the convergence of the iterates is necessarily geometric since, then,* $\|A^n\| = \|A\|^n$.

In some cases, the convergence rate can be improved by Newton's method. We'll illustrate it first in one dimension. Assume f is smooth and we want to solve the fixed point equation $f(x_0) = x_0$, $f'(x_0) \neq 0$ and we have a close enough starting point, x_1. We write $x_1 = x_0 + \varepsilon$, $f(x) - x = g$. Then,

$$0 = g(x_0) = g(x_0 + \varepsilon) + g'(x_0 + \varepsilon)\varepsilon + ... \tag{397}$$

$$\varepsilon_n = -\frac{g(x_n)}{g'(x_n)} + ... \tag{398}$$

We iterate

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \tag{399}$$

Assuming $x_k = x_0 + \varepsilon_k$ we have

$$g(x_k) = g'(x_0)\varepsilon_k + \frac{1}{2}g''(x_0)\varepsilon_k^2 + \cdots; \; g'(x_k) = g'(x_0) + g''(x_0)\varepsilon_k + \cdots \tag{400}$$

while

$$\varepsilon_{k+1} = \varepsilon_k + x_{k+1} - x_k = \varepsilon_k - \frac{g(x_k)}{g'(x_k)} = \varepsilon_k - \frac{g'(x_0)\varepsilon_k + \frac{1}{2}g''(x_0)\varepsilon_k^2}{g'(x_0) + g''(x_0)\varepsilon_k} + \cdots$$
$$= \varepsilon_k - \left(\varepsilon_k - \frac{g''(x_0)}{2g'(x_0)}\varepsilon_k^2 + \cdots\right) \Rightarrow \varepsilon_{k+1} = \frac{g''(x_0)}{2g'(x_0)}\varepsilon_k^2 \tag{401}$$

hence we expect (to be proved, of course)

$$\varepsilon_{n+1} - \varepsilon_n = O(\varepsilon_n^2) \Rightarrow \varepsilon_n = O(\varepsilon_0^{2^n}) \quad (!) \tag{402}$$

In an infinite dimensional space, we would use the iteration

$$X_{n+1} = -L^{-1}(X_n)G(X_n); \quad L := (DG)(X_n) \tag{403}$$

and DG is assumed to be invertible. A similar argument would show that $\|X_{n+1} - X_n\| = O(X_n^2)$. The iteration

$$Z_{n+1} = Z_n - [DG(Z_n)]^{-1}G(X_n) \tag{404}$$

is known as the Newton-Kantorovich iteration. This is useful in at number of cases. For "practical purposes" in "actual calculations": at times we can explicitly write the inverse operator, and then the convergence is improved. An example is calculating the square root (the method was known to the Babylonians!): $G = x^2 - \lambda$, $G' = 2x$,

$$x_{n+1} = x_n - \frac{x_n^2 - \lambda}{2x_n} = \frac{1}{2}\left(x_n + \frac{\lambda}{x_n}\right) \tag{405}$$

It is easy to check that the error satisfies $\varepsilon_{n+1} \leqslant \varepsilon_n^2/2$, $\varepsilon_n = 2(\varepsilon_0/2)^{2^n}$. It turns out that the solution of the recurrence can be written in closed for. The recurrence is equivalent to

$$\frac{x_{k+1} - \sqrt{\lambda}}{x_{k+1} + \sqrt{\lambda}} = \left(\frac{x_k - \sqrt{\lambda}}{x_k + \sqrt{\lambda}}\right)^2 \Rightarrow \frac{x_k - \sqrt{\lambda}}{x_k + \sqrt{\lambda}} = \left(\frac{x_0 - \sqrt{\lambda}}{x_0 + \sqrt{\lambda}}\right)^{2^n} \tag{406}$$

which, with $x_0 = 1$, always converges, and quadratically so.

Exercise 1. *Let $\lambda = -1$, $x_0 = 2$, $\tau = \arctan(4/3)/\pi$ and $t_n = \pi(2^n\tau \mod 1)$. Show that x_n in* (406) *is given by*

$$x_n = \frac{t_n}{\sqrt{t_n^2 + 1} - 1}$$

Is the sequence x_n periodic?

***Can you show that arctan(x), $x \in \mathbb{Q}$, cannot be a rational multiple of π , except if $x \in \{-1, 0, 1\}$?*

In a theoretical setting, this is useful when in a fixed point problem when the underlying operator in *not contractive* but has bounded norm when acting from a ball into a smaller ball, and the radius of the ball risks to shrink to zero if we straightforwardly iterate the operator. This is the case in the Siegel domain, due to small denominators.

18.1 The Siegel and Brjuno conditions

Definition 38. *(a) A point* $\boldsymbol{\lambda} \in \mathbb{C}^n$ *is of* **Siegel** *type* (C, ν) *where* C *and* ν *are positive constants, if for all* $j = 1, 2, ..., d$ *and* $\mathbf{k}$ *with* $k_i + 1 \in \mathbb{N}, |\mathbf{k}| \geqslant 2$ *we have*

$$|\lambda_j - \mathbf{k}\boldsymbol{\lambda}| \geqslant C|\mathbf{k}|^{-\nu} \tag{407}$$

(b) The optimal condition, beyond which convergence is not expected in general, is the **Brjuno condition**. *Let*

$$\omega_n = \inf\{|\lambda_j - \mathbf{k}\boldsymbol{\lambda}| : k_i \in \mathbb{N}, i, j = 1, ..., d, |\mathbf{k}| \in [2, 2^n]\} \tag{408}$$

The condition is

$$-\sum_{k=0}^{\infty} \frac{\ln \omega_k}{2^k} < \infty \tag{409}$$

Note 27. It can be shown quite straightforwardly that the Brjuno condition implies the Siegel condition. It can also be shown that the Siegel condition holds on a set of full measure if $\nu > (d-2)/2$.

In our problem, for a Newton iteration, the straightforward approach is to write

$$L\mathbf{h}_{n+1} = \mathbf{F}(\mathbf{w} + \mathbf{h_{n+1}}) = \mathbf{F}(\mathbf{w} + \mathbf{h_n}) + D\mathbf{F}(\mathbf{w} + \mathbf{h_n})(\mathbf{h}_{n+1} - \mathbf{h}_n) + O((\mathbf{h}_{n+1} - \mathbf{h}_n)^2) \tag{410}$$

and discard at each stage $O((\mathbf{h}_{n+1} - \mathbf{h}_n)^2)$. The precision of the iteration becomes quadratic, as it should in a Newton method, provided of course we invert the linear operator and write

$$\mathbf{h}_{n+1} = (L - D\mathbf{F}(\mathbf{w} + \mathbf{h_n}))^{-1}\Big[\mathbf{F}(\mathbf{w} + \mathbf{h_n}) - D\mathbf{F}(\mathbf{w} + \mathbf{h_n})\mathbf{h}_n\Big] \tag{411}$$

This is certainly a possible approach, but a quite awkward one, because $L - D\mathbf{F}(\mathbf{w} + \mathbf{h_n})$ is not diagonal anymore, and the coefficient of the monomial $\mathbf{w^k}$ depends now on the set of coefficients of lower order monomials in a rather messy way. A better way is to use the procedure described in §17.4.4, which produces a convergence with rate α^{2^n} only using L and already calculated $\mathbf{h}_n$s, seen below.

Proposition 39. *If* $\mathbf{F}$ *has a zero of order* n*, then* $\mathbf{h}$ *has a zero of order* n *and* $\mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w})) - \mathbf{F}(\mathbf{w})$ *has a zero of order* $2n - 1$.

Proof. The order of the zero of $\mathbf{h}$ is obvious from (414). We have

$$\mathbf{F}(\mathbf{w} + \mathbf{h}(\mathbf{w})) - \mathbf{F}(\mathbf{w}) = D\mathbf{F}(\mathbf{w})\mathbf{h}(\mathbf{w})(1 + o(1)) \tag{412}$$

where $D\mathbf{F}$ has a zero of order $n-1$ and $\mathbf{h}(\mathbf{w})$ has a zero of order n, and the statement follows. ☐

This suggests that the conjugation map, iterated in this way, has a convergence comparable to Newton's method.

18.1.1 The iteration under the Siegel condition

We could work with a condition closer to Brjuno's, essentially in the same way, but at the price of complicating the algebra quite a bit. We assume instead that $\boldsymbol{\lambda}$ is of Siegel (C,ν) type.

Note 28. *In the following, as usual, the symbol $\lesssim$ means less equal, up to a multiplicative constant the value of which is irrelevant.*

Proposition 40. *If*

$$\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{F}_{\rm n}(\mathbf{x});\;\; \text{and } \mathbf{x} = \mathbf{w} + \mathbf{h}_{\rm n}(\mathbf{w}) \tag{413}$$

where

$$L\mathbf{h}_{\rm n} = \mathbf{F}_{\rm n}(\mathbf{w}) \tag{414}$$

then, the equation for $\mathbf{w}$ *is*

$$\dot{\mathbf{w}} = \Lambda\mathbf{w} + \mathbf{F}_{n+1} := \Lambda\mathbf{w} + \big[I + (D\mathbf{h}_{\rm n})(\mathbf{w})\big]^{-1}\big[\mathbf{F}_{\rm n}(\mathbf{w} + \mathbf{h}_{\rm n}(\mathbf{w})) - \mathbf{F}_{\rm n}(\mathbf{w})\big] \tag{415}$$

Proof. Straightforward verification. □

Lemma 41. *Assume $\mathbf{f} \in C^{\omega}(\mathbb{D}_R)$ and $|\hat{K}\mathbf{f}_{\mathbf{k}}\mathbf{w}^{\mathbf{k}}\mathbf{e}_j| \lesssim |\mathbf{k}^{\mu}||\mathbf{f}_{\mathbf{k}}\mathbf{w}^{\mathbf{k}}|$ $(\mu > 0)$. Then, if δ_n is small enough (smaller than a constant depending on μ, R, C) and $R'/R = e^{-\delta_n}$ we have*

$$|||\hat{K}\mathbf{f}|||_{R'} \lesssim \delta_n^{-b}|||\mathbf{f}|||_R;\;\; b = \mu + d + 1 \tag{416}$$

Proof. We can check by induction that $\sum_{|\mathbf{k}|=n} 1 \leqslant dn^d$. If $|||\mathbf{f}||| = M$, then, in particular, $|f_{\mathbf{k}}| \leqslant M/R^{|\mathbf{k}|}$. Then,

$$\sum_{\mathbf{k}>0} |f_{\mathbf{k}}||\mathbf{k}|^{\mu} {R'}^{|\mathbf{k}|} \leqslant M \sum_{\mathbf{k}>0} |\mathbf{k}|^{\mu}\left(\frac{R'}{R}\right)^{|\mathbf{k}|} \leqslant dM\sum_{k=0}^{\infty} k^{\mu+d}e^{-k\delta_n}$$
$$\leqslant dM\int_0^{\infty} k^{\mu+d}e^{-k\delta_n}\,dk = d\,M\,\Gamma(\mu+d+1)\,\delta_n^{-(\mu+d+1)} \lesssim \delta_n^{-b} \tag{417}$$

□

Corollary 42. *We have*

$$|||\mathbf{h}_n|||_{R'} \lesssim \delta_n^{-b}|||\mathbf{F}_{\rm n}(\mathbf{w})|||_R \tag{418}$$

Proposition 43.

$$|||\mathbf{h}_{n+1}|||_{R'} \lesssim \delta_n^{-\gamma}|||\mathbf{h}_n|||_R^2\;\;\; \gamma = \nu + d + 3 \tag{419}$$

Proof. Recall the definition (415). On a disk of radius $< R$ we write

$$|\mathbf{F}_{\mathrm{n}}(\mathbf{w}+\mathbf{h}_{\mathrm{n}}(\mathbf{w}))-\mathbf{F}_{\mathrm{n}}(\mathbf{w})| \leqslant \sup|D\mathbf{F}_n|\sup|\mathbf{h}_n| \leqslant \sup|DL\mathbf{h}_n|\sup|\mathbf{h}_n| \quad (420)$$

here we used $L\mathbf{h}_n = \mathbf{F}_n$ and the sup is taken on any disk of radius $< R$. On the other hand $\mathbf{F}_n = L\mathbf{h}_n$ and we use $\mathbf{h}_{n+1} = L^{-1}\mathbf{F}_{n+1}$. On the disk of radius R' we have, using Lemma 41,

$$|||\mathbf{h}_{n+1}|||_{R'} \lesssim |||L^{-1}DL\mathbf{h}_n|||_{R'}\,|||\mathbf{h}_n|||_R \lesssim \delta_n^{-\gamma}|||\mathbf{h}_n|||_R^2; \quad (421)$$

if, say, $\sup\left|\left[I+(D\mathbf{h}_{\mathrm{n}})(\mathbf{w})\right]^{-1}\right| < 3/2$. ☐

Let $R_n = R_0\prod_{i=1}^n(1-\exp(-\delta_n))$.

Corollary 44. *For some C_1 we have*

$$|||\mathbf{h}_{n+1}|||_{R_0\prod e^{-\delta_n}} \leqslant \left(C_1^n\prod\delta_n^{-\gamma}\right)|||\mathbf{h}_0|||^{2^n} \quad (422)$$

We choose a decreasing sequence of δs s.t. $\prod_{n=0}^{\infty}(1-\delta_n) > 0$, for instance $\delta_n = n^{-2}$. Then, in view of the telescopic nature of the product,

$$\prod_{n=2}^{\infty}(1-n^{-2}) = 1/2 \quad (423)$$

implying

$$|||\mathbf{h}_n|||_{R_n} \leqslant C^n(n!)^{2\gamma}|||\mathbf{h}_0|||_{R_0}^{2^n} \quad (424)$$

Corollary 45. *If $|||\mathbf{h}_0|||$ is small then $R_n \geqslant R_0/const.$ (see Note 29) and for large n,*

$$|||\mathbf{h}_n|||_{R_0/const} \lesssim 2^{-2^n} \quad (425)$$

The composition

$$(I+\mathbf{h}_1)\circ(I+\mathbf{h}_2)\circ\cdots\circ(I+\mathbf{h}_2)\cdots \quad (426)$$

is convergent and maps the equation

$$\dot{\mathbf{x}} = \Lambda\mathbf{x}+\mathbf{F}(\mathbf{x}) \quad (427)$$

to

$$\dot{\mathbf{w}} = \Lambda\mathbf{w} \quad (428)$$

Note 29. It is now a matter of algebra to show that one can consistently choose $|||\mathbf{h}_0|||$ small enough s.t. all the inequalities above hold *inductively*, for all n ;"const" above is not necessarily 2 since in the first few iterations, we may have to shrink the ball by more than what is suggested by (423).

Note 30. We could cast this in a contractive mapping setting by rescaling $\mathbf{w}$ at every iteration.

18.1.2 Simple model planar system

Consider the system

$$\dot{x} = ax; \ \dot{y} = by + xy^2 \tag{429}$$

Exercise 2. *For which a, b is the pair (a, b) in the Poincaré domain? For which a, b is the pair (a, b) in the Siegel domain? In each case, for which a, b is the monomial xy^2 above resonant?*

The substitution $y = 1/z$ linearizes the system and it can be solved in closed form.

Question: Is this an acceptable linearization transformation in the sense of the transformation we studied after §17.6?

If $a \neq -b$, then

$$\frac{a+b}{C_2\,(a+b)\,\mathrm{e}^{-bt} - C_1 \mathrm{e}^{at}} \tag{430}$$

while if $a = -b$

$$x(t) = C_1 \mathrm{e}^{at}; \ y(t) = \frac{e^{-at}}{C_2 - C_1 t} \tag{431}$$

To find the linearization transformation explicitly, we solve for e^{at}, e^{bt} in terms of $x(t)$ and $y(t)$. Indeed, in this case the equations in the new variables should be $x_1' = ax_1, y_1' = by_1$. We take $C_1 = C_2 = 1$ and get

$$e^{at} = x(t); \ e^{bt} = \frac{y(t)(a+b)}{x(t)y(t) + a + b} \tag{432}$$

if $a + b \neq 0$; the linearizing transformation is then

$$x_1 = x; \ y_1 = \frac{y}{1 + (a+b)^{-1} xy} \tag{433}$$

which is analytic at zero if $a + b \neq 0$. If $b = -a$ the system is resonant ($\lambda_2 = n\lambda_1 + (n+1)\lambda_2$) and xy^2 is a resonant monomial. The general solution is

$$x(t) = C_2 \mathrm{e}^{at}; \ y(t) = \frac{\mathrm{e}^{-at}}{C_1 - C_2 t} \tag{434}$$

The system is hyperbolic, unless $a \in i\mathbb{R}$. The linearizing transformation is

$$x_1 = x; \ y_1 = \exp(-a^{-1} W(-ay^{-1} e^{-a})) \tag{435}$$

where W is the Lambert function, with the convergent expansion at the origin

$$y_1 = y\Big(1 - \frac{\ln(\ln z)}{\ln(z)} + \frac{\ln(\ln z)}{(\ln z)^2} + \frac{(\ln(\ln z))^2}{2(\ln(z))^2} - \frac{(\ln(\ln z))^2}{2(\ln(z))^3} - \frac{\ln(\ln z)}{(\ln z)^3} - \frac{(\ln(\ln z))^3}{6(\ln z)^3} + \cdots\Big) \tag{436}$$

where $z = -ay^{-1}e^{-a}$. Note that the transformation is not defined in a neighborhood of the origin, but on a Riemann surface.

Exercise 3. *Examine the connection with the Hartman-Grobman theorem.*

Note 31. In nonresonant cases, this system can be linearized by lifting the order, in some cases. Indeed let $q = xy^2$. Then, $\dot{q} = axy^2 + 2bxy^2 = (a+2b)xy^2$ and we get

$$\dot{x} = ax; \;\; \dot{y} = by + q; \;\; \dot{q} = (a+2b)q + 2xyq \tag{437}$$

with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & a+2b \end{pmatrix}$$

If $1, 1, 1$ is not a resonance, that is

$$a \neq a+b+(a+2b); \; b \neq a+b+(a+2b); \; a+2b \neq a+b+(a+2b) =\leftrightarrow (a \neq -b \,\&a \neq -3b)$$

this is linearizable once more and this allows for a convenient study of the equilibrium, in these cases.

Exercise 4. *Show that if $a = -b$, then the system is not linearizable.*

Of course then, the substitution preserved the resonance $a = -b$.

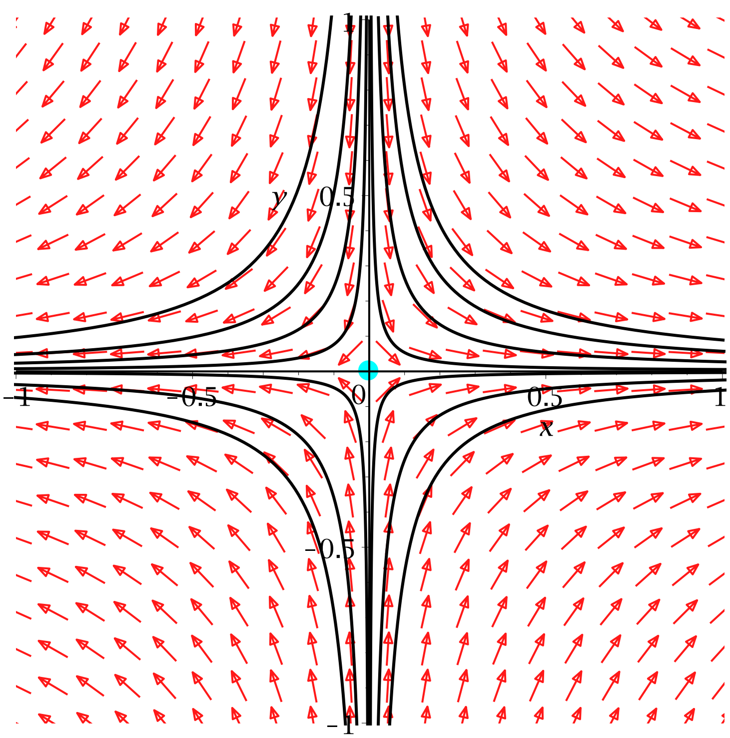

Figure 12: Contour plot of $x' = 2x; \; y' = -2y + xy^2$

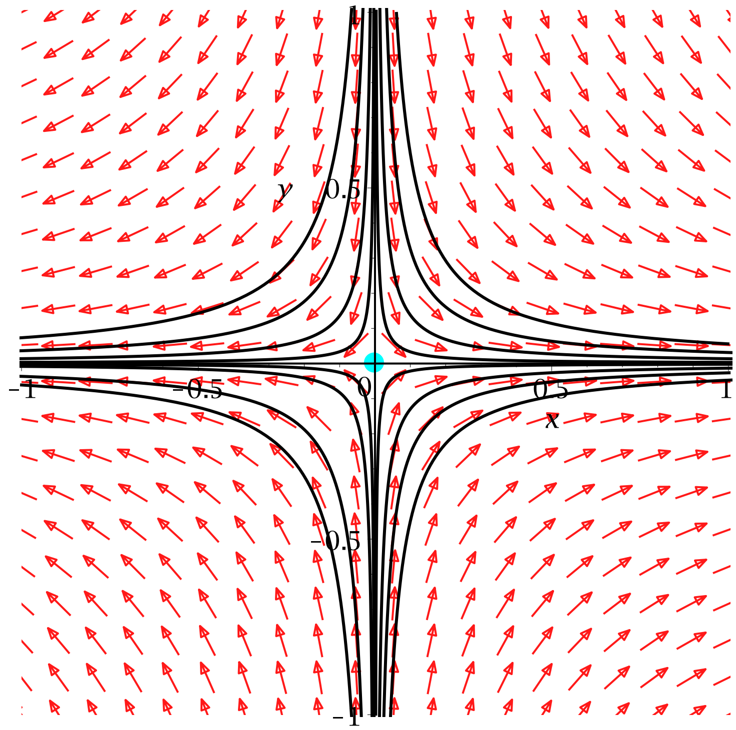

Figure 13: Contour plot of $x' = 2x;\ y' = -2y - xy^2$

A real valued system with eigenvalues $\pm i$ and with the resonant monomial xy^2 after diagonalization is

$$x' = y;\ y' = -x - x^3 + x^2y + xy^2 - y^3 \tag{438}$$

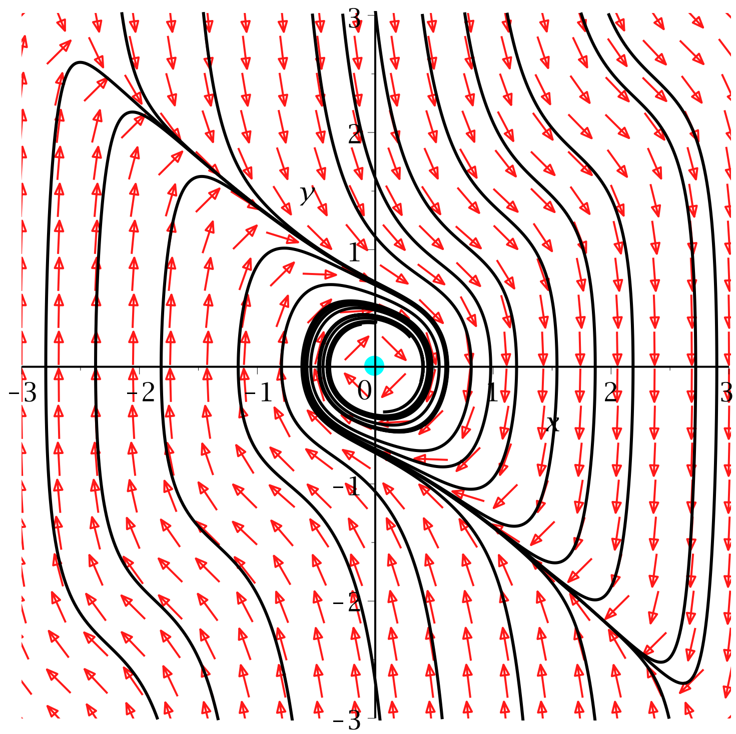

Figure 14: Contour plot of $x' = y;\ y' = -x - x^3 + x^2y + xy^2 - y^3$

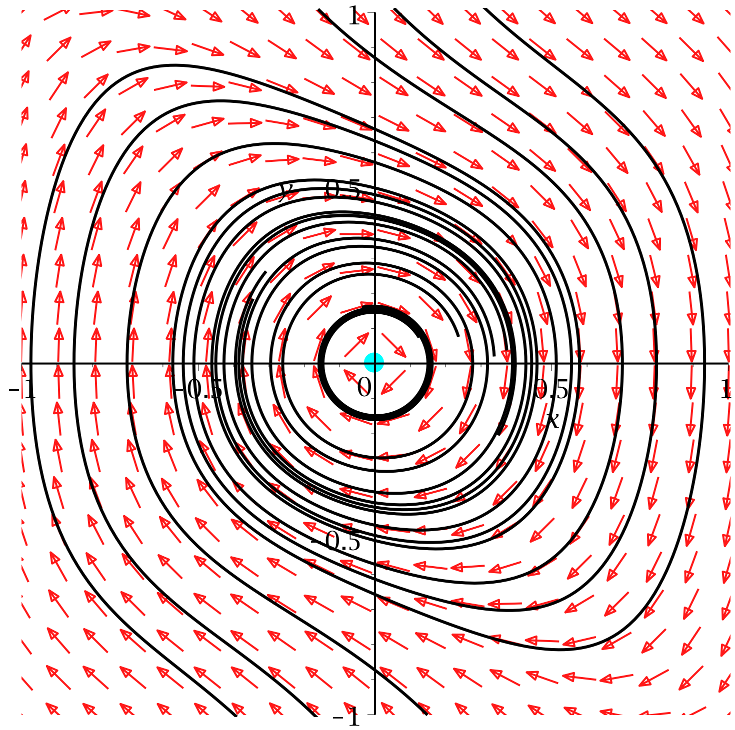

Figure 15: Contour plot of $x' = y;\ y' = -x - x^3 + x^2y + xy^2 - y^3$

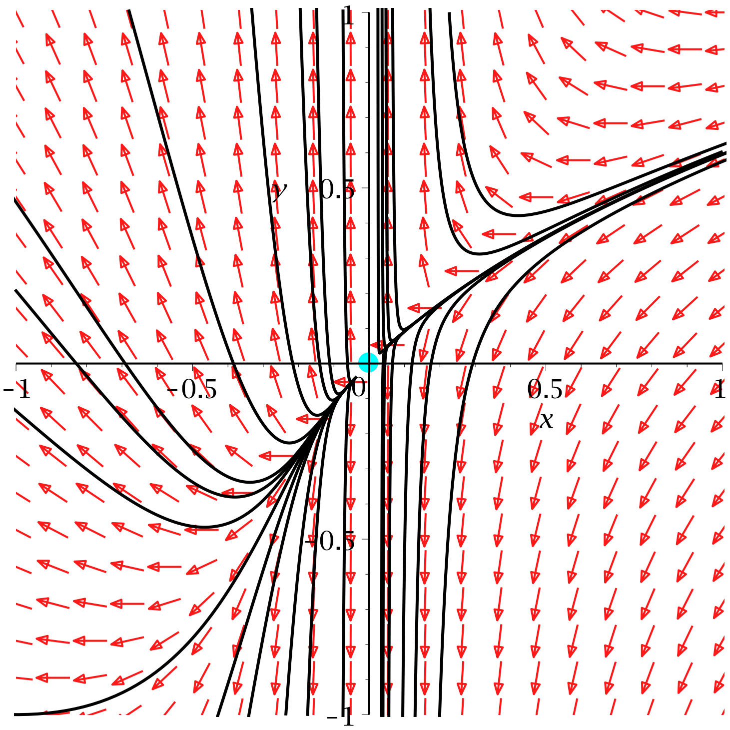

Figure 16: Contour plot of $x' = -x^2$; $y' = y - x$

18.1.3 A 0 eigenvalue resonant example

The Euler system

$$\dot{x} = -x^2 \tag{439}$$
$$\dot{y} = y - x \tag{440}$$

with linearized part

$$\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \tag{441}$$

and $\lambda_1 = 1, \lambda_2 = 0$. Note that a zero eigenvalue produces infinitely many resonances: $0 = 0n + n\lambda_2$. This cannot be linearized by extending the system to higher dimensions. If we take $q = -x^2$, we run into another system with zero eigenvalue and so on. This in itself is not a proof that it cannot be linearized, but based on the exact solution, we can make this assessment. The equation for the trajectories near $x = 0, y = 0$ is

$$y' + x^{-2}y - x^{-1} = 0 \tag{442}$$

which we have seen before: the singularity at 0 is irregular. The general solution of (439) is

$$x = \frac{1}{t + C_1}; \; y(t) = e^t \left[e^{C_1} \mathrm{Ei}(1, C_1 + t) + C_2 \right] \tag{443}$$

If the system were linearizable, then the general solution would be some analytic function of finitely many variables of the form $z_i = e^{\alpha_i} t^j$. This is clearly not the case, given the divergence of the series of y, or even the fact that $x = 1/t$ is a special solution; we shouldn't get non-analytic solutions if the system were linearizable.

while the solution of (442) is

$$y(x) = e^{1/x} \mathrm{Ei}(-1/x) + Ce^{1/x} \tag{444}$$

The Hamiltonian

$$H(x, y) = ye^{-1/x} - \mathrm{Ei}(-1/x) \tag{445}$$

has the same trajectories, as can be checked by writing the associated system (578). The trajectories near zero have exponential behavior and divergent series. We can linearize the system, but not in a very useful way: we can write $\mathrm{Ei}(-1/x) = \xi^2; y = \eta^2 e^{1/x(\xi)}$, but this is not close to the identity, neither does it help very much (except that here we have an explicit solution).

19 Planar systems

Assuming now we are studying a hyperbolic system in a neighborhood of an equilibrium. In a small neighborhood of the equilibrium, the system is equivalent by changes of coordinates to a linear system. So the local behavior is dictated by the types of flows associated to linear systems.

Let

$$x' = Bx \tag{446}$$

where B is a 2×2 matrix with constant coefficients.

19.1 Distinct eigenvalues

In this case, the system can be diagonalized, and it is equivalent to a pair of trivial first order ODEs

$$x' = \lambda_1 x \tag{447}$$

$$y' = \lambda_2 y \tag{448}$$

19.1.1 Real eigenvalues

The change of variables that diagonalizes the system has the effect of rotating and rescaling the phase portrait of (447). The phase portrait of (447) can be

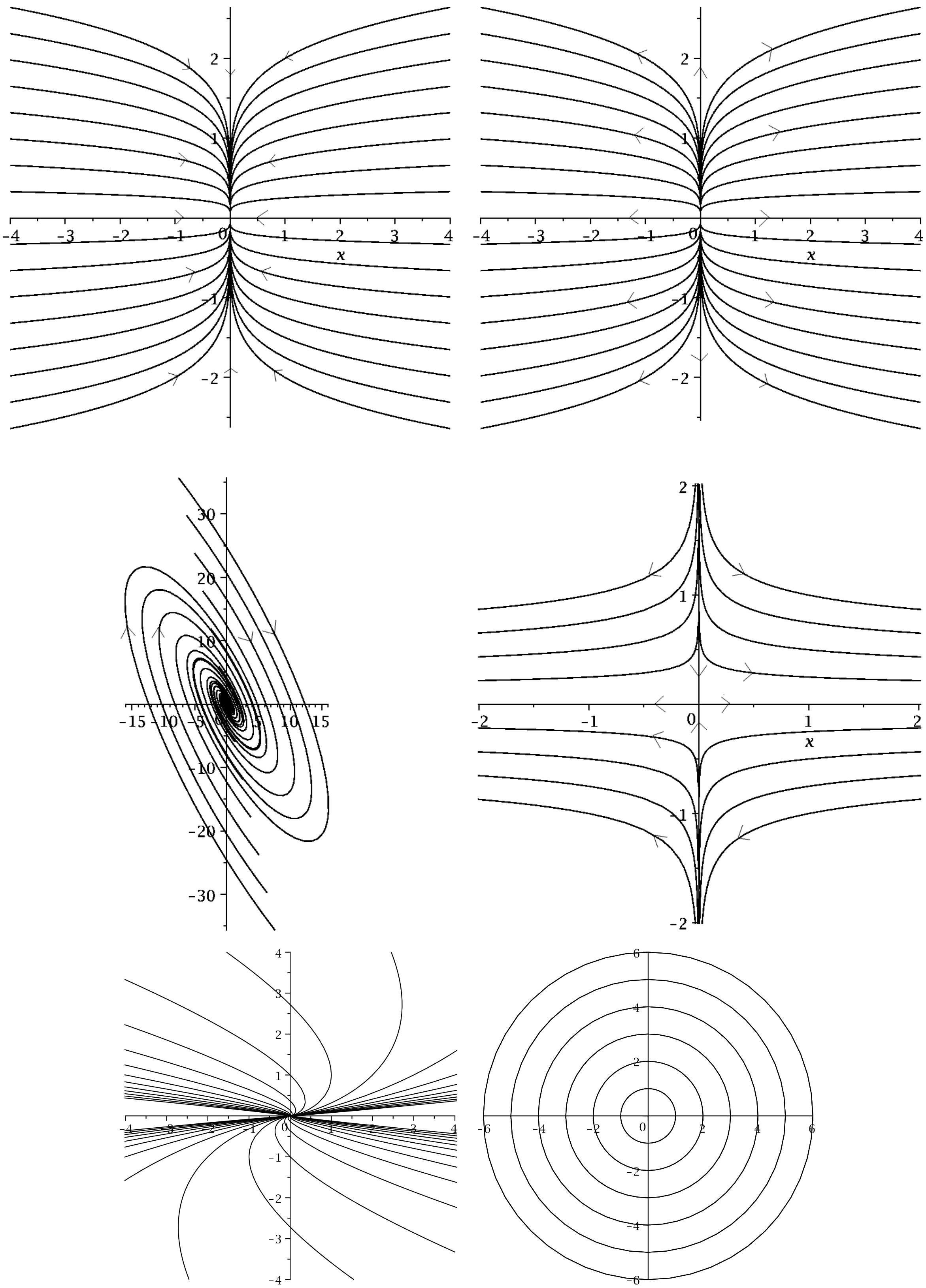

Figure 17: All types of linear equilibria in 2d, modulo euclidian transformations and rescalings: sink, source, spiral sink, saddle, nontrivial Jordan form, center resp. In the last two cases, the arrows point according to the sign of λ or ω, resp.

fully described, since we can solve the system in closed form, in terms of simple functions:

$$x = x_0 e^{\lambda_1 t} \tag{449}$$

$$y = y_0 e^{\lambda_2 t} \tag{450}$$

On the other hand, we have

$$\frac{dy}{dx} = \frac{\lambda_2}{\lambda_1}\frac{y}{x} = a\frac{y}{x} \Rightarrow y = C|x|^a \tag{451}$$

where we also have as trajectories the coordinate axes: $y = 0$ ($C = 0$) and $x = 0$ ("$C = \infty$"). These trajectories are generalized parabolas. If $a > 0$ then the system is either (i) **a sink**, when both λ's are negative, in which case, clearly, the solutions converge to zero. See Fig. 17, or (ii) **a source**, when both λ's are positive, in which case, the solutions go to infinity.

The other case is that when $a < 0$; then the eigenvalues have opposite sign. Then, we are dealing with a **saddle**. The trajectories are generalized hyperbolas,

$$y = C|x|^{-|a|} \tag{452}$$

Say $\lambda_1 > 0$. In this case there is a line, the x axis, along which solutions converge to zero. This is the **stable manifold**. The y axis is an **unstable manifold**, which would become stable if the direction of time is reversed. Other trajectories go to infinity both forward and backward in time. In the other case, $\lambda_1 < 0$, the figure is essentially rotated by $\pi/2$.

19.1.2 Complex eigenvalues

In this case we just keep the system as is,

$$x' = ax + by \tag{453}$$

$$y' = cx + dy \tag{454}$$

We solve for y, assuming $b \neq 0$ (check the case $b = 0$!), introduce in the second equation and we obtain a second order, constant coefficient, differential equation for x:

$$x'' - (a + d)x' + (ad - bc)x = 0 \quad or \tag{455}$$

$$x'' - \mathrm{tr}(B)x' + \det(B)x = 0 \tag{456}$$

If we substitute $x = e^{\lambda t}$ in (455) we obtain

$$\lambda^2 - \mathrm{tr}(B)\lambda + \det(B) = 0 \tag{457}$$

and, evidently, since $\lambda_1 + \lambda_2 = \mathrm{tr}(B)$ and $\lambda_1\lambda_2 = \det(B)$, this is the same equation as the one for the eigenvalues of B. The eigenvalues of B have been assumed complex, and since the coefficients we are working with are real, the roots are complex conjugate:

$$\lambda_i = \alpha \pm i\omega \tag{458}$$

The real valued solutions are

$$x = Ae^{\alpha t} \sin(\omega t + \varphi) \tag{459}$$

where A and φ are free constants. Substituting in

$$y = b^{-1}x' - ab^{-1}x \tag{460}$$

we get

$$y(t) = Ae^{\alpha t}b^{-1}[(\alpha - 1)\cos(\omega t + \varphi) - \omega \sin(\omega t + \varphi) \tag{461}$$

which can be written, as usual,

$$y(t) = A_1 e^{\alpha t} \sin(\omega t + \varphi_1) \tag{462}$$

If $\alpha < 0$, then we get a **spiral sink**. If $\alpha > 0$ then we get a spiral source, where the arrows are reverted.

A special case is that when $\alpha = 0$. This is the only non-hyperbolic fixed point with distinct eigenvalues. In this case, show that for some c we have $x^2 + cy^2 = A^2$, and thus the trajectories are ellipses. In this case, we are dealing with a **center**. We need more information about a nonlinear system to determine the nonlinear behavior.

19.2 Repeated eigenvalues

In 2d this case there is exactly one eigenvalue, and it must be real, since it coincides with its complex conjugate. Then the system can be brought to a Jordan normal form; this is either a diagonal matrix, in which case it is easy to see that we are dealing with a sink or a source, or else we have

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{463}$$

In this case, we obtain

$$\frac{dx}{dy} = \frac{x}{y} + \frac{1}{\lambda} \tag{464}$$

with solution

$$x = ay + \lambda^{-1} y \ln |y| \tag{465}$$

As a function of time, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = e^{\lambda t} \Big[I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \Big] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{466}$$

$$x(t) = (At + B)e^{\lambda t} \tag{467}$$
$$y(t) = Ae^{\lambda t} \tag{468}$$

We see that, in this case, only the x axis is a special solution (the y axis is not), and thus, all solutions approach (as $t \to \infty$ or $t \to -\infty$ for $\lambda < 0$ or $\lambda > 0$ respectively) the x axis.

Note 32. The eigenvalues of a matrix depend continuously on the coefficients of the matrix. In two dimensions you can see this by directly solving $\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$. Thus, if a linear or nonlinear system depends on a parameter α (scalar or not) and the equilibrium is hyperbolic when $\alpha = \alpha_0$, then the real parts of the eigenvalues will preserve their sign in a neighborhood of $\alpha = \alpha_0$. The type of equilibrium is the same and the local phase portrait changes smoothly unless the real part of an eigenvalue goes through zero.

Note 33. When conditions are met for a diffeomorphic local linearization at an equilibrium, then we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi \begin{pmatrix} u \\ v \end{pmatrix} \tag{469}$$

where the equation in (u, v) is linear and the matrix φ is a diffeomorphism. We then have

$$\begin{pmatrix} x \\ y \end{pmatrix} = (D\varphi) \begin{pmatrix} u \\ v \end{pmatrix} + o(u, v) \tag{470}$$

which implies, in particular that the phase portrait very near the equilibrium is changed through a linear transformation.

19.3 Stable and unstable manifolds in 2d

Assume that g is differentiable, and that the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = g \begin{pmatrix} x \\ y \end{pmatrix} \tag{471}$$

has an equilibrium at zero, which is a saddle, that is, the eigenvalues of $(Dg)(0)$ are $-\mu$ and λ, where λ and μ are positive. We can make a linear change of variables so that $(Dg)(0) = \mathrm{diag}(-\mu, \lambda)$. Consider a linearization tangent to the identity, that is, with $Dg(0) = I$. We call the linearized variables (u, v).

Theorem 18. *Under these assumptions, in a disk of radius $\varepsilon > 0$ near the origin there exist two functions $y = f_+(x)$ and $x = f_-(y)$ passing through the origin, tangent to the axes at the origin and so that all solutions with initial conditions $(x_0, f_+(x_0))$ converge to zero as $t \to \infty$, while the initial conditions $(f_-(y_0), y_0)$ converge to zero as $t \to -\infty$. The graphs of these functions are called the* **stable and unstable manifolds, resp**. *All other initial conditions necessarily leave this disk as time increases, and also if time decreases.*

Proof. We show the existence of the curve f_+, the proof for f_- being the same, by reverting the signs. We have

$$\begin{aligned} x(t) &= \varphi_1(u(t), v(t)) \\ y(t) &= \varphi_2(u(t), v(t)) \end{aligned} \tag{472}$$

where (u, v) satisfy $u' = -\mu u$ and $v' = \lambda v$.

Consider a point $(\varphi_1(u_0, 0), \varphi_2(u_0, 0))$. There is a unique solution passing through this point, namely $(\varphi_1(u_+(t), 0), \varphi_2(u_+(t), 0))$ where $u_+(0) = u_0, v_+(0) = 0$. Since $u_+(t) \to 0$ as $t \to \infty$ and φ is continuous, we have

$$(\varphi_1(u_+(t), 0), \varphi_2(u_+(t), 0)) \to 0$$

as $t \to \infty$. We now write $(u, v) = \Phi(x, y)$. Along the decaying solution, we have $v = 0$. Since $\Phi = I + o(1)$, we have $\partial\Phi_2/\partial y = 1$ at $(0, 0)$, and the implicit function theorem shows that $\Phi_2(x, y) = 0$ defines a differentiable function $y = f(x)$ near zero, and $y'(0) = 0$ (by implicit differentiation, check). Note that $y = f_+(x)$ is equivalent to $v = 0$ and initial conditions with $v_0 = 0$ evolve to the origin, implying the conclusion for $(x_0, f_+(x_0))$. The proof for $(f_-(y_0), y_0)$ is similar. For other solutions we have, from (472), that x, y exits any small enough disk (check). □

19.4 Further examples, [8],[14]

19.4.1 Stable and unstable manifolds in an exactly solvable model

Consider the system

$$x' = x + y^2 \tag{473}$$

$$y' = -y \tag{474}$$

The linear part of this system at $(0, 0)$ is

$$x' = x \tag{475}$$

$$y' = -y \tag{476}$$

The associated matrix is simply

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{477}$$

with eigenvalues 1 and -1. They are resonant with the lowest degree of resonance 3. Then, the conditions of a differentiable homeomorphism, Theorem 7 are satisfied (but not, of course, those of analytic equivalence. Nonetheless, it will turn out that the system can be analytically linearized.)

Locally, near zero, the phase portrait of the system (477) is thus the prototypical saddle.

Insofar as the field lines go, we have

$$\frac{dx}{dy} = -\frac{x}{y} - y \tag{478}$$

a linear inhomogeneous equation that can be solved by variation of parameters, or more easily noting that, by homogeneity, $x = ay^2$ must be a particular solution for some a, and we check that $a = -1/3$. The general solution of the homogeneous equation is clearly $xy = C$. It is interesting to make it into a homogeneous second order equation by the usual method. We write

$$\frac{1}{y}\frac{dx}{dy} = -\frac{x}{y^2} - 1 \tag{479}$$

and differentiate once more to get

$$\frac{d^2x}{dy^2} = -2\frac{x}{y^2} \tag{480}$$

which is an Euler equation, with indicial equation $(\lambda - 2)(\lambda + 1) = 0$, and thus the general solution is

$$x(y) = ay^2 + \frac{b}{y} \tag{481}$$

where the constants are not arbitrary yet, since we have to solve the more stringent equation (478). Inserting (481) into (478) we get $a = -1/3$. Thus, the general solution of (479) is

$$3xy + y^3 = C \tag{482}$$

which can be, of course, solved for x.

To linearize the system we note that

$$y(t) = C_1 e^{-t},\ x(t) = -\frac{1}{3}e^{-2t} + C_2 e^t = -\frac{1}{3C_1}y(t)^2 + C_2 e^t \tag{483}$$

where we solve, as in the previous sections, for e^t and e^{-t} in terms of x and y:

$$e^{-t} = y/C_1;\ e^t = \frac{1}{C_1}x(t) + \frac{1}{3}y(t) \tag{484}$$

and thus we expect that the transformation

$$x_1 = x + \frac{1}{3}y^2,\ y_1 = y \tag{485}$$

linearizes the system. Indeed, we have

$$x_1' = x_1,\ y_1' = -y_1 \tag{486}$$

Note 34 (Global linearization)**.** The linearizing change of coordinates is thus

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = I \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2/3 \\ 0 \end{pmatrix} \tag{487}$$

and in particular we see that the transformation is, to leading order, the identity. The unstable manifold is $y_1 = 0 = y$ and the stable one is $x_1 = 0$, the parabola $x = -y^2/3$.

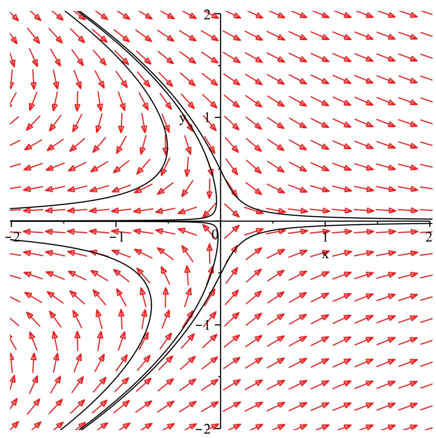

Figure 18: Phase portrait of (473)

Without using this explicit solution, the phase portrait can be obtained in the following way: we note that near the origin, the system is diffeomorphic to the linear part, thus we have a saddle there. There is a particular solution with $x = -1/3y^2$ and the field can be completed by analyzing the field for large x and y. This separates the initial conditions for which the solution ends up in the right half plane from those confined to the left half plane.

19.5 A limit cycle

Up to now we looked at equilibria, fixed points of the flow, which, along some direction(s), attract solutions as $t \to \infty$ or $t \to -\infty$. Fixed points are of course special, degenerate, trajectories. In nonlinear systems, solutions may be attracted by more structured trajectories: limit cycles.

As we will see, written in x, y the system is not resonant. This allows us to linearize the system near zero; zero is a spiral source.

This, of course, is not the complete story. What happens for larger (x, y) We follow again [8], but with a different starting point. Let's look at the simple system

$$r' = r(1 - r^2)/2 \tag{488}$$

$$\theta' = 1 \tag{489}$$

where, later, we will think of (r, θ) as polar coordinates.

Obviously, we can solve (488) in closed form. The flow clearly has no fixed point, since $\theta' = 1 \neq 0$.

To solve the first equation, note that if we multiply by $2r$ we get

$$2rr' = r^2(1 - r^2) \tag{490}$$

or, with $u = r^2$,

$$u' = u(1-u) \tag{491}$$

The exact solution is

$$r = \pm(1 + Ce^{-t})^{-1/2};\ \text{and also } r = 0;\ \pm 1, \text{ as special constant solutions} \tag{492}$$

$$\theta = t + t_0 \tag{493}$$

We see that all solutions that start away from zero converge to one as $t \to \infty$. We now interpret r and θ as polar coordinates and write the equations for x

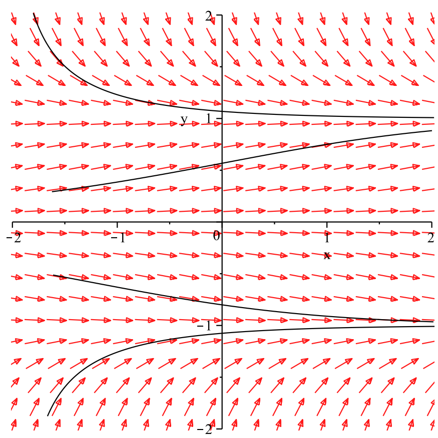

Figure 19: Phase portrait of (488)

and y. We get

$$x' = r'\cos\theta - r\sin\theta\theta' = \frac{1}{2}r(1-r^2)\cos\theta - r\sin\theta$$
$$= \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \tag{494}$$

$$y' = r'\sin\theta + r\cos\theta\theta' = \frac{1}{2}r(1-r^2)\sin\theta + r\cos\theta$$
$$= x + \frac{1}{2}y - \frac{1}{2}(x^2y + y^3) \tag{495}$$

thus the system

$$x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \tag{496}$$

$$y' = x + \frac{1}{2}y - \frac{1}{2}(x^2y + y^3) \tag{497}$$

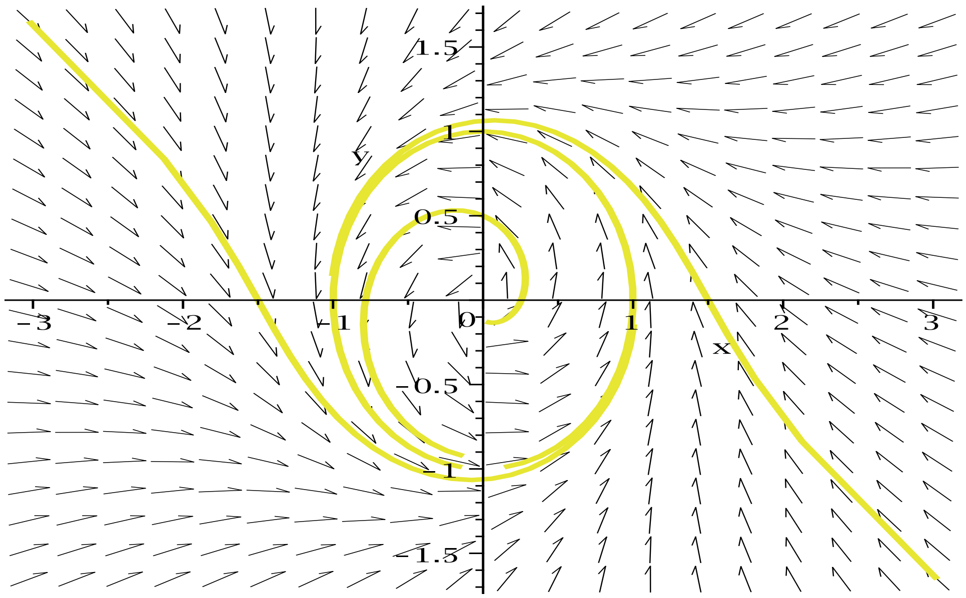

Figure 20: Phase portrait of (496), (497).

which looks rather hopeless, but we know that it can be solved in closed form.

To analyze this system, we see first that at the origin the matrix is

$$\begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \tag{498}$$

with eigenvalues $1/2 \pm i$. Thus the origin is a spiral source.

Exercise 1. (a) Show that the only equilibrium of (496), (497) is $(0,0)$.

(b) Although solvable in polar coordinates, show that there is no regular enough expression for the solutions of (496), (497), say as an implicit representation $F(x,y) = C$ with $F \in C(\mathbb{R}^2)$. (Note however that the system $\dot{y} = 2y, \dot{x} = x$ does not have a continuous implicit representation, but it has a meromorphic one, $y/x^2 = C$.)

Now we know the solution globally, by looking at the solution of (488) and/or its phase portrait.

We note that $r = 1$ is a solution of (488), thus the unit circle is a trajectory of the system (496). It is a closed curve, all trajectories tend to it asymptotically. This is a **limit cycle**.

19.6 Application: constant real part, imaginary part of analytic functions

Assume for simplicity that f is entire. The transformation $z \to f(z)$ is associated with the planar transformation $(x, y) \to (u(x, y), v(x, y))$ where $f = u + iv$. The grid $x = const, y = const$ is transformed into the grid $u = const, v = const$. We can first look at what this latter grid is transformed back into, by the transformation.

We take first $v(x(t), y(t)) = const$. We have

$$\frac{\partial v}{\partial x} x'(t) + \frac{\partial v}{\partial y} y'(t) = 0 \tag{499}$$

which we can write, for instance, as the system

$$x' = \frac{\partial v}{\partial y} \tag{500}$$

$$y' = -\frac{\partial v}{\partial x} \tag{501}$$

which, in particular, is a Hamiltonian system. We have a similar system for u. We can draw the curves $u = const, v = const$ either by solving this implicit equation, or by analyzing (500), or even better, by combining the information from both. Let's take, for example $f(z) = z^3 - 3z^2$. Then, $v = 3x^2 y - y^3 - 6xy$. It would be rather awkward to solve $v = c$ for either x or y. The system of equations reads

$$x' = -6x + 3x^2 - 3y^2 \tag{502}$$

$$y' = 6y - 6xy \tag{503}$$

Note that $\nabla u = 0$ is equivalent to $z' = 0$ and so is $\nabla v = 0$. For equilibria, we thus solve $3z^2 - 6z = 0$ which gives $z = 0; z = 2$. Near $z = 0$ we have

$$x' = -6x + o(x, y) \tag{504}$$

$$y' = 6y + o(x, y) \tag{505}$$

which is clearly a saddle point, with x the stable direction and y the unstable one. At $x = 2, y = 0$ we have, denoting $x = 2 + s$,

$$s' = 6s + o(s, y) \tag{506}$$

$$y' = -6y + o(s, y) \tag{507}$$

another saddle, where now $y = 0$ is the stable direction. We note that $y = 0$ is, in fact, a special trajectory, and it belongs to the nonlinear unstable/stable manifold at the equilibrium points. Note also that the nonlinear stable manifold at zero is the same as th unstable one at 2: this is a heteroclinic orbit, or a heteroclinic connection.

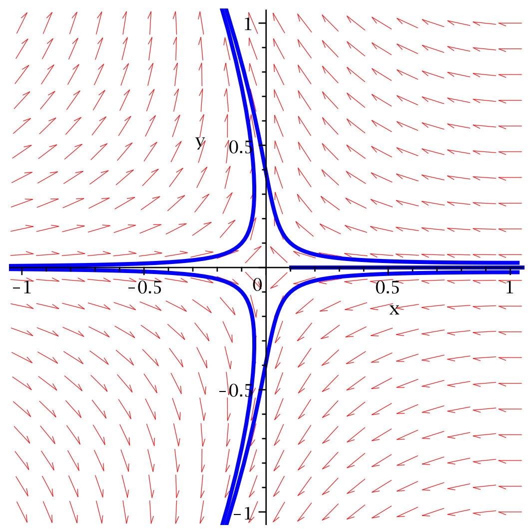

Figure 21: Phase portrait of (502) near $(0, 0)$.

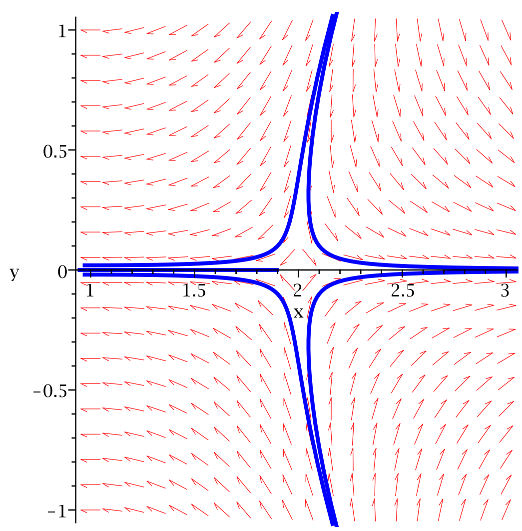

Figure 22: Phase portrait of (502) near $(0, 2)$.

We draw the phase portraits near $x = 0$ or near $x = 2$, we mark the special trajectory, and look at the behavior of the phase portrait at infinity. Then we "link" smoothly the phase portraits at the special points, and this should suffice for having the phase portrait of the whole system.

For the behavior at infinity, we note that if we write

$$\frac{dy}{dx} = \frac{y(1-6x)}{-6x+3x^2-3y^2} \tag{508}$$

we have the special solution $y = 0$, and if $|x| \gg 1, |y| \gg 1$, then the nonlinear terms dominate and we have

$$\frac{dy}{dx} \approx \frac{-6yx}{3x^2-3y^2} \tag{509}$$

By homogeneity, we look for special solutions of the form $y = ax$ (which would be asymptotes for the various branches of $y(x)$. We get, to leading order,

$$a = \frac{-6a}{3-3a^2} \tag{510}$$

We obtain

$$a = 0, a = \pm\sqrt{3} \tag{511}$$

We also see that, if $x = o(y)$, then $y' = o(1)$ as well. This would give us information about the whole phase portrait, at least qualitatively.

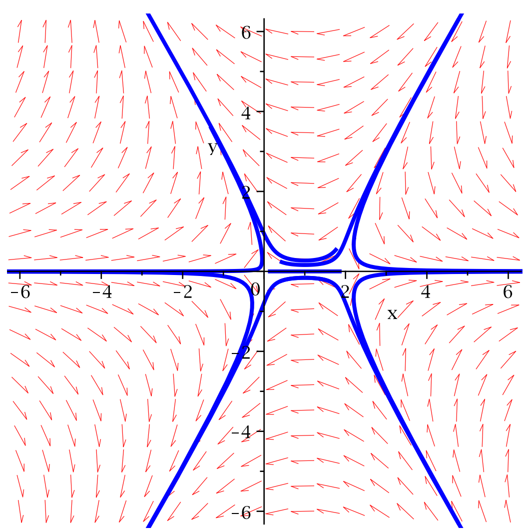

Figure 23: Phase portrait of (502), $v = const$.

Exercise 2. Analyze the phase portrait of $u(x, y) = const$.

The two phase portraits, plotted together give Note how the fields intersect at right angles, except at the saddle points. The reason, of course, is that $f(z)$ is a conformal mapping wherever $f' \neq 0$.

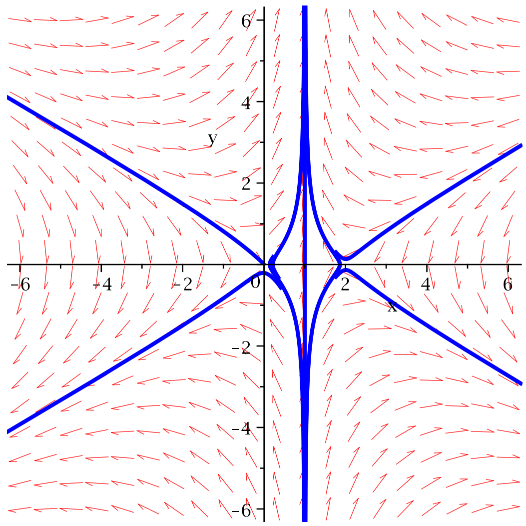

Figure 24: Phase portrait of $u = const$

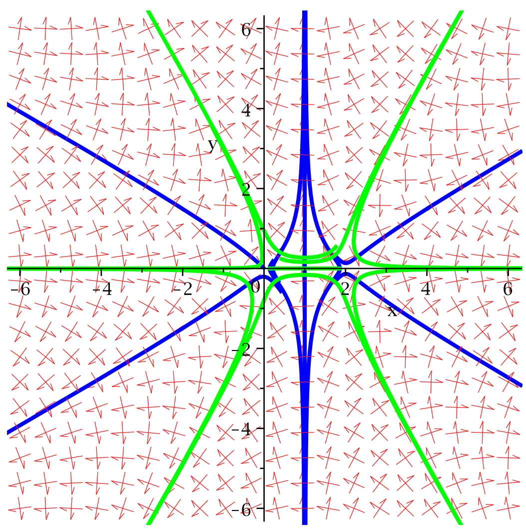

Figure 25: Phase portrait of $u = const$, and $v = const$.

Exercise 3. Draw the global phase portrait of the approximations of the pendulum, $x'' + x - x^3/6 = 0$, $x'' + x - x^3/3 + x^5/5 = 0$. Find the equilibria, local and global behavior. Find out if there are limit cycles. Discuss the conection sith the physical pendulum, $x'' + \sin(x) = 0$.

Exercise 4. Draw the global phase portrait of the damped pendulum, $x'' +$

$ax' + \sin x = 0$, where $a > 0$ is the air friction coefficient. Discuss what happens as $a \to 0$ and how this relates to the undamped pendulum, $a = 0$. Discuss also the bifurcation that occurs at $a = 0$ and $x = 0$ (for $a < 0$, the physical interpretation could be that we are looking backwards in time. Also, note any global bifurcations, that is changes in the global topology.

20 Bifurcations

Bifurcations occur when a change in a parameter induces topological changes in the phase portrait. These can be local, global (or both). There are many physical systems that are modeled by bifurcating dynamical systems, including reaction-diffusion equations, pattern formation, laser dynamics and so on.

Local bifurcations refer to the situations when a parameter crosses a value where the stability of in a neighborhood of a local equilibrium (or coalescing ones) changes. Global bifurcations affect higher (than zero...) dimensional attractors, such as limit cycles.

As we know, the phase portrait of a system depending on a parameter changes its **topological** structure near an equilibrium only if at least one eigenvalue becomes purely imaginary. This may happen if one eigenvalue (or both, but generically one) becomes zero, or else they pass through a point where they are nonzero, imaginary and complex conjugate to each other (since we are dealing with real-valued equations).

The classification is made by looking at the *normal form.* We keep all terms that cannot be eliminated topologically (using Hartman-Grobman) when the parameter changes in a neighborhood of the bifurcation point.

For instance, near $a = 0$,

$$x' = x^2 + r; \quad y' = -y$$

will represent a general system of the form

$$x' = a + bx^2 + O(x^3); \quad y' = -y + O(y^2)$$

20.1 Some types of bifurcations

We study the normal forms that are quadratic, or when the quadratic term i missing due, say, to a symmetry, cubic. They are classified according to this degree, and also according to the position of the "external" parameter.

$$x' = x^2 + r; \quad y' = -y; \quad \textbf{saddle-node bifurcation}$$

Here, for $r < 0$ there are two equilibria (a saddle and a node) that coalesce when $r = 0$ (when we have a "saddle-node"), while there is no equilibrium when $r > 0$.

$$x' = rx - x^2; \quad y' = -y; \quad \textbf{transcritical bifurcation}$$

When r goes through zero, two equilibria coalesce, and after the coalescence we still have two equilibria.

$$x' = rx - x^3; \quad y' = -y; \quad \textbf{supercritical pitchfork bifurcation}$$

One unstable equilibrium ($r < 0$) bifurcates into an unstable one and two stable ones.

$$x' = rx + x^3; \quad y' = -y; \quad \textbf{subcritical pitchfork bifurcation}$$

We see that in the pitchfork bifurcation, the quadratic term is absent (this is not so rare in applications due to symmetries of the system).

Of course, in the systems above, much of the information is contained in the x part, and we may in some sense ignore the y one, since the equations are decoupled.

$$x' = \beta x - y + \lambda x(x^2 + y^2); \quad y' = x + \beta y + \lambda y(x^2 + y^2) \ \textbf{Hopf bifurcation}$$

where $\beta = 0$ is the bifurcation point. Here, eigenvalues become imaginary, but nonzero.

20.2 Normal form of the saddle-node bifurcation

Consider a simple system which illustrates the first case, an eigenvalue going through zero, prototypical for *saddle-node bifurcations*,

$$x' = x^2 + r \tag{512}$$

$$y' = -y \tag{513}$$

Of course, we can solve this explicitly, but we choose not to, because solvable equations are infrequent. We first note that the only possible equilibria are $(\pm\sqrt{-r}, 0)$. Clearly, there are two of them if $r < 0$, one if $r = 0$ and none if $r > 0$. For $r = 0$, the equilibrium is non-hyperbolic and needs to be studied separately. For $r < 0$, at $x = \pm r$, we see that the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \pm 2\sqrt{-r} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{514}$$

Thus the point $(-\sqrt{-r}, 0)$ is a node, while $(\sqrt{-r}, 0)$ is a saddle.

Let's draw the complete phase portrait in the three regimes, $r < 0$, $r = 0$ and $r > 0$. Again, the portrait is determined by the set of equilibria, limit cycles, and by the behavior at infinity. The three lines $x = \pm\sqrt{-r}$ and $y = 0$ are special solutions of the system. We see that there are no limit cycles, since trajectories do not cross except at the equilibria, and the lines $x = \pm\sqrt{-r}$, never crossed, delimit regions where the sign of x' is constant.

Behavior at infinity: for x very large, we have $x' \approx x^2$ and $y' = -y$, and thus

$$\frac{dy}{dx} \approx \frac{-y}{x^2} \tag{515}$$

with the solution $y = Ce^{1/x}$. Thus in the far x field, the trajectories are expected to approach horizontal lines. How do we prove this rigorously? One way is to note that for any $\alpha > 1$ $(x^2 - r)^{-1} \leqslant \alpha x^{-2}$ if x is large enough.

Thus, we can write

$$\frac{y'(x)}{y(x)} \geqslant -\frac{\alpha}{x^2} \tag{516}$$

where we can integrate both sides and get

$$y(x) \geqslant C_{x_0,y_0} e^{\alpha/x} \tag{517}$$

where C_{x_0,y_0} is a constant depending on the initial condition (x_0, y_0). Similarly,

$$y(x) \leqslant C_{x_0,y_0} e^{\alpha'/x} \tag{518}$$

If instead x is bounded and $y \to \infty$, the direction field points straight to the origin, so there the trajectories essentially vertical lines. Piecing all this together, we get the phase portrait depicted below.

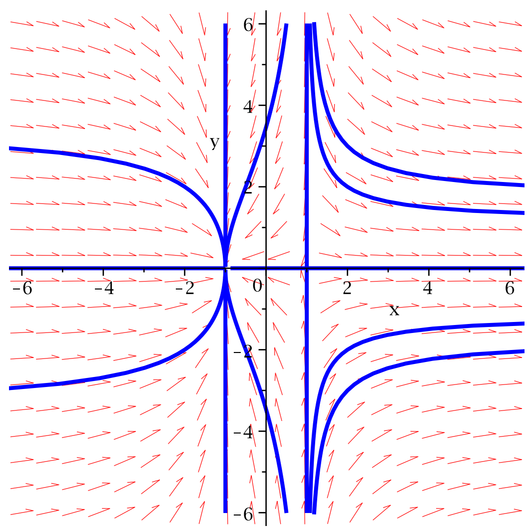

Figure 26: Phase portrait of (519) for $r = -1$

For $r = 0$ the system simply becomes

$$x' = x^2 \tag{519}$$

$$y' = -y \tag{520}$$

Clearly, the line $x = 0$, a special solution, is attracting, while the line $y = 0$ is repelling for $x > 0$ and attracting (since the field points towards the origin) for $x < 0$. So we see that, in some sense, the origin is now half-node, half saddle.

[h!]

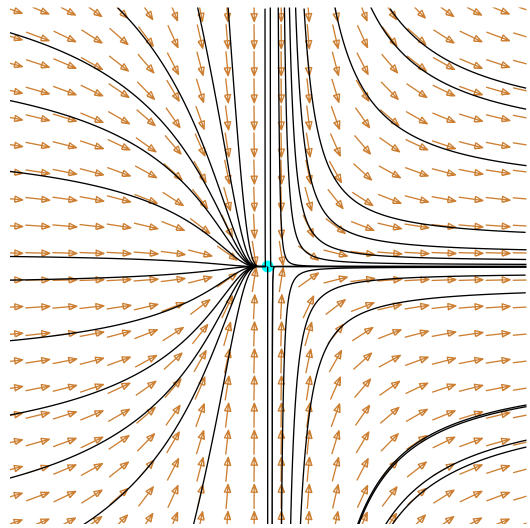

Figure 27: Phase portrait of (519) for $r = 0$

All nearby trajectories are attracted to zero if they start in the closed left half plane, and repelled otherwise.

The far-field picture is clearly the same as in the case $r < 0$, so we can piece together these informations to draw the phase portrait. We note that in this case, of course, the explicit solutions $y = Ce^{1/x}$, $C \in \mathbb{R}$ and $x = 0$, can be easily used to draw the phase portrait. This type of behavior as a function of r, at least in this example, explains the choice of name, saddle-node bifurcation.

Stable and center manifold

For hyperbolic systems, say near a saddle point, we have defined stable and unstable manifolds; these are manifolds invariant under the flow and tangent to the positive/negative eigenvectors.

Here one eigenvalue is zero. What manifolds are invariant under the flow and tangent to the eigenvectors? We have the y axis as a stable manifold, tangent to the direction $(0, 1)$, the eigenvector corresponding to the eigenvalue 1. The eigenvalue zero has $(1, 0)$ as eigenvector. $Ce^{1/x}$ are exact trajectories. *All of them* are tangent to Ox if $x < 0$, and one, with $C = 0$, that is Ox itself is tangent for $x > 0$. There is a continuum of invariant manifolds for $x < 0$, and thus in general since they can all be continued by $x = 0$ in the right half plane. These are called center manifolds, and in general none is privileged, except in an analytic setting like this one we could pick the analytic manifold $y = 0$. However, from the point of view of solution behavior, it is not distinguished.

“Linearization”

Note that the exact solution is $x = 1/(A - t), y = Be^{-t}$. Solving $y = e^{-t}$ for t, inserting in the first equation and solving for $1/A$ we get

$$x_1 = \frac{x}{1 - x \ln y} \tag{521}$$

with the property $x_1' = 0$. In the coordinates x_1, y the system is linear. The transformation from (x, y) to (x_1, y) has to be done separately in each quadrant,

$$x_1 = \frac{|x|}{1 - |x| \ln |y|} \tag{522}$$

and is quite singular, however.

*

Finally, for $r > 0$ there are no equilibria. We see that $x' > 0$ for all x. Trajectories extend from $-\infty$ to $+\infty$ in x. The behavior in the far field is the same as in the previous examples. The trajectories have horizontal lines as asymptotes for $x \to \pm\infty$ and, in the upper half plane, the asymptote for $x < 0$ lies above the one for $x > 0$, since $y' < 0$ there. We can now draw the phase portrait.

As we see, the node in the left half plane approaches the saddle, touches it at which time we have a half-node half-saddle picture, and then the equilibrium vanishes and the curves in the left half plane “spill over” in the right half plane.

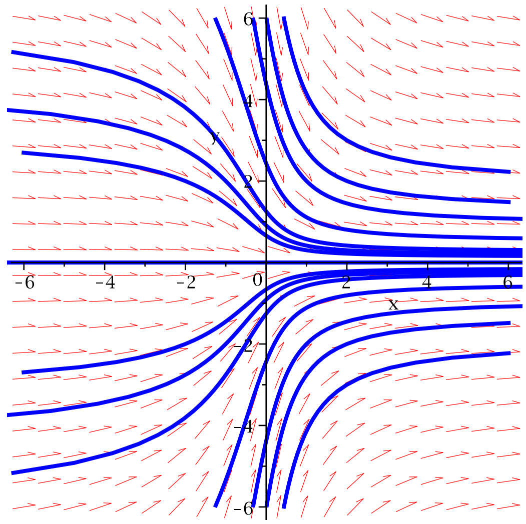

Figure 28: Phase portrait of (519) for $r > 0$

20.3 Transcritical bifurcation

In this type of bifurcation, there are two equilibria for all values of $r \neq 0$, a saddle and a node and one for $r = 0$. The node and saddle are interchanged when r changes sign.

Typically, for this and some other bifurcations, the y part is ignored, and for a good reason, as we mentioned there is effectively no y participation. The reason for which this type of bifurcation is called transcritical is from the way things look as a function of the parameter for the x-only system. To have however a unified picture in mind, and to recall that we are after all dealing with two dimensional systems for which it does *happen* that the normal form makes y "idle" we will look at the two dimensional system,

$$\begin{aligned} x' &= rx - x^2 \\ y' &= -y \end{aligned} \tag{523}$$

For $r \neq 0$ there are two equilibria, and for $r = 0$ only one; the two equilibria collide as before, but the outcome is different.

Take $r < 0$. Clearly, the origin, marked in blue, is a node. The other equilibrium, $x = r, y = 0$ is a saddle ($r - 2r = -r > 0$).

The global picture is obtained as before: the rays: $\{(-t, 0) : t < r\}$, $\{(t, 0) : t \in (r, 0)\}$,$\{(t, 0) : t > 0\}$, $\{(0, \pm t^2) : t > 0\}$ are special trajectories; in the far field, the trajectories are almost horizontal.

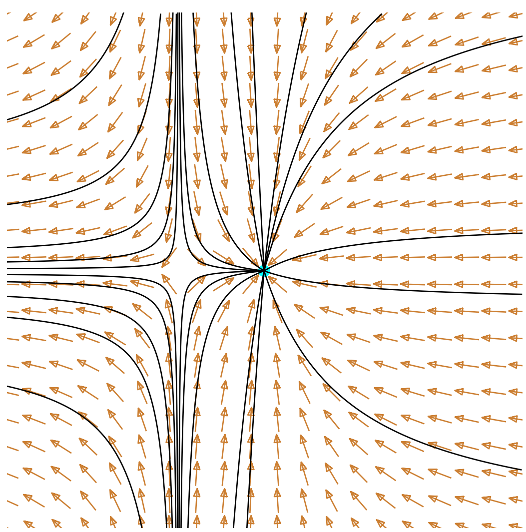

Figure 29: Transcritical phase portrait, $r < 0$

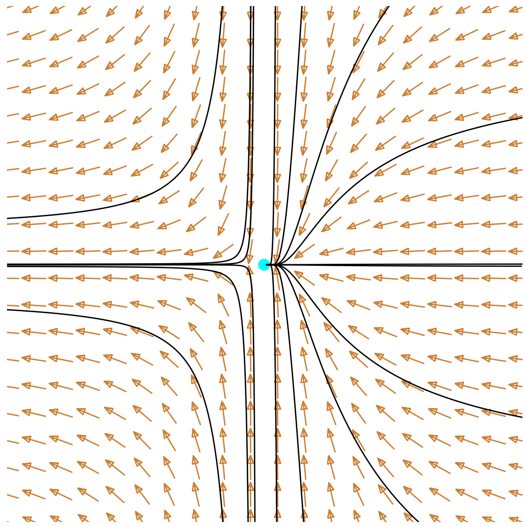

Figure 30: Transcritical phase portrait, $r = 0$

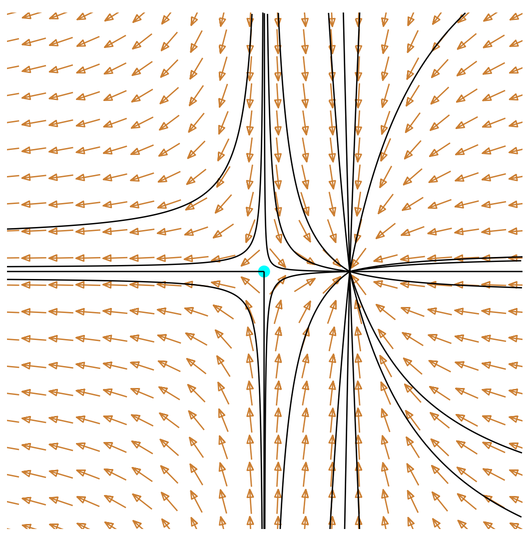

Figure 31: Transcritical phase portrait, $r > 0$

Here, we see that a saddle-node becomes a “half saddle-half node” and then it becomes a node-saddle and the types of equilibria are interchanged.

20.4 Normal form of the pitchfork bifurcation

Here we are dealing with a system with symmetries, the quadratic term is missing. The normal form of the supercritical pitchfork bifurcation is

$$\begin{aligned} x' &= rx - x^3 \\ y' &= -y \end{aligned} \tag{524}$$

whereas the subcritical one has the normal form

$$\begin{aligned} x' &= rx + x^3 \\ y' &= -y \end{aligned} \tag{525}$$

We look only at the supercritical case, the subcritical one being analyzed similarly.

20.4.1 Supercritical case

The field is an odd function of x and y, and stays odd for all (or only small, maybe) values of r. The name "pitchfork" will become clear in a moment.

In case 1) $r > 0$, we have three equilibria, $x = 0$ and $x = \pm\sqrt{r}$.

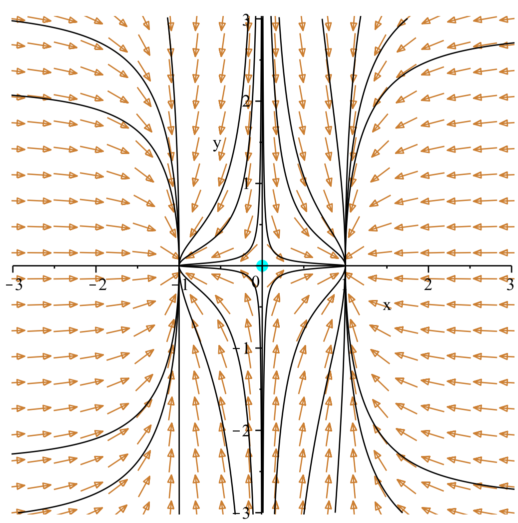

Figure 32: Pitchfork phase portrait, $r > 0$

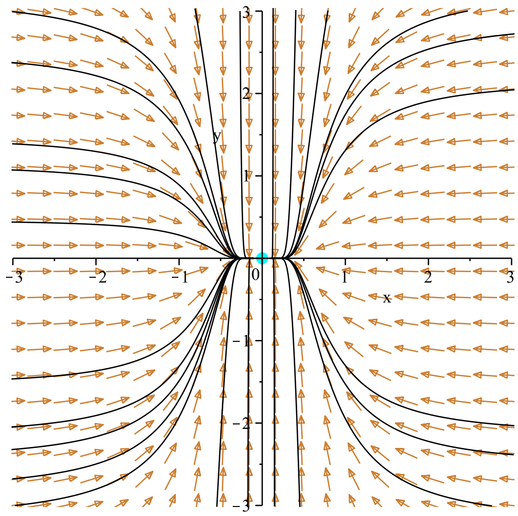

Figure 33: Pitchfork phase portrait, $r = 0$

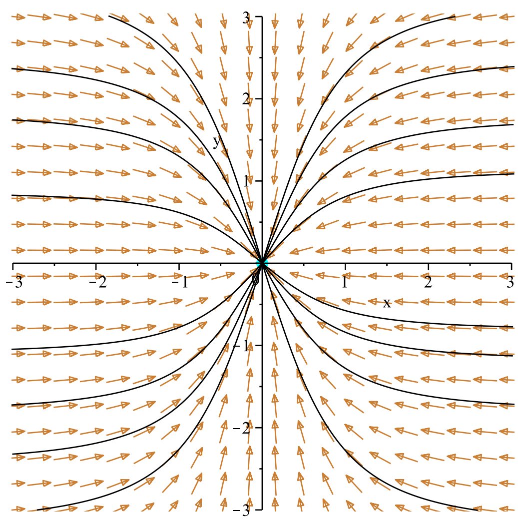

Figure 34: Pitchfork phase portrait, $r < 0$

It is clear that the origin is a saddle whereas the other two equilibria, symmetric, are nodes (sinks).

If, 2), $r = 0$, clearly we only have one equilibrium, and it is a node because $-x^3$ always points towards the origin.

By explicit solution, we see that the trajectories are given by $y = Ce^{-1/(2x^2)}$, which explains the fact that the phase portrait almost seems to have a continuum of nodes near zero.

Finally, in case 1) $r < 0$, we have only one equilibrium and it is a node. the number of equilibria changes.

Note that we can extend artificially the number of variables, to transform the two dimensional parameter-dependent problem into a three-dimensional parameter-free one,

$$x' = rx - x^3 \quad (526)$$

$$y' = -y \quad (527)$$

$$r' = 0$$

Clearly now the change in behavior is seen as a chenge in the 3d phase portrait, as a function of the initial condition in r.

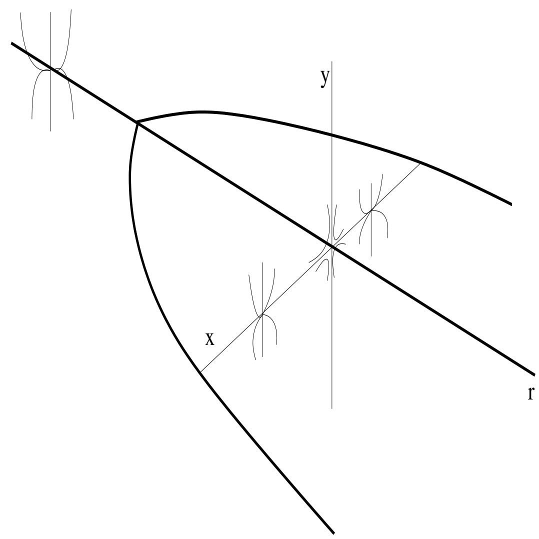

Figure 35: Pitchfork 3d phase portrait.

20.5 Normal form of the Hopf bifurcation

In this case, we are looking at a system for which passage through the critical value of the parameter (β) implies nonzero, purely imaginary eigenvalues. Take first $\sigma = -1$:

$$x' = \beta x - y - x(x^2 + y^2) \quad (528)$$

$$y' = x + \beta y - y(x^2 + y^2) \quad (529)$$

The origin is an equilibrium for all β, and it is the only one (this is best seen in polar coordinates, (531) below) where $\theta' > 0$. At the origin, the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = B \begin{pmatrix} x \\ y \end{pmatrix}; \quad B = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \tag{530}$$

where the equation for the eigenvalues of the matrix B is $(\beta - \lambda)^2 + 1 = 0$, and thus $\lambda_\pm = \beta \pm i$. At $\beta \neq 0$ the equilibrium is hyperbolic, a spiral sink if $\beta < 0$ and a spiral source for $\beta > 0$. A change of phase portrait, a bifurcation, should occur at $\beta = 0$.

For a simple analysis of the phase portrait, we rewrite the system in polar coordinates.

$$r' = \beta r - r^3 \tag{531}$$

$$\theta' = 1 \tag{532}$$

For $\beta < 0$, $\beta r - r^3 = 0$ has only one solution, $r = 0$. In (x, y), all solutions converge to $(0, 0)$ (since $r' < 0$) while spiraling.

In the far field, we have

$$r' \approx -r^3 \tag{533}$$

$$\theta' = 1 \tag{534}$$

with solution

$$r = (2\theta + 2C)^{-1/2} \tag{535}$$

For r to be very large, we must have θ very close to $-C$. That is, asymptotically the curves in the far field (x, y) plane have radial lines as asymptotes. The spiraling ceases there.

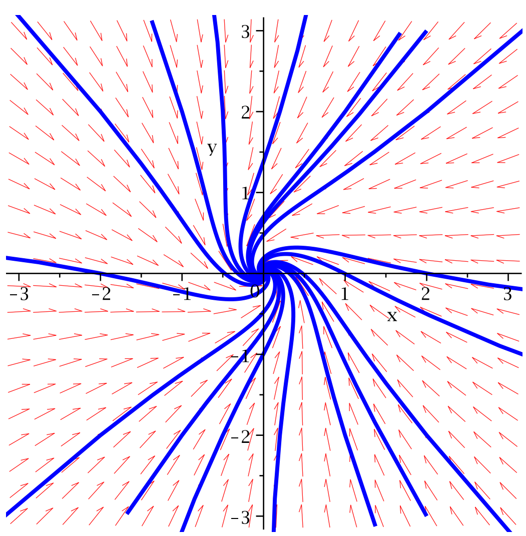

Figure 36: Phase portrait of (530) for $\beta = -1$

When $\beta = 0$, $r = 0$ is still the only solution of $\beta r - r^3 = 0$. Since again $r' < 0$, all trajectories go to the origin. The origin is approached at a very slow rate, $O(r^3)$, there is a lot of spiraling going on in that region. We see a tendency of a limit cycle being born.

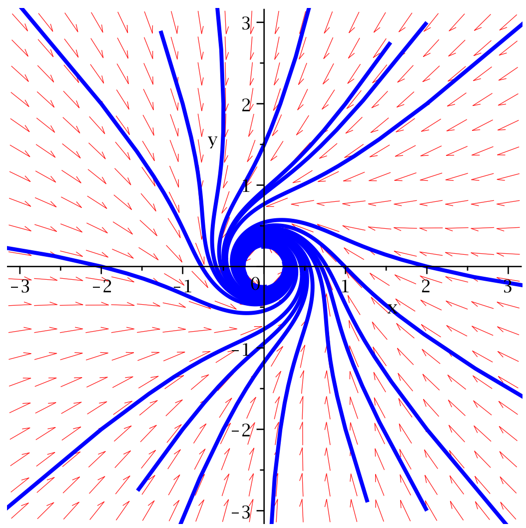

Figure 37: Phase portrait of (530) for $\beta = 0$

For $\beta > 0$, we have three solutions of $\beta r - r^3 = 0$, 0 and $\pm\sqrt{\beta}$ (the minus solution is "unphysical" for us). $r = 0$ is repelling and $r = \sqrt{\beta}$ is attracting. This means, in (x, y) that $x^2 + y^2 = \beta$ is *limit cycle*. We note that it approaches the origin as $\beta \to 0$. The spiral sink changes into a spiral source plus a limit cycle. It is probably worth looking at the exact solution, which can be obtained

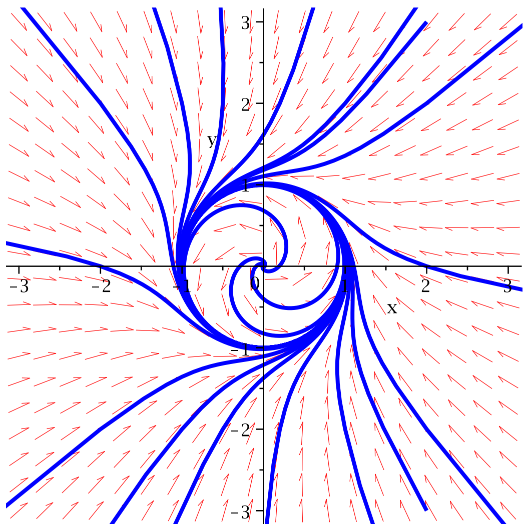

Figure 38: Phase portrait of (530) for $\beta = 1$

from the polar representation. For $\beta > 0$ we have

$$x(t) = \frac{\sqrt{\beta}\cos(t)}{\sqrt{1 + Ce^{-2\beta t}}};\ y(t) = \frac{\sqrt{\beta}\sin(t)}{\sqrt{1 + Ce^{-2\beta t}}} \tag{536}$$

For $\beta < 0$ the solution is

$$x(t) = \frac{\sqrt{|\beta|}\cos(t)}{\sqrt{e^{2|\beta|(t)} + C}};\ y(t) = \frac{\sqrt{|\beta|}\sin(t)}{\sqrt{e^{2|\beta|(t)} + C}} \tag{537}$$

while for $\beta = 0$ we get

$$x(t) = \frac{\cos t}{\sqrt{2t + C}};\ y(t) = \frac{\sin t}{\sqrt{2t + C}} \tag{538}$$

21 Bifurcations in more general systems. The central manifold theorem

Here we follow [7]. The setting is that of differential systems depending on a parameter,

$$\mathbf{x}' = \mathbf{f}_\mu(x), \quad x \in \mathbb{R}^n,\ \mu \in \mathbb{R}^m \tag{539}$$

where we assume sufficient smoothness of $\mathbf{f}_\mu$. To simplify the notation we will drop the boldface fonts wherever this is not confusing. Equilibrium solutions are given by the (constant) solutions of the equation

$$\mathbf{f}_\mu(x) = 0 \tag{540}$$

The equilibrium points depend smoothly on μ, by the implicit function theorem, as long as $D_x f_\mu$ is invertible, that is, as long as it has no zero eigenvalue. If $(\det D_x f_\mu)(x_0, \mu_0) = 0$, several branches of equilibria may form/disappear. These points (x_0, μ_0) are bifurcation points. For example, in the pitchfork bifurcation example, $\mu = r$ and $(0, 0)$ is the only bifurcation point. In that case, the equilibria coalesce.

A crucial notion here is that of *transversality*. In one dimension, $y = f(x)$ crosses the x axis transversally at x_0 if $f(x_0) = 0$ and $f'(x_0) \neq 0$. In d dimensions, two manifolds intersect transversally if the tangent spaces at the intersection point span $\mathbb{R}^d$ (there is no loss in dimension). It is clear that transversal intersections are generic. In particular, two manifolds Σ_1 and Σ_2 of dimensions d_1 and d_2 intersect transversally along a manifold of dimension $d_1 + d_2 - d$. Equivalently, the codimension of $\Sigma_1 \cap \Sigma_2$ is $(d - d_1) + (d - d_2)$. Two surfaces in 3d intersect generically along a line, two generic curves do not intersect, and a curve and a manifold generically intersect at a point, etc.

For the vector field $\mu + x^2$ thought of as a family of curves in $\mathbb{R}$, the curve for $\mu = 0$ intersects the x axis non-transversally at $x = 0$.

However, if we lift the number of dimensions to include μ in the picture, we have a *transversal* intersection of the surface $F(x, \mu) = x^2 + \mu$ with the (x, μ) coordinate plane.

Also, it is clear that transversal intersections are stable in the following sense. If two manifolds intersect transversally, then any small perturbation of the manifolds will also have a transversal intersection. On the contrary, if two manifolds intersect non-transversally, then their generic perturbations will intersect transversally. Let f be a C^r vector field on $\mathbb{R}^n$ vanishing at the origin ($f(0) = 0$) and let $A = (DF)(0)$. We denote as usual by $\sigma_{u,c,s}$ the parts of the spectrum (eigenvalues) for which $\mathrm{Re}\lambda > 0$, $= 0$, < 0 respectively.

Denote the generalized eigenspaces of $\sigma_{u,c,s}$ by $E^{u,c,s}$ respectively. By definition, the stable manifold is a set invariant under the flow which is tangent to E^s, the unstable one is tangent E^u whereas the center manifold *also invariant under the flow* is tangent to E^c.

We remember that, in hyperbolic systems (for which therefore the center manifold is absent) the stable/unstable manifolds are unique.

We see that center manifolds need not be unique (typically they are not) on the simple example (519),

$$x' = x^2 \tag{541}$$

$$y' = -y \tag{542}$$

Clearly $(0, 0)$ is a non-hyperbolic fixed point, with 0 eigenvalue in the x direction. We have a unique stable manifold at $(0, 0)$: here we look for an invariant set tangent to the vertical axis, and in this case it is the vertical axis itself. How about sets tangent to the center direction, $x = 0$? See Figure 20.2. We can solve for the trajectories

$$\frac{dy}{dx} = -\frac{y}{x^2} \tag{543}$$

i.e, $y = Ce^{1/x}$. We see that there is no such trajectory for $x > 0$, but *for all* C, the trajectories in the left half plane are tangent to the real line. Any of these would be a center manifold.

Theorem 19 (Center manifold theorem for flows). *There exist C^r stable and unstable manifolds (invariant under the flow and tangent to E^s, E^u) W^s and W^u respectively, and these are unique. There is a (generally nonunique) center manifold W^c, and it is C^{r-1}.*

Corollary 46. *We can take a set of local coordinates, $\tilde{x}, \tilde{y}, \tilde{z}$, corresponding to the local splitting $\mathbb{R}^d = W^c \times W^s \times W^u$, so that, topologically, the general system is equivalent to*

$$\tilde{x}' = \tilde{f}(\tilde{x}) \tag{544}$$

$$\tilde{y}' = -\tilde{y} \tag{545}$$

$$\tilde{z}' = \tilde{z} \tag{546}$$

Let us take the special case when W^s is empty. We bring the linear part at the equilibrium of our general system to the block diagonal form

$$x' = Cx + f(x, y) \tag{547}$$

$$y' = Hy + g(x, y) \tag{548}$$

where C is the part of the matrix whose eigenvalues have zero real part while H is the rest of the matrix, the "hyperbolic" part. The center manifold is tangent to E^c, and we can thus write it in locally in the form of the graph of a function, $y = h(x)$. Indeed, at the equilibrium $Hy + g(x, y) = 0$ and H has no zero eigenvalue, thus the implicit function theorem applies. Substituting into (547) we get

$$x' = Cx + f(x, h(x)) \tag{549}$$

Q: Does this give us the center manifold?

On the other hand, $h(x) = o(x)$ for small x, since it $Dh = 0$ there. Thus, we expect, and shall prove later, that the flow provided by (549)) is a good approximation of $\tilde{x}' = \tilde{f}(\tilde{x})$, which would evolve inside the center manifold. The following holds.

Theorem 20 (Henry, Carr). *If the origin $x = 0$ of* (549) *is locally asymptotically stable/unstable, then the origin of* (547) *is also locally asymptotically stable/unstable.*

21.1 The saddle-node bifurcation: general case

We follow [7]. We remember that the normal form we were aiming at was $a + x^2$, or, of course, more generally $\mu - \mu_0 \pm (x - x_0)^2$. Consider now the system (539), and assume that at $\mu = \mu_0, x = x_0$ there is an equilibrium in which one eigenvalue is zero and nondegenerate. The center manifold theorem would then allow us to reduce the study to the case where the system is one-dimensional. More precisely, there is a 2d center manifold Σ in $\mathbb{R}^n \times \mathbb{R}$ through (x_0, y_0) so that (1) Σ is tangent to the plane spanned by the 0 eigenvector and the direction of μ,

(2) For any r, Σ is C^r in a neighborhood of (x_0, y_0),

(3) The vector field of (539) is tangent to Σ,

and

(4) There is a neighborhood U of (x_0, y_0) in Σ which is invariant under the flow.

If we restrict (539) to Σ, we get a one-parameter family of equations on the one dimensional curves $\Sigma_\mu := \{z \in \Sigma : \mu = const =: \mu\}$. This is the reduction of the bifurcation problem. We now need to impose conditions that imply that the bifurcation type of this one-dimensional system is the same as that for the normal form $\mu - \mu_0 \pm (x - x_0)^2$. These are: $\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$ (transversality in the μ direction), and $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$, that is the equilibrium is quadratic.

More precisely, the following theorem holds.

Theorem 21. *Consider the setting above, under the following assumptions:*

(SN1) $M = D_x f(x_0, \mu_0)$ has a simple eigenvalue 0 with right eigenvector v and left eigenvector w ($wM = 0 \leftrightarrow M^T w = 0$). M has k eigenvalues with negative real parts and $(n - k - 1)$ with positive real parts.

(SN2) $w \cdot D_\mu f(x_0, \mu_0) \neq 0$.

(SN3) $w \cdot (v \cdot D_x^2 f(x_0, \mu_0) v) \neq 0$. (Note that $v \cdot D_x^2 f(x_0, \mu_0) v$ is a vector since f is a vector.)

Then there is a smooth curve of equilibria in $\mathbb{R}^n \times \mathbb{R}$ passing through (x_0, μ_0) and tangent to the hyperplane $\mathbb{R}^n \times \{\mu_0\}$. Depending on the signs in (SN1), (SN2) there are no equilibria near (x_0, μ_0) for $\mu < \mu_0$ ($\mu > \mu_0$ resp.). The two equilibria near (x_0, μ_0) are hyperbolic, and have stable manifolds of dimension k and $k + 1$, resp. The conditions (SN1) and (SN2) are generic, in the sense of forming an open dense set in the family of vector fields with an equilibrium with zero eigenvalue at (x_0, μ_0).

21.2 Transcritical and pitchfork bifurcations

We need appropriate changes in the assumptions. They are natural, if you think of the shape of the normal form:

(A) Transcritical bifurcation. Here we must have $f_\mu(0) = 0$ for all μ, and thus $D_\mu f$ cannot be nonzero anymore. This condition is replaced by (SN2') $w \cdot (\partial^2 f / \partial \mu \partial x) v \neq 0$ at $\mu = \mu_0$.

(B) Pitchfork bifurcation (one dimension). Here we are dealing with systems with symmetry in which f is odd. Thus, now we cannot have $D_x^2 f \neq 0$. Then, (SN3) is replaced by (SN3'), $D_x^3 f \neq 0$

Under these assumptions a theorem similar to the one in the previous section holds.

21.3 Hopf bifurcations

Consider now a system of the form (539) for which, at some (x_0, y_0) $D_x f$ has exactly one pair of nonzero imaginary eigenvalues, and the systems is hyperbolic otherwise, near (x_0, y_0). Then, by the implicit function theorem, the equilibrium position varies smoothly with μ, unlike in most other bifurcations. We expect however, by looking at what we called the normal form, a qualitative change in the structure of the equilibrium to occur at μ_0: a spiral sink is transformed into a spiral source plus a limit cycle.

By changes of variables (straightforward but rather lengthy [7]), the block affected by the bifurcation can be brought to the form

$$x' = (d\mu + a(x^2 + y^2)x - (\omega + c\mu + b(x^2 + y^2))y + \text{higher order terms} \quad (550)$$
$$y' = (\omega + c\mu + b(x^2 + y^2)x) + (d\mu + a(x^2 + y^2))y + \text{higher order terms} \quad (551)$$

(essentially, the quadratic terms can be eliminated). If we momentarily discard the higher order terms, this takes the following form in polar coordinates

$$r' = (d\mu + ar^2)r \quad (552)$$
$$\theta' = (\omega + c\mu + br^2) \quad (553)$$

The phase portrait of (552) does not differ substantially from the one we used before, where br^2 was missing. If a, d are nonzero, then there are periodic orbits of the (x, y) system lying along the parabola $\mu = -ar^2/d$; the surface of periodic orbits has quadratic tangency with the plane $\mu = 0$ in $\mathbb{R}^2 \times \mathbb{R}$.

The Hopf bifurcation theorem essentially says that the higher order terms do not change this picture locally.

Theorem 22 (Hopf, 1942)**.** *Suppose that the system $x' = f_\mu(x)$, $(x, \mu) \in \mathbb{R}^n \times \mathbb{R}$, has an equilibrium at (x_0, μ_0) and*

–(H1) $D_x f(\mu_0, x_0)$ has a unique pair of purely imaginary nonzero eigenvalues.

Then, there exists a smooth curve of equilibria $(x(\mu), \mu)$ with $x(\mu_0) = x_0$. The two eigenvalues which are imaginary at (x_0, μ_0), $\lambda(\mu)$ and $\overline{\lambda}(\mu)$ vary smoothly with μ.

Assume furthermore that

$$\frac{d}{d\mu}(\mathrm{Re}(\lambda(\mu)))\Big|_{\mu=\mu_0} = d \neq 0 \quad (554)$$

Then, there exists a unique three dimensional center manifold passing through (x_0, μ_0) in $\mathbb{R}^n \times \mathbb{R}$, and a smooth change of coordinates preserving the planes

$\mu = const.$ for which the Taylor expansion on the center manifold is given by (635). If $a \neq 0$, then there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0)$ and $\overline{\lambda}(\mu)$ agreeing to second order with the paraboloid $\mu = -(a/d)(x^2 + y^2)$. If $a < 0$, the periodic solutions are repelling.

22 Appendix

22.1 Solution to Exercise 3

The equation for Y_k is

$$kY_k + (Y_k J - JY_k) = R_k + \sum_{j=1}^{k-1} Y_j A_{k-j-1}; \quad R_k = A_{k-1} J \tag{555}$$

We consider the family of Banach spaces indexed by $\mu > 0$,

$$\mathcal{B}_\mu = \{\mathsf{Y} = (Y_l)_{l\in\mathbb{N}} : |||\mathsf{Y}|||_\mu := \sup_{j\in\mathbb{N}} \mu^{-j}\|Y_j\| < \infty\}$$

Note that, since $A(z)$ is analytic, the series $\sum_{l\in\mathbb{N}} A_l z^l$ converges, implying that, for some $C > 0$,

$$\sup_{j\in\mathbb{N}} \|A_j C^{-j}\| < \infty \tag{556}$$

Thus the vector $\mathsf{R} := (R_l)_{l\in\mathbb{N}}$ is in $\mathcal{B}_\mu$ for all $\mu > C$.

The function $\mathcal{C}$ given by $\mathcal{C}X = XJ - JX$ is evidently a linear function on $\mathbb{C}^{n^2}$, thus given by a matrix; since $\|\mathcal{C}X\| \leqslant 2\|J\| \|X\|$ by the triangle inequality, its norm is bounded by

$$\|\mathcal{C}\| \leqslant 2\|J\| \tag{557}$$

The function M_k given by

$$M_k X =: kX + \mathcal{C}X$$

is a linear function on $\mathbb{C}^{n^2}$, and thus it is also given by a matrix. We have shown that M_k is invertible, since $M_k X = 0 \Leftrightarrow X = 0$. Thus, for every k, M_k^{-1} exists (and evidently has finite norm).

We now also note that, if $k > 2\|J\|$ we have

$$\|M_k\|^{-1} \leqslant \frac{1}{k - 2\|J\|} \tag{558}$$

Indeed,

$$M_k^{-1} = k^{-1}(1 - k^{-1}\mathcal{C})^{-1} \tag{559}$$

Thus the series

$$\sum_{l=0}^{\infty} \mathcal{C}^l / k^l \tag{560}$$

converges for all $k > 2\|J\|$. This is called a Neumann series, and you can check that it converges to $(1 - k^{-1}\mathcal{C})^{-1}$.

Thus,

$$\|(1 - k^{-1}\mathcal{C})^{-1}\| \leqslant \sum_{l=0}^{\infty} k^{-l}(2\|J\|)^l = \frac{1}{1 - 2k^{-1}\|J\|} \tag{561}$$

and (558) follows.

Therefore,

$$\sup_{k\in\mathbb{N}} \|M_k^{-1}\| = \max\{\max_{k\leqslant 2\|J\|+1} \|M_k^{-1}\|, \sup_{k\geqslant 2\|J\|+2} (1 - 2k^{-1}\|J\|)^{-1}\}$$
$$= \max\{\max_{k\leqslant 2\|J\|+1} \|M_k^{-1}\|, 1/2\} = a_1 < \infty \tag{562}$$

Then the operator $\hat{\mathsf{T}}$ defined by

$$(\hat{\mathsf{T}}\mathsf{Y})_j = M_j^{-1} Y_j \tag{563}$$

is bounded in $\mathcal{B}_\mu$, and

$$\|\hat{\mathsf{T}}\| = a_1 \tag{564}$$

We define the (linear) operator $\hat{\mathsf{L}}$ on $\mathcal{B}_\mu$, $\mu > C$, by

$$(\hat{\mathsf{L}}\mathsf{Y})_j = \sum_{j=1}^{k-1} Y_j A_{k-j-1}; \quad j \geqslant 1 \tag{565}$$

This is well defined on $\mathcal{B}_\mu$ and

$$\|\hat{\mathsf{L}}\| \leqslant \frac{1}{\mu - C} \tag{566}$$

Indeed, since $\|Y\|_j \leqslant \mu^j |||Y||| =: N\mu^j$, we have

$$\|\sum_{j=1}^{k-1} Y_j A_{k-j-1}\| \leqslant N \sum_{j=1}^{k-1} \mu^j C^{k-j-1} \leqslant NC^{k-1} \frac{\mu^k}{C^k(\mu/C - 1)} = \frac{N\mu^k}{\mu - C} \tag{567}$$

Now, the system (555) can be written compactly as

$$\mathsf{Y} = \hat{\mathsf{T}}\mathsf{A} + \hat{\mathsf{T}}\hat{\mathsf{L}}\mathsf{Y} \tag{568}$$

This is a linear nonhomogeneous equation for Y. For it to be contractive, we need $\|\hat{\mathsf{T}}\hat{\mathsf{L}}\| \leqslant \|\hat{\mathsf{T}}\| \, \|\hat{\mathsf{L}}\| < 1$.

This is the case if

$$\frac{a_1}{\mu - C} < 1 \tag{569}$$

i.e., if $\mu > \mu_1 = C + a_1$. Thus $\mathsf{Y} \in \mathcal{B}_{\mu_1}$, implying that $\|Y_j\| \leqslant N\mu_1^j$ for some N and all j, and therefore the series

$$\sum_{j=1}^{\infty} Y_j z^j \tag{570}$$

converges (obviously to an analytic function) for $|z| < 1/\mu_1$, and therefore $Y(z)$ is analytic at zero as required.

23 Gradient and Hamiltonian systems

23.1 Gradient systems

These are quite special systems of ODEs, Hamiltonian ones arising in conservative classical mechanics, and gradient systems, in some ways related to them, arise in a number of applications. They are certainly nongeneric, but in view of their origin, they are common.

A system of the form

$$X' = -\nabla V(X) \tag{571}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞, is called, for obvious reasons, a gradient system. A critical point of V is a point where $\nabla V = 0$.

These systems have special properties, easy to derive.

Theorem 23. *For the system (571), if V is smooth, we have (i) If c is a regular point of V, then the vector field is perpendicular to the level hypersurface $V^{-1}(c)$ along $V^{-1}(c)$.*

(ii) A point is critical for V iff it is critical for (571).

(iii) At any equilibrium, the eigenvalues of the linearized system are real.

More properties, related to stability, will be discussed in that context.

Proof. □

(i) It is known that the gradient is orthogonal to level surface.

(ii) This is clear essentially by definition.

(iii) The linearization matrix elements are $a_{ij} = -V_{x_i,x_j}$ (the subscript notation of differentiation is used). Since V is smooth, we have $a_{ij} = a_{ji}$, and all eigenvalues are real.

23.2 Hamiltonian systems

If $\mathbf{F}$ is a conservative field, then $\mathbf{F} = -\nabla V$ and the Newtonian equations of motion (the mass is normalized to one) are

$$q' = p \tag{572}$$

$$p' = -\nabla V \tag{573}$$

where $q \in \mathbb{R}^n$ is the position and $p \in \mathbb{R}^n$ is the momentum. That is

$$q' = \frac{\partial H}{\partial p} \tag{574}$$

$$p' = -\frac{\partial H}{\partial q} \tag{575}$$

where

$$H = \frac{p^2}{2} + V(q) \tag{576}$$

is the Hamiltonian. In general, the motion can take place on a manifold, and then, by coordinate changes, H becomes a more general function of q and p. The coordinates q are called generalized positions, and q are the called generalized momenta; they are canonical coordinates on the phase on the cotangent manifold of the given manifold.

An equation of the form (574) is called a Hamiltonian system.

Exercise 1. *Show that a system $x' = F(x)$ is at the same time a Hamiltonian system and a gradient system iff the Hamiltonian H is a harmonic function.*

Proposition 47. *(i) The Hamiltonian is a constant of motion, that is, for any solution $X(t) = (p(t), q(t))$ we have*

$$H(p(t), q(t)) = const \tag{577}$$

where the constant depends on the solution.

(ii) The constant level surfaces of a smooth function $F(p, q)$ are solutions of a Hamiltonian system

$$q' = \frac{\partial F}{\partial p} \tag{578}$$

$$p' = -\frac{\partial F}{\partial q} \tag{579}$$

Proof. (i) We have

$$\frac{dH}{dt} = \nabla_p H \frac{dp}{dt} + \nabla_q \frac{dq}{dt} = -\nabla_p H \nabla_q + \nabla_q H \nabla_p = 0 \tag{580}$$

(ii) This is obtained very similarly. □

23.2.1 Integrability: a few first remarks

Hamiltonian systems (with time-independent Hamiltonian) in one dimension are integrable: the solution can be written in closed form, implicitly, as $H(y(x), x) = c$; in terms of t, once we have $y(x)$ of course we can integrate $x' = G(y(x), x) := f(x)$ by quadratures (using separation of variables). Note that for an equation of the form $y' = G(y, x)$, this is *equivalent* to the system having a *constant of motion*. The latter is defined as a function $K(x, y)$ defined globally in the phase space, (perhaps with the exception of some isolated points where it may have "simple" singularities, such as poles), and with the property that $K(y(x), x) = const$ for any given trajectory (the constant can depend on the trajectory, but not on x). Indeed, in this case we have

$$\frac{d}{dx}K(y(x), x) = \frac{\partial K}{\partial y}y' + \frac{\partial K}{\partial x} = 0$$

or

$$y' = -\frac{\partial K}{\partial x} / \frac{\partial K}{\partial y}$$

and the trajectories are the same as those of

$$\dot{x} = \frac{\partial K}{\partial y}; \quad \dot{y} = -\frac{\partial K}{\partial x} \tag{581}$$

which is a Hamiltonian system.

23.2.2 Dependence on initial conditions

Consider the system

$$y' = F(y, x); \; y(0) = y_0; \; y_0 \in \mathbb{R}^n \tag{582}$$

Proposition 48. *If F is smooth, then in a neighborhood of $(0, y_0)$, $y(x; y_0)$ is smooth in both x and y_0.*

Proof. We can prove this by extending the system (582) to include y_0. Maybe more transparently we can use the contraction mapping principle as follows. The proof is standard, so we only sketch it.

We write (582) in integral form,

$$y = y_0 + \int_0^x F(y(s), s)ds = \mathcal{N}(y, x; x_0) \tag{583}$$

and check that for small ε it is a contraction in the sup norm in a ball in $C(\mathbb{D}_\varepsilon \times B)$, the functions continuous in x and y_0, where B is a ball of radius $2\|y_0\|$.

Thus y is continuous in y_0. Now we differentiate formally w.r.t. y_0. Denoting by M the matrix $D_{y_0} y$ we get the matrix equation

$$M' = [D_y F(y(x; y_0))M; \quad M(0) = I \Leftrightarrow M(x) = I + \int_0^x D_y F(y(s; y_0))M(s)ds \tag{584}$$

where $y(x; y_0)$ is taken as a *known function*, which is continuous in x, y_0. This equation is also contractive in the space of matrix valued continuous functions in the sup norm is ε is small. We can continue in this way and see that the derivatives of all order exist and are continuous. It is straightforward to check that $y = \int \frac{\partial y}{\partial \tau} d\tau$ where τ is one of the components of y_0. The existence and continuity of higher order derivatives is checked similarly. □

With lower regularity we can for instance prove the following. Write the differential equation in integral form,

$$x = x_0 + \int_0^t F(x(s))ds = \mathcal{N}(x, x_0) \tag{585}$$

Theorem 24. Assume F is uniformly Lipschitz in x in an open set $\mathcal{O}$ and let K be a compact set contained in $\mathcal{O}$. Then there exists a $T = T(K)$ s.t. for any $x_0 \in K$ the solution $x(t, x_0)$ exists and is in $\mathcal{O}$ for all $t, |t| \leqslant T$ and x is continuous (thus uniformly continuous) in $(x_0, t) \in K \times [-T, T]$.

Proof. Consider the integral equation with initial conditions in a neighborhood of x_0.

$$x = x_0 + \xi + \int_0^t F(x(s))ds = \mathcal{N}(x, x_0) \tag{586}$$

Let κ be the Lipschitz constant of F in $\mathcal{O}$, that is,

$$|F(x) - F(x')| \leqslant \kappa |x - x'|, \ \forall x, x' \in \mathcal{O} \tag{587}$$

Note first that, by the compact covering theorem there is a δ s.t. $\forall x \in K$ and x' s.t. $d(x, x') \leqslant \delta$ we have $x' \in \mathcal{O}$. Define $K' = \{x \in \mathcal{O} | d(x, K) \leqslant \delta\}$ and let $M = \max_{x \in K'} |F|$. Finally, choose T s.t. $MT \leqslant \delta/3$ and $\kappa T \leqslant 1/2$ and $|\xi| < \delta/3$.

Consider the integral equation (586) in the Banach space $\mathcal{B}$ of functions continuous in $|t| \leqslant T$ and in $\xi, \xi + x_0 \in K, |\xi| < \delta/3$, in the sup norm. Take the closed ball $B = \{x \in \mathcal{B} | \|x - x_0\| \leqslant \delta/3\}$. The conditions above ensure that

$$(x, t, x_0 + \xi) \in B \times [-T, T] \times K \Rightarrow \mathcal{N}(x; x_0) \in B \text{ and } \mathcal{N} \text{ is contractive in } B \tag{588}$$

and the result follows. □

23.3 Example

As an example for both systems, we study the following problem: draw the contour plot (constant level curves) of

$$F(x, y) = y^2 + x^2(x - 1)^2 \tag{589}$$

and draw the lines of steepest descent of F.

For the first part we use Proposition 47 above and we write

$$x' = \frac{\partial F}{\partial y} = 2y \tag{590}$$

$$y' = -\frac{\partial F}{\partial x} = -2x(x - 1)(2x - 1) \tag{591}$$

The critical points are $(0, 0), (1/2, 0), (1, 0)$. It is easier to analyze them using the Hamiltonian. Near $(0, 0)$ H is essentially $x^2 + y^2$, that is the origin is a center, and the trajectories are near-circles. We can also note the symmetry $x \to (1 - x)$ so the same conclusion holds for $x = 1$, and the phase portrait is symmetric about 1/2.

Near $x = 1/2$ we write $x = 1/2 + s$, $H = y^2 + (1/4 - s^2)^2$ and the leading Taylor approximation gives $H \sim y^2 - 1/2s^2$. Then, 1/2 is a saddle (check). Now we can draw the phase portrait easily, noting that for large x the curves essentially become $x^4 + y^2 = C$ "flattened circles". Clearly, from the interpretation of the problem and the expression of H we see that *all* trajectories are closed.

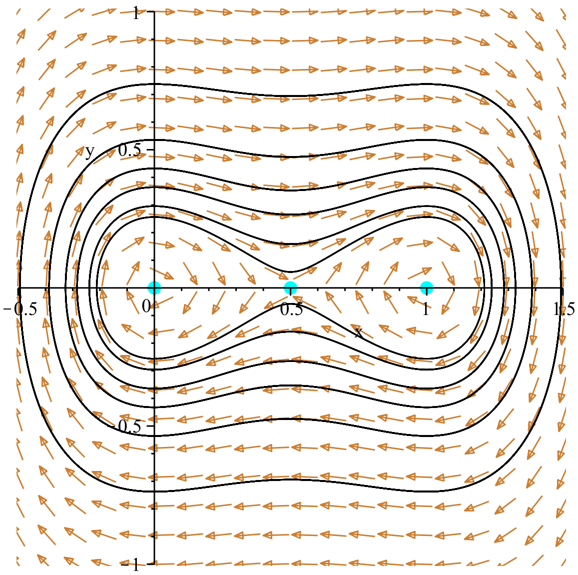

Figure 39:

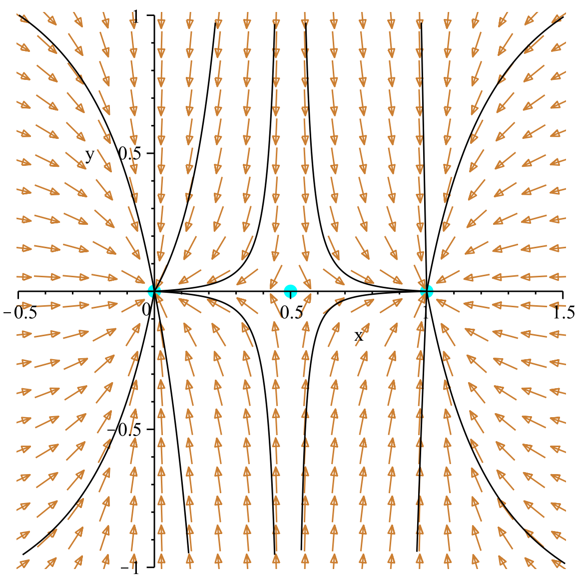

Figure 40:

The perpendicular lines solve the equations

$$x' = -\frac{\partial F}{\partial x} = -2x(x-1)(2x-1) \tag{592}$$

$$y' = -\frac{\partial F}{\partial y} = -2y \tag{593}$$

We note that this equation is separated. In any case, the two equation obviously share the critical points, and the sign diagram can be found immediately from the first figure.

Exercise 2. *Find the phase portrait for this system, and justify rigorously its qualitative features. Find the expression of the trajectories of (592). I found*

$$y = C\left(\frac{1}{(x-1/2)^2} - 4\right)$$

24 Flows, revisited

Often in nonlinear systems, equilibria are of higher order (the linearization has zero eigenvalues). Clearly such points are not hyperbolic and the methods we have seen so far do not apply.

There are no general methods to deal with all cases, but an important one is based on Lyapunov (or Liapunov, or Lyapounov,...) functions.
Definition. A flow is a smooth map

$$(X, t) \to \Phi_t(X)$$

A differential system

$$\dot{x} = F(x);\ \ F \in \mathbb{R}^d \tag{594}$$

generates a flow

$$(X, t) \to x(t; X)$$

where $x(t; X)$ is the solution at time t with initial condition X.

The derivative of a function G along a vector field F is, as usual,

$$D_F(G) = \nabla G \cdot F$$

Clearly

$$\frac{d}{dt}G(x(t)) = D_{F(x(t))}G = \nabla G(x(t)) \cdot F(x(t))$$

24.1 Lyapunov stability

Consider the system (594) and assume $x = 0$ is an equilibrium.

Then

1. $x_e = 0$ is Lyapunov stable (or simply stable) if starting with initial conditions near 0 the flow remains in a neighborhood of zero. More precisely, the condition is: for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x_0| < \delta$ then $|x(t)| < \varepsilon$ for all $t > 0$.

2. $x_e = 0$ is asymptotically stable if furthermore, trajectories that start close to the equilibrium converge to the equilibrium. That is, the equilibrium x_e is asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ so that if $|x_0| < \delta$, then $\lim_{t\to\infty} x(t) = 0$.

24.2 Lyapunov functions

Let X^* be a fixed point of (594). A Lyapunov function for (594) is a function defined in a neighborhood $\mathcal{O}$ of X^* with the following properties

(1) L is differentiable in $\mathcal{O}$.

(2) $L(X^*) = 0$ (this can be arranged by subtracting a constant).

(3) $L(x) > 0$ in $\mathcal{O} \setminus \{X^*\}$.

(4) $D_F L \leqslant 0$ in $\mathcal{O}$.

A strict Lyapunov function is a Lyapunov function for which

(4') $D_F L < 0$ in $\mathcal{O}$.

Finding a Lyapunov function is often nontrivial. In systems coming from physics, the energy is a good candidate. In general systems, one may try to find an exactly integrable equation which is a good approximation for the actual one in a neighborhood of X^* and look at the various constants of motion of the approximation as candidates for Lyapunov functions.

Theorem 25 (Lyapunov stability)**.** *Assume X^* is a fixed point for which there exists a Lyapunov function L. Then*

(i) X^* is stable.

(ii) If L is a strict Lyapunov function then X^* is asymptotically stable.

Proof. (i) Consider a small ball $B \ni X^*$ contained in $\mathcal{O}$. Let α be the minimum of L on ∂B. By the definition of a Lyapunov function, (3), $\alpha > 0$. Consider the following subset:

$$\mathcal{U} = \{x \in B : L(x) < \alpha\} \tag{595}$$

From the continuity of L, we see that $\mathcal{U}$ is an open set. Of course, the set

$$\mathcal{U}^c = \{x \in B : L(x) > \alpha\} \tag{596}$$

is also open in the relative topology of B , and disjoint from $\mathcal{U}$. Clearly, $X^* \in \mathcal{U}$. Let $X \in \mathcal{U}$. The function $t \mapsto x(t; X)$ is continuous, thus the forward image of $[0, t_0]$ is connected (and compact). Thus for any t_0, $x([0, t_0]; X)$ is completely contained in $\mathcal{U}$, completely contained in $\mathcal{U}^c$ (impossible) or it intersects $\partial\mathcal{U}$. But an intersection is impossible since by monotonicity, $L(x(t)) \leqslant L(X) < \alpha$ for all t. This proves stability.

(ii)

1. Note first that X^* is the only critical point in $\mathcal{O}$ since $\frac{d}{dt}L(x(t; X_1^*)) = 0$ for any fixed point.

2. Note that if $X \in \mathcal{U}$ then $x([0, t_0]; X) \subset \mathcal{U}$, by the above, and the trajectories exist for any $t_0 \in \mathbb{R}^+$. Thus the closure of the trajectories $x(\mathbb{R}; X)$ with $X \in \mathcal{U}$ are contained in $\overline{B}$, a compact set, and thus therefore contain limit points, i.e., points x^* s.t. $x(t_n, X) \to x^*$ for some sequence $t_n \uparrow \infty$. Any limit point x^* is strictly inside $\mathcal{U}$ since $L(x^*) < L(x(t); X) < \alpha$.

3. Let x^* be a limit point of a trajectory $x(t; X)$ where $X \in \mathcal{U}$. Then, by 1 and 2, if $x^* \in \mathcal{U} \neq X^*$, then x^* is a regular point of the field.

4. We want to show that $x^* = X^*$. We will do so by contradiction. Assuming $x^* \neq X^*$ we have $L(x^*) =: \lambda > 0$, again by (3) of the definition of L.
5. By 3 the trajectory $\{x(t; x^*) : t \geqslant 0\}$ is well defined and is contained in $\mathcal{B}$.
6. We then have $L(x(t; x^*)) < \lambda \forall t > 0$.
7. We look at the increasing sequence t_n in 2. For any n, the set

$$\mathcal{V} = \{X : L(x(t_{n+1} - t_n; X))\} < \lambda \tag{597}$$

   contains x^* and is open, so

$$L(x(t_{n+1} - t_n; X_1)) < \lambda \tag{598}$$

   for all X_1 close enough to x^*.
8. Let n be large enough so that $x(t_m; X) \in \mathcal{V}$ for all $m \geqslant n$.
9. Note that, by existence and uniqueness of solutions at regular points we have

$$x(t_{n+1}; X) = x(t_{n+1} - t_n; x(t_n; X)) \tag{599}$$

10. On the one hand $L(x(t_{n+1})) \downarrow \lambda$ and on the other hand we got $L(x(t_{n+1})) < \lambda$. This is a contradiction.

□

24.3 Examples

Hamiltonian systems, in Cartesian coordinates often assume the form

$$H(q, p) = p^2/2 + V(q) \tag{600}$$

where p is the collection of spatial coordinates and p are the momenta. If this ideal system is subject to external dissipative forces, then the energy cannot increase with time. H is thus a Lyapunov function for the system. If the external force is $F(p, q)$, the new system is generally not Hamiltonian anymore, and the equations of motion become

$$\dot{q} = p \tag{601}$$

$$\dot{p} = -\nabla V + F \tag{602}$$

and thus

$$\frac{dH}{dt} = pF(p, q) \tag{603}$$

which, in a dissipative system should be nonpositive, and typically negative. But, as we see, $dH/dt = 0$ along the curve $p = 0$.

For instance, in the ideal pendulum case with Hamiltonian

$$H = \frac{1}{2}\omega^2 + (1 - \cos\theta) \tag{604}$$

The associated Hamiltonian flow is

$$\theta' = \omega \tag{605}$$

$$\omega' = -\sin\theta \tag{606}$$

Then H is a global Lyapunov function at $(0, 0)$ for (607) (in fact, this is true for any system with *nonnegative* Hamiltonian). This is clear from the way Hamiltonian systems are defined.

Then $(0, 0)$ is a stable equilibrium. But, clearly, it is not asymptotically stable since $H = const > 0$ on any trajectory not starting at $(0, 0)$.

If we add air friction to the system (607), then the equations become

$$\theta' = \omega \tag{607}$$

$$\omega' = -\sin\theta - \kappa\omega \tag{608}$$

where $\kappa > 0$ is the drag coefficient. Note that this time, if we take $L = H$, the same H defined in (604), then

$$\frac{dH}{dt} = -\kappa\omega^2 \tag{609}$$

The function H is a Lyapunov function, but it is not strict, since $H' = 0$ if $\omega = 0$. Thus the system is stable. It is however intuitively clear that furthermore the energy still decreases to zero in the limit, since $\omega = 0$ are isolated points on any trajectory and we expect $(0, 0)$ to still be asymptotically stable. In fact, we could adjust the proof of Theorem 25 to show this. However, as we see in (603), this degeneracy is typical and then it is worth having a systematic way to deal with it. This is one application of Lasalle's invariance principle that we will prove next.

25 Some important concepts

We start by introducing some important concepts.

Definition 49. 1. An entire solution $x(t; X)$ is a solution which is defined for all $t \in \mathbb{R}$.

2. A positively invariant set $\mathcal{P}$ is a set such that $x(t, X) \in \mathcal{P}$ for all $t \geqslant 0$. Solutions that start in $\mathcal{P}$ stay in $\mathcal{P}$. Similarly one defines negatively invariant sets, and invariant sets.

3. The basin of attraction of a fixed point X^* is the set of all X such that $x(t; X) \to X^*$ when $t \to \infty$.

4. Given a solution $x(t;X)$, the set of all points x^* such that solution $x(t_n;X) \to x^*$ for some sequence $t_n \to \infty$ is called the set of ω-limit points of $x(t;X)$. At the opposite end, the set of all points x^* such that solution $x(-t_n;X) \to x^*$ for some sequence $t_n \to \infty$ is called the set of α-limit points. These may of course be empty.

   That is,

$$\omega(X) := \{x : \lim_{n\to\infty} x(t_n, X) = x \text{ for some sequence } t_n \to +\infty\} \tag{610}$$

   and, similarly, the α-limit set is defined as

$$\alpha(X) := \{x : \lim_{n\to\infty} x(t_n, X) = x \text{ for some sequence } t_n \to -\infty\}. \tag{611}$$

Proposition 50. Assume X belongs to a closed, positively invariant set $\mathcal{P}$ s.t., with $K = \mathcal{P}$, the hypotheses of Theorem 24 are satisfied. Then, the ω-limit set $\omega(X)$ is a closed invariant set: solutions with initial condition in $\omega(X)$ are **entire**. A similar statement holds for the α-set.

Proof. 1. (Closure) We show the complement of $\omega(X)$ is open. Let $b \in \omega(X)^c$. Then for some $\varepsilon > 0$, $d(x(t,X),b) \geqslant \varepsilon$ for all t. If $|b' - b| < \varepsilon/2$, then by the triangle inequality, $\liminf_{t\to\infty} d(x(t,X),b') > \varepsilon/2 > 0$ for all t.

2. By Theorem 24 the function $x(t,x_0)$ exists for any $x_0 \in \mathcal{P}$, $|t| \leqslant T$ and is uniformly continuous for all $x_0 \in \mathcal{P}$ and $|t| \leqslant T$. Since $\mathcal{P}$ is a compact set in $\mathcal{O}$ and positively invariant, for any $\tau \geqslant 0$ the function $x(\tau, x_0)$ exists for any $x_0 \in \mathcal{P}$ and is uniformly continuous in $x_0 \in \mathcal{P}$. This is by definition, and the fact that x is continuous in $x_0 \in \mathcal{P}$.

3. Note that for any τ the limit $\lim_{n\to\infty} x(t_n + \tau, X)$ exists and thus belongs to $\omega(X)$. This is the case because $x(t_n + \tau, X) = x(\tau; x(t_n))$ and by uniform continuity of x in the initial condition.

4. As a consequence, note now that for any $|t| \leqslant T$ and $x^* \in \omega(X)$ we have $x(t,x^*) \in \omega(X)$. Then $x(t,x^*)$ exists for all t since the set $\{\tau : x(\tau, x^*)\exists\& \in \omega(X)\}$ is open and closed.

□

26 Lasalle's invariance principle

Theorem 26. Let X^* be an equilibrium point for $x' = F(x)$ and let $L : \mathcal{U} \to \mathbb{R}$ ($\mathcal{U}$ open) be a Lyapunov function at X^*. Let $\mathcal{P} \subset \mathcal{U}$ be compact, positively invariant containing X^*. Assume there is no entire trajectory in $\mathcal{P} - \{X^*\}$ along which L is constant. Then X^* is asymptotically stable, and $\mathcal{P}$ is contained in the basin of attraction of X^*.

Proof. Since $\mathcal{P}$ is compact and positively invariant, then $X \in \mathcal{P} \Rightarrow \omega(X) \subset \mathcal{P}$. If $\omega(X) = \{X^*\}$, $\forall X$, the assumption follows easily (check!). So, we may assume there is an $x^* \neq X^*$ which is also an ω-limit point of some $x(t;X)$. By Proposition 50, the trajectory $x(t;x^*)$ is entire. Since L is nondecreasing along trajectories, we have $L(x(t;X)) \to \alpha = L(x^*)$ as $t \to \infty$. (This is clear for the subsequence t_n, and the rest follows by inequalities: check!) Since $x(t,x^*) = \lim x(t_n + t, X)$, by continuity, $L(x(t,x^*)) = \alpha$, contradiction. □

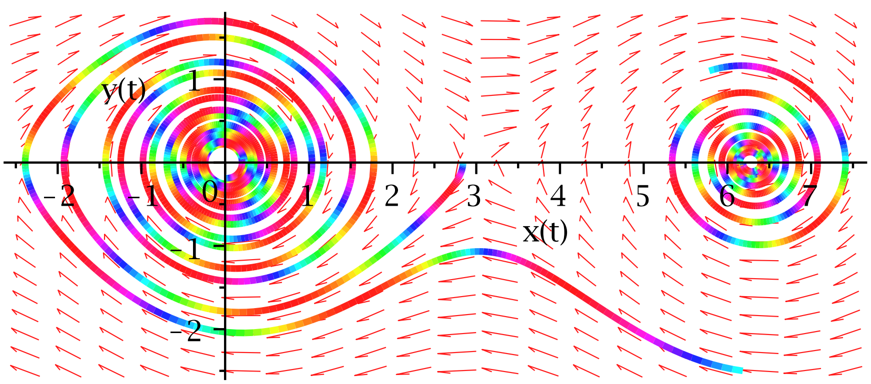

Figure 41:

26.1 Example: analysis of the pendulum with drag

Of course this is a simple example, but the way Lasalle's invariance principle is applied is representative of many other problems.

Intuitively, it is clear that any trajectory that starts with $\omega = 0$ and $\theta \in (-\pi, \pi)$ should asymptotically end up at the equilibrium point $(0,0)$ (other trajectories, which for the frictionless system would rotate forever, may end up in a different equilibrium, $(2n\pi, 0)$. For zero initial ω, the basin of attraction of $(0,0)$ should exactly be $(-\pi, \pi)$. In general, the energy should be less than precisely the one in this marginal case, $H = 1 - \cos(\pi) = 2$. Then, the region $\theta_0 \in (-\pi, \pi)$, $H < 1 - \cos(\pi) = 2$ should be the basin of attraction of $(0,0)$.

So let $c \in (0,2)$, and let

$$\mathcal{P}_c = \{(\theta, \omega) : H(\theta, \omega) \leqslant c, \text{ and } |\theta| \leqslant \arccos(1-c) \in (-\pi, \pi)\} \tag{612}$$

In H, θ coordinates, this is simply a closed rectangle and since (H, θ) is a continuous map, its preimage in the (ω, θ) plane is closed too.

Now we show that $\mathcal{P}_c$ is closed and forward invariant. If a trajectory were to exit $\mathcal{P}_c$, it would mean, by continuity, that for some t we have $H = c + \delta$ for a small $\delta > 0$ (ruled out by $\dot{H} \leqslant 0$ along trajectories) or that $|\theta| > \arccos(1-c)$ for some t which implies, from the formula for H the same thing: $H > c$.

Now there is no nontrivial entire solution (that is, other than $X^* = (0,0)$) along which $H = const$. Indeed, $H = const$ implies, from (609) that $\omega = 0$ identically along the trajectory. But then, from (606) we see that $\sin\theta = 0$ identically, which, within $\mathcal{P}_c$ simply means $\theta = 0$ identically. Lasalle's theorem applies, and all solutions starting in $\mathcal{P}_c$ approach $(0,0)$ as $t \to \infty$.

The phase portrait of the damped pendulum is depicted in Fig. 41

27 Gradient systems and Lyapunov functions

Recall that a gradient system is of the form (571), that is

$$X' = -\nabla V(X) \tag{613}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞ and a critical point of V is a point where $\nabla V = 0$. We have the following result:

Theorem 27. *For the system (571): (i) If c is a regular value of V, then the vector field is orthogonal to the level set of $V^{-1}(c)$.*

(ii) If a critical point X^* is an isolated minimum *of V, $V(X) - V(X^*)$ is a strict Lyapunov function at X^*, and then X^* is asymptotically stable.*

(iii) Any $\alpha-$ limit point of a solution of (571), and any $\omega-$ limit point is an equilibrium.

Note 35. (a) By (iii), any solution of a gradient system tends to a limit point or to infinity.

(b) Thus, descent lines of any smooth manifold have the same property: they link critical points, or they tend to infinity.

(c) We can use some of these properties to determine for instance that a system is not integrable. We write the associated gradient system and determine that it fails one of the properties above, for instance the linearized system at a critical point has an eigenvalue which is not real. Then there cannot exist a smooth H so that $H(x, y(x))$ is constant along trajectories.

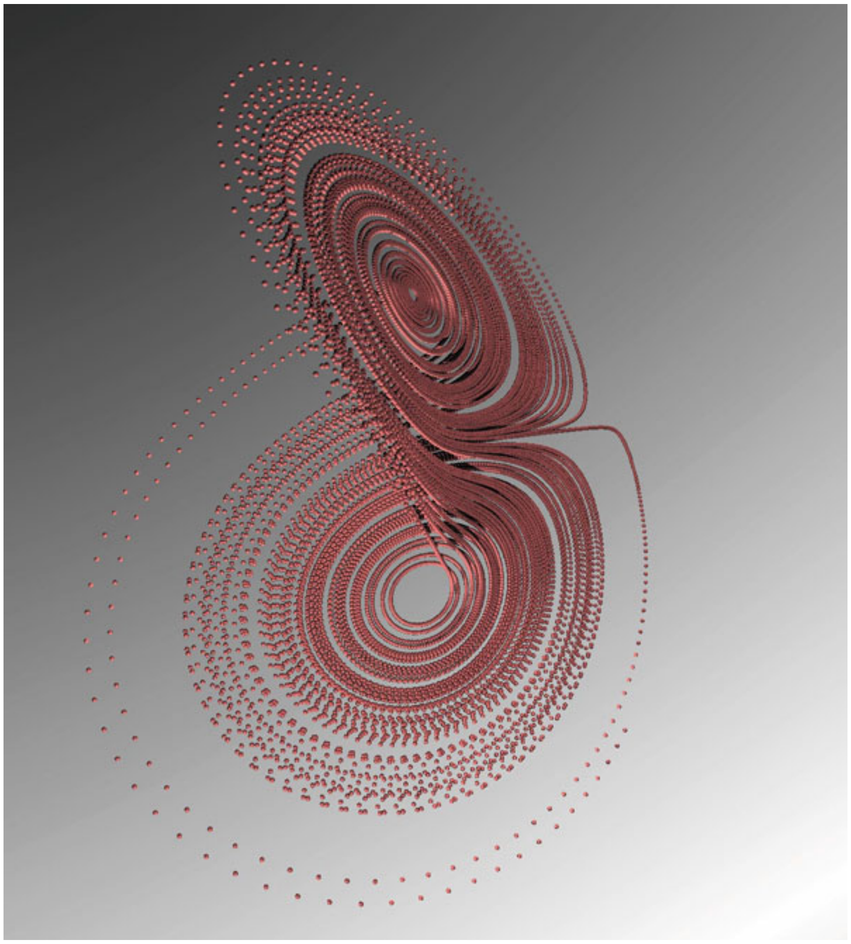

Figure 42: The Lorenz attractor

Proof. (i) is straightforward.

(ii) If an equilibrium point is isolated, then $\nabla V \neq 0$ in a set of the form $|X - X^*| \in (0, a)$. Then $-|\nabla V|^2 < 0$ in this set. Furthermore, $V(X) - V(X^*) > 0$ for all X with $|X - X^*| \in (0, a)$. Then, in this neighborhood, V is a strict Lyapunov function.

(iii) Note that if x is a point where $\nabla V = 0$, then x is an equilibrium. If Ω is an orbit along which V is constant, then $dV/dt = 0 = -|\nabla V|^2$ at any point along the trajectory, so all points are equilibria. There are no nontrivial limit sets.

□

28 Sections; the flowbox theorem

Consider first a planar system $x' = f(x)$ with f smooth, and a point x_0 such that $f(x_0) \neq 0$. A section through x_0 is a curve which is transversal to the flow, and passes through x_0. To be specific, take a unit vector $\mathbf{n}$ at x_0 which is orthogonal to $f(x_0)$, say $(-F_2(x_0), F_1(x_0))/|f(x_0)|$. We draw a line segment in the direction of $\mathbf{n}$,

$$\mathcal{S} = \{h(u) := x_0 + u\mathbf{n} | u \in (-\varepsilon, \varepsilon)\} \tag{614}$$

Once more, since f is continuous, for small δ there is a small ε so that we have $\mathbf{n} \cdot (-F_2(h(u)), F_1(h(u)))/|f(h(u))| \geqslant 1 - \delta$ if $u \in (-\varepsilon, \varepsilon)$. That is, the field is transversal to the section in a small neighborhood of x_0. By the same estimate, $\mathbf{n}\cdot(-F_2(h(u)), F_1(h(u)))/|f(h(u))|$ has constant sign along $\mathcal{S}$, which means that the field and the flow cross $\mathcal{S}$ in the same direction throughout $\mathcal{S}$. See the left side of fig. 45.

Definition 51. *The segment $\mathcal{S}$ defined above is called local section at x_0.*

28.0.1 The flowbox theorem for planar system; geometric approach

There is a diffeomorphic change of coordinates *in some neighborhood of* x_0, $x \leftrightarrow z$ so that in coordinates z the field is simply $\dot{z} = \mathbf{e}_1 := (1, 0)$.

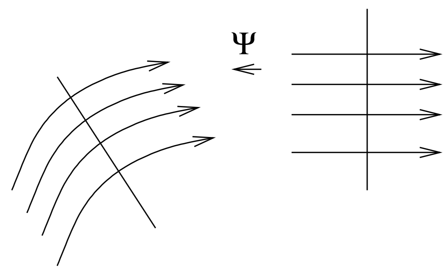

Figure 43: Flowbox and transformation

To straighten the field, we construct the following map, from a neighborhood of x_0 of the form

$$\mathcal{N} = \{\Psi(t, u) := x(t; h(u)) : |t| < \delta, u \in (-\varepsilon, \varepsilon)\}$$

where ε and δ are sufficiently small. Then, $(t, u) \mapsto x(t; h(u))$ is a diffeomorphism since the Jacobian of the transformation at $(0, 0)$ is

$$\det \begin{pmatrix} \dfrac{\partial \Psi_1}{\partial t} & \dfrac{\partial \Psi_1}{\partial u} \\ \\ \dfrac{\partial \Psi_2}{\partial t} & \dfrac{\partial \Psi_2}{\partial u} \end{pmatrix} = \det \begin{pmatrix} F_1 & n_1 \\ F_2 & n_2 \end{pmatrix} = |F(x_0)| \neq 0 \tag{615}$$

Clearly, the inverse image of trajectories through Ψ are straight lines, (t, u_0), as depicted. The associated flow in the set $\Psi^{-1}(\mathcal{N})$ is

$$\frac{dt}{dt} = 1; \quad \frac{du}{dt} = 0 \tag{616}$$

28.0.2 The flowbox theorem, in general

Consider a vector field F at a regular point, say 0, with $F(0) \neq 0$. Without loss of generality we can assume that $F(0) = \alpha e_1$ where e_1 is the first unit vector and by rescaling time we can assume $\alpha = 1$. We seek a local diffeomorphism $x = w + h(w), h = o(w)$ s.t.

$$\dot{w} = e_1 \tag{617}$$

This is the case if

$$\dot{w} + \frac{\partial h}{\partial w}\dot{w} = e_1 + \frac{\partial h}{\partial w}e_1 = \dot{x} = F(w + h(w)) \tag{618}$$

and thus, since $F(0) = e_1$, we have

$$\frac{\partial h}{\partial w}e_1 = F(w + h(w)) - F(0) = g(w + h(w)) \tag{619}$$

which is equivalent to the system

$$\frac{\partial h_j}{\partial w_1} = g_j(w + h(w)) \tag{620}$$

which we write in integral form

$$h_j(w_1, ..., w_n) = \int_0^{w_1} g_j(s + h_1(s, w_2, ..., w_n), ..., w_n + h_n(s, w_2, ..., w_n)))ds \tag{621}$$

which is contractive for small w. We see that $h_j(0, w_2, ..., w_n) = 0$. This is built into the initial conditions in the integral equation, and also natural, since $w_1 = w_1(0) + t; w_i = w_i(0)$.

28.0.3 Local versus global

Given the linearization theorem above, any system would be Hamiltonian, locally, near a regular point of the field. A system is a properly Hamiltonian one if the Hamiltonian is defined in a wide enough region of the phase space.

29 Limit sets, Poincaré maps, the Poincaré Bendixson theorem

In two dimensions, there are typically two types of limit sets: equilibria and periodic orbits (which are thereby limit cycles). Exceptions occur when a limit set contains a number of equilibria, as we will see in examples.

The Poincaré-Bendixson theorem states that if $\omega(X)$ is a nonempty compact limit set of a *planar system of ODEs* containing *no equilibria*, then $\omega(X)$ is a closed orbit. We will return to this important theorem and prove it.

Beyond two dimensions however, the possibilities are far vaster and limit sets can be quite complicated. Fig. 42 depicts a limit set for the Lorenz system, in three dimensions. Note how the trajectories seem to spiral erratically around two points. The limit set here has a fractal structure.

We begin the analysis with the two dimensional case, which plays an important role in applications.

We have already studied the system $r' = 1/2(r - r^3)$ in Cartesian coordinates. There the circle of radius one was a periodic orbit, and a limit cycle. All trajectories, except for the trivial one $(0,0)$ tended to it as $t \to \infty$.

We have also analyzed many cases of nodes, saddle points etc, where trajectories have equilibria as limit sets, or else they go to infinity.

A rather exceptional situation is that where the limit sets contain equilibria. Here is one example

29.1 Example: equilibria on the limit set

Consider the system

$$x' = \sin x(-\cos x - \cos y) \tag{622}$$

$$y' = \sin y(\cos x - \cos y) \tag{623}$$

The phase portrait is depicted in Fig. 44.

Exercise 1. *Find the equilibria of this field and their type. Justify the qualitative elements in Fig. 44.*

In the example above, we see that the limit set is a collection of fixed points and orbits, none of which periodic.

29.1.1 Closed orbits

A closed orbit is a solution whose trajectory is a closed curve with *no equilibria* on it. Let $\mathcal{C}$ be such a trajectory.

Note that the trajectory $x(t, X)$ is differentiable, the flow is always in the direction of the field, since

$$\dot{x}(t) = F(x(t))$$

and furthermore, the speed is, as we see from the above

$$|\dot{x}(t)| = |F(x(t))|$$

Since trajectories and f are smooth and there are no equilibria along $\mathcal{C}$, $|\dot{x}(t)| = |F(x)|$ is bounded below, and $\mathcal{C}$ is traversed in finite time. That is, starting at a point $x_1 \in \mathcal{C}$, after a (finite) time T, then, the solution returns to x_1. From that time on, the solution must repeat itself identically, by uniqueness of solutions. It then means that the solution is periodic, and there is a smallest τ so that $\Phi_{t+\tau}(x_1) = \Phi(x_1)$. This τ is called the period of the orbit.

Proposition 52. (i) If x_1 and x_2 lie on the same solution curve, then $\omega(x_1) = \omega(x_2)$ and $\alpha(x_1) = \alpha(x_2)$.

(ii) If $\mathcal{P}$ is a closed, positively invariant set and $x_2 \in \mathcal{P}$, then $\omega(x_2) \subset \mathcal{P}$; similarly for negatively invariant sets and $\alpha(x_2)$.

(iii) A closed invariant set, and in particular a limit set, contains the $\alpha-$limit and the $\omega-$limit of every point in it.

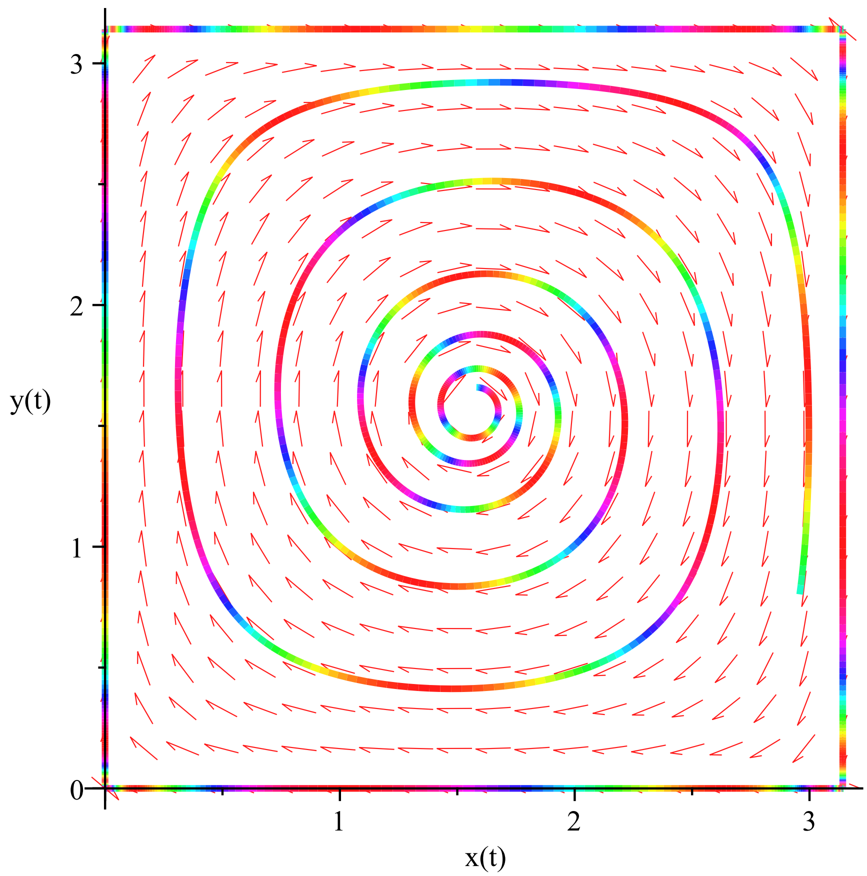

Figure 44: Phase portrait for (622).

Proof. Exercise. □

Exercise 2. *Show that τ is the same for any two points x_1, x_2 on $\mathcal{C}$.*

29.2 Time of arrival

We consider all solutions in the domain $\mathcal{O}$ where the field is defined and a section $\mathcal{S}$. Some of the trajectories intersect $\mathcal{S}$. Since the trajectories are continuous, if $x(t, z_0)$ intersects $\mathcal{S}$, then there is a first time of arrival, the smallest t so that $x(t, z_0) \in \mathcal{S}$.

This time of arrival is continuous in z_0, as shown in the next proposition.

Proposition 53. *Let $\mathcal{S}$ be a local section at x_0 and assume $x(t_0; z_0) = x_0$. Let $\mathcal{W}$ be a neighborhood of z_0. Then there is an open set containing z_0, $\mathcal{U} \subset \mathcal{W}$*

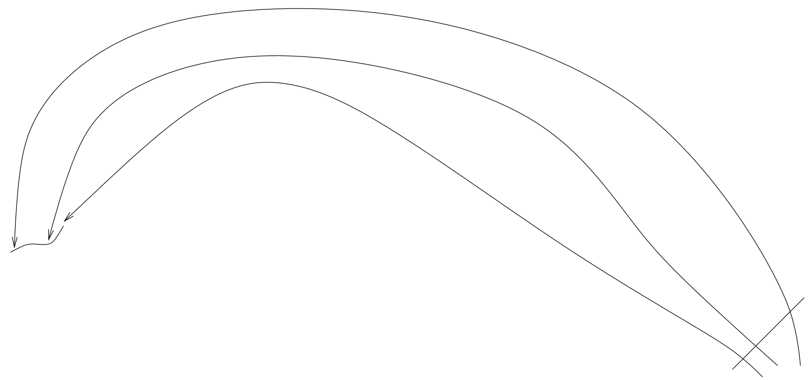

Figure 45: Time of arrival function

and a differentiable function $\tau : \mathcal{U} \to \mathbb{R}$ *such that* $\tau(z_0) = t_0$ *and*

$$x(\tau(X), X) \in \mathcal{S} \tag{624}$$

for each $X \in \mathcal{U}$.

Note 36. *In some sense, a small subsegment of the section* $\mathcal{S}$ *is carried* backwards *smoothly through the field arbitrarily far, assuming that the backward flow exist for a sufficiently long time, and that the subsegment is small enough.*

Proof. A point x_1 belongs to the line ℓ containing $\mathcal{S}$ iff $x_1 = x_0 + u\mathbf{n}$ for some u. Since $\mathbf{n}$ is orthogonal to $F(x_0)$ we see that $x_1 \in \ell$ iff $(x_1 - x_0) \cdot F(x_0) = 0$.

We look now at the more general function

$$G(z, t) = (x(t; z) - x_0) \cdot F(x_0) \tag{625}$$

We have, by assumption

$$G(z_0, t_0) = 0 \tag{626}$$

We want to see whether we can apply the implicit function theorem to

$$G(x, t) = 0 \tag{627}$$

For this we need to check $\frac{\partial}{\partial t} G\big|_{(z_0,t_0)}$. But this equals

$$x'(t; x_0) \cdot F(x_0)\big|_{(t_0,z_0)} = |F(x_0)|^2 \neq 0 \tag{628}$$

Then, there is a neighborhood of t_0 and a differentiable function $\tau(x)$ so that

$$G(x, \tau(x)) = 0 \tag{629}$$

□

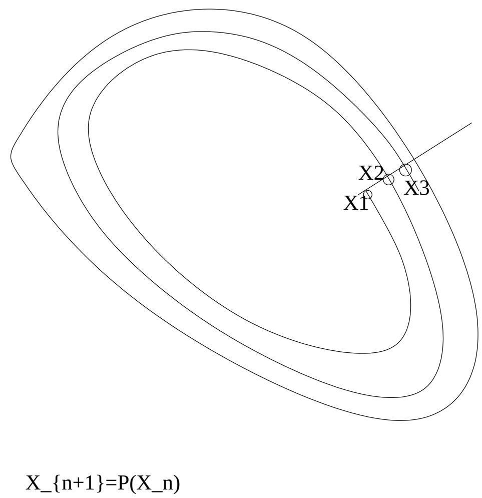

Figure 46: A Poincaré map.

29.3 The Poincaré map

The Poincaré map is a useful tool in determining whether closed trajectories (that is, periodic orbits) are stable or not. This means that taking an initial close enough to the periodic orbit, the trajectory thus obtained would approach the periodic orbit or not.

The basic idea is simple, we look at a section containing a point on the periodic orbit, and then follow the successive re-intersections of the perturbed orbit with the section. Now we are dealing with a discrete map $x_{n+1} = P(x_n)$. If $P(x_n) \to x_0$, the point on the closed orbit, then the orbit is asymptotically stable. See Figure 50.

It is often not easy to calculate the Poincaré map; in general it can't quite be easier than calculating the trajectories, but it is a very useful concept, and it has many theoretical applications; furthermore, we often don't need fully explicit knowledge of P.

Let's define the map P rigorously.

Consider a periodic orbit $\mathcal{C}$ and a point $x_0 \in \mathcal{C}$. We have

$$x(\tau; x_0) = x_0 \tag{630}$$

where τ is the period of the orbit. Consider a section $\mathcal{S}$ through x_0. Then according to Proposition 53, there is a neighborhood of $\mathcal{U}$ of x_0 and a continuous function $\tau(x)$ close to the period τ such that $x(\tau(X), X) \in \mathcal{S}$ for all $X \in \mathcal{U}$. Then certainly $\mathcal{S}_1 = \mathcal{U} \cap \mathcal{S}$ is an open set in $\mathcal{S}$ in the induced topology. The return map is thus defined on $\mathcal{S}_1$. It means that for each point in $X \in \mathcal{S}_1$ there is a point $P(X) \in \mathcal{S}$, so that $x(\tau(X); X) = P(X)$ and $\tau(X)$ is the smallest time with this property. Note that now $\tau(x)$ is not a period, though it is "very close to one": the trajectory does not return to the same point.

This is the Poincaré map associated to $\mathcal{C}$ and to its section $\mathcal{S}$.

This can be defined for planar systems as well as for higher dimensional ones, if we now take as a section a subset of a hyperplane through a point $x_0 \in \mathcal{C}$. The statement and proof of Proposition 53 generalize easily to higher dimensions.

In two dimensions, we can identify the segments $\mathcal{S}$ and $\mathcal{S}_1$ with intervals on the real line, $u \in (-a, a)$, and $u \in (-\varepsilon, \varepsilon)$ respectively, see also Definition 51. Then P defines an analogous transformation of the interval $(-\varepsilon, \varepsilon)$, which we still denote by P though this is technically a different function, and we have

$$P(0) = 0$$

$$P(u) \in (-a, a), \quad \forall u \in (-\varepsilon, \varepsilon)$$

We have the following easy result, the proof of which we leave as an exercise.

Proposition 54. *Assume that $x' = F(x)$ is a planar system with a closed orbit $\mathcal{C}$, let $x_0 \in \mathcal{C}$ and $\mathcal{S}$ a section at x_0. Define the Poincaré map P on an interval $(-\varepsilon, \varepsilon)$ as above, by identifying the section with a real interval centered at zero. If $|P'(x_0)| < 1$ then the orbit $\mathcal{C}$ is asymptotically stable.*

Example 37. Consider the planar system

$$r' = r(1 - r) \tag{631}$$

$$\theta' = 1 \tag{632}$$

In Cartesian coordinates it has a fixed point, $x = y = 0$ and a closed orbit, $x = \cos t, y = \sin t; x^2 + y^2 = 1$. Any ray originating at $(0, 0)$ is a section of the flow. We choose the positive real axis as $\mathcal{S}$. Let's construct the Poincaré map. Since $\theta' = 1$, the return time is 1, for any $x \in \mathbb{R}^+$ we have $x(2\pi; X) = x(0, X)$. We have $P(1) = 1$ since 1 lies on the unit circle. In this case we can calculate explicitly the solutions, thus the Poincaré map and its derivative.

We have

$$\ln r(t) - \ln(r(t) - 1) = t + C \tag{633}$$

and thus

$$r(t) = \frac{Ce^t}{Ce^t - 1} \tag{634}$$

where we determine C by imposing the initial condition $r(0) = x$: $C = x/(x-1)$. Thus,

$$r(t) = \frac{xe^t}{1 - x + xe^t} \tag{635}$$

and therefore we get the Poincaré map by taking $t = 2\pi$,

$$P(x) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}} \tag{636}$$

Direct calculation shows that $P'(1) = e^{-2\pi}$, and thus the closed orbit is stable. We could have seen this directly from (636) by taking $t \to \infty$.

Note that here we could calculate the orbits explicitly. Thus we don't quite need the Poincaré map anyway, we could just look at (635). When explicit solutions, or at least an explicit formula for the closed orbit is missing, calculating the Poincaré map can be quite a challenge.

30 Monotone sequences in two dimensions

There are two kinds of monotonicity that we can consider. One is monotonicity along a solution: $x_1, ..., x_n$ is monotone along the solution if $x_n = x(t_n, X)$ and t_n is increasing in n. Or, we can consider monotonicity along a segment, or more generally a piece of a curve. On a piece of a smooth curve, or on an interval we also have a natural order (or two rather), by arclength parameterization of the curve: $\gamma_2 > \gamma_1$ if γ_2 is farther from the chosen endpoint. To avoid this rather trivial distinction (dependence on the choice of endpoint) we say that a sequence $\{\gamma_n\}_n$ is monotone along the curve if γ_n is inbetween γ_{n-1} and γ_{n+1} for all n. Or we could say that a sequence is monotone if it is either increasing or else decreasing.

If we deal with a trajectory crossing a curve, then the two types of monotonicity need not coincide, in general. But for sections, they do.

Proposition 55. *Assume $x(t; X); t \in [0, \tau]$ is a solution of a planar system $x' = F(x)$, s.t. f is regular and nonzero in a sufficiently large region. Let $\mathcal{S}$ be a local section. Then monotonicity along the solution $x(t; X)$ assumed to intersect $\mathcal{S}$ at $x_1, x_2, ...$ (finitely or infinitely many intersection) and along $\mathcal{S}$ coincide.*

Note that all intersections are taken to be with $\mathcal{S}$, along which, by definition, they are always transversal.

Proof. We assume we have three successive distinct intersections with $\mathcal{S}$, x_1, x_2, x_3 (if two of them coincide, then the trajectory is a closed orbit and there is nothing to prove).

We want to show that x_3 is not inside the interval (x_1, x_2) (on the section, or on its image on $\mathbb{R}$). Consider the curve $\mathcal{C}_1 = \{x(t; x_1) : t \in [0, t_2]\}$ where t_2 is the first time of re-intersection of $x(t; x_1)$ with $\mathcal{S}$. By definition $x(t_2 - t_1; x_1) = x_2$. $\mathcal{C}_1$ is a smooth curve, with no self-intersection (since the field is assumed regular along the curve) thus of finite length. If completed with the line segment $\mathcal{J}$ linking x_1 and x_2, $\mathcal{C}_1 \cup \mathcal{J}$ is a closed continuous curve. By Jordan's lemma, we can define the inside int $\mathcal{C}$ and the outside of the curve, $D =$ext $\mathcal{C}$. Note that the field has a definite direction along $[x_1, x_2]$, by the definition of a section. Note also that it points towards ext $\mathcal{C}$, since $x(t; x_1)$ exits int $\mathcal{C}$ at $t = t_2$. Then, no trajectory can enter int $\mathcal{C}$. Indeed, it should intersect either $x(t; x_1)$ or else $[x_1, x_2]$. The first option is impossible by uniqueness of solutions. The second case is ruled out since the field points outwards from $\mathcal{J}$. Thus $x(t_3, x) = x_3$ must lie in ext $\mathcal{C}$, thus outside $[x_1, x_2]$. □

The next result shows points towards limiting points being special: parts of closed curves, or simply infinity.

Proposition 56. *Consider a planar system and $z \in \omega(x)$ (or $z \in \alpha(x)$), assumed a regular point of the field. Consider a local section $\mathcal{S}$ through a regular point $\tilde{z}$. Then the intersection of $\{\Phi_t(z) : t > 0\} \cap \mathcal{S}$ has at most one point (note that we are dealing with $\Phi_t(z)$ and not $\Phi_t(x)$).*

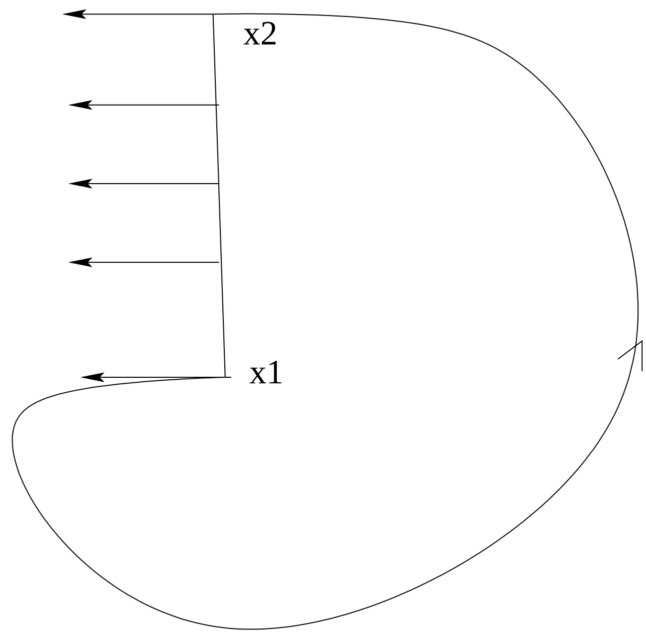

Figure 47: Monotone sequence theorem

Proof. Assume there are two distinct intersection points $x(t_1, z) = z_1$ and $x(t_2, z) = z_2$ on $\mathcal{S}$. By Proposition 50, $\{\Phi_t(z) : t > 0\} \subset \omega(x)$; in particular, z_1 and z_2 are also in $\omega(x)$. There are then infinitely many points on $\Phi_t(x)$ arbitrarily close to z_1 and infinitely many others arbitrarily close to z_2, by the definition of $\omega(x)$.

We can assume without loss of generality that the points that converge to z_1 lie on $\mathcal{S}$. Indeed, this can be arranged by a small change in t_j as follows:

The first arrival times at $\mathcal{S}$ for the trajectory $\Phi_t(z_1)$ is clearly zero. By the continuity of τ, if j is large enough s.t. $\Phi_{t_j}(x)$ is close to z_1, then $\tau(\Phi_{t_j}(x))$ exists by Proposition 53; $\tau(\Phi_{t_j}(x))$ and is arbitrarily small if j is large enough. Thus, by choosing $t_j + \tau(\Phi_{t_j}(x))$ instead of t_j, we can arrange that $\Phi_{t_j}(x) \in \mathcal{S}$. Similarly, we can arrange that the points converging to z_2 are on $\mathcal{S}$.

Also w.l.o.g. (rotating and translating the figure) we can assume that $\mathcal{S} = (-a, b) \in \mathbb{R}$ and $[z_1, z_2] \subset (-a, b)$. We know that $x(t_j, X)$, where t_j are the increasing times when $x(t_j, X) \in \mathcal{S}$, are monotone in $\mathcal{S} = (-a, b)$. Thus they converge. But then, by definition of convergence, they cannot be arbitrarily close to two distinct points.

□

31 The Poincaré-Bendixson theorem

Theorem 28 (Poincaré-Bendixson)**.** *Let $\Omega = \omega(x)$ be a nonempty compact limit set of a planar system of ODEs, containing no equilibria. Then Ω is a closed orbit.*

Proof. First, recall that Ω is invariant. Let $y \in \Omega$. Then $\Phi_t(y)$ is contained Ω, and then $\Phi_t(y)$ has infinitely many accumulation points in Ω. Let z be one of them and let t_j be s.t. $x(t_j, y) = z + o(1)$. Let $\mathcal{S}$ be a section though z. As in the Proposition 56, we can assume that $x(t_j, y) \in \mathcal{S}$. By Proposition 56, $x(t_j, y) = x(t_{j'}, y) \forall j, j'$, and thus the trajectory is periodic and $t_{j+1} - t_j = T$ is the period.

We have to now show that Ω is a closed orbit (and not a collection of distinct ones).

We take a section through y, and consider the sequence $x(t_j, X)$ of points on $x(t, X)$ approaching y. By the continuity of the Poincaré map and continuity w.r.t. initial conditions, $t_{j+1} - t_j = T + o(1)$ for large j. Thus any $t = t_j + s$ for some j and $s \in (o(1), T + O(1))$, By continuity w.r.t. initial conditions, $x(t_j + s, X) = x(s, y) + o(1)$, thus the distance between $x(s, y)$ and Ω) is zero. □

Exercise 1. *Where have we used the fact that the system is planar? Think how crucial dimensionality is for this proof.*

32 Applications of the Poincaré-Bendixson theorem

Definition. *A limit cycle is a closed orbit γ which is the ω-set, or an $\alpha-$ set of a point $X \notin \gamma$. These are called ω limit cycles or α limit cycles respectively.*

As we see, closed orbits are limit cycles only if *other* trajectories approach them arbitrarily. There are of course closed orbits which are not limit cycles. For instance, the system $x' = -y, y' = x$ with orbits $x^2 + y^2 = C$ for any C clearly has no limit cycles.

$\omega-$ limit cycles have at least one-sided stability.

Corollary 57. *Assume γ is an $\omega-$limit cycle. Then there is a one-sided (or two-sided) neighborhood $\mathcal{N}$ of γ s.t. $X \in \mathcal{N} \Rightarrow \omega(X) = \gamma$.*

Proof. Take a section $\mathcal{S}$ through any point on γ. Similar to the construction for the monotonicity proof, we take the region $\mathcal{R}$ bounded by the trajectory from x_j to x_{j+2}, see Fig. 49. Note that any point X starting on $\mathcal{S}$ in an open neighborhood of some X in (x_j, x_{j+1}) has the property $\omega(X) = \gamma$. Indeed, the blue region in the figure has, by assumption no equilibrium and, because of non-intersection of trajectories and continuity of the return time, if j is large enough, will cross the section $\mathcal{S}$ in a time $t_{j+1} - t_j + o(1)$ somewhere in (x_{j+1}, x_{j+2}), and in general will cross $\mathcal{S}$ in (x_k, x_{k+1}) for all $k > j$. The rest is immediate. □

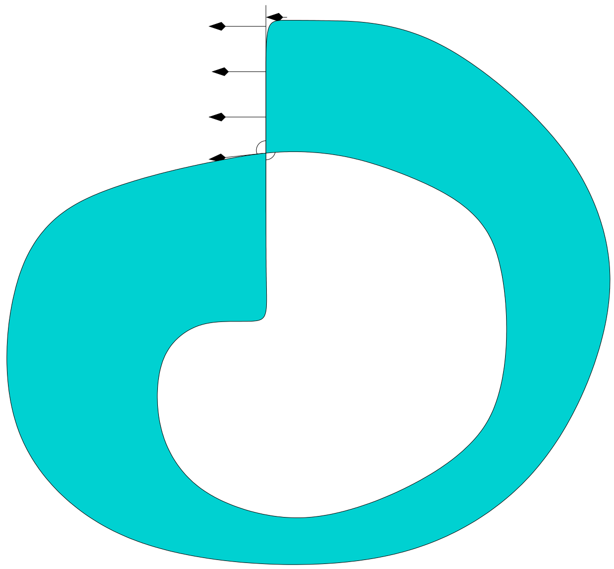

Figure 48: One-sided stability

Corollary 58. *Assume $\omega(X) = \gamma$, $\gamma \not\ni X$ is a limit cycle. Then there exists a neighborhood $\mathcal{O}$ of X s.t. $\forall X' \in \mathcal{O}$ we have $\gamma = \omega(X')$.*

Proof. Let t_0 be large enough so that $\Phi_t(X) \in \mathcal{N}$, the one-sided neighborhood of stability of γ, for all $t \geqslant t_0$. Take any $t_1 > t_0$ and a small enough neighborhood $\mathcal{O}_1$ of $x_1 = \Phi_{t_1}(X)$, so that, in particular, $\mathcal{O}_1 \subset \mathcal{N}$. Clearly, $\Phi_{-t_1}(x_1) = X$. As $\operatorname{diam}(\mathcal{O}_1) \to 0$, we have $\operatorname{diam}(\Phi_{-t_1}(\mathcal{O}_1)) \to 0$ as well, by continuity with respect to initial conditions. Also by continuity of Φ_t, and noting that $\Phi_{-t}(Z) = (\Phi_t)^{-1}(Z)$, we see that $\mathcal{O}_2 := \Phi_{-t_1}(\mathcal{O}_1)$ is an open set, which clearly contains X. By construction, $\omega(X') = \gamma$ for all $X' \in \mathcal{O}_2$. □

Corollary 59. *If a planar system has a first integral J that is not constant in any open set, then it has no limit cycles.*

Proof. Indeed, if $\gamma = \omega(X)$ is a limit cycle for some X, then by Corollary 58 there is a neighborhood $\mathcal{O}_X$ so that $\omega(X') = \gamma$ for all $X' \in \mathcal{O}_X$. We know that J is constant along any trajectory. Let $Y_0 \in \gamma$. By continuity, $J(X') = J(Y_0)$ for any $X' \in \mathcal{O}_X$. □

Corollary 60. *Let $\mathcal{P}$ be a* compact*, simply connected, positively invariant set. Then $\mathcal{P}$ contains at least a limit cycle or an equilibrium.*

Proof. Assume to get a contradiction that there were no equilibria or limit cycles in $\mathcal{P}$. By invariance, $\mathcal{P}$ must contain the ω limit set Ω of any $X \in \mathcal{P}$. By Poincaré-Bendixson, $\omega(X)$ is a closed curve which is not a limit cycle, and thus $X \in \omega(X)$, and $\omega(X)$ is a closed orbit. Take now X_1 in $\operatorname{int}(\omega(X))$; then similarly

$\omega(X_1)$ is a closed orbit and $X_1 \in \omega(X_1) \subsetneq \omega(X)$. We can form, by induction, a nested sequence of closed orbits $\omega(X_j)$, each of them strictly contained in the interior of the previous one. We consider now the set of all such nested sequences and let ν be the inf of the areas of the regions inside these $\omega(X_j)$. If $\nu \neq 0$, then we take a nested sequence of closed orbits whose areas converge to ν and let $\mathcal{P}$ be the intersection of all $\omega(X_j) \cup \text{int}(\omega(X_j))$. This is a compact, simply connected invariant set $\mathcal{P}$. If $\mathcal{P}$ has nonempty interior, then for any point X in $\text{int}(\mathcal{P})$, $\omega(X)$ is a closed trajectory also contained in $\text{int}(\mathcal{P})$ (why?) and this $\omega(X)$ necessarily has area $< \text{area}(\mathcal{P}) < \nu$, contradiction. If instead $\mathcal{P}$ has empty interior, then for any $X \in \mathcal{P}$, $\omega(X)$ cannot be a curve, as smooth curves have nonempty interior. Then $\omega(X)$ is an equilibrium, contradiction. □

Corollary 61. *Let γ be a closed orbit and $\mathcal{U}$ its interior. Then $\mathcal{U}$ contains at least an equilibrium.*

Proof of the Corollary. We first show that if there is no equilibrium in $\mathcal{U}$ then there are infinitely many limit cycles. Indeed, $\mathcal{P} = \mathcal{U} \cup \gamma$ is positively invariant and then it must contain a limit cycle. If γ itself is the only limit cycle or equilibrium in $\mathcal{P}$, then, since $\mathcal{P}$ is also negatively invariant, γ is also the $\alpha-$limit set of any point in $\mathcal{P}$, but this would violate monotonicity along sections (check!). If there were finitely many limit cycles in $\mathcal{P}$ then there would be one of minimal area, impossible by the arguments in Corollary 60.

Thus, that there are infinitely many limit cycles γ_n in U. We can furthermore assume they are contained in each other, since each limit cycle contains an equilibrium or yet another limit cycle (strict inclusion). Now we can repeat the last part of the proof of Corollary 60, since a limit cycle is, in particular, a closed orbit. (At the end of that proof, $\omega(X)$ cannot be a closed orbit, otherwise, once more, it would contain an even smaller one.) □

Corollary 62. *If K is positively (or negatively) invariant, then it contains an equilibrium.*

Proof. Combine Corollaries 60 and 61. □

33 The Painlevé property

As mentioned on p.2, Hamiltonian systems (with time-independent Hamiltonian) in one dimension are integrable: the solution can be written in closed form, implicitly, as $H(y(x), x) = c$; in terms of t, once we have $y(x)$ of course we can integrate $x' = G(y(x), x) := f(x)$ in closed form, by separation of variables. The classification of equations into integrable and nonintegrable, and in the latter case finding out whether the behavior is chaotic plays a major role in the study of dynamical systems.

As usual, for an $n-$th order differential equation $x' = f(x)$, a constant of motion is a function $K(u_1, ..., u_n, t)$ with a predefined degree of smoothness (analytic, meromorphic, C^r etc.) and with the property that for any solution $y(t)$ we have

$$\frac{d}{dt}K(y(t), y'(t), ..., y^{(n-1)}(t), t) = 0$$

There are multiple precise definitions of integrability, and no one perhaps is comprehensive enough to be widely accepted. For us, let us think of a system as being integrable, relative to a certain regularity class of first integrals, if there are sufficiently many global constants of motion so that a particular solution can be found by knowledge of the values of the constants of motion.

If f is analytic, it is usually required that K is analytic too, except perhaps for *isolated* singularities (in particular, single-valued; e.g., the log does not have an isolated singularity at zero, whereas $e^{1/x}$ does).

We note once more that an integral of motion needs to be defined in a wide region. The existence of local constants along trajectories follows immediately either from the flowbox theorem, or from the implicit function theorem: indeed, if $x' = f(x)$ is a system of equations near a regular point, x_0, then evidently there exists a local solution $x(t; x_0) = \phi_t(x_0)$. It is easy to check that $D_{x_0}x_{|t=0} = I$, so we can write, near $x_0, t = 0$, $x_0 = K(x, t) = \Phi_{-t}(x)$. Clearly K is constant along trajectories. Not a very explicit function, admittedly, but smooth, at least locally. K is thus obtained by integrating the equation backwards in time. This is not a very useful constant of motion however, since in general it is only defined for small t: typically for larger t singularities will arise.

Assume now that f is an analytic function, so that it makes sense to extend the equation to $\mathbb{C}$.

If t is in the complex domain, we can in principle circumvent possible singularities, and define K by analytic continuation around singularities. When is this possible? If the singularities are always isolated, and in particular solutions are single valued, it does not matter which way we go. But if these are, say, square root branch points, if we avoid the singularity on one side we get $+\surd$ and on the other $-\surd$. There is no obvious way to prescribe a systematic path of analytic continuation since the Riemann surface is solution-dependent. We will see in the next section that this is typically not a mere failure to find a systematic prescription.

On the other hand, if we impose the condition that the equation have only isolated singularities (at least, those depending on the initial condition, or *movable*, then we have a single valued global constant of motion, take away some lower dimensional singular manifolds in $\mathbb{C}^2$.

Such equations are said to have the Painlevé property (PP) and are integrable, at least in the sense above. But it turns out, in those considered so far in applications, that more is true: they were all ultimately reduced to linear equations.

Failure of the Painlvé property and nonintegrability

In the case the movable singularities of solutions of a meromorphic equation are branch points we don't expect simple, closed form solutions which are single-

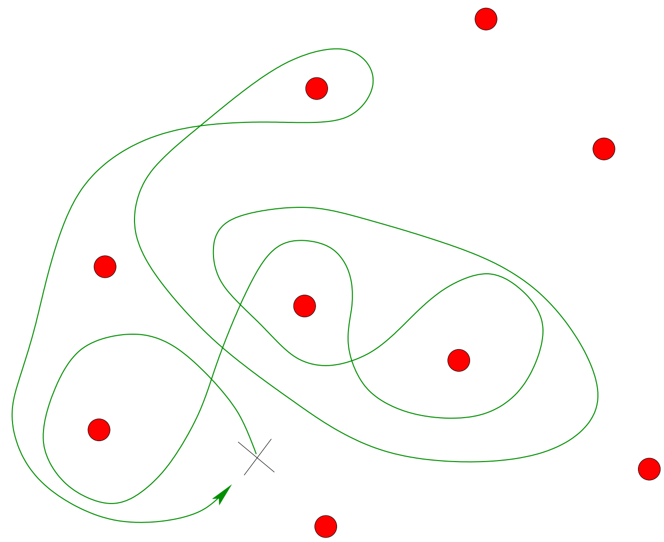

Figure 49: A continuation path for a solution with movable singularities

valued in $\mathbb{C}$. Indeed, assume that the solution $y(z)$ is given by $\Phi(z, y(z)) = 0$ where Φ is nontrivial, and analytic (or meromorphic) in $\mathbb{C}^2$. Then Φ should be constant along the trajectory. We follow the solution y on a Riemann surface avoiding the singularities. Since the solution is not single valued, after surrounding one singularity we end up with $y_1(z)$, a solution of the same ODE, but *a different one*. The expectation is that by wandering "randomly" around branch points we generate a family of solutions dense in the space of all solutions (*dense branching*). This is because of the huge amount of freedom we have in choosing the continuation path. In case of dense branching– the typical situation in fact– then $\Phi(z, y)$ takes the same value on a dense set of y and thus it does not depend on y; of course it cannot depend only on x and thus Φ is a number, contradiction. One of course has to check whether in a particular ODE dense branching occurs, but this is generically the case and failure of the PP is a "red flag" when trying to solve equations in any explicit way.

33.1 The Painlevé equations

33.2 Spontaneous singularities: The Painlevé's equation $\mathrm{P_I}$

Let us analyze local singularities of the Painlevé equation $\mathrm{P_I}$,

$$y'' = y^2 + x \tag{637}$$

The standard existence and uniqueness theorem guarantees that there is a unique solution in any region where y is bounded, and this solution is analytic.

In a neighborhood of a point where y is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)x^{p-2} = A^2(x - x_0)^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(x - x_0)^{-2}$: $y = 6(x-x_0)^{-2} + c_{-1}(x-x_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -x_0/10, c_3 = -1/6$ and c_4 is undetermined, thus free. Choosing a c_4, all others are uniquely determined.

Series solutions at movable singularities for neighboring equations

It is convenient to make the substitutions $y(x) = 6(x - x_0)^{-2} + \delta(x)$ where for consistency we should have $\delta(x) = o((x - x_0)^{-2})$ and taking $x = x_0 + z$ we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{638}$$

In this form, the singular point of y is now placed at $z = 0$ and zero is a singularity of the equation.

Note that with the standard substitution that we used for Frobenius systems, $u_1 = \delta, u_2 = z\delta'$ we get the system

$$\begin{aligned} u_1' &= z^{-1}u_2 \\ u_2' &= 12z^{-1}u_1 + z^{-1}u_2 + zu_1^2 + z^2 + zx_0 \end{aligned} \tag{639}$$

which can be extended to an autonomous system by adding $\dot{z} = z$, and P1 becomes equivalent to

$$\begin{aligned} \dot{u_1} &= u_2 \\ \dot{u_2} &= 12u_1 + u_2 + z^2u_1^2 + z^3 + z^2x_0 \\ \dot{z} &= z \end{aligned} \tag{640}$$

where 0 is a critical point of the field. The linearized matrix at zero M of this system and its corresponding diagonal form D are given by

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 12 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{641}$$

we see that the system is resonant, with eigenvalues in the Siegel domain.

Typically therefore we do not expect local analytic linearization. Typical would mean here that we allow for generic nonlinear monomials instead of the specific ones.

However, for P1, by "accident" the solutions of (638) are locally analytic. Substituting

$$\delta(z) = c_0 + c_1 z + ...$$

in (638) we get

$$-12c_0 z^{-2} - 12c_1 z^{-1} + ... = 0$$

forcing $c_0 = c_1 = 0$. Thus a power series would have the form

$$\delta(z) = \sum_{k=2}^{\infty} c_k z^k \tag{642}$$

which, used in (638), gives

$$[-10c_2 - x_0] + (-6c_3 - 1)\, z + 8c_5 z^3 + \left(18c_6 - {c_2}^2\right) z^4 + (30c_7 - 2c_2c_3)\, z^5 \\ + \left(-2c_2c_4 - {c_3}^2 + 44c_8\right) z^6 + \cdots = 0 \tag{643}$$

We note that the coefficient of z^2 is zero, and c_4 is undetermined. For any value of c_4, the recurrence for c_k determines uniquely these coefficients.

The fact that c_4 is not determined is due to 4 being an eigenvalue of the linear part, and that *there are no obstructing monomials.* If we take a modification of (638), for instance

$$\delta'' = \frac{12}{z^2}\delta + z + az^2 + x_0 + \delta^2 \tag{644}$$

we get as before $c_0 = 0, c_1 = 0$ and

$$[-10c_2 - x_0] + (-6c_3 - 1)\, z + az^2 + 8c_5 z^3 + \left(18c_6 - c_2^2\right) z^4 + \cdots = 0 \tag{645}$$

Now an equation for c_4 is still missing, and the term z^2 cannot be eliminated (unless $a = 0$, which is the original P_1.) As in the linear case, we expect $\log z$ to appear in the expansion. Indeed, substituting

$$\delta(z) = \sum_{k=2}^{6} c_k z^k + Az^4 \ln z \tag{646}$$

in (644) we get

$$-10c_2 - x_0 + (-6c_3 - 1)\, z + (7A + a)\, z^2 + 8c_5 z^3 + \left(18c_6 - c_2^2\right) z^4 + \cdots = 0 \tag{647}$$

which is now solvable (at least to order 6).

Existence of a convergent power series for δ in (638)

To show that there indeed is a convergent such power series solution we substitute Note now that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (638) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximately by a linear equation. It is then natural to separate out the large terms from the small terms and write a fixed point equation for the solution based on this separation. We write (638) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{648}$$

and integrate as if the right side were known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to be *relatively smaller*, by construction this integral equation is expected to be contractive.

Click here for Maple file of the formal calculation ($y'' = y^2 + x$)

The indicial equation for the Euler equation corresponding to the left side of (648) is $r^2 - r - 12 = 0$ with solutions $4, -3$ (same as the eigenvalues of the linearized matrix, of course). By the method of variation of parameters we thus get

$$\delta = \frac{D}{z^3} - \frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \tag{649}$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$; C is arbitrary. To find δ formally, we would simply iterate (649) in the following way: We take $r := \delta^2 = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $r = \delta_0^2$ and compute δ_1 from (649) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + \ldots \tag{650}$$

This series is actually convergent. To see that, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z s u^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \tag{651}$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ε such that this is a contractive equation for u in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \varepsilon$. We conclude that δ is analytic and that y is meromorphic near $x = x_0$. **Note.** The analysis above does *not* prove that the solutions are meromorphic functions in $\mathbb{C}$ (why not?).

Exercise 1. Show that $y'' = y^2 + P(x)$ where P is a polynomial has the Painlevé property **iff** $P(x) = ax + b$ where, modulo elementary changes of variables, $a = 1, b = 0$.

Click here for Maple file of the formal calculation, for $y'' = y^2 + x^2$

33.2.1 The six Painlevé equations

This is the list of the six Painlevé equations:

$$
\begin{aligned}
P_1 \quad w'' &= 6w^2 + z \\
P_2 \quad w'' &= 2w^3 + zw + \alpha \\
P_3 \quad w'' &= \frac{1}{w}(w')^2 - \frac{1}{z}w' + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \\
P_4 \quad w'' &= \frac{1}{2w}(w')^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \\
P_5 \quad w'' &= \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{1}{z}\frac{dw}{dz} \\
&\quad + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \\
P_6 \quad w'' &= \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz} \\
&\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right)
\end{aligned}
\tag{652}
$$

A nonintegrable example

The following is a nonintegrable case of the Abel class of ODEs:

$$y' = y^3 + x \tag{653}$$

We claim that all singularities are branch points. First, we note that the standard existence and uniqueness theorem guarantees that the solution of (653) is analytic in any region where y is bounded; for a point x_0 to be singular we need that $y \to \infty$ as $x \to x_0$. Near the singular point we must have $y' \sim y^3$ which by direct integration gives $y \sim \pm(2x)^{-1/2}$.

To show that this is indeed the behavior near a point z_0 where y blows up, we take $y = 1/u, x = z_0 + z$ and get

$$\frac{dz}{du} = -\frac{u}{1 + u^3 z_0 + u^3 z} \tag{654}$$

in this presentation $u = 0, z = 0$ is a regular point of the ODE, and the solution is analytic. It is clear that $z = -\frac{u^2}{2}(1 + h(u))$ where h is analytic and $h(0) = 0$. We leave it as a straightforward exercise to check that this implies the existence of two solutions of (653) in the form $y = \pm(x - x_0)^{-1/2} H(\sqrt{x - x_0})$ where H is analytic.

Exercise 2. Show that an equation of the form

$$y' = P(y) + Q(x) \tag{655}$$

where P and Q are polynomials has the Painlvé property iff P is quadratic, in which case the equation is Riccati, thus integrable.

34 Asymptotics of ODEs: first examples

Asymptotic behavior typically refers to behavior near an irregular singular point.

Remember from Frobenius theory that regular singular points (say $z = 0$) of an nth order ODE are characterized by the order of the poles relative to the order of differentiation. Homogeneous equations with regular singularities are of the form

$$y^{(n)} + \frac{A_1(z)}{z} y^{(n-1)} + \cdots \frac{A_j(z)}{z^j} y^{(n-j)} + \cdots \frac{A_n(z)}{z^n} y = 0 \tag{656}$$

where $A_j(z)$ are analytic at zero.

A singularity is regular **iff** there is a fundamental system of solutions in the form of a finite combination of terms of the form $z^a, \mathcal{A}(z), \ln^j z$ where a may be complex, $j \leqslant n-1$, $\mathcal{A}(z)$ analytic.

Thus the general solution at an irregular singular point is *not* given by a convergent power series. Two things can happen:

- Solutions do not have power-like behavior (usually this means exponential behavior).
- Series exist but are divergent.

Consider first the very simple ODE

$$y' = Ay/z^p; \quad p > 1 \tag{657}$$

near $z = 0$. The general solution is

$$y = C \exp(-Az^{-p+1}/(p-1)) \tag{658}$$

Note that this function has no power series at $z = 0$ (in $\mathbb{C}$); the behavior is exponential.

Most often, irregular singularities are placed at infinity (to characterize a singularity at infinity, make the substitution $z = 1/x$). Then, in first order equations with coefficients behaving polynomially, infinity is an irregular singular point if the equation is of the form $y' = Ax^q(1 + o(1))y, \ q > -1$. Equation (657), after the transformation $z = 1/x$ becomes

$$y' = ax^q y; \ a = -A, q = p - 2 \tag{659}$$

and infinity is an irregular singular point if $q > -1$, and the solution is given by

$$y = C \exp\left(\frac{ax^{q+1}}{q+1}\right) \tag{660}$$

For the second new phenomenon, consider the equation

$$y' = -y + 1/x; \quad y \to \infty \tag{661}$$

We can make it homogeneous by multiplying by x and differentiating once more. By taking $z = 1/x$ you convince yourself that the resulting equation is second order with a fourth order pole at zero.

Eq. (661) has a power series solution. Indeed, inserting

$$y = \sum_{k=0}^{\infty} c_k/x^k \tag{662}$$

in (661) we get $c_k = (k-1)c_{k-1}; \quad c_1 = 1 \ \Rightarrow c_k = (k-1)!$ and thus

$$y = \sum_{k=0}^{\infty} k!/x^{k+1} \tag{663}$$

The domain of convergence of this expansion in *empty*.

Many equations for special functions have an irregular singularity at infinity. Typical equations

1. Bessel:
$$y'' + x^{-1}y' + (1 - \alpha^2/x^2)y = 0 \tag{664}$$

2. Parabolic cylinder functions
$$y'' + (\nu + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \tag{665}$$

3. Airy functions
$$y'' = xy \tag{666}$$
as well as many nonlinear ones

4. Elliptic functions
$$y'' = y^2 + 1 \tag{667}$$

5. Painlevé P1
$$y'' = 6y^2 + x \tag{668}$$
etc.

It is important to understand the behavior of irregular singularities. Start again from the example (657). It is clear that the singularity remains irregular if z^{-p} is replaced by $z^{-p} + \cdots$ where $\cdots$ are terms with *higher* powers of z.

The WKB method

Given the exponential behavior at an irregular singular point, it is natural to make an *exponential substitution* $y = e^w$. Of course, at the end, we re-obtain the solution we had before. Only, the equation for w' will admit power-like, instead of exponential behavior.

This substitution works in much more generality, and it is behind what is known as the WKB method.

Let's illustrate this on second order equations with rational coefficients:

$$y'' + R_1 y' + R_2 y = 0; \quad R_1, R_2 \text{ rational} \tag{669}$$

A Liouville transformation $y = \exp(-\frac{1}{2}\int R_1)$ transforms (671) into

$$u'' + (R_2 - \frac{1}{2}R_1 - \frac{1}{4}R_1^2)u = 0 \tag{670}$$

and thus we can assume without loss of generality that the equation was of the form

$$y'' - Ry = 0; \quad R = P/Q \ P, Q \text{ polynomials} \tag{671}$$

to start with. The singularity at infinity is irregular iff $\deg P \geqslant \deg Q - 1$. The substitution suggested by the previous discussion is $y = e^W$. This gives

$$W'^2 + W'' = R \tag{672}$$

It is easy to see theat the dominant balance is $W'^2 \sim R$. Then, $W \sim x^a$, $a = \deg P - \deg Q$. Since the differential equation can be written in integral form, the asymptotic behavior is differentiable. This means

$$W'^2 \sim x^{2a-2} \gg x^{a-2} = W'' \tag{673}$$

The balance $W'^2 \gg W''$ is quite universal in WKB-like problems. Then we write the equation as

$$f = \pm\sqrt{R - f'}; \quad f = W' \tag{674}$$

and iterate under the assumption (673). This implies $f' \ll R$, and with the plus sign we get

$$f^{[n+1]} = \sqrt{R - f'^{[n]}}; \quad f^{[0]} = 0 \tag{675}$$

We get

$$\begin{aligned} f^{[1]} &= \sqrt{R} + \cdots \\ f^{[2]} &= \sqrt{R} - \frac{1}{4}\frac{R'}{R} + \cdots \\ f^{[3]} &= \sqrt{R} - \frac{1}{4}\frac{R'}{R} - \frac{5R'^2 - 4RR''}{R^{5/2}} + \cdots \end{aligned} \tag{676}$$

We see that, if $R \sim x^a$, then

$$R'/R \sim x^{-1}; \quad \frac{5R'^2 - 4RR''}{R^{5/2}} \sim x^{-2-a/2} \tag{677}$$

confirming the asymptotic nature of the expansion: the successive corrections have more and more negative powers of x. By integration

$$W = \int_a^x \sqrt{R(s)}ds - \frac{1}{4}\ln R + \frac{1}{8}\frac{R'}{R^{\frac{3}{2}}} + \frac{1}{32}\int_a^x \frac{R(s)'^2}{R(s)^{\frac{5}{2}}}ds \tag{678}$$

giving

$$y \sim R^{-\frac{1}{4}} e^{\int_a^x \sqrt{R(s)ds}} \left(1 + \frac{1}{8}\frac{R'}{R^{\frac{3}{2}}} + \frac{1}{32}\int_a^x \frac{R(s)'^2}{R(s)^{\frac{5}{2}}}ds + \cdots\right) \tag{679}$$

Similarly, the minus sign results in

$$y \sim R^{-\frac{1}{4}} e^{-\int_a^x \sqrt{R(s)ds}} \left(1 - \frac{1}{8}\frac{R'}{R^{\frac{3}{2}}} - \frac{1}{32}\int_a^x \frac{R(s)'^2}{R(s)^{\frac{5}{2}}}ds + \cdots\right) \tag{680}$$

34.1 Example: the Airy equation (666)

Substituting $y = e^w$ in (666) we get

$$w'' + w'^2 = x \tag{681}$$

or, choosing the plus sign,

$$f = \sqrt{x - f'} = \sqrt{x} - \frac{f'}{2\sqrt{x}} - \frac{f'^2}{8x^{3/2}} + \cdots \tag{682}$$

The sequence of iterations (676) gives

$$\begin{aligned} f^{[0]} &= \sqrt{x} \\ f^{[1]} &= \sqrt{x} - \frac{1}{4x} \\ f^{[2]} &= \sqrt{x} - \frac{1}{4x} - \frac{5}{32}x^{-5/2} \end{aligned} \tag{683}$$

etc. In terms of w, we get

$$w = C_1 + \frac{2}{3}x^{3/2} - \frac{1}{4}\ln x + \frac{5}{48}x^{-3/2} \tag{684}$$

and thus

$$y \sim Ce^{\frac{2}{3}x^{\frac{3}{2}}}x^{-1/4}\left(1 + \frac{5}{48}x^{-3/2} + \cdots\right) \tag{685}$$

We justify this asymptotic expansion next.

34.1.1 Rigorous justification of the asymptotics for (666)

Theorem 29. *There exist two linearly independent solutions of* (666) *with the (two) asymptotic behaviors (corresponding to different choices of sign)*

$$y_{\pm} \sim e^{\pm\frac{2}{3}x^{\frac{3}{2}}} x^{-1/4}(1 + o(1)) \text{ as } x \to +\infty \tag{686}$$

A similar analysis can be performed for $x \to -\infty$.

Proof. It is enough to show that $w_{\pm} - [\pm\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\ln x] \to 0$ as $x \to \infty$. We choose the sign $+$, for which the analysis is slightly more involved. Define $w = \sqrt{x} + g$ and consider the equation for g,

$$g' + 2\sqrt{x}g = -\frac{1}{2\sqrt{x}} - g^2 \tag{687}$$

or, more generally

$$g' + 2\sqrt{x}g = H(x) \tag{688}$$

The differential equation (688) with initial condition $g(x_0) = 0$ (chosen for simplicity) where $x_0 > 0$ will be chosen large, is equivalent to

$$g(x) = \mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} H(s)\, \mathrm{e}^{4/3\, s^{3/2}} ds \tag{689}$$

In our specific case, we have

$$H(x) := -\frac{1}{2\sqrt{x}} - g(x)^2 \tag{690}$$

and thus

$$g(x) = -\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds - \mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} g^2(s) \mathrm{e}^{4/3\, s^{3/2}} ds \tag{691}$$

What is the expected behavior of the first integral? We can see this by L'Hospital (which, you can check, applies). We have

$$\frac{\left(\displaystyle\int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds\right)'}{\left(\dfrac{\mathrm{e}^{4/3\, x^{3/2}}}{4x}\right)'} = \frac{1}{1 - 2x^{-3/2}} \to 1 \ (x \to +\infty) \tag{692}$$

and thus

$$= -\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds \sim -\frac{1}{4x} \quad x \to +\infty \tag{693}$$

Let's more generally, look at the behavior of

$$\int_{x_0}^{x} s^n e^{As^m} ds \tag{694}$$

where $m > 0$ We look at the value of l for which, by L'Hospital, we would get

$$\frac{\left(\int_{x_0}^x x^n e^{Ax^m}\right)'}{\left(x^l e^{Ax^m}\right)'} = \frac{x^{n+1-l-m}}{Am + lx^{-m}} \to C \tag{695}$$

where $C \neq 0$ is some constant. We need $l = n - m + 1$ and $C = (Am)^{-1}$. In particular,

$$\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^x s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds \sim \frac{1}{2} x^{-a-1/2} \ \ x \to +\infty \tag{696}$$

Since the behavior of the first term of (689) is $-1/(4x)$, consistent with our formal WKB analysis and thus g should be $O(1/x)$, this suggests we write $g = u/x$. We get

$$\begin{aligned} &u(x) \\ &= -x\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^x \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds - x\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^x u^2(s) s^{-2} \mathrm{e}^{4/3\, s^{3/2}} ds =: \mathcal{N}u \end{aligned} \tag{697}$$

We analyze this equation in $L^\infty[x_0, \infty)$. We first need bounds on the main ingredients of (697), that is on integrals of the form

$$\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^x s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds \tag{698}$$

which are valid on $[x_0, \infty)$ and not merely as $x \to \infty$. The asymptotic information (696) is helpful, but concrete bounds would provide more information (though this is not necessary if we merely want to prove an expansion as $x \to \infty$). From the limiting information, it follows that for any $A > 1$, if x_0 is large enough, we have

$$\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^x s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds < A \frac{1}{2x^{a+1/2}}, \tag{699}$$

To find a specific x_0, we look at

$$f(x) = \int_{x_0}^x s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds - A\mathrm{e}^{4/3\, x^{3/2}} \frac{1}{2x^{a+1/2}} \tag{700}$$

We note that $f(x_0) < 0$. Calculating f', we get

$$\begin{aligned} f'(x) &= x^{-a} \mathrm{e}^{4/3\, x^{3/2}} - Ax^{-a} \mathrm{e}^{4/3\, x^{3/2}} \left(1 - \frac{a + 1/2}{2x^{3/2}}\right) \\ &= -x^{-a} \mathrm{e}^{4/3\, x^{3/2}} \left[A - 1 - \frac{A(a + 1/2)}{2x^{3/2}}\right] \end{aligned} \tag{701}$$

It is clear that $f' < 0$ if $x > x(A)$ where

$$x(A)^{3/2} = \frac{A(1+2a)}{4(A-1)} \tag{702}$$

Thus we proved

Lemma 63. *If $A > 1$, with $x(A)$ as given in* (702) *we have*

$$\int_{x(A)}^{x} s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds < A\mathrm{e}^{4/3\, x^{3/2}} \frac{1}{2x^{a+1/2}} \tag{703}$$

for all $x > x(A)$.

We will now write (697) in contractive form in a suitable ball in $L^\infty[x_0, \infty)$. We will make some choices of A, x_0 etc, to write down something specific. The proof of the theorem is completed by the following result.

Lemma 64. *Let $A = 2$ and $x_0 \geqslant x(A), 2\}$. Consider the ball*

$$B = \{u : \sup_{x \geqslant x_0} |u(x)| \leqslant 1\} \tag{704}$$

Then $\mathcal{N}$ is contractive in B, and thus (697) *has a unique solution u_0 there.*

Proof of the lemma. It is straightforward to check that $\mathcal{N}B \subset B$. We have

$$|\mathcal{N}(u_2 - u_1)| = \left| x\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} (u_2 - u_1)(u_2 + u_1) s^{-2} \mathrm{e}^{4/3\, s^{3/2}} ds \right|$$
$$\leqslant \|u_2 - u_1\| \frac{2|x|}{|x|^{5/2}} \leqslant \frac{2}{|x_0|^{3/2}} \|u_2 - u_1\| \leqslant 2^{-1/2} \|u_2 - u_1\| \tag{705}$$

□

On the other hand, as $x \to \infty$, using (696) and the fact that $\|u_0\| < 1$, we have

$$g = -\frac{1}{4x} + o(1/x) \quad \text{as} \quad x \to \infty \tag{706}$$

□

Nonlinear ODEs

Some nonlinear ODEs admit special solution that have asymptotic expansions at infinity. Such is the case of many Painlevé equations, the Abel equation discussed before, and more generally ODEs having that can be brought to the form

$$y' = \Lambda y + x^{-1} B y + F(1/x, y); \quad F(z, y) \text{ analytic near } (0,0),\ F = o(x^{-m}, y^2) \tag{707}$$

where m is large enough and Λ and B are constant matrices.

Consider Abel's equation (653) in the limit $x \to +\infty$. We first find the asymptotic behavior of solutions formally, and then justify the argument. We use the method of *dominant balance* that we will discuss in detail later. As x becomes large, y, y', or both need to become large if the equation (653) is to hold. Assume first that the balance is between y' and x and that $y \ll x$. If $y' \sim x$ then we have $y \sim x^2/2$ and $y^3 \sim x^6/8$, and this is inconsistent since it would imply $x^8/8 = O(x)$. Now, if we assume $x \ll y^3$ then the balance would be $y' \approx y^3$, implying $y \sim -\frac{1}{2}(x-x_0)^{-2}$; but this is small for $x - x_0 \gg 1$, which conflicts with what we assumed, $x \ll y^3$. We have one possibility left: $y = \alpha x^{1/3}(1+o(1))$, where $\alpha^3 = 1$, which assuming differentiability implies $y' = O(x^{-2/3})$ which is now consistent. We substitute

$$y = \alpha x^{1/3}(1+v(x)) \tag{708}$$

in (653); for definiteness, we choose $\alpha = e^{i\pi/3}$, though any cube root of -1 would work. We get

$$\alpha x^{1/3} v' + 3xv + 3xv^2 + xv^3 + \frac{\alpha}{3}x^{-2/3} + \frac{\alpha}{3}x^{-2/3}v = 0 \tag{709}$$

Now a consistent balance is between $3xv$ and $-\frac{\alpha}{3}x^{-2/3}$ meaning that $v = O(x^{-5/3})$. This makes the nonlinear terms small and, for the purpose of justifying the analysis, we don't need to further expand v. We now aim at writing (709) in a suitable integral form. We first place the formally largest term(s) containing v *and* v' on the left side and the smaller terms as well as the terms not depending on v on the right side:

$$\alpha x^{1/3} v' + 3xv = h(x, v(x)); \;\; -h(x,v(x)) := 3xv^2 + xv^3 + \frac{\alpha}{3}x^{-2/3} + \frac{\alpha}{3}x^{-2/3}v \tag{710}$$

We treat (710) as a linear inhomogeneous equation, and solve it thinking for the moment that h is given.

This leads to

$$v = \mathcal{N}(v);$$
$$\mathcal{N}(v) := Ce^{-\frac{9}{5\alpha}x^{5/3}} + \frac{1}{\alpha}e^{-\frac{9}{5\alpha}x^{5/3}} \int_{x_0}^{x} e^{\frac{9}{5\alpha}s^{5/3}} s^{-1/3} h(s, v(s))ds \tag{711}$$

We chose the limits of integration in such a way that the integrand is maximal when $s = x$: if $x \to +\infty$, then $x^{-1/3}e^{\frac{9}{5\alpha}x^{5/3}} \to \infty$, and our choice corresponds indeed to this prescription.

The largest of the terms not containing v on the right side of (711) comes from the term $\frac{\alpha}{3}x^{-2/3}$ in h, and is of the order $\frac{1}{3}x^{-5/3}(1+o(1))$. Indeed, (696) gives

$$\frac{\int_a^x e^{bs^m}/s^n ds}{e^{bx^m}/x^n} \sim b^{-1}m^{-1}x^{1-m}; \;\; x \to +\infty \tag{712}$$

Again by dominant balance, we expect $v = O(x^{-5/3})$. Thus, it is natural to choose x_0 large enough and introduce the Banach space

$$\{f : \|f\| := \sup_{x>x_0} |x^{5/3} f(x)| < \infty\} \tag{713}$$

or the region $|x| > x_0$ in a sector $\mathcal{S}$ in the complex domain where $\mathrm{Re}\left(\frac{1}{\alpha}x^{5/3}\right) > 0:\ \arg x \in (-\frac{\pi}{10}, \frac{\pi}{2})$:

$$\mathcal{B} = \{f : \|f\| := \sup_{x\in\mathcal{S}} |x^{5/3} f(x)| < \infty\} \tag{714}$$

and within this space a ball of size large enough $-\frac{2}{3}-$ to accommodate for the largest term on the right side, $\frac{\alpha}{3}x^{-2/3}$:

$$B_1 := \{f \in \mathcal{B} : \|f\| \leqslant \frac{2}{3}\} \tag{715}$$

Lemma 65. *For given C, if x_0 is large enough, then the operator $\mathcal{N}$ is contractive in B_1 and thus* (711) *(as well as* (710)*) has a unique solution there.*

Proof. We first check that $\mathcal{N}(B_1) \subset B_1$, by estimating each term in $\mathcal{N}$. By (712) we have for large enough x_0, $|\mathcal{N}x^{-m}| = \frac{1}{3}|x|^{-m-1}(1+o(1))$. In particular, $|\mathcal{N}\frac{\alpha}{3}x^{-2/3}| \leqslant \frac{\alpha}{9}|x|^{-5/3}(1+o(1))$. The contribution of the other terms are much smaller. For instance, $|xv^2| < Cx^{1-5/2}\|v\|$ we have $|\mathcal{N}(xv^2)| = C|x|^{-5/2}(1+o(1))$.

To show contractivity, we note that, for $k > 1$,

$$|\mathcal{N}(v_2^k - v_1^k)| \leqslant k\|v_2 - v_1\||\mathcal{N}\left[x^{-5/3}2(2/3)^{k-1}x^{-5(k-1)/3}\right]$$

□

35 Elements of eigenfunction theory–material complementary to Coddington-Levinson

35.1 Properties of the Wronskian of a system

Lemma 66. *Let A be a matrix on $\mathbb{C}^n$. We have*

$$\det(I + \varepsilon A) = 1 + \varepsilon \mathrm{Tr}\, A + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 \tag{716}$$

Proof 1. The property is obvious for

$$\begin{pmatrix} 1+\varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & 1+\varepsilon a_{22} \end{pmatrix} \tag{717}$$

For the general case, use induction and row expansion. □

Proof 2. Note that $\det B = \prod_j (1 + b_j)$, where b_j are the eigenvalues of B (repeated if the multiplicity is not one). If $(I + \varepsilon A)v = \mu v$ then $\varepsilon A v = (\mu - 1)v$ that is, $v = v_j$ is an eigenvector of A: $Av_j = a_j v$. Thus $(1+\varepsilon a_j)v_j = (I+\varepsilon A)v_j = \mu v_j \Rightarrow \mu = (1 + \varepsilon a_j)$. The property now follows. □

35.1.1 The Wronskian

The definition is

$$W[f_1, ..., f_n] = \begin{vmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix} \tag{718}$$

Lemma 67. *Let*

$$M' = AM \tag{719}$$

be a matrix equation in $\mathbb{C}^n$*. We have*

$$\det M(t) = \det M(0) \exp\left(\int_0^t \mathrm{Tr} A(s) ds\right) \tag{720}$$

Proof. We have (just by differentiability)

$$M(t+\varepsilon) - M(t) = A(t)M(t)\varepsilon + o(\varepsilon) \tag{721}$$

and thus

$$\begin{aligned} M^{-1}(t)M(t+\varepsilon) &= I + M^{-1}(t)A(t)M(t)\varepsilon + o(\varepsilon) \\ \Rightarrow \det\left(M^{-1}(t)M(t+\varepsilon)\right) &= \det\left(I + M^{-1}(t)A(t)M(t)\varepsilon + o(\varepsilon)\right) \\ &= 1 + \mathrm{Tr}(A)\varepsilon + o(\varepsilon) \end{aligned} \tag{722}$$

and thus

$$\begin{aligned} \frac{\det M(t+\varepsilon)}{\det M(t)} = 1 + \mathrm{Tr}(A)\varepsilon + o(\varepsilon) \Rightarrow \frac{\det M(t+\varepsilon) - \det M(t)}{\varepsilon} \\ = \det M(t)\mathrm{Tr}(A(t)) + o(1) \Rightarrow (\det M(t))' = \det M(t)\mathrm{Tr}(A(t)) \end{aligned} \tag{723}$$

and the result follows by integration. ☐

Note that an equation of the kind we are considering,

$$Lf = p_0(t)f^{(n)} + p_1(t)f^{(n-1)} + \cdots + p_n(t)f = \lambda f \tag{724}$$

has the matrix equation counterpart

$$M' = AM \tag{725}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ -\frac{p_n}{p_0} & -\frac{p_{n-1}}{p_0} & -\frac{p_{n-2}}{p_0} & ... & -\frac{p_1}{p_0} \end{pmatrix} \tag{726}$$

and

$$M = \begin{pmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)} & ... & f_n^{(n-1)} \end{pmatrix} \tag{727}$$

Clearly, $\mathrm{Tr} A = -p_1/p_0$. Thus we have

Corollary 68. *The Wronskian W of a fundamental system for* (724) *satisfies*

$$W(t) = W(0) \exp\left(-\int_0^t \frac{p_1(s)}{p_0(s)} ds\right) \tag{728}$$

36 Discrete dynamical systems

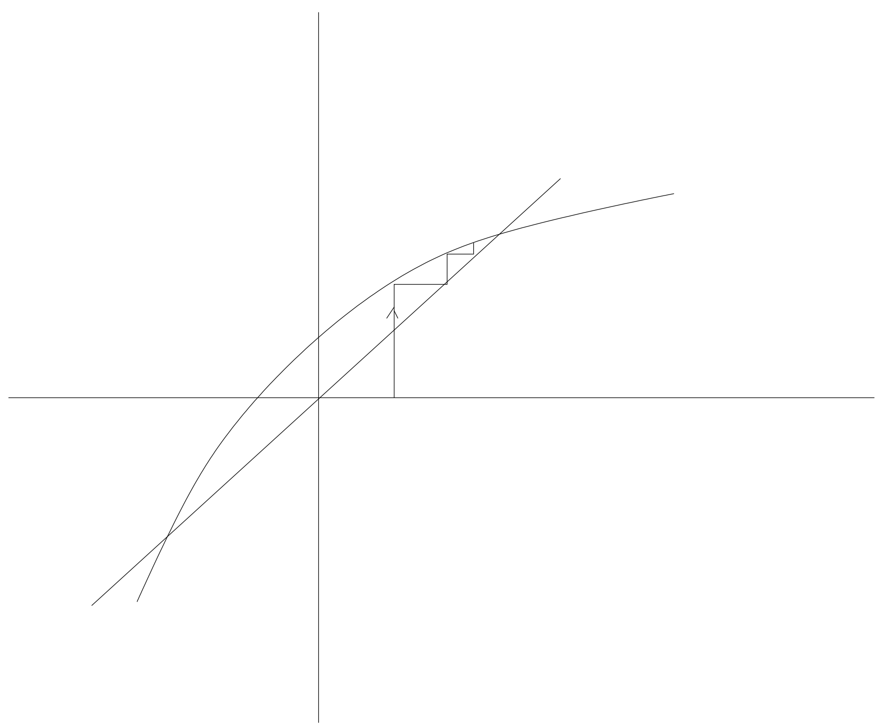

Figure 50:

The study of the Poincaré map leads naturally to the study of discrete dynamics. In this case we have closed trajectory, x_0 a point on it, $\mathcal{S}$ a section through x_0 and we take a point x_1 near x_0, on the section. If x_1 is sufficiently close to x, it must cross again the section, at x_1', still close to x_1, after the *return time* which is then close to the period of the orbit. The application $x_1 \to x_1'$ defines the Poincaré map, which is smooth on the manifold near x_0.

The study of the behavior of differential systems is near closed orbits is often more easily understood by looking at the properties of the Poincaré map.

In one dimension first, we are dealing with a smooth function f, where the iterates of f are what we want to understand.

We write $f^n(x) = f(f(...(f(x))))$ n times. The *orbit* of a point x_0 is the sequence $\{f^n(x_0)\}_{n\in\mathbb{N}}$, assuming that $f^n(x_0)$ is defined for all n. In particular, we may assume that $f: J \to J$, where $J \subset \mathbb{R}$ is an interval, possibly the whole line.

The effects of the iteration are often easy to see on the graph of the iteration, in which we use the bisector $y = x$ to conveniently determine the new point. We have $(x_0, 0) \to (x_0, f(x_0)) \to (f(x_0), f(x_0)) \to (f(x_0), f(f(x_0)))$, where the two-dimensionality and the "intermediate" step helps in fact drawing the iteration faster: we go from x_0 up to the graph, horizontally to the bisector, vertically back to the graph, and repeat this sequence.

There are simple iterations, for which the result is simple to understand globally, such as

$$f(x) = x^2$$

where it is clear that $x = 1$ is a fixed point, if $|x_0| < 1$ the iteration goes to zero, and it goes to infinity if $|x_0| > 1$.

Local behavior near a fixed point is also, usually, not difficult to understand, analytically and geometrically.

Theorem 30. *(a) Assume f is smooth, $f(x_0) = x_0$ and $|f'(x_0)| < 1$. Then x_0 is a sink, that is, for x_1 in a neighborhood of x_0 we have $f^n(x_1) \to x_0$.*

(b) If instead we have $|f'(x_0)| > 1$, then x_0 is a source, that is, for x_1 in a small neighborhood $\mathcal{O}$ of x_0 we have $f^n(x_1) \notin \mathcal{O}$ for some n (this does not mean that $f^m(x_1)$ cannot return "later" to $\mathcal{O}$, it just means that points very nearby are repelled, in the short run.)

Proof. We show (a), (b) being very similar. Without loss of generality, we take $x_0 = 0$. There is a $\lambda < 1$ and ε small enough so that $|f'(x)| < \lambda$ for $|x| < \varepsilon$. If we take x_1 with $|x_1| < \varepsilon$, we have $|f(x_1)| = |f'(c)||x_1| < \lambda|x_1|(< \varepsilon)$, so the inequality remains true for $f(x_1) : |f(f(x_1))| < \lambda|f(x_1)| < \lambda^2|x_1|$ and in general $f^n(x_1) = O(\lambda^n) \to 0$ as $n \to \infty$.

In fact, it is not hard to show that, for smooth f, the evolution is essentially geometric decay.

□

When the derivative is one, in absolute value, the fixed point is called neutral or indifferent. It does not mean that it can't still be a sink or a source, just that we cannot resort to an argument based on the derivative, as above.

Example 38. *We can examine the following three cases:*

(a) $f(x) = x + x^3$.

(b) $f(x) = x - x^3$

(c) $f(x) = x + x^2$.

It is clear that in the first case, any positive initial condition is driven to $+\infty$. Indeed, the sequence $f^n(x_1)$ is increasing, and it either goes to infinity or else it has a limit. But the latter case cannot happen, because the limit should satisfy $l = l + l^3$, that is $l = 0$, whereas the sequence was increasing.

The other cases are analyzed similarly: in (a), if $x_0 < 0$ then the sequence still diverges. Case (c) is more interesting, since the sequence converges to zero if $x_1 < 0$ is small enough and to ∞ for all $x_1 > 0$. We leave the details to the reader.

It is useful to see what the behavior of such sequences is, in more detail.

Let's take the case (c), where $x_1 < 0$. We have

$$x_{n+1} = x_n + x_n^2$$

where we expect the evolution to be slow, since the relative change is vanishingly small. We then approximate the true evolution by a differential equation

$$(d/dn)x = x^2$$

giving

$$x_n = (C - n)^{-1}$$

We can show rigorously that this is the behavior, by taking $x_n = -1/(n+c_0)+\delta$, $\delta_{n_0} = 0$ and we get

$$\delta_{n+1} - \delta_n = \frac{1}{n^2(n+1)} - \frac{2}{n}\delta_n + \delta_n^2 \tag{729}$$

and thus

$$\delta_n = \sum_{j=n_0}^{n} \left(\frac{1}{j^2(j+1)} - \frac{2}{j}\delta_j + \delta_j^2 \right) \tag{730}$$

Exercise 1. *Show that (730) defines a contraction in the space of sequences with the property $|\delta_n| < C/n^2$, where you choose C carefully.*

Exercise 2. *Find the behavior for small positive x_1 in (b), and then prove rigorously what you found.*

36.1 Bifurcations

The local number of fixed points can only change when $f'(x_0) = 1$. As before, we can assume without loss of generality that $x_0 = 0$.

We have

Theorem 31. *Assume $f(x, \lambda)$ is a smooth family of maps, that $f(0, 0) = 0$ and that $f_x(0, 0) \neq 1$. Then, for small enough λ there exists a smooth function $\varphi(\lambda)$, also small, so that $f(\varphi(\lambda), \lambda) = \varphi(\lambda)$, and the character of the fixed point (source or sink) is the same as that for $\lambda = 0$.*

Exercise 3. *Prove the theorem, using the implicit function theorem.*

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