Class notes, Functional Analysis 7212

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Contents

1 Banach Algebras

In what follows, X will be a compact Hausdorff space.

A Banach algebra $\mathfrak{B}$ is an algebra over $\mathbb{C}$ with a norm s.t. $\|fg\| \leqslant \|f\|\|g\|$, which implies that multiplication is continuous. The Banach algebra $\mathfrak{B}$ is **unital** if it has an **identity** $1 \in \mathfrak{B}$ such that $1f = f1 = f$ for any f and $\|1\| = 1$.

Examples 1.0.1.
1. $GL(n, \mathbb{C})$ with the usual matrix norm, $\|M\| = \sup\{\|Mu\| : \|u\| = 1\}$, is a noncommutative unital Banach algebra.
2. $C(X)$ with multiplication and the sup norm is a canonical example of a unital commutative Banach algebra.
3. $L^1(\mathbb{R}^+)$ with multiplication given by convolution,
$$(F * G)(p) = \int_0^p F(s)G(p-s)ds \tag{1}$$
is an important example of a commutative Banach algebra **without identity**.

Exercise 1. Prove the statements in part 2. of Example 1.0.1 above.

Exercise 2. The purpose of this exercise is to show that any Banach algebra can be embedded (as an ideal of codimension one) in a unital Banach algebra.

1. In this step we start with a Banach algebra without an identity and adjoin one.

   Let $\mathfrak{B}$ be a Banach algebra without an identity and define $\mathfrak{B}_1 = \mathfrak{B} \oplus \mathbb{C}$ with multiplication and norm given by

   $$(f, \alpha) \cdot (g, \beta) := (fg + \beta f + \alpha g, \alpha\beta); \ \|(f, \alpha)\| = \|f\| + |\alpha|$$

   Show that $\mathfrak{B}_1$ is a unital Banach algebra, and that $(f, 0)$ is an ideal in $\mathfrak{B}_1$ which is isometrically isomorphic to $\mathfrak{B}$.

2. Now we start with a Banach algebra $\mathfrak{B}$ with an identity 1. Note that $\|1\| \geqslant 1$ simply follows from $\|fg\| \leqslant \|f\|\|g\|$ (and of course $\|1\| < \infty$ since $1 \in \mathfrak{B}$). Assuming that $\|1\| > 1$ we want to show that there is an equivalent Banach algebra norm $\|\cdot\|'$ on $\mathfrak{B}$ s.t. $\|1\|' = 1$. The idea, that we will see again in various shapes is to use an "operator norm".

   Recall first that the norm of a bounded linear operator A defined on a Banach space is defined as $\|A\| = \sup_{\|u\|=1} \|Au\|$ and, if A, B are bounded linear operators, then $\|A + B\| \leqslant \|A\| + \|B\|$ and $\|AB\| \leqslant \|A\|\|B\|$.

   Check that for any fixed $x \in \mathfrak{B}$, $y \mapsto xy$ defines a linear bounded operator on $\mathfrak{B}$, and define $\|x\|'$ to be the norm of x as a linear operator. Check first that $\|x\|' \leqslant \|x\|$. Check that $\|x\|' \geqslant \|x\|/\|1\|$. Thus $\|\cdot\|$ and $\|\cdot\|'$ are equivalent and convince yourself that $\mathfrak{B}$ endowed with $\|\cdot\|'$ is a unital Banach algebra.

Finally, in example 1.0.1 2. above, we could adjoin a unit as a distribution. Which?

In what follows, we will **assume** (as the exercise above shows, without loss of generality) that $\mathfrak{B}$ **is unital**. It is the existence of a norm that leads to many interesting properties.

Note 1.0.2. The norm in a Banach algebra makes possible the development of analytic functional calculus. If F is a function analytic in the unit disk $\mathbb{D}$, $F(z) = \sum_{k \geqslant 0} c_k z^k$, $|z| < 1$ we can define $F(f)$ in the open unit ball at zero $B_1(0)$, through the norm-convergent series

$$F(f) = \sum_{k \geqslant 0} c_k f^k; \ \ \forall f, \ \|f\| < 1 \tag{2}$$

Let us first look at the invertible elements of $\mathfrak{B}$.

Proposition 1.0.3. The open ball of radius one around the identity of $\mathfrak{B}$, $B_1(1)$, consists of invertible element; if $f \in B_1(1)$, then $\|f^{-1}\| \leqslant (1 - \|1 - f\|)^{-1}$

Proof. If $f \in B_1(1)$ then $f = 1 - h$ with $\|h\| < 1$. The natural candidate for $f^{-1} = (1-h)^{-1}$ is $y = \sum_{k=0}^{\infty} h^k$ (the **von Neumann series** for $(1-h)^{-1}$).

We check that the series indeed converges, and $y(1-h) = (1-h)y = 1$. For the first part, note that, with obvious notations,

$$\left\| \sum_{k=m}^{n} h^k \right\| \leqslant \sum_{k=m}^{n} \|h\|^k \leqslant \frac{\|h\|^{m+1}}{\|1-h\|}$$

implying that the series is Cauchy. We can check that $\|1 - (1-h)\sum_{k=0}^{n} h^k\| = \|h^{n+1}\| \leqslant \|h\|^{n+1}$ implying the result. $\square$

Definition 1.0.4. Let $\mathfrak{B}$ be a Banach algebra. We denote by $\mathcal{G}_l$ and $\mathcal{G}_r$ the left and right invertible elements of $\mathfrak{B}$ and by $\mathcal{G} = \mathcal{G}_l \cap \mathcal{G}_r$ the group of invertible elements of $\mathfrak{B}$.

Here is a very useful identity that will come handy on a number of occasions.

Proposition 1.0.5 (Second resolvent identity)**.** Assume f, g are invertible elements of an algebra. Then,

$$g^{-1} - f^{-1} = f^{-1}(f-g)g^{-1}$$

Proof. This follows simply from $g - f = f^{-1}fg - f^{-1}gg^{-1}$. $\square$

Note 1.0.6. A similar identity holds if g^{-1} (f^{-1}) are right (left) inverses, respectively.

Corollary 1.0.7. In a Banach algebra, $\mathcal{G}_l$, $\mathcal{G}_r$ and $\mathcal{G}$ are open.

Proof. It suffices to show that $\mathcal{G}_r$ is open. Assume $f \in \mathcal{G}_r$ and δ has small norm: $\|\delta\| \leqslant \varepsilon/\|f^{-1}\|, \varepsilon < 1$ more precisely. We have $f + \delta = f(1 + f^{-1}\delta)$ whose right inverse is $(1 + f^{-1}\delta)^{-1}f^{-1}$, provided $(1 + f^{-1}\delta)$ is invertible. By Proposition 1.0.3 this is so provided $\|f^{-1}\delta\| < 1$ which is implied by the condition above. Note also that, for small δ, we have by Proposition 1.0.3,

$$\|(f+\delta)^{-1}\| \leqslant \|f^{-1}\|/(1 - \|f^{-1}\delta\|) \leqslant \|f^{-1}\|/(1-\varepsilon) \tag{3}$$

$\square$

Corollary 1.0.8. If $\mathfrak{B}$ is a Banach algebra, then $\mathcal{G}$ is a topological group.

Proof. We only need to check that the map $f \mapsto f^{-1}$ is continuous. This follows immediately from the second resolvent formula, since, using the notations in the proof of Corollary 1.0.7,

$$\|(f+\delta)^{-1} - f^{-1}\| = \|(f+\delta)^{-1}\delta f^{-1}\| \leqslant \|f^{-1}\|^2(1-\varepsilon)^{-1}\|\delta\|$$

$\square$

Proposition 1.0.9. Let $\mathfrak{B}$ be a Banach algebra and let $\mathcal{G}_0$ be the connected component of the identity in $\mathcal{G}$. Then $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$ which is both open and closed. The group $\mathcal{G}/\mathcal{G}_0$ is discrete, and the connected components of $\mathcal{G}$ are the cosets of $\mathcal{G}_0$.

Definition 1.0.10. $\Lambda_{\mathfrak{B}} = \mathcal{G}/\mathcal{G}_0$ is called the abstract index group for $\mathfrak{B}$. The natural homomorphism γ from $\mathcal{G}$ to $\Lambda_{\mathfrak{B}}$ is called the abstract index.

Proof of Proposition 1.0.9. By continuity of multiplication, if $f \in \mathfrak{B}$ then $f\mathcal{G}_0$ is connected. Assuming now $f, g \in \mathcal{G}_0$, $f\mathcal{G}_0$ is a connected set in $\mathcal{G}$ which contains f and fg. It follows that $\mathcal{G}_0 \cup f\mathcal{G}_0$ is connected in $\mathcal{G}$, hence $\mathcal{G}_0 \cup f\mathcal{G}_0 \subset \mathcal{G}_0$ implying that $\mathcal{G}_0$ is closed under multiplication. Similarly, $f^{-1}\mathcal{G}_0 \cup \mathcal{G}_0 \subset \mathcal{G}_0$ implying that $\mathcal{G}_0$ is a subgroup of $\mathcal{G}$. In the same way, for any $f \in \mathcal{G}$, the group $f\mathcal{G}_0 f^{-1}$ is connected and contains the identity, hence $f\mathcal{G}_0 f^{-1} = \mathcal{G}_0$. Taking any component C of $\mathcal{G}$ and $f \in C$ we see that $f\mathcal{G}_0 \subset C$ is both closed and open (why?), hence $f\mathcal{G}_0 = C$. Finally, using again the fact that $\mathcal{G}_0$ is open and closed, it must be that $\mathcal{G}/\mathcal{G}_0$ is a discrete group. $\square$

1.1 The exponential map

The log, $f \mapsto \log(1+f)$, is defined in $B_1(0)$ by

$$\log(1+f) = -\sum_{n>0} n^{-1}(-f)^n \tag{4}$$

Define the exponential map by the norm-convergent series

$$\exp f = \sum_{n=0}^{\infty} \frac{1}{n!} f^n \tag{5}$$

Lemma 1.1.1. 1. If $\|1-g\| < 1$, then $\exp \ln g = g$. In particular, $B_1(1) \subset \exp \mathfrak{B}$.

2. If $f, g \in \mathfrak{B}$ commute, $fg = gf$, then

$$\exp(f+g) = \exp(f)\exp(g) \tag{6}$$

Proof. Both follow from rearranging norm-convergent sums. □

Note 1.1.2. Commutativity is essential for (6) to hold in any generality. It is a **common mistake** to forget about this condition.

Theorem 1.1.3. In a Banach algebra $\mathfrak{B}$, the group $\mathcal{G}_0$ equals the set $\mathcal{F}$ of finite products of elements of $\exp\mathfrak{B}$.

Proof. Note first that $\exp\mathfrak{B} \subset \mathcal{G}$, since $\exp(f)\exp(-f) = \exp(f-f) = 1$. Further, noting that $1 = \exp(0) \in \exp\mathfrak{B}$, and that $\exp\mathfrak{B}$ is arc-connected ($\{\exp(tg) : t \in [0,1]\}$ connects 1 to $\exp g$) we see that $\exp\mathfrak{B} \subset \mathcal{G}_0$. Recalling that $\mathcal{G}_0$ is a group, check that $\mathcal{F}$ is a subgroup of $\mathcal{G}_0$. Since $B_1(1) \subset \mathcal{F}$, we see that $\mathcal{F}$ is open. Since the cosets of $\mathcal{F}$ are open it follows [1] that $\mathcal{F}$ is closed; this completes the proof, since $\mathcal{G}_0$ is connected. □

Now, use this theorem and Lemma 1.1.1 to prove this nice identity:

Corollary 1.1.4. If $\mathfrak{B}$ is a commutative Banach algebra, then $\exp\mathfrak{B} = \mathcal{G}_0$.

Exercise 3. Characterize $\mathcal{G}, \mathcal{G}_0$ and $\Lambda_{\mathfrak{B}}$ when $\mathfrak{B}$ is the Banach algebra generated by an element of $GL(n)$.

1.2 The index group of $\mathfrak{B} = C(X)$

In this case, the group $\mathcal{G}$ consists precisely of the continuous functions that do not vanish, $f : X \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{G}_0$, then by Corollary 1.1.4 $f = \exp g$ for some $g \in C(X)$ and the function $F := (g, \lambda) \mapsto \exp(\lambda g)$ is clearly continuous and has the property $F(\cdot, 0) = 1$, $F(\cdot, 1) = f$, showing that f is homotopic to the constant function 1. Conversely, of course, if f is homotopic to 1, then $f \in \mathcal{G}_0$. More generally, essentially the same argument shows that f_1 and f_2 are equivalent modulo $\mathcal{G}_0$ iff they are homotopic. This proves the following.

[1] And the reason is written in invisible ink here:

Proposition 1.2.1. The abstract index group for $C(X)$ is the group of homotopy classes of continuous maps from X to $\mathbb{C}^*$

1.2.1 $\pi^1(X)$

By definition, the first cohomotopy group $\pi^1(X)$ of a compact Hausdorff space X is the group of homotopy classes of continuous maps from X to the group $\mathbb{T} = \{z : |z| = 1\}$.

Theorem 1.2.2. If X is a compact Hausdorff space, then the abstract index group Λ for $C(X)$ and $\pi^1(X)$ are naturally isomorphic.

Proof. See Douglas. □

Corollary 1.2.3. If X is a compact Hausdorff space, then Λ is naturally isomorphic to $H^1(X, \mathbb{Z})$, the first Čech cohomology group with integer coefficients.

Proof. See Douglas. □

1.3 Multiplicative functionals

Definition 1.3.1. Let $\mathfrak{B}$ be a Banach algebra. A multiplicative linear functional $\varphi : \mathfrak{B}$ is a linear functional s.t.

1. $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in \mathfrak{B}$ and
2. $\varphi(1) = 1$.

The set of all linear multiplicative functionals on $\mathfrak{B}$ is denoted by $M_{\mathfrak{B}}$ or simply M.

We note that continuity is not imposed as an assumption, as it will follow from 1,2.

Proposition 1.3.2. For any $\varphi \in M_{\mathfrak{B}}$ we have $\|\varphi\| = 1$; here $\mathfrak{B}$ and $M_{\mathfrak{B}}$ are as in Definition 1.3.1.

Proof. Let $\mathfrak{K}$ be the kernel of φ,

$$\mathfrak{K} = \{f \in \mathfrak{B} : \varphi(f) = 0\} \tag{7}$$

Note that if h is an invertible element of $\mathfrak{B}$, then $h \notin \mathfrak{K}$, because assuming otherwise we would get

$$1 = \varphi(1) = \varphi(h \cdot 1/h) = \varphi(h)\varphi(1/h) = 0 \tag{8}$$

By linearity and multiplicativity we have $\varphi(f - \varphi(f)1) = 0$. We can thus write any $f \in \mathfrak{B}$ as $f = \lambda 1 + k$ for some $\lambda \in \mathbb{C}$ and $k \in \mathfrak{K}$. Note that $\mathfrak{K} \neq \mathfrak{B}$, since $\varphi(1) = 1$. Thus, in the definition of the norm of φ

$$\|\varphi\| = \sup_{x \neq 0} \frac{|\varphi(x)|}{\|x\|} \tag{9}$$

we may assume $x \notin \mathfrak{K}$, that is $x = \lambda + k$ for some $\lambda \in \mathbb{C}^*$ and $k \in \mathfrak{K}$. Note also that for any $k \in \mathfrak{K}$ we must have $\|1 + k\| \geqslant 1$ (since the opposite inequality would imply invertibility of k). Hence

$$\|\varphi\| = \sup_{k \in \mathfrak{K}, \lambda \neq 0} \frac{|\lambda|}{\|\lambda + k\|} = \sup_{k \in \mathfrak{K}} \frac{1}{\|1 + k/\lambda\|} = 1 \tag{10}$$

achieved when $k = 0$. □

1.3.1 Multiplicative functionals on $C(X)$

Clearly, for any $x \in X$ and $f \in C(X)$, the map $\varphi_x := f \mapsto f(x)$ is in $M_{C(X)}$. Importantly, the converse is also true:

Proposition 1.3.3. The map $\psi := x \mapsto \varphi_x$ establishes a homeomorphism from X onto $M_{C(X)}$ endowed with the weak-* topology [2].

Note 1.3.4. Recall that by the Riesz-Markov-Kakutani representation theorem, the dual $C(X)^*$ of $C(X)$ is the space of Radon (regular Borel) measures on X. The weak-* topology on $C(X)^*$ is the **vague** topology. A generalized sequence of measures μ_α converges vaguely to a measure μ iff

$$\int_X f d\mu_\alpha \to \int_X f d\mu, \;\; \forall f \in C(X)$$

From this perspective, multiplicative functionals on $C(X)$ are hence exactly the Dirac masses at points in X, as seen in the exercise below.

Exercise 4. 1. Assume then that μ is a Radon measure on X s.t. $\int 1 d\mu = 1$ and $\int fg d\mu = (\int f d\mu)(\int g d\mu)$ for all $f, g \in C(X)$. Show that there is some $x_0 \in X$ s.t. μ is the Dirac mass at x_0.

[Hint: show first that $\mu(A \cap B) = \mu(A)\mu(B)$ for any measurable sets A, B; you may then use Exercise 11 on p. 220 in Folland (there are simpler arguments), but then you have to include your solution to that exercise.]

2. Assume $\mathfrak{B} = C_0(X)$ where now X is **locally** compact Hausdorff. Note that this $\mathfrak{B}$ is not unital. Adjoin a unit, and call this extension still $\mathfrak{B}$. What is $M_{\mathfrak{B}}$?

3. **"MultiplicativeFun": bonus problem** Let now $\mathfrak{B} = C(\mathbb{R})$, with the usual sup norm. Clearly for any $x \in \mathbb{R}$ $\varphi := f \mapsto f_x$ is a linear multiplicative functional. Prove or disprove: "These are all the linear multiplicative functionals on $C(\mathbb{R})$."

Proof of Proposition 1.3.3. Let $\mathfrak{K}$ be the kernel of φ, cf. (7). We first show that, for some $x_0 \in X$, we have

$$\mathfrak{K} = \{f \in C(X) : f(x_0) = 0\} \tag{11}$$

To get a contradiction assume that

$$(\forall x \in X)(\exists f \in \mathfrak{K})(f(x) \neq 0)$$

Note that, by continuity, $f(x) \neq 0$ in some open neighborhood $\mathcal{O}_x$, and by assumption $\cup_{x\in X}\mathcal{O}_x = X$. Then, by compactness, there is a finite set $\{x_1, ..., x_n\}$ s.t. $\cup_{i\leqslant n}\mathcal{O}_{x_i} = X$; let the associated functions be $f_1, ..., f_n$ and define

$$h = \sum_{i=1}^{n} f_i \overline{f}_i = \sum_{i=1}^{n} |f_i|^2 \tag{12}$$

By assumption $h > 0$ on X, and by continuity and compactness $\min_X h = a > 0$ and therefore $1/h \in C(X)$, a contradiction (see (8)). Also as in that proof, note now that for any $f \in C(X)$ we have $f - \varphi(f)1 \in \mathfrak{K}$, hence with x_0 as in (11) we have

$$0 = (f - \varphi(f)1)(x_0) \;\text{ hence }\; \varphi(f) = f(x_0) \tag{13}$$

as desired.

To establish the homeomorphism, observe first that if $x \neq y$ in X, by Urysohn's lemma there is an $f \in C(X)$ s.t. $f(x) = 0, f(y) = 1$. Then $\varphi_x(f) = f(x) = 0$ and $\varphi_y(f) = f(y) = 1$, implying $\varphi_x \neq \varphi_y$ and ψ is a bijection. For continuity, we take a net x_α converging to x in X. Then, for any $f \in C(X)$ we have $\varphi_{x_\alpha}(f) = f(x_\alpha) \to f(x)$ by continuity, establishing pointwise convergence. $\square$

We now check that, w.r.t. the weak-* topology, M is a compact Hausdorff space.

Proposition 1.3.5. If $\mathfrak{B}$ is a Banach algebra, then $M_{\mathfrak{B}}$ is weak-* compact in the unit ball $(\mathfrak{B}^*)_1$.

Proof. It suffices to prove that $M_{\mathfrak{B}}$ is closed which follows if limits of nets preserve 1 and 2 of Definition 1.3.1. Let $\{\varphi_\alpha\}_{\alpha\in A}$ be a net in M weak-* convergent in $(\mathfrak{B}^*)_1$ to φ. Clearly, $\varphi_\alpha(1) = 1$ implies $\varphi(1) = 1$. Now, with $f, g \in \mathfrak{B}$ we have

$$\varphi(fg) = \lim_\alpha \varphi_\alpha(fg) = \lim_\alpha \varphi_\alpha(f)\varphi_\alpha(g) = \varphi(f)\varphi(g) \tag{14}$$

Noting also that $(\mathfrak{B}^*)_1$ is Hausdorff [3] we see that M is a compact Hausdorff space. $\square$

[2] Recall that this is equivalently characterized as the topology of pointwise convergence of functionals.
[3] The reason is written in invisible ink here:

Note now that $f \in \mathfrak{B}$ induces a continuous linear functional on $(\mathfrak{B}^*)$ through $\hat{f}(\varphi) = \varphi(f)$ (recall the notion of double-dual). The restriction of $\hat{f}$ to $M \subset (\mathfrak{B}^*)_1$, $\hat{f}|_M$ is a weak-* continuous linear functional on M. In fact, it turns out that it is also multiplicative, and this restriction is the important Gelfand transform.

Definition 1.3.6. Let $\mathfrak{B}$ be a Banach algebra. If $M_{\mathfrak{B}} \neq \emptyset$, then the Gelfand transform Γ is the function defined on $\mathfrak{B}$ with values in $C(M)$ given by

$$\Gamma(f) = \hat{f}|_M; \text{ i.e., for any } \varphi \in M,\ \Gamma(f)(\varphi) = \varphi(f) \tag{15}$$

Proposition 1.3.7.

1. Γ is an algebra homomorphism.
2. $\|\Gamma f\|_\infty \leqslant \|f\|$ for any $f \in \mathfrak{B}$.

Proof. 1. We only need to check multiplicativity which follows from

$$\Gamma(fg)(\varphi) = \varphi(fg) = \varphi(f)\varphi(g) = [(\Gamma f)(\varphi)][(\Gamma g)(\varphi)] = [(\Gamma f)(\Gamma g)](\varphi) \tag{16}$$

2. Let $f \in \mathfrak{B}$. Then,

$$\|\Gamma f\|_\infty = \left\|\hat{f}|_M\right\|_\infty \leqslant \left\|\hat{f}\right\|_\infty = \|f\| \tag{17}$$

(Why is the last equality true?) □

Exercise 5. Let $\mathfrak{B} = L(\mathbb{C}, n), n \geqslant 1$ with the usual matrix norm. What is $M_{\mathfrak{B}}$?

Note 1.3.8. Γ is a homomorphism between $\mathfrak{B}$ and a subalgebra of $C(M)$, which is clearly commutative. It means that when $\mathfrak{B}$ is not commutative, there is loss of information in the Gelfand transform representation. The kernel of Γ contains at least all elements of the form $fg - gf \in \mathfrak{B}$. There are simple Banach algebras for which $M_{\mathfrak{B}} = \emptyset$

1.4 Spectrum of an element relative to a Banach algebra

Let $\mathfrak{B}$ be a Banach algebra, commutative or not. The spectrum of $f \in \mathfrak{B}$ is defined as in operator theory,

$$\sigma_{\mathfrak{B}}(f) = \{\lambda \in \mathbb{C} : (f - \lambda) \text{ is not invertible in } \mathfrak{B}\} \tag{18}$$

We will simply write $\sigma_{\mathfrak{B}}(f) = \sigma(f)$ when no confusion is possible.

Similarly, the resolvent set is defined by

$$\rho_{\mathfrak{B}}(f) = \mathbb{C} \setminus \sigma_{\mathfrak{B}}(f)$$

Finally, the spectral radius of f is

$$r_{\mathfrak{B}}(f) = \sup_{\lambda \in \sigma_{\mathfrak{B}}(f)} |\lambda|$$

Note 1.4.1. We can think that $f - \lambda$ acts as a linear operator on the Banach **space** $\mathfrak{B}$, by $(f-\lambda)(g) = (f-\lambda)g$. We have however two operator candidates here $(f-\lambda)g$ and $g(f-\lambda)$. In the noncommutative case, the spectrum of f is generally *larger* than either of the two operatorial interpretations above. Nonetheless, this interpretation makes the following result an expected one.

Proposition 1.4.2. Let f be an element of the Banach algebra $\mathfrak{B}$. Then, $\sigma_{\mathfrak{B}}(f)$ is a compact subset of the disk of radius $\|f\|$, $\mathbb{D}_{\|f\|}(0) \subset \mathbb{C}$, and $r(f) \leqslant \|f\|$.

Proof. To show that $\sigma_{\mathfrak{B}}(f)$ is closed, or equivalently that $\rho_{\mathfrak{B}}(f)$ is open, consider the function $F := \lambda \mapsto f - \lambda$. The statement $f - \lambda$ is invertible is equivalent to $F(\lambda) \in \mathcal{G}$. Noting now that $\mathcal{G}$ is open and that F is continuous we see that the resolvent set is open.

For the bound on $\sigma_{\mathfrak{B}}(f)$ we take $|\lambda| > 0$. We write

$$\lambda - f = \lambda(1-h); \;\; h := f/\lambda$$

and note that, if $|\lambda| > \|f\|$, then $\|h\| < 1$ and $(1-h)$ is invertible. $\square$

Definition 1.4.3. Let $F : \mathbb{C} \to \mathfrak{B}$ where $\mathfrak{B}$ is a Banach algebra. We say that F is differentiable at $\lambda_0 \in \mathbb{C}$ if there is an element of $\mathfrak{B}$, denoted by $F'(\lambda_0)$, such that

$$\lim_{\lambda \to \lambda_0} \left[(\lambda - \lambda_0)^{-1} (F(\lambda) - F(\lambda_0)) \right] = F'(\lambda_0)$$

where the limit is understood as a norm limit. In the same setting, F is analytic in a domain $\mathcal{D} \subset \mathbb{C}$ if it is differentiable at any λ_0 in $\mathcal{D}$.

Exercise 6.

1. Prove the following extension of Liouville's theorem: If $F : \mathbb{C} \to \mathfrak{B}$ is analytic in $\mathbb{C}$ (entire) and bounded, then it is a constant. If additionally $\lim_{|\lambda| \to \infty} F(\lambda) = 0$ (norm-limit), then F is the zero function.

2. Let f be an element of the Banach algebra $\mathfrak{B}$. Use the first part of this exercise to show that the assumption $f - \lambda$ invertible for any λ (that is $\sigma_{\mathfrak{B}}(f) = \emptyset$) leads to a contradiction.

Definition 1.4.4. Let $\mathfrak{A}$ be an algebra with unit. Then $\mathfrak{A}$ is said to be a **division algebra** it zero is the only noninvertible element.

Theorem 1.4.5 (The Gelfand-Mazur Theorem)**.** Let $\mathfrak{B}$ be a Banach algebra. If $\mathfrak{B}$ is also a division algebra then $\mathfrak{B}$ is isometrically isomorphic to $\mathbb{C}$ [4].

Proof. Let $\mathcal{S} = \mathfrak{B} \setminus \mathcal{G}$ be the set of noninvertible (singular) elements of $\mathfrak{B}$, let $f \in \mathfrak{B}$ and $\lambda_f \in \sigma(f)$ ($\sigma(f) \neq \emptyset$ by the exercise). Then $f - \lambda_f 1$ is noninvertible, hence it is zero, implying $f = \lambda_f 1$.

Note next that for any $f \in \mathfrak{B}$, $\sigma(f) = \{\lambda_f\}$. Indeed, given $\lambda \in \mathbb{C}$, if $\lambda \neq \lambda_f$, then $f - \lambda 1 = (\lambda_f - \lambda)1$ which is invertible, hence $\lambda \notin \sigma(f)$.

Let $\psi : \mathfrak{B} \to \mathbb{C}$ be given by $\psi(f) = \lambda_f$, which is well defined by the above. Now: 1. $f + g = \lambda_f 1 + \lambda_g 1 = (\lambda_f + \lambda_g)1$; 2. $fg = (\lambda_f 1)(\lambda_g)1 = \lambda_f \lambda_g 1$, and 3. $f \neq 0 \Rightarrow \psi(f) \neq 0$Thus, for any $\lambda \in \mathbb{C}$ we have $\lambda = \lambda_f$ if $f = \lambda 1$ we check that φ is an algebra isomorphism. It is also isometric since $\|\psi(f)\| = \|\lambda_f 1\| = |\lambda_f| \|1\| = |\lambda_f|$. ☐

Definition 1.4.6 (Reminder: ideals)**.** Let $(R, +, \cdot)$ be a ring. A **left ideal** of R, $I \subset R$ is an additive subgroup R s.t $\forall (r, j) \in R \times I$, $rj \in I$. A **right ideal** is defined similarly, with jr replacing rj. A **two-sided ideal** (sometimes simply called ideal) is a left ideal that is also a right ideal. The whole ring is of course an ideal; it is called the **unit ideal**. Any (left, right or two-sided) ideal that is not the unit ideal is called a **proper** ideal. A proper ideal I is called a **maximal ideal** if the only ideal strictly containing I is the unit ideal. The **left ideal generated by** $S \subset R$, denoted RS is the smallest left ideal containing R, and it is given by

$$RS = \left\{ \sum_{i=1}^{n} r_i s_i : n \in \mathbb{N}, r_i \in R, s_i \in S \right\}$$

Similarly for the **right ideal generated by** $S \subset R$, SR. The **two-sided ideal generated by** $S \subset R$, denoted RSR is the smallest ideal containing R, and it is given by

$$RSR = \left\{ \sum_{i=1}^{n} r_i s_i r_i' : n \in \mathbb{N}, r_i, r_i' \in R, s_i \in S \right\}$$

[4] The isometry is unique since there are no nontrivial automorphisms of $\mathbb{C}$ as a Banach algebra over $\mathbb{C}$.

Note 1.4.7 (Ideals and congruence relations)**.** Ideals of R are in a bijection with congruence relations on R: Given an ideal I of R, let $f \sim g$ if $f - g \in I$. Then $\sim$ is a congruence relation on R. Conversely, given a congruence relation $\sim$ on R, let $I = \{f : f \sim 0\}$. Then I is an ideal of R.

Given an ideal of R, one forms the **quotient ring** (check it is a ring) R/I which is the set of all equivalence classes $[f]$ where $[f] = [g]$ iff $f - g \in I$.

Note 1.4.8. Recall that if $\mathcal{B}$ is a Banach space and $C \subset \mathcal{B}$ is a closed subspace, then the set of equivalence classes $[f]$ mod C form a Banach space with the norm

$$\|[f]\| = \inf_{h\in[f]} \|h\| = \inf_{c\in C} \|f + c\|$$

A linear map L between two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ is a contraction if $\|Lf\|_2 \leqslant \|f\|_1$

Proposition 1.4.9. Let $\mathfrak{B}$ be a Banach algebra and $\mathfrak{M}$ a closed two-sided proper ideal of $\mathfrak{B}$. Then $\mathfrak{B}/\mathfrak{M}$ is a Banach algebra, and the map $f \to [f]$ is a contraction.

Proof. Based on the discussions above, $\mathfrak{B}/\mathfrak{M}$ is an algebra and a Banach space. We only need to check that the norm induced on $\mathfrak{B}/\mathfrak{M}$ is a Banach algebra norm. First, $\|[1]\| \leqslant 1$. But we had $h \in \mathfrak{M}$ s.t. $\|1 - h\| < 1$ then h would be invertible, implying $hh^{-1} \in \mathfrak{M}$ and $\mathfrak{M} = \mathfrak{B}$, a contradiction.

Finally, let $f, g \in \mathfrak{B}$. Since for any $h, h_1 \in \mathfrak{M}$ we have $fh_1 + gh + hh_1 \in \mathfrak{M}$, we have

$$\begin{aligned}\|[fg]\| = \inf_{h\in\mathfrak{M}} \|fg + h\| &\leqslant \inf_{h,h_1\in\mathfrak{M}} \|fg + fh_1 + gh + hh_1\| = \inf_{h,h_1\in\mathfrak{M}} \|(f + h)(g + h_1)\| \\ &\leqslant \inf_{h,h_1\in\mathfrak{M}} \|(f + h)\|\|(g + h_1)\| = \inf_{h\in\mathfrak{M}} \|(f + h)\| \inf_{h_1\in\mathfrak{M}} \|(g + h_1)\| = \|[f]\|\|[g]\| \quad (19)\end{aligned}$$

Clearly, $f \to [f]$ is a contraction. ☐

The next result establishes the important correspondence between multiplicative functionals and maximal ideals, in a **commutative** Banach algebra.

Proposition 1.4.10. In a **commutative** Banach algebra $\mathfrak{B}$ there is a (natural) bijection between $M_{\mathfrak{B}}$ and the set of maximal ideals[5] on $\mathfrak{B}$. These maximal ideals are closed.

Proof. Let $\varphi \in M_{\mathfrak{B}}$ and $\mathfrak{K}$ be its kernel. You can easily check that $\mathfrak{K}$ is an ideal, and that it is closed. To check that it is maximal, we let $f \notin \mathfrak{K}$ and claim that the linear span of $(\mathfrak{K}, f)$ is $\mathfrak{B}$. By assumption, $\varphi(f) \neq 0$. We see that $(1 - f/\varphi(f)) \in \mathfrak{K}$. Since

$$1 = (1 - f/\varphi(f)) + f/\varphi(f)$$

we see that 1 is in the linear span of $(\mathfrak{K}, f)$, and any ideal containing $\mathfrak{K} \cup \{f\}$ contains 1, hence it equals $\mathfrak{B}$.

In the opposite direction, take $\mathfrak{M}$ a maximal ideal in $\mathfrak{B}$. We first check that $\mathfrak{M} = \overline{\mathfrak{M}}$. Clearly, $\overline{\mathfrak{M}}$ is an ideal. We show that it is proper, for this clearly implies $\overline{\mathfrak{M}} = \mathfrak{M}$.

We must have $\mathfrak{M} \cap \mathcal{G} = \emptyset$ (otherwise $1 \in \mathfrak{M}$ entailing $\mathfrak{M} = \mathfrak{B}$). But this forces $\|1 - f\| \geqslant 1$ and hence $1 \notin \overline{\mathfrak{M}}$, and $\overline{\mathfrak{M}}$ is a proper ideal as claimed.

With this information, we know that $\mathfrak{C} = \mathfrak{B}/\mathfrak{M}$ is a (commutative, certainly) Banach algebra. Since $\mathfrak{M}$ is maximal, $\mathfrak{C}$ is a division algebra [6]. Hence, by the theorem of Gelfand-Mazur, there is an isomorphic isometry $\mathcal{I}$ from $\mathfrak{C}$ onto $\mathbb{C}$. Let π be the natural homomorphism from $\mathfrak{B}$ onto $\mathfrak{C}$. Hence, $\mathcal{I} \circ \pi : \mathfrak{B} \to \mathbb{C}$ is a multiplicative functional on $\mathfrak{B}$.

It remains to check that the correspondence $\varphi \leftrightarrow \ker\varphi$ is one-to one, which follows if distinct multiplicative functionals have distinct kernels. So, assume $\ker \varphi_1 = \ker \varphi_2 = \mathfrak{M}$, a maximal ideal as we have seen. Note that for any $f \in \mathfrak{B}$, $\varphi_1(f) - \varphi_2(f)$ is in $\mathfrak{M}$ since it equals $(f - \varphi_1(f)) - (f - \varphi_2(f))$ and is a multiple of 1, so it must be zero.

□

Note 1.4.11. In the commutative case, we will use the notation M or $M_{\mathfrak{B}}$ for either multiplicative functionals or the maximal ideal space.

In fact the commutative case is summarized in the theorem below.

Theorem 1.4.12 (Gelfand). Let $\mathfrak{B}$ be a commutative Banach algebra, M is its maximal ideal space, and $\Gamma : \mathfrak{B} \to C(M)$ be the Gelfand transform. Then,

1. M is nonempty.
2. Γ is an algebra homomorphism.
3. Γ is a contraction: $\|\Gamma f\|_\infty \leqslant \|f\|$ for all f in $\mathfrak{B}$
4. f is invertible in $\mathfrak{B}$ iff Γf is invertible in $C(M)$.

Proof. 1. follows from the maximal ideal theorem and the fact that 0 is always a proper ideal in a Banach algebra.

[5] Automatically two-sided.
[6] And here is the reason, in white ink:

2,.3. were proved already.

4. In one direction, it is clear: if f^{-1} exists, then $\Gamma(f^{-1})$ is the inverse of $\Gamma(f)$. Hence, assume f is not invertible. Then, $f\mathfrak{B}$ is a proper ideal, since $1 \notin f\mathfrak{B}$, and hence it is contained in some maximal ideal $\mathfrak{M}$. Let φ be the linear multiplicative functional whose kernel is $\mathfrak{M}$, cf. Proposition 1.4.10. But then $\Gamma(f)$ noninvertible in $C(M)$ ($f \in$ker(φ), meaning $\varphi(f) = 0$ and hence $\Gamma(f)(\varphi) = 0$). □

Note 1.4.13. 1. In the commutative case, we will use the notation M or $M_{\mathfrak{B}}$ for either multiplicative functionals or the maximal ideal space.

2. In general, the correspondence between $\mathfrak{B}$ and $C(M)$ may not be one-to-one, not even when $\mathfrak{B}$ is commutative.

Corollary 1.4.14. Let $\mathfrak{B}$ be a commutative Banach algebra. Then the spectrum $\sigma(f)$ of $f \in \mathfrak{B}$ equals the range of Γf, implying also $r(f) = \|\Gamma f\|_\infty$

Proof. This is straightforward since $f - \lambda$ is not invertible in $\mathfrak{B}$ iff $\Gamma f - \lambda$ is not invertible in $C(M)$ iff $\lambda \in \text{ran}\Gamma f$. □

Exercise 7. Check that the closed subalgebra $\mathfrak{B}_f$ generated by $\{1, f, (f-\lambda)^{-1} : \lambda \in \rho(f)\}$ is commutative and that $\rho_{\mathfrak{B}_f}(f) = \rho_{\mathfrak{B}}(f)$.

Corollary 1.4.15. Let $\mathfrak{B}$ be a Banach algebra and $f \in \mathfrak{B}$. Assume $F : \mathcal{D} \to \mathbb{C}$ is analytic and the domain $\mathcal{D}$ contains $\mathbb{D}_{\|f\|}(0)$. Then,

$$\sigma(F(f)) = F(\sigma(f)) = \{F(\lambda) : \lambda \in \sigma(f)\}$$

Proof. Recall that $F(f)$ is defined as the norm-convergent series $\sum_{k=0}^{\infty} c_k f^k$ where $F(z) = \sum_{k=0}^{\infty} c_k z^k$ for $z \in \mathbb{D}_{\|f\|}(0)$. This implies that $F(f) \in \mathfrak{B}_f$ defined in Exercise 7, and, in proving this corollary, we can assume without loss of generality that $\mathfrak{B}$ is commutative. From this point on, the proof is straightforward. We note first that, by continuity, $\Gamma(F(f)) = F(\Gamma(f))$ (where is the right side defined?) and thus, by the previous corollary,

$$\sigma(F(f)) = \text{ran}[\Gamma(F(f))] = \text{ran}[F(\Gamma(f))] = F(\text{ran}(\Gamma f)) = F(\sigma(f))$$

□

Theorem 1.4.16 (Beurling-Gelfand, spectral radius theorem)**.** Let f be an element of a Banach algebra $\mathfrak{B}$. Then $r_{\mathfrak{B}}(f) = \lim_{n\to\infty} \|f^n\|^{1/n}$.

Proof. As before we can assume without loss of generality that the Banach algebra is commutative, by replacing $\mathfrak{B}$ with the algebra generated by $\{1, f, (\lambda - f^n)^{-1} : \lambda \in \rho_{\mathfrak{B}}(f^n), n \in \mathbb{N}\}$ (check the details here!).

In one direction, by Corollary 1.4.15 $\sigma(f^n) = \sigma(f)^n$, hence $r(f)^n = r(f^n) \leqslant \|f^n\|$, implying

$$r(f) \leqslant \liminf_{n\to\infty} \|f^n\|^{1/n} \tag{20}$$

In the opposite direction, the series

$$\sum_{n=0}^{\infty} \lambda^{-n} f^n$$

is norm-convergent to $(1 - \lambda^{-1} f)^{-1}$ if $\limsup_{n\to\infty} \|f^n\|^{1/n} < |\lambda|$ and thus

$$r(f) \geqslant \limsup_{n\to\infty} \|f^n\|^{1/n} \tag{21}$$

The existence of the limit and the result now follow from (20) and (21). ☐

Here is an important consequence of these results.

Corollary 1.4.17. Let $\mathfrak{B}$ be a commutative Banach algebra. Then the Gelfand transform Γ is an isometry iff $(\forall f \in \mathfrak{B})(\|f^2\| = \|f\|^2)$.

Proof. By Corollary 1.4.14 we have $r(f) = \|\Gamma f\|_\infty$. On the other hand, $r(f) = \lim_{n\to\infty} \|f^n\|^{1/n} = \|f\|$ (because: white ink) ☐

Exercise 8. Check that the linear operator from $L^1([-1,1])$ to $L^1([-1,1])$ given by

$$Bf = \int_0^x f(s)ds$$

is bounded. Characterize the range of B, i.e., the function space $B[L^1([-1,1])]$.

Let $\mathfrak{B}$ be the Banach algebra of the bounded linear maps from $L^1([-1,1])$ to $L^1([-1,1])$. Find $\sigma_{\mathfrak{B}}(B)$. For each $\lambda \in \sigma_{\mathfrak{B}}(B)$ use your analysis knowledge to explain why it belongs to $\sigma_{\mathfrak{B}}(B)$.

Due: Jan 29.

Here, as a reminder, is the relevant formulation for us of Stone-Weierstrass:

Theorem 1.4.18. Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed unital self-adjoint algebra of $C(X)$ which separates the points of X, then $\mathcal{A} = C(X)$.

Now we will analyze the most general closed unital self-adjoint algebra of $C(X)$, and show that they can still be identified with $C(Y)$ for some compact Hausdorff space. In fact, any such $\mathcal{A}$ is the collection of functions in $C(X)$ which are constant on some partition of X.

Definition 1.4.19. In the following, X a compact Hausdorff space, $\mathcal{A}$ is a closed unital subalgebra of $C(X)$, $\varphi_x \subset M_{\mathcal{A}}$ is the evaluation functional in $M_{\mathcal{A}}$ defined by $\varphi_x(f) = f(x)$, $\eta = x \to \varphi_x$ is the map that associates to any point in X the evaluation functional φ_x.

Proposition 1.4.20. The map $\eta = x \mapsto \varphi_x$ is continuous from X to $M_{\mathcal{A}}$, with the w-* topology. (Self-adjointness of $\mathcal{A}$ is not required here)

Proof. Similar to a proof we have seen before. Let $\{x_\alpha\}_{\alpha \in A}$ be a net in X converging to x. The result follows from continuity: take any $f \in \mathcal{A}$ and note that

$$\lim_\alpha \eta(x_\alpha)(f) = \lim_\alpha \varphi_{x_\alpha}(f) = \lim_\alpha f(x_\alpha) = f(x) = \varphi_x(f) = \eta(x)(f)$$

□

Proposition 1.4.21. If $\mathcal{A}$ is self-adjoint, then the function η maps X onto $M_{\mathcal{A}}$.

Proof. In other words, any $\varphi \in M_{\mathcal{A}}$ is an evaluation functional $\varphi_x, x \in X$, for some (non-unique, perhaps) $x \in X$, and this is what we will show. In the proof, try to recognize any strategies and objects we have used before.

Let $\varphi \in M_{\mathcal{A}}$, $f \in \mathcal{A}$ and define K_f to be the set of points where $\varphi(f) = f(x)$,

$$K_f = \{x \in X : \varphi(f) = f(x)\}$$

These are clearly compact, and not only they are nonempty, but any finite collection of them has a nonempty intersection. For, assuming to get a contradiction, that $f_1, ..., f_n$ is a finite collection in $\mathcal{A}$ s.t.

$$\bigcap_{j=1}^n K_{f_j} = \emptyset$$

we can define an everywhere positive function h by (we use self-adjointness of $\mathfrak{A}$ here)

$$h = \sum_{j=1}^n |f_j - \varphi(f_j)|^2$$

Clearly, $\varphi(h) = 0$. Positivity entails $\min_{x\in X} h(x)/\|h\| = \varepsilon > 0$; thus h is invertible in $\mathcal{A}$ since $0 \leqslant 1 - h/\|h\| \leqslant 1 - \varepsilon$, hence $\|1 - h/\|h\|\| < 1$. This in turn implies that $\varphi(h) \neq 0$, a contradiction.

Finally, now using the finite intersection property, any

$$x \in \bigcap_{f\in\mathcal{A}} K_f$$

has the property $\varphi = \eta(x)$ completing the proof. □

Proposition 1.4.22. If $\mathcal{A}$ is self-adjoint, then the Gelfand transform Γ is an isometric isomorphism between $\mathcal{A}$ and $C(M_{\mathcal{A}})$.

Proof. Let $f \in \mathcal{A}$. For the isometry: by continuity, there is an x_0 s.t. $\|f\|_\infty = |f(x_0)|$. Thus,

$$\|f\|_\infty \geqslant \|\Gamma f\|_\infty = \sup_{\varphi\in M_{\mathcal{A}}} |(\Gamma f)(\varphi)| \geqslant |(\Gamma f)(\eta(x_0))| = |f(x_0)| = \|f\|_\infty$$

It remains to prove that Γ is onto, since we know already that Γ is an algebra homomorphism. For this, we check the conditions of the Stone-Weierstrass theorem for the algebra $\mathfrak{A} := \Gamma(\mathcal{A}) \subset C(M_{\mathcal{A}})$.

Clearly, $1 \in \mathfrak{A}$ since $\Gamma 1 = 1$, $\mathfrak{A}$ is closed in the sup norm since Γ is an isometry, and it separates points[7].

Now, if u is real-valued, then $\forall\lambda \notin \mathbb{R}$, $u - \lambda$ is invertible, which means $\Gamma u - \lambda$ is invertible, hence Γu is real valued too. This implies, by linearity, $\Gamma(\overline{f}) = \overline{\Gamma f}$. Thus, since $\mathcal{A}$ is self-adjoint, $\mathfrak{A}$ is self-adjoint. Now, by Stone-Weierstrass, $\mathfrak{A} = C(M_{\mathcal{A}})$. □

We now prove an elementary theorem about compositions of continuous maps. Let X and Y be compact Hausdorff spaces and $\theta \in C(X,Y)$ be surjective.

Proposition 1.4.23. The linear map $\theta^* : C(Y) \to C(X)$ defined by $\theta^* f = f \circ \theta$ is an isometric isomorphism onto the subalgebra of $C(X)$ consisting of functions that are constant on each compact $K_y = \theta^{-1}(y)$.

Proof. The fact that θ is onto immediately shows that $\|f\|_Y = \|f \circ \theta\|_X$. It remains to show that we can write any g which is constant on each K_y in the form $f \circ \theta$ for some **continuous** f. Choose such a g and consider the **set** function $h = g \circ \theta^{-1} : \mathcal{P}(Y) \to \mathcal{P}(\mathbb{C})$ [8] and note that $h(\{y\}) = \{c\}$ for some $c \in \mathbb{C}$. Thus $f := \{(y,c) : c \in h(\{y\}, y \in Y)\}$ is a function, $g = f \circ \theta$ and we only have to check f is continuous. Let $C \subset \mathbb{C}$ be closed, and note that $f^{-1}(C) = (g \circ \theta^{-1})^{-1}(C) = \theta(g^{-1}(C)$ is compact because $g^{-1}(C)$ is compact and θ is continuous. □

[7] If $\varphi_1 \neq \varphi_2$ are in $C(M_{\mathfrak{A}})$, by definition there must be af $f \in \mathcal{A}$ s.t. $\varphi_1(f) \neq \varphi_2(f)$.

[8] Though this is not used here, it is worth remembering that, in general, surjective functions have right inverses.

Proposition 1.4.24. Let $\mathfrak{A}$ be a closed unital self-adjoint subalgebra of $C(X)$ and η as in Definition 1.4.19. Then η^* defined in Proposition 1.4.21 is an isometric isomorphism of $C(M_{\mathfrak{A}})$ onto $\mathfrak{A}$ and a left inverse of the Gelfand transform, $\eta^* \circ \Gamma = I$.

Proof. For the left inverse property note that $((\eta^* \circ \Gamma)(f))(x) = (\Gamma f)(\eta_x) = f(x)$. Now Γ maps $\mathfrak{A}$ onto $C(M_{\mathfrak{A}})$ and Proposition 1.4.23 shows that η^* maps $C(M_{\mathfrak{A}})$ onto $\mathfrak{A}$. □

Definition 1.4.25. Let X be any nonempty set and $\mathfrak{A}$ a set of functions on X. The relation: $x_1 \sim x_2$ iff $(\forall f \in \mathfrak{A})(f(x_1) = f(x_2))$ is an equivalence relation on X. Then $\sim$ induces a partition on X into sets on which each function in $\mathfrak{A}$ is constant. We denote this partition by $\Pi_{\mathfrak{A}}$.

Combining the results in the previous Propositions we arrive at the following classification result.

Theorem 1.4.26. If X is a compact Hausdorff space and $\mathfrak{A}$ is a closed, self-adjoint, unital subalgebra of $C(X)$, then $\mathfrak{A}$ consists of the functions in $C(X)$ which are constant on $\Pi_{\mathfrak{A}}$.

. □

Note 1.4.27. Of course, if $\mathfrak{A}$ separates points, then the partition consists of singleton sets, and $\mathfrak{A} = C(X)$.

Exercise 9. Organize $\Pi_{\mathfrak{A}}$ in Theorem 1.4.26 as a compact Hausdorff space $X_\sim$ so that $\mathfrak{A}$ is isometrically isomorphic to $C(X_\sim)$ and discuss the relation between the maximal ideals of $\mathfrak{A}$ and your $X_\sim$.

Due: Feb 5.

1.5 Examples

1.5.1 Trigonometric polynomials

In the following $\mathbb{T}$ is the circle group $\{z : |z| = 1\}, \chi_n = z^n = e^{in\varphi}$ Note that $\overline{\chi_n} = \chi_{-n}$.

Definition 1.5.1. The set of trigonometric polynomials is defined as

$$\mathcal{P}_+ = \left\{ \sum_{n=-N}^{N} c_n \chi_n : N \in \mathbb{N}, \{c_j\}_{|j|\leqslant N} \subset \mathbb{C} \right\} \tag{22}$$

The set of **analytic** trigonometric polynomials is defined as

$$\mathcal{P} = \left\{ \sum_{n=0}^{N} c_n \chi_n : N \in \mathbb{N}, \{c_j\}_{|j|\leqslant N} \subset \mathbb{C} \right\} \tag{23}$$

It is straightforward to characterize the Banach algebra $\mathcal{A}$ which is the closure of $\mathcal{P}$ in the sup norm. Indeed $\mathcal{P}$ separates points, contains the identity and is selfadjoint. By Stone-Weierstrass, the closure of $\mathcal{P}$ is $C(\mathbb{T})$. Hence, the maximal ideal space is $\mathbb{T}$ itself.

The Banach algebra $\mathcal{A}$ generated by $\mathcal{P}_+$ is the closure in sup norm, $\overline{\mathcal{P}_+}$. Since the unital algebra $\mathcal{P}_+$ is not self-adjoint, $\mathcal{A} = \overline{\mathcal{P}_+}$, and its maximal ideals, are more interesting.

What is $\mathcal{A}$? Of course, polynomials are analytic in $\mathbb{C}$ (they are entire functions). By the maximum principle we see that convergence in the sup norm of a sequence of analytic functions on the boundary of a domain $\mathcal{D} \subset \mathbb{C}$ implies uniform convergence in the interior of $\mathcal{D}$ as well, therefore to a function analytic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$. In the opposite direction when this domain is $\mathbb{D}$ (and much more generally[9]), any analytic function continuous up to $\mathbb{T}$ is the uniform limit of polynomials, namely of their truncated Maclaurin series. Thus $\overline{\mathcal{P}_+}$ is the **Banach algebra $\mathcal{A}$ of boundary values of functions which are analytic in $\mathbb{D}$ and continuous up to $\mathbb{T}$**. All of these are given by one-sided Fourier series.

What is $M_{\mathcal{A}}$? Clearly, for every $z \in \overline{\mathbb{D}}$, the evaluation functional φ_z is linear, multiplicative and, by the arguments above, continuous. Conversely, let $\varphi \in M_{\mathcal{A}}$. We have $z = \varphi(\chi_1) \in \mathbb{C}$ by definition, and since $\|\chi_1\| = 1$ and (by Proposition 1.3.2) $\|\varphi\| = 1$ we see that $z \in \overline{\mathbb{D}}$. Now, for any polynomial $P = \sum_{k=0}^{N} a_k \chi_k = \sum_{k=0}^{N} a_k \chi_1^k$ we have $\varphi(P) = P(z) = \varphi_z(P)$ because of the simple calculation

$$\varphi(P) = \sum_{k=0}^{N} a_k [\varphi(\chi_1)]^k = P(z) = \varphi_z(P)$$

By continuity and the density of $\mathcal{P}_+$ it follows that $\varphi = \varphi_z$ on $\mathcal{A}$. It is also clear that η is a continuous bijection between $M_{\mathcal{A}}$ and $\overline{\mathbb{D}}$ and in this sense we can identify $M_{\mathcal{A}}$ with $\overline{\mathbb{D}}$.

In this example of a function algebra, $M_{\mathcal{A}}$ is the natural domain of existence of the functions in $M_{\mathcal{A}}$: continuous boundary values of analytic functions in the disk evidently extend analytically to the disk. Note also that a function in $\mathcal{A}$ is invertible iff it does not vanish in $\overline{\mathbb{D}}$, another way to view that $\mathbb{T}$ was "too small". Going back to general trig polynomials, which generate the full $C(\mathbb{T})$, invertibility of a function simply means it does not vanish in $\mathbb{T}$, and this suggests that there is no further meaningful, nontrivial extension operator of a function in $C(\mathbb{T})$ to a larger set. (Think of the possible natural definitions of such an extension operator, and why we are making this assertion.) When we get to study operators, will see that any "extrapolation" operator densely defined on polynomials is "maximally pathological": not only are these unbounded, but they are not even closable.

[9] Look up Mergelyan's theorem.

1.6 The Shilov boundary theorem

Theorem 1.6.1. Let $\mathcal{B}$ be a Banach algebra and $\mathcal{A}$ a closed subalgebra of $\mathcal{B}$. Then, for any $f \in \mathcal{A}$, $\partial\sigma_{\mathcal{A}}(f) \subset \partial\sigma_{\mathcal{B}}(f)$.

Proof. First, it is clear that $\sigma_{\mathcal{A}}(f) \supset \sigma_{\mathcal{B}}(f)$, since invertibility of $f - \lambda$ relative to $\mathcal{A}$ implies invertibility in $\mathcal{B}$. Let $\lambda_0 \in \partial\sigma_{\mathcal{A}}(f)$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence in $\rho_{\mathcal{A}}(f)$ converging to λ_0.

By the second resolvent formula, in any Banach algebra if B is an invertible element and $(1 + (B - A)B^{-1})$ is invertible, then A is an invertible element (see details in Corollary 1.0.7). With $B = f - \lambda_n$ and $A = f - \lambda_0$ which is not (as elements of $\mathcal{A}$), $\|B - A\| = |\lambda - \lambda_0|$. Then, it must be that

$$|\lambda_0 - \lambda_n|\|(f - \lambda_n)^{-1}\| \geqslant 1$$

Hence $\|(f - \lambda_n)^{-1}\| \geqslant |\lambda_0 - \lambda_n|^{-1} \to \infty$ as $n \to \infty$. However, all these λ_n are in $\rho_{\mathcal{B}}(f)$ and hence the limit, λ_0 is in $\overline{\rho_{\mathcal{B}}(f)}$. Hence, it suffices to show that $\lambda_0 \in \sigma_{\mathcal{B}}(f)$. But otherwise $\lambda_0 \in (\rho_{\mathcal{B}}(f))^{\circ}$ which would imply $\|f - \lambda\|$ bounded in some neighborhood of λ_0 which contains all λ_n for large n, a contradiction. □

Look up the notion of Shilov boundary, for instance in Exercises 2.26–2.29 in Douglas. This is an important concept which can be seen as a wide generalization of the maximum principle.

1.7 Further examples

1.7.1 The convolution algebra $\ell^1(\mathbb{Z})$

Discrete convolution is defined as usual as

$$f * g = \sum_{-\infty}^{\infty} f(\cdot - k)g(k), \quad (f, g \in \ell^1(\mathbb{Z}))$$

You know from real analysis that $*$ is commutative and associative, and $\|f * g\|_1 \leqslant \|f\|_1\|g\|_1$, hence $(\ell^1(\mathbb{Z}), +, *)$ is a Banach algebra, and it is unital with $1 = e_0 = \delta_{\cdot,0}$, the Kronecker symbol. In fact, the Fourier transform is an isomorphism of $(\ell^1(\mathbb{Z}), +, *)$ with a Banach algebra $\mathcal{A} \subset C(\mathbb{T})$, namely, using the notations introduced for trig polynomials,

$$\mathcal{A} = \{f \in C(\mathbb{T}) : f = \sum_{n\in\mathbb{Z}} c_n\chi_n, \|f\| = \sum_{n\in\mathbb{Z}} |c_n| < \infty\}$$

We can of course use this isomorphism to carry over any part of the analysis to $(\mathcal{A}, \|\cdot\|, +, \cdot)$.

Note 1.7.1. Convergence in $\|\cdot\|$ implies convergence in the sup norm since $\|f\|_\infty \leqslant \|f\|$.

Theorem 1.7.2. M_A consists of evaluation functionals and ψ is a surjective homeomorphism from $\mathbb{T}$ to $M_{\ell^1(\mathbb{Z})}$.

Thus we can identify $M_{\ell^1(\mathbb{Z})}$ with $\mathbb{T}$.

Proof. Injectivity is immediate. Next, let $\varphi \in M_{\ell^1(\mathbb{Z})}$, and set $z = \varphi(e_1)$. Then,

$$1 = \|e_1\| \geqslant |\varphi(e_1)| = |z| = \frac{1}{|z^{-1}|} = \frac{1}{|\varphi(e_{-1})|} \geqslant \frac{1}{\|e_{-1}\|} = 1$$

and thus $z \in \mathbb{T}$. Then, for P a trig polynomial, it follows, as before, that $\varphi(P) = P(z)$. By Note 1.7.1 $\varphi(f) = f(z)$ for all $f \in \mathcal{A}$. $\square$

Proposition 1.7.3. The Gelfand transform Γ is not surjective on $C(\mathbb{T})$, and is not an isometry on its image $\Gamma(A)$.

Proof. We know from real analysis that the set of continuous functions for which the Fourier series converges at any given point is meager[10], see my real analysis course notes and also Folland, Exercise 35 p. 269. , whereas any function in $\mathcal{A}$ has uniformly and absolutely convergent Fourier series. The discrepancy is due to the strictly stronger norm on $\mathcal{A}$. Since $\mathcal{A}$ contains all trig polynomials, $\mathcal{A}$ is dense in $C(\mathbb{T})$, so Γ cannot be an isometry even on its image, $\mathcal{A} \subset C(\mathbb{T})$, where $C(\mathbb{T})$ is understood of course with the sup norm.

$\square$

We note that the algebra $\mathcal{A}$ has an involution ($f^* = \overline{f}$, complex conjugation) but it is not a C*-algebra (that we'll study in the sequel) with respect to it since the norm is not compatible with this involution (it fails the condition $\|ff^*\| = \|f\|^2$. It will turn out that in commutative C*-algebras Γ is always an isometric isomorphism.

It was an important question in analysis whether for any function f with absolutely convergent Fourier series $1/f$, when continuous, also has an absolutely convergent Fourier series. The result was quite nontrivial and was first proved by Wiener, in a pretty involved and delicate way. We can state and prove this result in a straightforward way using the tools of the Gelfand transform.

Theorem 1.7.4. Let $\varphi \in \mathcal{A}$ and assume $\varphi \neq 0$ on $\mathbb{T}$. Then $\dfrac{1}{\varphi} \in \mathcal{A}$.

Proof. This follows immediately from the fact that φ is invertible in $\mathcal{A}$ iff $\Gamma\varphi$ (which can be identified with φ) is invertible in $C(\mathbb{T})$. $\square$

Check out Wiener's Tauberian theorems, e.g. on Wiki, for some of its cool applications.

[10]This is caused by the unboundedness of the Dirichlet kernel.

1.7.2 The return of Real Analysis: the case of L^∞

(Review as needed L^∞ and essup.) In this last example, the Gelfand transform is an isometric isomorphism onto $C(M)$, but M is a "monster" compact Hausdorff space of cardinality $2^{\mathfrak{c}}$ whose existence depends heavily on the axiom of choice. First we define the essential range of functions in L^∞ (see also Exercise 11 p. 187 of Folland).

Definition 1.7.5. Let f be measurable on a measure set (X, μ). The essential range of f is defined as

$$\mathcal{R}(f) = \{\lambda \in \mathbb{C} : (\forall \varepsilon > 0)(\mu(\{x \in X : |f(x) - \lambda| < \varepsilon\}) > 0\}$$

Proposition 1.7.6. For any $f \in L^\infty$, $\mathcal{R}(f)$ is a compact subset of $\mathbb{C}$ and

$$\|f\|_\infty = \sup\{|\lambda| : \lambda \in \mathcal{R}(f)\}$$

(values on the essup circle are essentially attained). Furthermore, $\sigma(f) = \mathcal{R}(f)$.

Proof. Assume $\lambda \notin \mathcal{R}(f)$. Then there is an $\varepsilon > 0$ s.t. $\mu(\{x \in X : |f(x) - \lambda| < \varepsilon\}) = 0$. The set of such λ is clearly open, hence their complement $\mathcal{R}(f)$ is closed. Note also that if $|\lambda| >$essup$(|f|)$, then $\lambda \notin \mathcal{R}(f)$, implying that $\mathcal{R}(f)$ is compact.

Let $\nu = \|f\|_\infty$ and assume that no λ on the circle $S = \partial\mathbb{D}_\nu(0)$ were in $\mathcal{R}(f)$. Each of these λ is contained in $[\mathcal{R}(f)]^c$ together with an open disk $\mathbb{D}_\varepsilon(\lambda), \varepsilon > 0$. Choose a finite set $\lambda_1, ..., \lambda_n$ so that $S \subset \cup_i \mathbb{D}_{\varepsilon_i}(\lambda_i)$. Then you can check $\cup_i \mathbb{D}_{\varepsilon_i}(\lambda_i)$ cover the set $\{\lambda : |\lambda| \in (\nu - \varepsilon', \nu + \varepsilon')$ for some $\varepsilon' > 0$. Hence $\|f\|_\infty \leqslant \nu - \varepsilon$, a contradiction.

For the last part, if $\lambda \notin \sigma(f)$, then $(f - \lambda)$ is invertible in L^∞, hence the set $\{\lambda : \|f - \lambda)\| < \varepsilon\}$ must have zero measure for some ε, hence $\lambda \in [\mathcal{R}(f)]^c$. The converse is proved similarly.

Theorem 1.7.7. Γ is an isometric isomorphism between L^∞ and $C(M_{L^\infty})$ which commutes with complex conjugation.

Proof. Proposition 1.7.6 and Corollary 1.4.14 imply that

$$\|\Gamma f\|_\infty = \|f\|_\infty$$

Hence $\Gamma(L^\infty)$ is a closed subalgebra of $C(M)$, and it is immediately checked that it separates points and contains 1. Since the essential range of a real-valued f is real, commutation with complex conjugation follows as in Proposition 1.4.22, and hence $\Gamma(L^\infty)$ is self-adjoint. By Stone-Weierstrass, $\Gamma(L^\infty) = C(M)$. ∎

Exercise 10. Solve problems 2.4, 2.6,2.11,2.26, 2.27 in Douglas. Due Feb. 12. Since these are "public" problems, I will just grade on completion. Make sure you look at all the problems in that section, you may find some/all of them quite interesting.

□

2 Bounded operators on Hilbert spaces

Please review Hilbert spaces as needed.

Operators on Hilbert spaces are simply linear maps between Hilbert spaces. Here we focus on operators from a Hilbert space $\mathcal{H}$ to itself, that is, elements of $L(H)$. This is most often all we need since typically one deals with Hilbert spaces with countable bases, and any two such spaces are isomorphic. In this section all operators are **bounded**, where the norm, or operator norm, of an operator B is defined as

$$\|B\| = \sup_{u,\|u\|=1} \|Bu\| = \sup_{x\neq 0} \frac{\|Bx\|}{\|x\|} \tag{24}$$

Later, when we introduce unbounded operators, we will see that these are only defined on a dense set in $\mathcal{H}$ and do not extend, not even linearly, to the whole of $\mathcal{H}$ [11].

2.1 Adjoints

Proposition 2.1.1. Let $T \in L(\mathcal{H})$. There exists a unique adjoint operator $T^* \in L(\mathcal{H})$ s.t. for any $(f,g) \in \mathcal{H}$ we have

$$(Tf,g) = (f,T^*g) \tag{25}$$

Proof. Fix $g \in \mathcal{H}$ and define the linear functional φ by $\varphi f = (Tf,g), f \in \mathcal{H}$. It is easy to check that φ is bounded, and then, by the Riesz representation theorem φ is given by an inner product, hence $(Tf,g) = (f,h)$ for a unique $h \in \mathcal{H}$, a function of g. We set $T^*g = h$. Linearity is checked immediately. Let's estimate the norm of T^*:

$$\|T^*g\|^2 = (T^*g,T^*g) = (TT^*g,g) \leqslant \|TT^*g\|\|g\| \leqslant \|T\|\|T^*g\|\|g\| \tag{26}$$

hence

$$\|T^*\| \leqslant \|T\| \tag{27}$$

For uniqueness, if we assume $(Tf,g) = (f,Ag) = (f,Bg)$ for $A,B \in L(\mathcal{H})$ and all $f,g \in \mathcal{H}$, it follows that $(f,Ag-Bg) = 0$ for all $f \in \mathcal{H}$ and hence $Ag = Bg$, and since g is arbitrary, then $A = B$. □

[11] Except in uninteresting, highly nonunique and pathological ways, using the axiom of choice.

Proposition 2.1.2 (Properties of the adjoint)**.** Let $\mathcal{H}$ be a Hilbert space and S, T any two bounded operators. Then

1. $(T^*)^* \equiv T^{**} = T$.
2. $\|T\| = \|T^*\|$
3. $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$, $\forall a, b \in \mathbb{C}$.
4. $(ST)^* = T^*S^*$
5. If T is invertible, then T^* is invertible, with inverse $(T^{-1})^*$.
6. $\|T^*T\| = \|T\|^2$

Proof. 1. This follows from this simple calculation:

$$(f, T^{**}g) = (T^*f, g) = \overline{(g, T^*f)} = \overline{(Tg, f)} = (f, Tg)$$

2. follows immediately from (27), and 3., 4. are straightforward.

5. This follows from

$$T^*(T^{-1})^* = (T^{-1}T)^* = I = (TT^{-1})^* = (T^{-1})^*T^*$$

6. For any unit vector, we have

$$(Tu, Tu) = (T^*Tu, u) \leqslant \|T^*Tu\| \leqslant \|T^*T\|$$

and taking the sup over u we get $\|T^*T\| \geqslant \|T\|^2$, while the opposite direction is immediate $\|T^*T\| \leqslant \|T^*\|\|T\| = \|T\|^2$.

∎

Note 2.1.3. 2. and 6. in Proposition 2.1.2 are essential when we establish that $L(\mathcal{H})$ is a C^* algebra.

Definition 2.1.4. Let $T \in L(\mathcal{H})$. Its range and kernel are defined as

$$\operatorname{ran} T = \{Tx : x \in \mathcal{H}\}; \quad \ker T = \{z \in \mathcal{H} : Tz = 0\}$$

The following result extends to unbounded operators where it will play a key role.

Proposition 2.1.5. Let $T \in L(\mathcal{H})$. Then $\ker T = (\operatorname{ran} T^*)^\perp$ and $\ker T^* = (\operatorname{ran} T)^\perp$.

Proof. Since $T^{**} = T$ it suffices to show the first equality. Let $z \in \ker T$. Then $(z, T^*y) = (Tz, y) = 0$ for all $y \in \mathcal{H}$ hence $z \perp \operatorname{ran} T^*$. The opposite inclusion follows from the same calculation, where z is now any element s.t. $z \perp \operatorname{ran} T^*$. □

Definition 2.1.6. An operator T is bounded below if

$$\inf_{u, \|u\|=1} \|Tu\| > 0 \quad \text{or, equivalently, } \exists \varepsilon > 0 \text{ s.t. } \|Tf\| \geqslant \varepsilon \|f\|$$

Proposition 2.1.7. $T \in L(\mathcal{H})$ is invertible iff it has dense range (equivalently, $\ker T^* = \{0\}$) and bounded below. The sup of the bounds below ε as in Definition 2.1.6 equals $1/\|T^{-1}\|$.

Note 2.1.8. By the open mapping theorem, if T is a bijection, then T^{-1} is also continuous, equivalently of finite norm, hence T is invertible.

Proof. If T is invertible, then clearly the range $(= \mathcal{H})$ is dense. We can calculate an $\varepsilon > 0$ as in Definition 2.1.6 as follows. For any $f \in \mathcal{H}$ we have

$$\|T^{-1}\|\|Tf\| \geqslant \|T^{-1}Tf\| = \|f\| \quad \text{hence } \|Tf\| \geqslant \varepsilon\|f\| \text{ if } \quad \varepsilon = 1/\|T^{-1}\|$$

In the opposite direction, assume T is bounded below and the range is dense. It follows that $\ker T = \{0\}$. Assume $y \in \mathcal{H}$, and let $\{Tx_n\}_{n\in\mathbb{N}}$ be in $\operatorname{ran} T$ and converge to y. But

$$\|T(x_n - x_m)\| \geqslant \varepsilon \|x_n - x_m\|$$

hence x_n converges to an $x \in \mathcal{H}$ and Tx_n converges to $Tx = y$, thus $y \in \operatorname{ran} T = \overline{\operatorname{ran} T} = \mathcal{H}$). We have $\|TT^{-1}u\| \geqslant \varepsilon\|T^{-1}u\|$ and taking the sup over unit vectors, we see that $\|T^{-1}\| \leqslant \varepsilon^{-1}$.
□

Corollary 2.1.9. $T \in L(\mathcal{H})$ is invertible if both T and T^* are bounded below.

Proof. If T is invertible, then T^* is invertible too, by Proposition 2.1.2, hence they are both bounded below.

In the other direction, if T, T^* are bounded below then $\ker T = \{0\}$, hence T is injective, and $(\operatorname{ran} T)^\perp = \ker T^* = \{0\}$ and $\operatorname{ran} T$ is dense, and Proposition 2.1.7 completes the proof. □

The simplest case of a spectral theorem is the diagonalization theorem for a normal matrix M (meaning, $MM^* = M^*M$). The algebra generated by a normal matrix is isomorphically isometric to that of continuous functions on its spectrum. (What does this even mean, since the spectrum is a finite set of points?). Another form of a spectral theorem is the existence of a unitary matrix U s.t. $U^*MU = D$ where D is a diagonal matrix. Equivalently, there exists a change of Hilbert basis s.t. the matrix becomes diagonal. A third one is the decomposition $M = \sum m_k P_k$ where P_k is an orthogonal projection (spectral projection) on the eigenspace of the eigenvalue m_k. They do follow from each other, but they are distinct enough to carry a special label, s.a. "spectral theorem, spectral projection version". Note that they do not hold unless indeed M is normal. Clearly, the situation cannot get better in infinite dimensions, but it does not get terribly worse either, properly interpreted. If you take the operator of multiplication by the variable $f(x) \mapsto xf(x)$ on $L^2[0,1]$, then it is false that there is a change of basis s.t. x becomes an infinitely dimensional diagonal matrix (why so?). But there is still a unitary transformation (the identity!) s.t. the operator becomes pointwise multiplication with some function (the identity function, here).

Definition 2.1.10. Let $T \in L(\mathcal{H})$. Then

1. T is normal if $TT^* = T^*T$.
2. T is unitary if $T^*T = TT^* = I$
3. T is self-adjoint if $T = T^*$.
4. The numerical range of T is the set $W(T) = \{(Tu, u) : \|u\| = 1\}$.
5. T is positive if $W(T) \subset [0, \infty)$.

Proposition 2.1.11. An operator $T \in L(\mathcal{H})$ is self-adjoint iff $W(T) \subset \mathbb{R}$.

Proof. Assume $W(T) \subset \mathbb{R}$. Here the *polarization identity*

$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x-iy\|^2 + i\|x+iy\|^2\right) \ \forall\, x,y \in \mathcal{H} \tag{28}$$

comes in handy. It can be checked by expanding out

$$(x,y) = \frac{1}{4}\left((x+y,x+y) - (x-y,x-y) + i(x+iy,x+iy) - i(x-iy,x-iy)\right) \tag{29}$$

with $x = Tf$ and $y = g$ you can check that, whenever $W(T) \subset \mathbb{R}$ we have

$$(Tf,g) = \overline{(Tg,f)} \ \ (\text{hence } (Tf,g) = (f,Tg))$$

implying self-adjointness. In the opposite direction we have

$$(Tf,f) = (f,Tf) = \overline{(Tf,f)}$$

□

Alternative proof. (credit: N. Bruno) Suppose that $W(T) \subset \mathbb{R}$. Note that this means for any $v \in \mathcal{H}$ that $(Tv, v) = (v, Tv)$. Let $f, g \in \mathcal{H}$. Then, we have that

$$(T(f+g), f+g) = (Tf, f) + (Tf, g) + (Tg, f) + (Tg, g) \tag{30}$$

which equals

$$(f+g, T(f+g)) = (f, Tf) + (f, Tg) + (g, Tf) + (g, Tg) \tag{31}$$

Therefore,

$$(Tf, g) + (Tg, f) = (f, Tg) + (g, Tf) \tag{32}$$

Rearranging we obtain that

$$\Im[(Tf, g)] = \Im[(g, Tf)] \tag{33}$$

By performing similar operations with $f + ig$, we obtain the real parts are also equal, so $(Tf, g) = (f, Tg)$ i.e. T is self-adjoint.

For the reverse direction we have,

$$(Tf, f) = (f, Tf) = \overline{(Tf, f)} \tag{34}$$

□

Proposition 2.1.12. For any $T \in L(\mathcal{H})$, the operator T^*T is positive.

Proof. This follows from a simple calculation: $(T^*Tf, f) = (Tf, Tf) = \|Tf\|^2 \geqslant 0$. □

Proposition 2.1.13. If T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$. If T is positive, then $\sigma(T) \subset [0, \infty)$.

Proof. By Corollary 2.1.9 it suffices to show that $T - z$ is bounded below if $z = a + ib \notin \mathbb{R}$. Assume otherwise. Then, $\inf\{|((T-z)u, u)| : \|u\| = 1\} = 0$. However,

$$((T-z)u, u) = (Tu, u) - (au, u) - ib(u, u)$$

hence $|((T-z)u, u)| \geqslant b$, a contradiction. The second part is proved similarly. □

Proposition 2.1.14. Let U be a unitary operator on a Hilbert space $\mathcal{H}$. Then, $\|Ux\| = \|x\|$ for any $x \in \mathcal{H}$ and $\sigma(U) \subset \mathbb{T}$.

Proof. For the first statement, note that

$$\|Ux\|^2 = (Ux, Ux) = (U^*Ux, x) = (x, x) = \|x\|^2$$

Now, by Proposition 2.1.2 6., $\|U\| = \|U^*\| = 1$, implying that $\sigma(U)$ and $\sigma(U^*)$ are contained in $\overline{\mathbb{D}}$. The result follows from Corollary 2.1.9 if we show that for any unitary U and $z \in \mathbb{D}$, $U - z$ in bounded below. Let $0 \neq x \in \mathcal{H}, z \in \mathbb{D}$, let $y = Ux$ and note that $x = U^*y$. We have

$$\|(U-z)x\| = \|y - zU^*y\| \geqslant \Big| \|y\| - |z|\|y\| \Big| = (1-|z|)\|y\| = (1-|z|)\,|\|x\|$$

□

Definition 2.1.15. Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H})$. Then,

1. T is idempotent if $T^2 = T$.
2. T is a projection if it is idempotent and selfadjoint.

Proposition 2.1.16. If J is idempotent, then J^* and $I - J$ is also idempotent, and $\operatorname{ran} J = \ker(I - J)$. In particular, J has closed range.

Proof. The first two properties are immediate. Now, if $f \in \operatorname{ran} J$, then $f = Jg$ for some g and $(I-J)f = (I-J)Jg = Jg - J^2 g = 0$. If $f \in \ker(I-J)$ then $f = Jf$ and $f \in \operatorname{ran} J$. □

Definition 2.1.17. Let M be a closed subspace of $\mathcal{H}$. Then, writing uniquely $f = f_M + f_{M^\perp}$ where $f_M \in M$ and $f_{M^\perp} \in M^\perp$, f_M is called the orthogonal projection of f on M.

Proposition 2.1.18. P is a projection iff there is a closed subspace $M \subset \mathcal{H}$ s.t. Pf is the orthogonal projection of f on M. If P is a projection, then P is the projection on the closed subspace $\operatorname{ran} P$.

Proof. Closure of $\operatorname{ran} P$ follows from Proposition 2.1.16. Since $I = P + (I-P)$ the result follows from self-adjointness and the fact that $(Pf, (I-P)f) = 0$ for any $f \in \mathcal{H}$.

In the opposite direction, let $M \subset \mathcal{H}$ be closed, and for any $f \in \mathcal{H}$ let $Pf = f_M$, with f_M as in the definition above. Clearly $P^2 = P$. Linearity and boundedness are clear. But we also have $(Pf, f) = (f_M, f_M + f_{M^\perp}) = (f_M, f_M) \geqslant 0$ proving that P is positive, hence self-adjoint.

□

As it is often the case, working with functions which are natural to some topological space is a convenient way to get topological information about the space. In particular, the geometric properties of closed subspaces of Hilbert spaces can often be conveniently expressed in terms of their associated projections.

Proposition 2.1.19. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}_1, \mathcal{M}_2$ closed subspaces, and P_1, P_2 the orthogonal projections onto them. Then $P_1 + P_2$ is a projection iff $M_1 \perp M_2$ and $P_1 + P_2 = I$ iff $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{M}_2$. The result extends to any number n of subspaces.

Proof. First we show that $P_1 + P_2$ is a projection iff $M_1 = \operatorname{ran}(P_1)$ and $M_2 = \operatorname{ran}(P_2)$ are mutually orthogonal. In one direction, assume $P_1 + P_2$ is a projection. Then

$$P_1 + P_2 = (P_1 + P_2)^2 = P_1^2 + P_2^2 + P_1P_2 + P_2P_1 = P_1 + P_2 + P_1P_2 + P_2P_1$$

Hence

$$P_1P_2 + P_2P_1 = 0 \tag{35}$$

Let $M_{1,2} = \operatorname{ran} P_{1,2}$. Note first that $y \in M_1 \cap M_2$ implies $y = -y$ hence $M_1 \cap M_2 = 0$. Now, if $x \in M_1$, then $P_1P_2x = -P_2x$, hence $\|P_1(P_2x)\| = \|P_2x\|$ and thus $P_2x \in M_1 \cap M_2$, hence $x \in \ker P_1$. Interchanging $1, 2$ implies the result.

In the opposite direction, $(\operatorname{ran} P_1)^{\perp} = \ker P_1$ and therefore $\operatorname{ran} P_2 = \mathcal{M}_2 \subset \ker P_1$, and similarly $\operatorname{ran} P_1 \subset \ker P_2$ implying (35), which in turn implies that $P_1 + P_2$ is an involution, and as a sum of two self-adjoint operators is self-adjoint, hence a projection. The next statement, and generalization to any $n \in \mathbb{N}$ are immediate. □

Exercise 11. Does this statement extend to arbitrary direct sums? What notions of convergence can we use when $\mathcal{H}$ is separable? What if it is not separable?

2.2 Example: a space of "diagonal" operators.

Let (X, Σ, λ) be a probability space[12]. Let $\mathcal{H} = L^2(\lambda)$ and consider the action of $L^\infty(\lambda)$ on $L^2(\lambda)$ by multiplication. Namely, for $\varphi \in L^\infty$ and $f \in L^2$, let M_φ be the operator defined by

$$M_\varphi f = \varphi f$$

You can check that

1. $\|M_\varphi\| \leqslant \|\varphi\|_\infty$
2. For any polynomial P we have $M_{P(\varphi)} = P(M_\varphi)$
3. $M_\varphi^* = M_{\bar{\varphi}}$ hence $MM^* = M^*M$
4. If φ is invertible in L^∞, then M_φ is invertible with inverse $M_{1/\varphi}$
5. $\mathfrak{M} = \{M_\varphi : \varphi \in L^\infty\}$ is a Banach algebra **in the operator norm**.

[12]More generally, one can start with a finite measure μ, and replace μ with the mutually absolutely continuous $\rho = \mu/\mu(X)$.

We now claim that $\|M_\varphi\| = \|\varphi\|_\infty$. Take any $z \in \mathcal{R}(\varphi)$ and let $\psi = \varphi - z$. For any $\varepsilon > 0$, the set $E = \{x \in X : |\psi(x)| < \varepsilon\}$ has positive measure. Clearly, $\chi_E \in L^2(\lambda)$. We have

$$\|M_\psi \chi_E\|_2^2 = \|\psi\chi_E\|_2^2 = \int_X |\psi|^2 {\chi_E}^2 d\lambda \leqslant \varepsilon^2 \int_X {\chi_E}^2 d\lambda = \varepsilon^2 \|\chi_E\|_2^2 \tag{36}$$

hence M_ψ is not bounded below, and $z \in \sigma(M_\varphi)$. Therefore $r(M_\varphi) \geqslant |z|$, hence $\|M_\varphi\| \geqslant \|\varphi\|_\infty$, completing the proof of the claim. We have shown the following.

Proposition 2.2.1. The Banach algebras $L^\infty(\lambda)$ and $\mathfrak{M}$ are (canonically) isometrically isomorphic.

Note 2.2.2. $\mathfrak{M}$ is a standard example of a W^*-algebra. When X is a compact subset of $\mathbb{C}$, $\mathfrak{M}$ is the result of applying the spectral theorem to the W^* algebra generated by a normal operator, where $\rho = \mu/\mu(X)$ and μ is the spectral measure.

Definition 2.2.3. An abelian algebra $\mathfrak{M}$ of operators on a Hilbert space $\mathcal{H}$ is maximal abelian if it is not properly contained in any larger abelian algebra of operators on $\mathcal{H}$.

Proposition 2.2.4. The Banach algebra $\mathfrak{M}$ in Proposition 2.2.1 is maximal abelian.

Proof. Notice first that, since λ is a finite measure, $L^\infty(X) \subset L^2(X)$.

We show if $T \in L(\mathcal{H})$ commutes with $\mathfrak{M}$ then $T \in \mathfrak{M}$, that is it is a multiplication operator with some ψ. Since that's what we expect, ψ should equal $T1$. Thus, let $\psi = T1$. For $\varphi \in L^\infty$ we have

$$T\varphi = TM_\varphi 1 = M_\varphi T1 = \varphi\psi$$

which shows that $\|\psi\varphi\|_2 \leqslant \|T\|\|\varphi\|_2$ on L^∞[13] It is now a simple measure theory exercise to show that, moreover, $\|\psi\|_\infty \leqslant \|T\|$. Indeed, if $a > \|T\|$ and $E = \psi^{-1}(a,\infty)$then $\lambda(E)^2 = \|\chi_E\|^2 = 0$ because

$$\|T\chi_E\|^2 = \|\psi\chi_E\|_2^2 = \int_X |\psi|^2|\chi_E|^2 d\lambda \geqslant a^2 \int_X |\chi_E|^2 d\lambda = a^2\|\chi_E\|_2^2$$

The result now follows by choosing a sequence of $a = a_n$ of the form $\|T\| + n^{-1}, n \in \mathbb{N}$. Using the density of L^∞ in L^2, we have $Tf = \psi f$ for any $f \in L^2$, hence $\psi \in \mathfrak{M}$. □

[13] Therefore, by density $\|\psi f\|_2 \leqslant \|T\|\|f\|_2$ in L^2, but we are not using this fact.

Exercise 12. Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{A} \in L(\mathcal{H})$ be a maximal abelian algebra. Show that $\mathfrak{A}$ is a Banach algebra (commutative, clearly).

Let $T \in L(\mathcal{H})$ and $\lambda \in \rho(T)$. Show that T commutes with $(T-\lambda)^{-1}$. (We proved a similar result for spectra relative to a Banach algebra.) Check that this implies

Proposition 2.2.5. Let $\mathcal{H}$ be a Hilbert space, $T \in L(\mathcal{H})$ and $\mathfrak{A}$ a maximal abelian algebra containing T. Then $\sigma(T) = \sigma_{\mathfrak{A}}(T)$.

The corollary below follows easily from Proposition 2.2.5, and can also be proved in a number of direct ways.

Corollary 2.2.6. If $\varphi \in L^\infty(\lambda)$, in the notations and conditions above, then $\sigma(M_\varphi) = \mathcal{R}(\varphi)$.

2.3 The shift operator on $\ell^2(\mathbb{Z})$

The shift operator is important in a number of areas of math.

Definition 2.3.1. The bilateral shift U is defined by $(Uf)(n) = f(n-1) \forall f \in \ell^2(\mathbb{Z})$ and $n \in \mathbb{Z}$. A similarly defined U on $\ell^2(\mathbb{N})$ is called left shift.

Note 2.3.2. A similarly defined U on $\ell^2(\mathbb{N})$ is called left shift. It behaves quite differently, because of the "one-sidedness" of $\mathbb{N}$.

It is easily checked that $\|U\| = 1$. Now,

$$(Uf, g) = \sum_{n\in\mathbb{Z}} f(n-1)\overline{g(n)} = \sum_{n\in\mathbb{Z}} f(n)\overline{g(n+1)} = (f, U^*g)$$

hence the adjoint of U is the operator defined by $(U^*f)(n) = f(n+1) \forall f \in \ell^2(\mathbb{Z})$ and $n \in \mathbb{Z}$. Clearly, $UU^* = U^*U = I$ and U is unitary. It follows from Proposition 2.1.14 that $\sigma(U) \subset \mathbb{T}$. Now we want to determine

1. the smallest closed subalgebra $\mathcal{A}$ of $\mathcal{H}$ generated by U;
2. the maximal abelian one $\mathfrak{M}$ containing U;
3. and find the spectrum of U relative to each.

We are in a mathematical setting quite similar to that of §1.5.1.

The inverse Fourier transform $\mathcal{F}^{-1}$ (it acts by mapping an $f \in \ell^2(\mathbb{Z})$ to $\mathcal{F}^{-1}f = \sum_{k\in\mathbb{Z}} f(k)e^{ik\varphi}$) is an isometric isomorphism, between $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$. The image $\hat{U} = \mathcal{F}U\mathcal{F}^{-1} \in L^2(\mathbb{T})$ acts by multiplication by $\chi = \chi_1$: $\hat{U}g = M_\chi g$. In this representation of the Hilbert space, the construction is much easier[14].

It is now easy to check that $\sigma(\hat{U})$ is the whole of $\mathbb{T}$. Furthermore, the algebra A generated by $\hat{U}$ is $\{M_{P(\chi)} : P \text{ analytic polynomial}\}$. All polynomials extend to $\mathbb{C}$, as entire functions. The example in §2.2 shows that the operator norm on A is the sup norm, and by the maximum principle, the closure $\mathcal{A}$ of A is the space $A_{\mathbb{D}}$ of boundary values of analytic functions in $\mathbb{D}$, continuous up to $\overline{\mathbb{D}}$, with the sup norm on $\overline{\mathbb{D}}$.

We also know that $\mathfrak{M} = \{M_\varphi : \varphi \in L^\infty(\mathbb{T})\}$ is maximal abelian.

Note 2.3.3.
1. $\sigma_{\mathcal{A}}(\hat{U}) = \overline{\mathbb{D}} \neq \sigma(\hat{U}) = \mathbb{T}$.
2. $\mathcal{A}$ is not selfadjoint, since $\chi_1 = \bar{\chi}$ does not extend analytically to $\mathbb{D}$, and thus $M_\chi^* = M_{\bar{\chi}} \notin \mathcal{A}$
3. By Stone-Weierstrass the Banach algebra generated by M_χ and M_χ^*, in the operator norm is $M_{C(\mathbb{T})} = \{M_\varphi : \varphi \in C(\mathbb{T})\}$. (Note also that $\|M_\varphi\| = \|\varphi\|_\infty$.)
4. $\sigma_{M_{C(\mathbb{T})}}(M_\chi) = \mathbb{T}$.
5. The maximal abelian algebra $\mathfrak{M}$ is also closed in operator norm, but is larger than the closure of A in operator norm.

Exercise 13. Due 02/24.
1. It is clear that $\mathcal{A} \supset A_{\mathbb{D}}$. Show that $\mathcal{A} \subset A_{\mathbb{D}}$.
2. In what topology can one obtain $\mathfrak{M}$ from $\{M_\varphi : \varphi \in C(\mathbb{T})\}$ by closure?

Definition 2.3.4. $\mathfrak{A}$ is an algebra with involution if there is an involution, $T \to T^*$, s.t.
1. $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$ for all $a, b \in \mathbb{C}$, and $(S, T) \in \mathfrak{A}$.
2. $(ST)^* = T^*S^*$ for all $(S, T) \in \mathfrak{A}$
3. $T^{**} = T$ for all $T \in \mathfrak{A}$

[14] This is an instance where isomorphisms are useful...

Definition 2.3.5. A Banach algebra with involution $\mathfrak{A}$ is a C^*-algebra if $\forall T \in \mathfrak{A}$, $\|T^*T\| = \|T\|^2$

Note 2.3.6.
1. Any closed, self-adjoint subalgebra of $L(\mathcal{H})$ is a C^*-algebra.
2. Conversely, it can be shown that any C^*-algebra is isomorphic to a C^*-subalgebra of $L(\mathcal{H})$, for some $\mathcal{H}$.
3. Relatedly, the definitions we introduced on $L(\mathcal{H})$ that are based on the adjoint (s.a.: self-adjoint, normal, unitary) extend to C^*-algebras, by using the involution map $*$ instead of taking the adjoint. All proofs that do not rely on the inner product extend as well. For instance, T is invertible iff T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Proposition 2.3.7. In a C^*-algebra, the involution map is an isometry.

Proof. We have seen this proof before:

$$\|T\|\|T^*\| \geqslant \|TT^*\| = \|T\|^2$$

□

hence $\|T\| \leqslant \|T^*\|$ which, applied to T^* also gives $\|T^*\| \leqslant \|T\|$.

Proposition 2.3.8. Let $\mathfrak{A}$ be a C^*-algebra. Then $T \in \mathfrak{A}$ is invertible iff T^*T and TT^* are invertible.

Proof. In one direction, this is clear by Note 2.3.6, 3. Assuming T^*T is invertible, you can check that $(T^*T)^{-1}T^*$ is a left inverse of T, while $T^*(TT^*)^{-1}$ is a right inverse, which implies now (as it would in any ring) that T is invertible. □

Proposition 2.3.9. If U is unitary in a C^*-algebra, then $\sigma(U) \subset \mathbb{T}$.

Proof. We have $\|U\| = \|U^*\| = 1$. From this point on, the proof follows that of Proposition 2.1.14. □

Proposition 2.3.10. In a C^*-algebra, if A is self-adjoint, then the spectrum of A is real and $U = e^{iA}$ is unitary.

Proof. Using the power series definition of e^{iA} we see that $U^* = (e^{iA})^* = e^{-iA}$ is in the commutative Banach algebra generated by A and from the properties of the exponential, $UU^* = U^*U = I$. By Corollary 1.4.15 we have $\mathbb{T} \supset e^{-i\sigma(A)}$, hence the spectrum of A is real. □

Proposition 2.3.11. Let $\mathfrak{B}$ and $\mathfrak{A} \subset \mathfrak{B}$ be C^*-algebras, and $T \in \mathfrak{A}$. Then $\sigma_{\mathfrak{A}}(T) = \sigma_{\mathfrak{B}}(T)$.

(Compare with Proposition 2.2.5.)

Proof. We need to check that invertibility in $\mathfrak{B}$ implies invertibility in $\mathfrak{A}$. By Proposition 2.3.8 we can assume that T is self-adjoint. It means that it is enough to prove the statement when T itself is self-adjoint, in which case the statement is equivalent to

$$A := \rho_{\mathfrak{A}}(S) \cap \mathbb{R} = \rho_{\mathfrak{B}}(S) \cap \mathbb{R} =: B \tag{37}$$

Let $a \in A$ and (a_1, a_2) (we assume both are finite; the proof is similar when the interval is semiinfinite) its connected component in A. Clearly, $(a_1, a_2) \subset B$. Necessarily both a_1 and a_2 are in ∂A, hence, by Shilov's theorem 1.6.1, $a_1, a_2 \in \partial B$ and hence (a_1, a_2) is also the connected component of a in B implying the result. □

Exercise 14 (Bonus). Due Mar. 3.

Let $K \subset \mathbb{C}$ be any compact set. Define a Hilbert space $\mathcal{H}$ and a normal operator N in $L(\mathcal{H})$ such that $\sigma(N)$ is exactly K.

(It follows that $K \subset \mathbb{C}$ is compact iff there is a Hilbert space $\mathcal{H}$ and a normal operator N in $L(\mathcal{H})$ such that $\sigma(N)$ is exactly K.)

Proposition 2.3.12. Let $\mathfrak{A}$ be a C^*-algebra and Γ the Gelfand transform. Then Γ commutes with complex conjugation: $\Gamma(T^*) = \overline{\Gamma(T)}$.

Proof. Let $T \in \mathfrak{A}$. Then $T = A_1 + iA_2$, where $A_1 = \frac{1}{2}(T + T^*)$ and $A_2 = -\frac{i}{2}(T - T^*)$ are self-adjoint (check that such a decomposition is unique). Since $\sigma(A_1), \sigma(A_2)$ are real, by a fundamental property of the Gelfand transform (see Theorem 1.4.12), $\Gamma(A_1)$ and $\Gamma(A_2)$ are real-valued. Now, $T^* = A_1 - iA_2$ and $\Gamma(T^*) = \Gamma(A_1) - i\Gamma(A_2) = \overline{\Gamma(T)}$. □

Theorem 2.3.13 (Gelfand-Naimark). If $\mathfrak{A}$ is a commutative C^*-algebra, then Γ is a $*$-isometric isomorphism of $\mathfrak{A}$ onto $C(M)$, where M is the maximal ideal space of $\mathfrak{A}$.

Proof. This follows from Stone-Weierstrass and Proposition 2.3.12, as soon as we show that Γ is an isometry. Recalling Corollary 1.4.14 and Theorem 1.4.16, and using Proposition 2.3.12, we see that

$$\|T\|^2 = \|T^*T\| = \lim_{k\to\infty} \|(T^*T)^{2^k}\|^{2^{-k}} = \|\Gamma(T^*T)\|_\infty = \||\Gamma(T)|^2\|_\infty = \|\Gamma(T)\|_\infty^2 \tag{38}$$

□

Proposition 2.3.14. If T is normal, then the C^*-algebra $\mathfrak{A}_T$ generated by T is commutative.

Proof. The algebra $A = \{P(T, T^*) : P \text{ polynomial}\}$ is contained in $\mathfrak{A}_T$ and is commutative. The norm closure of A is a commutative C^*-algebra containing T, hence it must equal $\mathfrak{A}_T$. □

In the following, $\mathcal{H}$ is a Hilbert space, T is a normal operator on $\mathcal{H}$, and $\mathfrak{A}_T$ is the (commutative, by Proposition 2.3.14) C^*-algebra generated by T, and M is the maximal ideal space of $\mathfrak{A}_T$.

Theorem 2.3.15 (Spectral Theorem for normal operators). M is homeomorphic to $\sigma(T)$ and Γ is a *-isometric isomorphism of $\mathfrak{A}_T$ onto $C(\sigma(T))$

Proof. The only thing left to prove is the homeomorphism of M and $\sigma(T)$. We define this map ψ from M to $\sigma(T)$ by $\psi(\varphi) = \Gamma(T)(\varphi)$. By Corollary 1.4.14 $\operatorname{ran}(\Gamma(T)) = \sigma(T)$ and ψ is onto. It is also injective. Indeed, assume $\psi(\varphi_1) = \psi(\varphi_2)$. This means by definition $\Gamma(T)(\varphi_1) = \Gamma(T)(\varphi_2)$, and by the definition of Γ, $\varphi_1(T) = \varphi_2(T)$. We note that $\varphi_1(T^*) = \varphi_2(T^*)$ since

$$\varphi_1(T^*) = \Gamma(T^*)(\varphi_1) = \overline{\Gamma(T)(\varphi_1)} = \overline{\Gamma(T)(\varphi_2)} = \Gamma(T^*)(\varphi_2) = \varphi_2(T^*)$$

Hence φ_1 and φ_2 agree on the dense algebra $A = \{P(T, T^*) : P \text{ polynomial}\}$, and by continuity, $\varphi_1 = \varphi_2$.To check that ψ is bi-continuous, we only need to check that it is continuous, since M and $\sigma(T)$ are compact Hausdorff spaces. If $\{\varphi_\alpha\}_{\alpha\in A}$ is a convergent net in M with limit φ, then

$$\lim_\alpha \psi(\varphi_\alpha) = \lim_\alpha \Gamma(T)(\varphi_\alpha) = \Gamma(T)(\varphi) = \psi(\varphi)$$

□

Note 2.3.16. 1. This is a very general form of the spectral theorem. In the result above, T could be a normal element of any C^*-algebra.

2. An important consequence is that Theorem 2.3.15 ensures the existence of continuous

functional calculus with normal operators: if f is a continuous function on $\sigma(T)$, then $\Gamma^{-1}(f)$ defines an element of the C^*-algebra that can be meaningfully called $f(T)$.

3. Indeed, you can check that, in all cases when we can define $f(T)$ directly, such as for polynomials, or more generally for functions F analytic on a ball containing the spectrum, this directly defined $F(T)$ coincides with $\Gamma^{-1}F$. But without this theorem, it is not obvious what to do when f is just some continuous function on $\sigma(T)$.

4. Let λ be any probability measure on $\sigma(T)$ and $\mathcal{H} = L^2(\sigma(T), \lambda)$. Then, $C(\sigma(T))$ is a is a commutative C^*-subalgebra of $L(\mathcal{H})$. When T is a normal operator on some Hilbert space $\mathcal{H}_0$ we have found a representation of T as a multiplication operator on some other Hilbert space $\mathcal{H}$.

5. However, $\mathcal{H}_0$ may have higher cardinality than $\mathcal{H}$, and thus may not be unitarily equivalent to it. Furthermore, it is not clear what particular function $\Gamma(T)$ is, nor what its properties are, save some basic ones. For all of this, we will need to work some more. But we can get some important general facts about operators at this stage already. Furthermore, for a given $f \in C(\sigma(T))$ $f(T) = \Gamma^{-1}(f)$ is canonical.

Note 2.3.17. Let $\mathcal{H}$ be a Hilbert space and $U \in L(\mathcal{H})$. It is easy to check that U is unitary iff it is an automorphism of $\mathcal{H}$.

Proposition 2.3.18. Let T be a normal operator on a Hilbert space $\mathcal{H}$. Then T is self-adjoint iff $\sigma(T) \subset \mathbb{R}$ and T is positive iff $\sigma(T) \subset [0, \infty)$.

Proof. In one direction we have already proven this. So, assume $\sigma(T) \subset \mathbb{R}$. Then $\Gamma(T)$ is real-valued. Lat $\mathfrak{A}$ be the commutative C^*-algebra generated by T. Since $\Gamma(T) = \overline{\Gamma(T)} = \Gamma(T^*)$, we have $T = T^*$. If $\sigma(T) \subset [0, \infty)$, then $\operatorname{ran}(\Gamma(T)) \subset [0, \infty)$, hence, with $g = \sqrt{\Gamma(T)} = \Gamma(S)$ for a unique S, we have $\Gamma(T) = g^2 = \overline{g}g = \Gamma(S^*S)$, hence $T = S^*S$ is positive. $\square$

In the proof we also showed the following.

Corollary 2.3.19. Let T be an operator on a Hilbert space $\mathcal{H}$. Then T is positive iff there is an $S \in L(\mathcal{H})$ such that $T = S^*S$.

Proposition 2.3.20. If P is a positive operator on a Hilbert space $\mathcal{H}$, then there exists a unique positive operator Q s.t. $Q^2 = P$, and Q commutes with every operator commuting with P.

Proof. We show that Q commutes with P. Indeed, $QP = QQ^2 = Q^2Q = PQ$. Then, in the commutative C^*-algebra $\mathfrak{A}_1$ generated by Q and T, we have $\Gamma(Q)^2 = \Gamma(\sqrt{P})^2$, and since both $\Gamma(Q)$ and $\Gamma(\sqrt{P})$ are nonnegative, they must be equal, hence $Q = \sqrt{P}$. ☐

Definition 2.3.21. Let $\mathcal{H}$ be a Hilbert space. Then V is an isometry if $\|Vf\| = \|f\|$ for any $f \in \mathcal{H}$ and it is a partial isometry if V restricted to the **initial space** $(\ker V)^{\perp}$ is an isometry.

Note that any projection is a partial isometry. It is an interesting elementary exercise (if you don't know the proof already) to show that the only (linear or nonlinear) isometries of the Euclidean space are given by the Euclidean group. Hence linear isometries are unitary operators. That this is false in infinite dimensions is illustrated by the next example.

2.3.1 Example: the shift operators on $\mathcal{H} = \ell^2(\mathbb{N})$

We let the right shift be defined on $\mathcal{H}$ by $(Rf)(n) = f(n-1)$ for $n > 0$ and $(Sf)(0) = 0$:

$$R := (a_1, a_2, a_3, \ldots) \mapsto (0, a_1, a_2, \ldots)$$

We could extend this operator to $\ell^2(\mathbb{Z})$ and use the Fourier machinery to extract its properties, but in view of the importance of the operator, we will analyze it as is. Clearly, R is an isometry, hence $\sigma(R) \subset \overline{\mathbb{D}}$ but it is not unitary since $e_1 := n \mapsto \delta_{1,n}$ is orthogonal to $\operatorname{ran} R$ (here, δ is the Kronecker symbol). $R^* = L$, where L is the left shift defined by

$$L := (a_1, a_2, a_3, \ldots) \mapsto (a_2, a_3, a_4, \ldots)$$

L is also a left inverse for R, $LR = I$. R is not a normal operator since $\operatorname{ran}(RL) \perp \{e_1\}$. The C^*-algebra generated by R, R^* is noncommutative.

We have: $\ker R = \{0\}$, $\ker L = \{e_1\}$. Now, we note that the (point) spectrum of L is $\mathbb{D}$. Indeed, for any $z \in \mathbb{D}$,

$$f_z = (1, z, z^2, \ldots) \in \mathcal{H} \quad \text{and} \quad Lf_z = zf_z$$

Hence $\sigma(L) = \overline{\mathbb{D}}$. For any $z \in \mathbb{D}$, $\operatorname{ran}(R - z)^{\perp} = \ker(L - z) \ni f_z$, hence $R - z$ is not invertible when $z \in \mathbb{D}$ and $\sigma(R) = \overline{\mathbb{D}}$. This type of spectrum is called **residual**. The **pure point** (or simply **point**) spectrum of R is empty. The pure point spectrum of an operator T is defined as

$$\sigma_{pp}(T) = \{\lambda : (\exists x \in \mathcal{H})(Tx = \lambda x)\}$$

Exercise 15. Due March 1.

The natural extension of R to $\ell^2(\mathbb{N})$ is UP where P is the projection on $\ell^2(\mathbb{N})$ and U is the bilateral shift on $\ell^2(\mathbb{Z})$. This is an example of a polar decomposition of operators, that we'll examine soon. Clearly $UP \neq PU$. Use the Fourier machinery we introduced before to obtain the results above about U and its adjoint.

Proposition 2.3.22. Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of same dimension [15]of a Hilbert space $\mathcal{H}$. Then there exist partial isometries V on $\mathcal{H}$ with initial space $\mathcal{M}$ and range $\mathcal{N}$.

Proof. Let U be unitary between $\mathcal{M}$ and $\mathcal{N}$. We let P be the projection on $\mathcal{M}$. Then $V = UP$ is such partial isometry, as you can check. □

The notion of a partial isometry below plays a crucial role in the theory of self-adjoint and normal unbounded operators.

Proposition 2.3.23. Let $\mathcal{H}$ be a Hilbert space and $V \in L(\mathcal{H})$. Then, the following are equivalent:

1. V is a partial isometry;
2. V^* is a partial isometry;
3. VV^* is a projection.
4. V^*V is a projection;

Furthermore, VV^* is the projection onto $\overline{\operatorname{ran} V} = (\ker V^*)^\perp$ and V^*V is the projection onto $\overline{\operatorname{ran} V^*} = (\ker V)^\perp$.

Combining the above, we see that if a partial isometry V is normal, then the initial space and range of V coincide; if V is self-adjoint, then V^2 is a projection; and if V is positive, then V is a projection.

Proof. We show that 1. is equivalent to 4. (hence 2. is equivalent to 3.). Noting that $\|Vx\| \leqslant \|x\|$ for any $x \in \mathcal{H}$, we see that, for any $x \in \mathcal{H}$ we have

$$((I - V^*V)x, x) = \|x\|^2 - \|Vx\|^2 \geqslant 0$$

hence $I - V^*V$ is positive, and $(I - V^*V)^{1/2}$ is well defined. If $x \in M := (\ker V)^\perp$, then $\|x\| = \|Vx\|$. Now, $\|(I - V^*V)^{1/2}x\|^2 = ((I - V^*V)x, x) = 0$, hence $(I - V^*V)^{1/2}x = 0$ implying

[15]By same dimension as usual we mean the existence of Hilbert bases indexed by the same set of indices A; it is the same as saying $\mathcal{M}$ and $\mathcal{N}$ are unitarily equivalent.

$(I - V^*V)x = 0$ and thus $V^*Vx = x$ and V^*V is the projection on M. Conversely, assume that V^*V is a projection on M and $x \in M$. Then, $V^*Vx = x$, which implies $\|Vx\| = x$. Now we show that $\ker(V^*V) = M^\perp = \ker V$. It is clear that $\ker V^*V \supset \ker V$. In the opposite direction, $x \in \ker V^*V$ implies $\|Vx\| = 0$. Combining these results, it follows that V is a partial isometry.

For 4$\Rightarrow$ 3, assume V^*V is a projection. Note that $V(V^*V) = V$ since the equality holds on $\ker V$ and V^*V is a projection on the initial space of V. In turn, this implies the result because $(VV^*)^2 = VV^*VV^* = VV^*$.

□

Theorem 2.3.24 (Polar decomposition)**.** Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H})$. Then, T can be written as $T = VP$ where P is a positive operator and V is a partial isometry. Imposing the extra condition $\ker P = \ker V$, the decomposition becomes unique.

Similarly, T can be written as $T = QW$ where Q is a positive operator and W is a partial isometry, and this decomposition is unique if $\operatorname{ran} W = (\ker Q)^\perp$.

Proof. The construction is natural, if we have in mind the much easier polar decomposition of a complex number. Let $P = \sqrt{T^*T}$. We see that

$$\|Pf\|^2 = (Pf, Pf) = (P^2 f, f) = (T^*Tf, f) = (Tf, Tf) = \|Tf\|^2 \tag{39}$$

We see first that $\ker P = \ker T$. Define $V = 0$ on $\ker P = (\operatorname{ran} P)^\perp$. For $g \in \operatorname{ran} P$, we have $g = Pf$ for some $f \in \mathcal{H}$ and we define $Vg = Tf$. Using (39) we get $\|Vg\| = \|Tf\| = \|Pf\| = \|g\|$, hence V is densely defined and bounded, and thus it extends uniquely to an operator in $L(\mathcal{H})$ which we still denote by V. This construction shows that V is a partial isometry, with $\ker V = \ker P$, and that $T = VP$.

For uniqueness, suppose that $T = WQ$ is a polar decomposition of T, with $\ker W = \ker Q$. Then, W^*W is the projection onto

$$(\ker W)^\perp = (\ker Q)^\perp = \overline{\operatorname{ran} Q} \tag{40}$$

By the above, we have $P^2 = T^*T = QW^*WQ = Q^2$, implying $P = Q$ by uniqueness of the positive square root, and the decomposition WQ is nothing else but $WP = VP$, and $W = V$ on $\operatorname{ran} P$. But $\ker W = \ker V$ since they both equal $\ker P = \ker Q$, meaning $W = V$.

The second part of the theorem can be obtained as a corollary of the first, or the proof of the first part can be mimicked to prove the second part, and we leave this to the reader. □

Note 2.3.25. The positive operator P in the polar decomposition belongs to any C^*-algebra containing T but V, in general, does not. Multiplication by the variable, M_x, on $L^2(-1,1)$ easily shows what can go wrong. In the decomposition $x = |x|V(x)$ where, $V(x) = x/|x|$ for $x \neq 0$ and say $V(0) := 1$ (or any other definition, as one point is immaterial) $|x|$ is continuous and thus $M_{|x|}$ is in the C^*-algebra generated by x while V is not, meaning M_V is not in the C^*-algebra generated by x.

Definition 2.3.26 (Reducing spaces)**.** Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H})$. A closed subspace M of $\mathcal{H}$ is an invariant subspace for T if $TM \subset M$ and a reducing subspace if moreover $T(M^\perp) \subset M^\perp$.

Proposition 2.3.27.

1. Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H})$, $\mathcal{M}$ a closed subspace of $\mathcal{H}$ and $P_\mathcal{M}$ the projection onto $\mathcal{M}$. Then, $\mathcal{M}$ is an invariant subspace for T iff $P_\mathcal{M} T P_\mathcal{M} = T P_\mathcal{M}$, and also iff $\mathcal{M}^\perp$ is an invariant subspace for T^*.
2. $\mathcal{M}$ is a reducing subspace for T iff $P_\mathcal{M} T = T P_\mathcal{M}$, and also iff $\mathcal{M}$ is an invariant subspace for T and for T^*.

Proof. All this is quite straightforward. If $\mathcal{M}$ is invariant for T and $f \in \mathcal{H}$, then clearly $TP_\mathcal{M} f \in \mathcal{M}$ implying $P_\mathcal{M} T P_\mathcal{M} = T P_\mathcal{M}$. In the opposite direction, if $P_\mathcal{M} T P_\mathcal{M} = T P_\mathcal{M}$ and $f \in \mathcal{M}$, then $Tf = TP_\mathcal{M} f = P_\mathcal{M} T P_\mathcal{M} \in \mathcal{M}$, hence $\mathcal{M}$ is an invariant subspace for T.

The projection on $\mathcal{M}^\perp$ is $I - P_\mathcal{M}$. The statement that $\mathcal{M}^\perp$ is an invariant subspace for T^* is equivalent to

$$T^*(I - P_\mathcal{M}) = (I - P_\mathcal{M}) T^* (I - P_\mathcal{M}) \tag{41}$$

in turn equivalent to $P_\mathcal{M} T^* = P_\mathcal{M} T^* P_\mathcal{M}$, and by taking the adjoint, to $P_\mathcal{M} T P_\mathcal{M} = T P_\mathcal{M}$.

To complete the proof, we note that, if $\mathcal{M}$ reduces T, then $P_\mathcal{M} T = P_\mathcal{M} T P_\mathcal{M} = T P_\mathcal{M}$. □

Next we look to extend functional calculus to measurable functional calculus.

3 W*-algebras and measurable functional calculus

The goal of this section is to extend functional calculus to L^∞ is to obtain the spectral theorem for normal operators in a form which is a direct generalization of the following finite dimensional form: If M is a normal matrix, then $M = \sum m_k P_k$ where m_k are the eigenvalues, and P_k are projections onto the eigenspace corresponding to m_k. In fact, continuous functional calculus allows us already to obtain this form, in case the spectrum is discrete.

Proposition 3.0.1. Let $\mathcal{H}$ be a Hilbert space and assume $T \in L(\mathcal{H})$ is a normal operator s.t. $\sigma(T)$ is a discrete set, $\{\lambda_1, ..., \lambda_n\}$. Then there exist n projections $P_1, ..., P_n$ s.t. $P_1 + \cdots P_n = I$ and

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n \tag{42}$$

Proof. Apply the spectral theorem in the form of Theorem 2.3.15, and let $\Gamma(T) = F$. Since $F - \lambda_i$ is not invertible iff $\lambda_i \in \sigma(T)$ then F is some bijection of $\sigma(T)$ (a.k.a. rearrangement). Thus, up to an isomorphism we can assume $F(x) = x$ for $x \in \sigma(T)$.

The functions $\pi_j = \chi_{\{\lambda_j\}}$ are continuous on $\sigma(T)$. We obviously have $\pi_i\pi_j = \delta_{ij}\pi_j$ and $\sum_{j=1}^n \pi_j = 1$, and $\Gamma^{-1}\pi_j = P_j$ are projections with $\sum_{j=1}^n P_j = I$. We also have $F = \sum_{j=1}^n \lambda_j\chi_j$, hence $T = \sum_{j=1}^n \lambda_j P_j$. ☐

With C^*-algebra tools, that's almost as far as we can go with the projector decomposition of operators. For operators with continuous spectrum, indicator functions are not continuous anymore. Hence the need to extend continuous functional calculus.

3.1 The strong and weak topologies of operators

Definition 3.1.1. Let $\mathcal{H}$ be a Hilbert space and $L(\mathcal{H})$ the algebra of operators on $\mathcal{H}$.

1. A net of operators $\{T_\alpha\}_{\alpha\in A}$ converges in the strong operator topology to $T \in \mathcal{H}$ if for any $f \in \mathcal{H}$ $\lim_\alpha T_\alpha f = T$.

2. A net of operators $\{T_\alpha\}_{\alpha\in A}$ converges in the weak operator topology to $T \in \mathcal{H}$ if for any $f, g \in \mathcal{H}$ $\lim_\alpha (T_\alpha f, g) = (Tf, g)$.

Note 3.1.2. The norm topology is strictly stronger than the strong operator topology, which in turn is strictly stronger than the weak topology.

Exercise 16. Show that, in both topologies introduced above, the C^*-algebra operations on $L(\mathcal{H})$ (taking the adjoint, multiplication and addition of operators) are continuous.

Due March 6.

Exercise 17. An even weaker convergence would be

"A net of operators $\{T_\alpha\}_{\alpha\in A}$ converges in the very weak operator topology to $T \in \mathcal{H}$ if for any $f \in \mathcal{H}$ $\lim_\alpha (T_\alpha f, f) = (Tf, f)$".

But is this strictly weaker than the weak topology?

Definition 3.1.3 ($\mathbf{W^*}$-algebras). Let $\mathcal{H}$ be a Hilbert space and $L(\mathcal{H})$ the algebra of operators on $\mathcal{H}$. A W^*-algebra is a a C^*-subalgebra of $L(\mathcal{H})$ which is closed in the weak operator topology.

Proposition 3.1.4. Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{M}$ be a self-adjoint subalgebra of $L(\mathcal{H})$. Then the closure of $\mathfrak{A}$ of $\mathfrak{M}$ in the weak operator topology is a W^*-algebra, and $\mathfrak{A}$ is commutative if $\mathfrak{M}$ is.

Proof. Straightforward. □

Corollary 3.1.5. Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H})$ be normal. Then the W^*-algebra $\mathfrak{M}_T$ generated by T is commutative. If Λ_T is the maximal ideal space of $\mathfrak{M}_T$, then the Gelfand transform is a *-isometric isomorphism of $\mathfrak{M}_T$ to $C(\Lambda_T)$.

Proof. Immediate. □

Note that the isomorphism is to a space of **continuous** functions, so we cannot expect Λ_T to be homeomorphic with $\sigma(T)$, except in very simple cases. For *separable* Hilbert spaces, and $T \in L(\mathcal{H})$ a normal operator, we will extend functional calculus to $L^\infty(\sigma(T))$ in such a way that the diagram below is commutative, where vertical arrows are inclusion maps.

$$\begin{array}{ccc} \mathfrak{C}_T & \xrightarrow{\Gamma} & C(\sigma(T)) \\ \downarrow & & \downarrow \\ \mathfrak{M}_T & \xrightarrow{\Gamma^*} & L^\infty(\sigma(T)) \end{array}$$

Proposition 3.1.6. Let (X, Σ, λ) be a probability space. Then $L^\infty(X, \lambda)$ is a maximal abelian W^*-algebra over $L^2(X, \lambda)$.

Proof. This follows from Propositions 2.2.1, 2.2.4, and 3.1.4. □

Proposition 3.1.7. If (X, Σ, λ) is a probability space, then the weak operator topology on $L^\infty(X, \lambda)$ (identified with M_{L^∞}) is the same as the w^* topology.

Proof. Recall (or check) that $f \in L^1$ iff $f = g\bar{h}$, with $g, h \in L^2$. Let φ_α be a w^* convergent net, and w.l.o.g, we can assume it converges to zero. Let $f \in L^1$ and g, h in L^2, $f = g\bar{h}$. The equivalence of the norms can be read from the equalities

$$0 = \lim_\alpha \int_X \varphi_\alpha f d\lambda = \lim_\alpha \int_X \varphi_\alpha g\bar{h} d\lambda = \lim_\alpha (M_{\varphi_\alpha} g, h) \tag{43}$$

□

Proposition 3.1.8. Let (X, Σ, λ) be a probability space, where λ is a regular Borel measure. Then $C(X)$ is w^*- dense in $L^\infty(X, \lambda)$, and in particular, the unit ball of $C(X)$ is w^*- dense in the unit ball of $L^\infty(X, \lambda)$.

Proof. Real analysis, see e.g. §7.2 in Folland. □

In the following results we assume that X is a metric Hausdorff space. Their application in the sequel will be to $X \subset \mathbb{C}$ compact.

Lemma 3.1.9. In a compact metric space X, the set of pointwise limits of decreasing sequences of continuous functions contains all characteristic functions of compact subsets of X.

Proof. Let ρ be the metric on X. For a compact set $K \subset X$ and $x \in X$, let $d(x, K) = \inf\{\rho(x, y) | y \in K\}$ be the usual distance between x and K. For $K \subset X$ compact, the set of functions $\{\varphi_n\}_{n \in \mathbb{N}}$ defined by $\varphi_n(x) = 0$ if $d(x, K) \geqslant 1/n$ and $1 - nd(x, K)$ if $d(x, K) < 1/n$ are clearly continuous, decreasing, and pointwise convergent to χ_K. □

For two measures λ_1 and λ_2, $\lambda_1 \sim \lambda_2$ means they are mutually absolutely continuous.

Proposition 3.1.10. Let (X, Σ) be a compact metric space and λ_1, λ_2 finite regular Borel measures on X. If Φ is a *-isometric isomorphism between $L^\infty(X, \lambda_1)$ and $L^\infty(X, \lambda_2)$ which is the identity on $C(X)$, then $\lambda_1 \sim \lambda_2$ and Φ is the identity.

Proof. We show that, by taking appropriate limits, Φ is the identity between $L^\infty(X, \lambda_1)$ and $L^\infty(X, \lambda_2)$; this immediately implies that the measures are mutually absolutely continuous.

(a) Since Φ is a *-isometric isomorphism, Φ preserves positivity. Hence, if a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $C(X)$ is decreasing and converges pointwise everywhere, so does $\{\Phi(\varphi_n)\}_{n \in \mathbb{N}}$, and since Φ is the identity on $C(X)$, Φ is the identity on the family $\mathcal{F}$ of all decreasing limits of continuous functions, hence, by Lemma 3.1.9 on all characteristic functions of compact sets.

(b) Let E be a measurable set in X. Using the regularity of λ_1, there is a family of compact sets $K_n' \subset E$, s.t. $K_n' \subset K_{n'}'$ if $n' > n$, s.t. $\lambda_1(E \setminus K_n') \to 0$ as $n \to \infty$. Likewise there is a family of compact sets $K_n'' \subset E$, s.t. $K_n'' \subset K_{n'}''$ if $n' > n$, s.t. $\lambda_2(E \setminus K_n'') \to 0$. The sets $K_n = K_n' \cup K_n''$

are compact, and increasing. The sequence $\{\chi_{k_n}\}_{n\in\mathbb{N}}$ is bounded, increasing, and converges to some function F. We have $\Phi(\chi_{K_n}) = \chi_{K_n}$ and $\lambda_{1,2}(\{x : F(x) \neq \chi_E\}) = 0$, hence $F = \chi_E$ in both $L^\infty(X, \lambda_1)$ and $L^\infty(X, \lambda_2)$, and thus Φ is the identity on all characteristic functions measurable sets. By density of simple functions in L^∞, Φ is the identity between $L^\infty(X, \lambda_1)$ and $L^\infty(X, \lambda_2)$. $\square$

Definition 3.1.11. Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{A}$ a subalgebra of $L(\mathcal{H})$.

1. A cyclic vector for $\mathfrak{A}$ is a vector f s.t. $\overline{\mathfrak{A}f} = \mathcal{H}$. (The closure is norm-closure.).
2. A separating vector for $\mathfrak{A}$ is a vector f s.t. $A \in \mathfrak{A}$ and $Af = 0$ implies $A = 0$.

Proposition 3.1.12. Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{A}$ a commutative subalgebra of $L(\mathcal{H})$. If f is a cyclic vector for $\mathfrak{A}$, then f is a separating vector for $\mathfrak{A}$.

Proof. If B is any element of $\mathfrak{A}$ for which $f \in \ker B$, then $\mathcal{H} = \overline{\mathfrak{A}f} \subset \ker B$ since $B(\mathfrak{A}f) = \mathfrak{A}Bf = 0$ and the result follows. $\square$

Note 3.1.13. Generally subalgebras do not have cyclic vectors. The simplest example is the commutative C^*-algebra $\mathfrak{C}_I$ generated by the identity matrix I in $\mathbb{C}^n$, $\{\varphi(I) : \varphi : \{1, ..., n\} \to \mathbb{C}\}$. If $A \in \mathfrak{C}_I$, then, for some $\varphi : \{1, ..., n\} \to \mathbb{C}$, A is a diagonal matrix with $\varphi(1)$ on the diagonal. Thus, for any nonzero $x \in \mathbb{C}^n$, the space generated by $\mathfrak{C}_I x$ is one-dimensional.

Exercise 18. A simple exercise due after the break, on March 16.

1. Let M be a normal matrix on $\mathbb{C}^n$. What is the exact condition for $\mathfrak{C}_M$ to have a cyclic vector?
2. Let $\mathcal{H} = L^2([0,1]) \oplus L^2([0,1])$; we write an element of $\mathcal{H}$ as $\langle \varphi, \psi \rangle$ where φ and ψ are in $L^2([0,1])$. Let T be the self-adjoint operator defined by $T\langle \varphi, \psi \rangle = \langle J\varphi, J\psi \rangle$, where $J(x) = x$ for all $x \in [0,1]$. Check that $\mathfrak{C}_T$ does not have a cyclic vector.
3. If $\mathcal{H} = L^2(K)$ where $K \subset \mathbb{C}$ is compact, then $M_{C(K)}$ is a commutative C^*-algebra on $\mathcal{H}$. Does it have a cyclic vector? If $\varphi \in C(K)$, what conditions on φ can you find for $\mathfrak{C}_T(M_\varphi)$ to have a cyclic vector?

Definition 3.1.14. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces.

1. $U \in L(\mathcal{H}_1, \mathcal{H}_2)$ is called unitary if it is onto, and for any $f, g \in \mathcal{H}_1$ we have $(Uf, Ug)_{\mathcal{H}_2} = (f, g)_{\mathcal{H}_1}$.
2. The notion of adjoint extends similarly: If $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ then there exists a unique $T^* \in L(\mathcal{H}_2, \mathcal{H}_1)$ s.t. for any $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ we have $(Tf, g)_{\mathcal{H}_2} = (f, T^*g)_{\mathcal{H}_1}$.

Exercise 19.

1. Check that all the statements in Proposition 2.1.2 hold.
2. Check that U in Definition 3.1.14 is unitary iff its adjoint U^* is unitary between $\mathcal{H}_2$ and $\mathcal{H}_1$, and that U^*U is the identity on $\mathcal{H}_2$ while UU^* is the identity on $\mathcal{H}_1$.
3. See if any item in Definition 2.1.10 admits a meaningful extension when $\mathcal{H}_1 \neq \mathcal{H}_2$.

Note 3.1.15.

1. The existence of a U as in 1. in the definition above clearly implies $\mathcal{H}_1$ and $\mathcal{H}_2$ are isometrically isomorphic.
2. Using the polarization identity, we see that a linear operator from $\mathcal{H}_1$ **onto** $\mathcal{H}_2$ is unitary iff for any $f \in \mathcal{H}_1$ we have $\|Uf\|_{\mathcal{H}_2} = \|f\|_{\mathcal{H}_1}$.
3. Compare this notion with what we found in §2.3.1 and the discussion immediately before it.

For normal operators with cyclic vectors we can already prove our desired extension to measurable functional calculus. The exercise above shows why there will be more work to cover the general case.

4 Spectral theorems

Proposition 4.0.1.

1. Let $\mathcal{H}$ be a Hilbert space, let T be a normal operator in $L(\mathcal{H})$ with spectrum $\Lambda = \sigma(T)$, and let $\mathfrak{C}_T$ the C^*-algebra generated by T. If $\mathfrak{C}_T$ has a cyclic vector, then there exists a positive regular Borel measure λ on $\mathbb{C}$ with support Λ and a unitary operator U from $\mathcal{H}$ onto $L^2(\Lambda, \lambda)$ such that the map Γ^* defined from $\mathfrak{M}_T$ to $L^\infty(\Lambda, \lambda)$ by $\Gamma^* A = UAU^*$ is a *-isometric isomorphism from $\mathfrak{M}_T$ onto $L^\infty(\Lambda, \lambda)$.
2. Moreover Γ^* restricted to $\mathfrak{C}_T$ is the Gelfand transform from $\mathfrak{C}_T$ onto $C(\Lambda)$.
3. We have $(UTU^*\varphi)(x) = x\varphi(x)$ for any $x \in \Lambda$ (that is, T is represented by multiplication by the identity function).

4. The following uniqueness statement holds.

   If $\tilde{\lambda}$ is a regular Borel measure on $\mathbb{C}$ and $\tilde{\Gamma}^*$ is a *-isometric isomorphism from $\mathfrak{M}_T$ onto $L^\infty(\Lambda, \tilde{\lambda})$ such that $\tilde{\Gamma}^*$ restricted to $\mathfrak{C}_T$ is the Gelfand transform from $\mathfrak{C}_T$ onto $C(\Lambda)$, then λ and $\tilde{\lambda}$ are mutually absolutely continuous, $L^\infty(\Lambda, \lambda) = L^\infty(\Lambda, \tilde{\lambda})$ and $\Gamma^* = \tilde{\Gamma}^*$.

Note. Contrast 3. above with the nonuniqueness we faced in the proof of Proposition 3.0.1.

Proof. Let f be a cyclic vector for $\mathfrak{C}_T$, $\|f\| = 1$. We define a linear functional on $C(\Lambda)$ by $\psi(\varphi) = (\varphi(T)f, f)$. Furthermore, this functional has norm $\leqslant 1$ and is positive (why?). By the Riesz representation theorem, there is a positive regular Borel measure on Λ such that

$$(\varphi(T)f, f) = \int_\Lambda \varphi d\lambda; \quad \forall \varphi \in C(\Lambda) \tag{44}$$

Note 4.0.2. There is an important physical interpretation to this definition. Note first that the right side of (44) is the classical expectation of a function φ when the probability measure on the space is λ.

In quantum mechanics, physical quantities (such as position, momentum energy, angular momentum and so on) are represented by (usually unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$. The "complete description" of the state of a system is its wave function $\psi \in \mathcal{H}$, $\|\psi\| = 1$, which contains **probabilistic** information about the system.

For one particle in $\mathbb{R}^3$, the position is multiplication by the function $x \mapsto x$, momentum (which classically is $p = mv$) is $\hat{p} = -i\hbar\nabla$, the Hamiltonian is $\dfrac{\hat{p}^2}{2m} + V$ where V is the potential and so on. For a self-adjoint operator A, the connection with experimental measurementis the following: $(A\psi, \psi)$ is the expectation of the physical quantity A when the system is described by the wave function ψ.

If the support of λ were not Λ, then there would exist an open set $\mathcal{O} \subset \Lambda$ s.t. $\lambda(\mathcal{O}) = 0$. Noting that f is a separating vector for $\mathfrak{C}_T$, this leads to a contradiction by taking a continuous function which is 1 on the closure of a small disk in $\mathcal{O}$ and zero outside $\mathcal{O}$:

$$\|\varphi(T)f\|^2 = (|\varphi(T)|^2 f, f) = \int_\Lambda |\varphi|^2 d\lambda = 0 \tag{45}$$

We now define a map U from $\mathfrak{C}_T$ to the dense subset $C(\Lambda) \subset L^2(\Lambda)$ by $U(\varphi(T)f) = \varphi$. The following calculation shows that U_0 is an isometry:

$$\|\varphi\|_2^2 = \int_\Lambda |\varphi|^2 d\lambda = (|\varphi(T)|^2 f, f) = \|\varphi(T)f\|^2 \tag{46}$$

We know that continuous functions are dense in L^2, and that $\mathfrak{C}_T f$ is dense in $\mathcal{H}$, since f is a cyclic vector. It follows that U extends uniquely to an isometric isomorphism from $\mathcal{H}$ to $L^2(\Lambda, \lambda)$. Furthermore, if we define Γ^* from $\mathfrak{M}_T$ to $L(\mathcal{H}_\Lambda)$, where $\mathcal{H}_\Lambda = L^2(\Lambda, \lambda)$ by $\Gamma^*(A) = UAU^*$.

Next, we show that $\Gamma^*|_{\mathfrak{C}_T} = \Gamma$, the Gelfand transform. This follows from the following calculation, where ψ is any continuous function on Λ:

$$[\Gamma^*(\psi(T))]\varphi = U\psi(T)U^*\varphi = U\psi(T)\varphi(T)f = U[(\psi\varphi)(T)f] = \psi\varphi = M_\psi\varphi$$

This implies $\Gamma^*\psi(T) = M_\psi$ on $L^2(\Lambda, \lambda)$, by density of $C(\Lambda)$ in $L^2(\Lambda, \lambda)$, and also proves 3. in the Proposition.

Next, use the w^* density of $C(\Lambda)$ in $L^\infty(\Lambda)$, and the following calculation, where the closures are in the weak-* operator topology, and w^* topologies, resp.

$$\Gamma^*(\mathfrak{M}_T) = \Gamma^*(\overline{\mathfrak{C}_T}) = \overline{C(\Lambda)} = L^\infty(\Lambda, \lambda)$$

showing that Γ^* is a *-isometric isomorphism mapping $\mathfrak{M}_T$ onto $L^\infty(\Lambda, \lambda)$.

Finally, for the uniqueness statement, note that any *-isometric isomorphism Γ_1^* from $\mathfrak{M}_T$ onto $L^2(\Lambda, \mu)$, where μ is a positive regular Borel measure on Λ induces a *-isometric isomorphism, $\Gamma^*\Gamma_1^{*-1}$ from $L^2(\Lambda, \mu)$ onto $L^2(\Lambda, \lambda)$, and Proposition 3.1.10 completes the proof. □

Lemma 4.0.3. Assume $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras and Φ is a *-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Then,

1. $\|\Phi\| \leqslant 1$
2. Φ is an isometry **iff** it is injective **iff** $\Phi(\mathfrak{A}) \subset \mathfrak{B}$ is a C^*-algebra and Φ is a *-isometric isomorphism from $\mathfrak{A}$ onto $\Phi(\mathfrak{A})$.

Proof. We start by showing that the third statement in 2 follows from the first two, which in turn follows from the fact that the range of an injective isometry is closed. Indeed, $\Phi(T_\alpha)$ is a convergent net in the range of Φ iff T_α is a net convergent to a T in $\mathfrak{A}$, and then $\Phi(T_\alpha) \to \Phi(T)$.

To show that Φ is a contraction, it suffices to check this on self adjoint elements, for, if this is the case, then for any $T \in \mathfrak{A}$ we have

$$\|T\|^2 = \|T^*T\| \geqslant \|\Phi(T^*T)\| = \|\Phi(T)^*\Phi(T)\| = \|\Phi(T)\|^2$$

Thus, let A be self-adjoint in $\mathfrak{A}$, and $\mathfrak{C}_A$ be the C^*-algebra generated by A, and note that $\mathfrak{a} = \overline{\Phi(\mathfrak{C}_A)}$ is a commutative C*-subalgebra of $\mathfrak{B}$. Applying the Gelfand-Naimark theorem, choose a linear multiplicative functional on $\mathfrak{a}$ ψ s.t. $|\psi(\Phi(A))| = \|\Phi(A)\|$. Now, $\psi \circ \Phi$ is a linear multiplicative functional on $\mathfrak{C}_A$, and thus $\|A\| \geqslant |\psi(\Phi(A))|$. Combining these, we find that indeed $\|\Phi(A)\| \leqslant \|A\|$.

Finally, we need to show that Φ is an isometry, which we show by contradiction. Assume $A \in \mathfrak{A}$, $\|A\| = 1$ and $\|\Phi(A)\| = 1 - \varepsilon < 1$. We can assume w.l.o.g that A is self-adjoint, otherwise we work with A^*A. We have $\sigma(A) \subset [0, 1]$. Define a continuous function on $[0, 1]$ s.t. $f(1) = 1$ and $f = 0$ on $[0, 1-\varepsilon]$. By Gelfand-Naimark, $\sigma(f(A)) = \operatorname{ran}\Gamma(f(A)) = f(\sigma(A))$. Hence, $1 \in \sigma(f(A))$, implying $f(A) \neq 0$. However, for P a polynomial $\Phi(P(A)) = P(\Phi(A))$ and by continuity $\Phi(f(A)) = f(\Phi(A))$. Noting that $\|\Phi(A)\| = 1 - \varepsilon$, we must have $\sigma(\Phi(A)) \subset [0, 1-\varepsilon]$.

Working now in $\mathfrak{a}$ we see that

$$\sigma(\Phi(f(A))) = f(\sigma(\Phi(A))) \subset f([0, 1-\varepsilon]) = 0$$

hence $\Phi(A) = 0$, contradiction.

$\square$

Lemma 4.0.4. Let $\mathcal{H}$ be a separable Hilbert space, and assume $\mathfrak{A}$ is a C^*-algebra in $L(\mathcal{H})$ with a separating vector f. Let $\mathcal{M} = \overline{\mathfrak{A}f}$.

1. Then, the mapping $\Phi : \mathfrak{A} \to L(\mathcal{M})$ defined by $\Phi(A) = A|_{\mathcal{M}}$ is a *-isometric isomorphism from $\mathfrak{A}$ onto the C^*-algebra $\Phi(\mathfrak{A})$.

2. Φ is **isospectral**: $\sigma(A) = \sigma(A|_{\mathcal{M}})$ for any $A \in \mathfrak{A}$.

Proof. Φ is clearly a $*$-homomorphism. We show that Φ is injective. Let $A \in \mathfrak{A}$ be such that $\Phi(A) = A|_{\mathcal{M}} = 0$. Since $f \in \mathcal{M}$ is a separating vector, then $A = 0$. By the previous lemma $\|\Phi\| \leqslant 1$ and $\Phi(\mathfrak{A})$ is a C*-algebra. We now show that Φ is isospectral. This is so by Proposition 2.3.11 which implies

$$\sigma_{L(\mathcal{H})}A = \sigma_{\mathfrak{A}}A = \sigma_{\Phi(\mathfrak{A})}(A|_{\mathcal{M}}) = \sigma_{L(\mathcal{M})}(A|_{\mathcal{M}})$$

$\square$

At this stage, we have reduced the construction of a measurable calculus for a normal operator to finding a separating vector for $\mathfrak{C}_T$.

Lemma 4.0.5. Let $\mathcal{H}$ be a separable Hilbert space and $T \in L(\mathcal{H})$ a normal operator.

1. Then there is a direct sum decomposition $\mathcal{H} = \oplus_{n \in J}\mathcal{H}_n$, where $J \subset \mathbb{N}$. (We assume $\mathcal{H}_n \neq \{0\}$, hence we take J to be an initial segment of $\mathbb{N}$ or $\mathbb{N}$ itself.) such that or each $n \in J$ there is an $f_n \in \mathcal{H}_n$ s.t. $\overline{\mathfrak{C}_T f_n} = \mathcal{H}_n$ (that is, f_n is a cyclic vector for $\mathfrak{C}_T$ restricted to $\mathcal{H}_n$).

2. The spaces $\mathcal{H}_n$ are invariant for T.

3. Let $f_j, j \in \mathbb{N}$ be the cyclic vectors given in Lemma 4.0.5. Then the vector $f = \oplus_j f_j$ is separating for $\mathfrak{C}_T$.

Proof. For 1. take any nonzero $f \in \mathcal{H}$, and note that $\mathfrak{C}_T f \neq \{0\}$ since $I \in \mathfrak{C}_T$ and $If = f$. Let $\mathcal{H}_1 = \overline{\mathfrak{C}_T f}$. Take now a nonzero $f_2 \in \mathcal{H}_1^{\perp}$, and let $\mathcal{H}_2 = \overline{\mathfrak{C}_T f_2}$, and so on. This process may stop after a finite number of steps, leading to the conclusion when J is finite. If it does not, the proof is left as an exercise, below. 2. follows from 1. For 3., assume that $Af = 0$. Then, by construction $Af_n = 0$ for any $n \in J$. Fix an n and any $v \in \mathcal{H}$. For $\varepsilon > 0$ let $B_\varepsilon \in \mathfrak{C}_T$ be s.t. $\|B_\varepsilon f_n - v\| < \varepsilon$. Since $0 = B_\varepsilon A f_n = AB_\varepsilon f_n$, you can check by passing to the limit $\varepsilon \to 0$ that $Av = 0$, hence $A|_{\mathcal{H}_n} = 0$, and since n is arbitrary, we have $A = 0$. $\square$

Exercise 20. Define appropriate chains of spaces for Zorn's lemma to complete the proof when J is infinite.

These results imply the following form of the spectral theorem.

Theorem 4.0.6 (Spectral theorem, multiplication operator form)**.** Let T be a bounded normal operator on the separable Hilbert space $\mathcal{H}$. Then, there exists $J \subset \mathbb{N}$, measures $\{\lambda_n\}_{n\in J}$ and a unitary operator U from $\mathcal{H}$ onto $\bigoplus_{n\in J} L^2(\sigma(T), \lambda_n)$, giving the **spectral representation** of T: for any

$$\langle \varphi_1, \cdots, \varphi_n, \cdots \rangle \in \bigoplus_{n\in J} L^2(\Lambda, \lambda_n)$$

and any $\lambda \in \Lambda$ and $n \in J$, we have

$$(UTU^*\varphi)_n(\lambda) = \lambda\varphi_n(\lambda)$$

Corollary 4.0.7. Let $\mathcal{H}$ be a separable Hilbert space and $T \in L(\mathcal{H})$ be normal.

1. There exists a probability space (M, λ), an $F \in L^\infty(M, \lambda)$, and a unitary map $U : \mathcal{H}$ onto $L^2(M, \lambda)$ so that for any $\varphi \in L^2(M, \lambda)$ we have

$$(UTU^*\varphi)(x) = F(x)\varphi(x) \ \ \forall x \in M$$

2. The C^*-algebra and W^*-algebra generated by T have a separating vector.

Proof. For 1, using Lemma 4.0.3, we choose for each n a cyclic vector f_n on $\mathcal{H}_n$ with $\|f_n\| = 2^{-n}$. Proposition 4.0.1 shows that the corresponding measures λ_n on Λ have the property $|\lambda_n| = 2^{-n}$. We now take $M = \Lambda^J$ with the product σ-algebra, the measure μ defined by $\mu = \mu_n$ on the nth component of M, and F defined as the identity function on the each component of M. Then $\mu(M) = 1$ and the result follows. $\square$

Theorem 4.0.8 (Spectral Theorem, Extended Functional Calculus). Let $\mathcal{H}$ be a separable Hilbert space and $T \in L(\mathcal{H})$ be normal.

Let f be a separating vector for $\mathfrak{C}_T$ and $\mathcal{M} = \overline{\mathfrak{C}_T f}$. Replacing $\mathcal{H}$ by the subspace $\mathcal{M}$ and the operators in $\mathfrak{C}_T$ and $\mathfrak{M}_T$ by their restriction to $\mathcal{M}$, Proposition 4.0.1 applies, providing a unitary equivalence from $\mathcal{M}$ onto $\mathcal{H}_\Lambda = L^2(\Lambda, \lambda)$. The Gelfand transform from $\mathfrak{C}_T$ onto $\mathcal{H}_\Lambda$ extends to a *-isometric isomorphism Γ^* from $\mathfrak{M}_T$ onto $L^\infty(\lambda)$, and $\Gamma^* A = UAU^*$.

Moreover, the measure λ is unique up to mutual absolute continuity while $L^\infty(\lambda)$ and Γ^* are unique.

Proof. Lemma 4.0.4 implies that $\sigma(T) = \sigma(T|_{\mathcal{M}})$. By construction, f is a cyclic vector for $\mathfrak{C}_{T|_{\mathcal{M}}}$. Now Proposition 4.0.1 applies finishing the proof. $\square$

4.1 Integration of normal operators

Definition 4.1.1. Let $\mathcal{H}$ be a separable Hilbert space and $T \in L(\mathcal{H})$ be normal. Let $\mathfrak{W}_T$ be the W-* algebra generated by T. With Γ^* as in Theorem 4.0.8 and $F \in \mathfrak{W}_T$ it is natural to define for a measurable $A \subset \Lambda$,

$$\int_A F d\lambda = \int_A \Gamma^* F d\lambda$$

4.2 Spectral projections

Definition 4.2.1. Let $\mathcal{H}$ be a Hilbert space and (X, M) be a measurable space. Then a projection-valued measure on (X, M) is a function P from the σ-algebra M to the projections in $L(\mathcal{H})$ s.t.

1. $P(X) = I$, the identity and $P(\emptyset) = 0$.
2. If $A, B \in M$, then $P(A \cap B) = P(A)P(B)$.
3. If $\{A_j\}_{j\in\mathbb{N}}$ are disjoint sets in M and $A = \cup_{j\in\mathbb{N}} A_j$, then

$$P(\cup_{j\in\mathbb{N}} A_j) = \sum_{j\in\mathbb{N}} P(A_j)$$

   where the sum converges in the strong operator norm.

Let $\Lambda \subset \mathbb{C}$, λ a measure with support Λ, and write as before $\mathcal{H}_\Lambda = L^2(\Lambda, \lambda)$. Note that the map $I(A) = \chi_A$ is a projection-valued measure for $\mathcal{H}_\Lambda = L^2(\Lambda, \lambda)$.

Definition 4.2.2. Let $\mathcal{H}$ be a separable Hilbert space and T be a normal operator on $\mathcal{H}$ with spectrum Λ. Let Γ^* be as in Theorem 4.0.6. Then, the spectral projection for T is defined as the map $P_A := P(A) = {\Gamma^*}^{-1}\chi_A$, where A is a Lebesgue measurable set in Λ.

Exercise 21. 1. Let $\mathcal{H}$ be a separable Hilbert space and T a normal operator on $\mathcal{H}$ with spectrum Λ. Let $(X, M) = (\Lambda, \mathcal{L})$, where $\mathcal{L}$ is the family of Lebesgue measurable sets in Λ. Check that the spectral projection in Definition 4.2.2 is a projection-valued measure.

2. Check that for any $\varphi \in \mathcal{H}$, the map $m(A) = (\varphi, P_A\varphi)$ is a Borel measure on Λ.

3. Let $f \in L^\infty(\Lambda)$. With m as above, show that

$$(\varphi, B\varphi) = \int f(\lambda) dm; \quad \forall \varphi \in \mathcal{H}$$

defines a unique bounded operator on $\mathcal{H}$.

Definition 4.2.3 (Integration with respect to spectral projections). Let B be defined as in Exercise 21. We write

$$B = \int f(\lambda) dP_\lambda \tag{47}$$

Exercise 22. Let

$$f_\varepsilon(x) = \frac{1}{2\pi i} \int_a^b \left(\frac{1}{x - \lambda - i\varepsilon} - \frac{1}{x - \lambda + i\varepsilon} \right) d\lambda$$

Check that $|f_\varepsilon|$ is uniformly bounded in $\varepsilon > 0$ and that $\lim_{\varepsilon \to 0^+} f_\varepsilon(x) = \frac{1}{2}(\chi_{[a,b]} + \chi_{(a,b)})$. This formula can be checked by direct integration, or by contour deformation. Use these facts to prove Stone's formula:

Theorem 4.2.4. *Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1} \right] d\lambda = \frac{1}{2}(P_{[a,b]} + P_{(a,b)})$$

where the limit is in the strong operator topology, and P denotes the spectral projection.

Theorem 4.2.5 (Spectral Theorem: projection-valued measure form). Let $\mathcal{H}$ be a separable Hilbert space, T a normal operator on $\mathcal{H}$ with spectrum Λ. Then

$$T = \int \lambda dP_\lambda$$

Proof. An easy exercise. □

Definition 4.2.6 (Types of spectrum). Let $\mathcal{H}$ be a separable Hilbert space and T a bounded operator on $\mathcal{H}$ with spectrum Λ, and P the spectral projection associated to T.

1. λ is in the **point spectrum** of T, $\sigma_p(T)$, if λ is an eigenvalue of T, that is, there is a nonzero $\varphi \in \mathcal{H}$ s.t. $T\varphi = \lambda\varphi$.

2. If λ is not an eigenvalue of T and $\operatorname{ran}(\lambda - T)$ is not dense, then λ belongs to the **residual spectrum** of T.

   Exercise 23. *Show that if λ is in the residual spectrum of T then λ is in the point spectrum of T^*, and that if λ is in the point spectrum of T, then $\lambda \in \sigma(T^*)$, and it can be in either its point or residual spectrum.*

3. Let λ be the spectral measure of a normal operator, and let $\lambda = \lambda_{pp} + \lambda_{ac} + \lambda_{sing}$ be the decomposition of λ w.r.t. the Lebesgue measure into the pure point part, the absolutely continuous part, and the singular continuous part. Since of course these parts are mutually singular, we have

   $$L^2(\Lambda, \lambda) = L^2(\Lambda, \lambda_{pp}) \oplus L^2(\Lambda, \lambda_{ac}) \oplus L^2(\Lambda, \lambda_{sing})$$

   Through Γ this induces a decomposition of $\mathcal{H}$:

   $$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$$

   Exercise 24. *Show that if T is normal, with the decomposition above, $T|_{\mathcal{H}_{pp}}$ has a complete set of eigenvectors.*

4. We have the following characterization of the spectrum of a normal operator:

   (a) $\sigma_{pp}(T) = \{\lambda : \lambda \text{ is an eigenvalue of } T\}$.

   (b) $\sigma_{cont}(T) = \sigma(T|_{\mathcal{H}_{cont}}) =: \sigma(T|_{\mathcal{H}_{sing} \oplus \mathcal{H}_{ac}})$

   (c) $\sigma_{ac}(T) = \sigma(T|_{\mathcal{H}_{ac}})$

   (d) $\sigma_{sing}(T) = \sigma(T|_{\mathcal{H}_{sing}})$

Also: λ is in the **essential spectrum** of T, $\sigma_{ess}(T)$, if, for any open neighborhood $\mathcal{O}$ of λ is infinite dimensional. Otherwise $\lambda \in \sigma_d(T)$, the **discrete spectrum** of T.

Exercise 25. *Show that*

$$\sigma_{cont}(T) = \sigma_{ac}(T) + \sigma_{sing}(T) \ \text{ and } \sigma(T) = \overline{\sigma_{pp}(T)} \cup \sigma_{cont}(T)$$

Exercise 26. 1. With the notations in the theorem above, show that $\sigma_{ess}(T)$ is always a compact set. Is $\sigma_d(T)$ necessarily compact?

2. Show that $\lambda \in \sigma_d(T)$ iff both conditions below hold:

   (a) λ is an isolated point of Λ, that is, there is an open set $\mathcal{O}$ containing λ such that $\Lambda \cap \mathcal{O} = \{\lambda\}$.

   (b) λ is a **eigenvalue of T of finite multiplicity**. This means $\{\varphi \in \mathcal{H} : T\varphi = \lambda\varphi\} \neq \{0\}$ and is finite-dimensional.

5 Bounded and unbounded operators

1. Let X, Y be Banach spaces and $\mathcal{D} \subset X$ a *linear* space, **not necessarily closed**.
2. A linear operator is a linear map $T : \mathcal{D} \to Y$.
3. $\mathcal{D}$ is the domain of T, sometimes written dom (T), or $\mathcal{D}\,(T)$.
4. The *range* of T, ran (T), is simply $T(\mathcal{D})$.
5. The *graph of T* is

$$\mathbb{G}(T) = \{[x, Tx] | x \in \mathcal{D}\,(T)\}$$

   where $[a, b]$ will denote the ordered pair of elements a, b whenever we risk confusion with the inner product. The graph will play an important role, especially for unbounded operators.

6. The kernel of T is

$$\ker\,(T) = \{x \in \mathcal{D}\,(T) : Tx = 0\}$$

5.1 Operations

1. $aT_1 + bT_2$ is defined on $\mathcal{D}\,(T_1) \cap \mathcal{D}\,(T_2)$.
2. Let $T_1 : \mathcal{D}\,(T_1) \to Y$ and $T_2 : \mathcal{D}\,(T_2) \to Z$, where $\mathcal{D}\,(T_1) \subset X$ and $\mathcal{D}\,(T_2) \subset Y$. Then, the composition is defined on $\mathcal{D}(T_2T_1) = \{x \in \mathcal{D}\,(T_1) : T_1(x) \in \mathcal{D}\,(T_2)\}$ with values in Z.

   In particular, if $\mathcal{D}\,(T)$ and ran(T) are both in the space X, then, inductively, $\mathcal{D}\,(T^n) = \{x \in \mathcal{D}\,(T^{n-1}) : T(x) \in \mathcal{D}\,(T)\}$. The domain of the composition may be trivial, $\{0\}$.

3. **Inverse**. The inverse is defined iff $\ker\,(T) = \{0\}$. This condition of course implies T is bijective from its domain to its range. Then $T^{-1} : \text{ran}\,(T) \to \mathcal{D}\,(T)$ is defined as the usual function inverse, and is clearly linear.

   Example 5.1.1. Let $\mathcal{S}(\mathbb{R})$ be the Schwarz space, with its Fréchet space topology induced by the family of seminorms $\|f\|_n = \sup_{x\in\mathbb{R}} |\partial^n f|$. Then $\partial : \mathcal{S} \to \mathcal{S}$ is injective. The seminorms $\sup_{x\in[0,1]} |\partial^n f|$, where the derivatives at the endpoints are understood as lateral derivatives, make $C^\infty([0,1])$ a Fréchet space where ∂ is not injective.

Exercise 27. Check that the Fréchet space topologies above do not come from a norm. One way is to use Arzelá-Ascoli to show that every closed and bounded set is compact, which, in a normed space, would imply that the space is finitely dimensional.

In this sense, in a Banach space of functions, the domain of ∂ is never closed.

4. **Closed operators**. Let $\mathcal{B}$ be a Banach space, $\mathcal{D} \subset \mathcal{B}$ and $T : \mathcal{D} \to \mathcal{B}$. T is called **closed** if $\mathbb{G}(T)$ is a closed set in $\mathcal{B} \times \mathcal{B}$. Note that this does not imply that the *domain* of T is closed.

5. **Closable operators**. Let $\mathcal{B}$ be a Banach space, $\mathcal{D} \subset \mathcal{B}$ and $T : \mathcal{D} \to \mathcal{B}$. T is called **closable** if
$$\overline{\mathbb{G}(T)} \text{ is the graph of an operator} \tag{48}$$
By linearity, closability is clearly equivalent to the condition
$$\Big(x_n \to 0 \text{ and } Tx_n \to y\Big) \Rightarrow y = 0 \tag{49}$$
In this case, the extension $\tilde{T}$ defined by $\tilde{T}x := y$ whenever $x_n \to x$ and $Tx_n \to y$ is consistent and defines a linear operator.

Exercise 28 (Due Mar 31)**.** (a) Define X on $\mathcal{S}$ by $(X\varphi)(x) = x\varphi(x)$; as you know (or can easily check), $\mathcal{S}$ is dense in $L^2(\mathbb{R})$. Show that X is closable and find the domain of its closure.

(b) Use the properties of the Fourier transform to show that $\partial : \mathcal{S} \to L^2(\mathbb{R})$ is closable, and find the domain of its closure.

6. Clearly, a bounded operator defined on a Banach space is closed, by the closed graph theorem.

7. It will turn out that symmetric operators on Hilbert spaces, meaning $(Ax, y) = (x, Ay)$ for all $x, y \in \mathrm{dom}\,(A)$, are closable.

   Operators that are not closable are pathological in a number of ways. Common operators are however "usually" closable. E.g., ∂ defined on a subset of *continuously differentiable functions* as a dense subset of, say $L^2(0,1)$, is closable (see Exercise 27 as well). Assume $f_n \overset{L^2}{\to} 0$ (in the sense of L^2) and $f_n' \overset{L^2}{\to} g$. Since $f_n' \in C^1$ we have
$$f_n(x) - f_n(0) = \int_0^x f_n'(s)ds = (f_n', \chi_{[0,x]}) \to (g, \chi_{[0,x]}) = \int_0^x g(s)ds \tag{50}$$
   Thus
$$(\forall x) \ \lim_{n\to\infty} (f_n(x) - f_n(0)) = 0 = \int_0^x g(s)ds \tag{51}$$
   implying $g = 0$.

6. As an example of non-closable operator, consider, say $L^2[0,1]$ (or any separable Hilbert space) with an orthonormal basis e_n. Define $Ne_n = ne_1$, extended by linearity, whenever

it makes sense (it is an unbounded operator). Then $x_n = e_n/n \to 0$, while $Nx_n = e_1 \neq 0$. Thus N is not closable.

Every infinite-dimensional normed space admits a nonclosable linear operator. The proof requires the axiom of choice and so it is in general non-constructive. See also Exercise 29 below.

The closure of an operator T based on graph closure is called the *canonical closure* of T.

5.1.1 Adjoints of unbounded operators.

1. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces (we will most often be interested in the case $\mathcal{H} = \mathcal{K}$), with scalar products $(,)_{\mathcal{H}}$ and $(,)_{\mathcal{K}}$.

2. $T : D(T) \subset \mathcal{H} \to \mathcal{K}$ is **densely defined** if $\overline{D(T)} = \mathcal{H}$.

3. Assume T is densely defined.

4. The adjoint of T is defined as follows. We look for those y for which

$$\exists\, v = v(y) \in \mathcal{H} \text{ s.t.} \forall\, x \in D(T),\ (y, Tx)_{\mathcal{K}} = (v, x)_{\mathcal{H}} \tag{52}$$

   Since $D(T)$ is dense, such a $v = v(y)$, if it exists, is unique.

5. We define $D(T^*)$ to be the set of y for which $v(x)$ as in (52) exists, and define $T^*(y) = v$. Note that $Tx \in \mathcal{K}$, $y \in \mathcal{K}$, $T^*y \in \mathcal{H}$.

6. An unbounded operator A is self-adjoint if $A = A^*$. Note: this means in particular *that A and A^* have the same domain*. We will return to this.

7. **Spectrum** . Let X be a Banach space and $\mathcal{D} \subset X$. The resolvent set $\rho(T)$ of a a densely defined operator $T : \mathcal{D} \to X$ is defined as the set $\rho(T)$ of $\lambda \in \mathbb{C}$ s.t.

$$(T - \lambda) \text{ is injective from } \mathcal{D} \text{ to } \operatorname{ran} T \tag{53}$$

$$\overline{\operatorname{ran}(T - \lambda)} = X \text{ and} \tag{54}$$

$$(T - \lambda)^{-1} \text{ extends as a bounded operator on } X \tag{55}$$

   The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$. There are various reasons for $(T - \lambda)^{-1}$ not to exist: $(T - \lambda)$ might not be injective, $(T - \lambda)^{-1}$ might be unbounded, or not densely defined. These possibilities correspond to different types of spectra, similar to the ones encountered in the bounded operator case.

8. The spectrum of unbounded operators, even closed ones, can be any closed set, including $\emptyset$ and $\mathbb{C}$.

Proposition 5.1.2. *Let X be a Banach space and $\mathcal{D} \subset X$ a dense linear space and $T : \mathcal{D} \to X$. The resolvent set $\rho(T)$ is an open subset of $\mathbb{C}$ (possibly empty).*

Proof. A useful exercise. For part of the proof the second resolvent formula is helpful. □

Let $T_1 = \partial$ be defined on $\mathcal{D}(T_1) = \{f \in C^1[0,1] : f(0) = 0\}$ [16] with values in the Banach space $C[0,1]$ (with the sup norm). (Note also that $\operatorname{dom} T$ is dense in $C[0,1]$.) Then the spectrum of T_1 is empty. (Does this contradict Exercise 6, 2.?)

Indeed, to show that the spectrum is empty, note that by assumption $(\partial - z)\mathcal{D}(T_1) \subset C[0,1]$. Now, $(\partial - z)f = g$, $f(0) = 0$ is a linear differential equation with a unique solution

$$f(x) = e^{xz} \int_0^x e^{-zs} g(s) ds$$

We can therefore check that f defined above is an inverse for $(\partial - z)$, by checking that $f \in C^1[0,1]$, and indeed it satisfies the differential equation. Clearly $\|f\| \leqslant const(z)\|g\|$.

At the opposite end, take $L^2(\mathbb{C})$ with the Lebesgue measure, and the operator M_I of multiplication with the identity function $I(z) = z$, densely defined on $\mathcal{S}(\mathbb{C})$. Then $\sigma(M_I) = \mathbb{C}$. This operator is normal if defined on $\{f \in L^2(\mathbb{C}) : M_I f \in L^2(\mathbb{C})\}$, as you can check by a modification of the arguments in Exercise 28. Another example of an operator with spectrum $\mathbb{C}$ is $T_0 = \partial$ defined on $\mathcal{D}(T_0) = C^1[0,1]$, a dense subset of the Banach space $C[0,1]$. Indeed, for any $z \in \mathbb{C}$, if $f(x; z) = e^{zx}$, then $T_0 f - zf = 0$.

We note that T_0 is closed too, since if $f_n \to 0$ then $f_n - f_n(0) \to 0$ as well.

9. Any non-closable operators has spectrum the whole of $\mathbb{C}$. Indeed, if T is not closable, you can check that neither is $T - \lambda$, $\lambda \in \mathbb{C}$. We then only have to show that a non-closable T is not invertible. Assume T^{-1} existed and let the sequence $x_n \to 0$ be s.t. $Tx_n = y_n \to y \neq 0$. Then, $T^{-1}Tx_n = T^{-1}y_n \to T^{-1}y$ but $T^{-1}Tx_n = x_n \to 0$, thus $T^{-1}y = 0$ But $0 \in \mathcal{D}(T)$ since $\mathcal{D}(T)$ is a linear space. Then, by linearity, $T(0) = 0 \neq y$ a contradiction.

10.

Proposition 5.1.3. *If $T : \mathcal{D}(T) \subset X \to Y$ is closed and injective, then T^{-1} is also closed.*

Proof. Define the involution $J : \mathcal{B} \times \mathcal{B}$ by $J(x, y) = (y, x)$. J is clearly an automorphism of $\mathcal{B} \times \mathcal{B}$ and $\mathbb{G}(T^{-1}) = J(\mathbb{G}(T))$. □

Corollary 5.1.4. Let $\mathcal{B}, \mathcal{B}'$ be Banach spaces, $\mathcal{D}$ dense in $\mathcal{B}$, and $T : \mathcal{D} \to \mathcal{B}'$. If $\sigma(T) \neq \mathbb{C}$, then T is closable.

Proof. Immediate, if we note that T is closed iff $T + \lambda I$ is closed for some/any λ. □

Proposition 5.1.5. *Let $\mathcal{B}, \mathcal{B}'$ be Banach spaces, $\mathcal{D}$ a linear subspace of $\mathcal{B}$ and $T : \mathcal{D} \to \mathcal{B}'$ be closed. If T is injective and onto, then T^{-1} is bounded.*

Proof. We see that T^{-1} is defined everywhere and it is closed. By the closed graph theorem, it is bounded. □

[16] $f(0) = 0$ can be replaced by $f(a) = 0$ for some fixed $a \in [0,1]$.

As an example of an operator with unbounded left inverse we have $(\mathcal{P}f)(x) := \int_0^x f$. The left inverse, $f \mapsto f'$ is unbounded from $\operatorname{ran}\mathcal{P} \to L^2$; however, $\operatorname{ran}\mathcal{P} = AC \cap L^2 \subsetneq L^2$ (it is only a dense subspace of L^2).

11. An interesting example is $\mathcal{E}$ defined by $(\mathcal{E}\psi)(x) = \psi(x+1)$. This is well defined and bounded (unitary) on $L^2(\mathbb{R})$. The "same" operator can be defined on the polynomials on $[0,1]$, an L^∞ dense subset of $C[0,1]$.

Exercise 29. In $L^2[0,1]$ consider the "extrapolation" operator $\mathcal{E}$ densely defined on polynomials by $\mathcal{E}(P)(x) = P(x+1)$. Show that $\mathcal{E}$ is a bijection from its domain to its range. Is $\mathcal{E}$ bounded? Is $\mathcal{E}$ closable? What is the domain of $\mathcal{E}^*$? Calculate the spectrum $\sigma(\mathcal{E})$ from the definition of σ. The solution is given in 11, 58, in the sequel.

Exercise 30. Show that $T_2 = \partial$ defined on $\mathcal{D}(T_2) = \{f \in C^1[0,1] : f(0) = f(1)\}$ has spectrum exactly $2\pi i\mathbb{Z}$.

12. Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces, $\mathcal{D}$ a dense linear space in $\mathcal{H}$ and $T\mathcal{H} \to \mathcal{H}$. Let $U : \mathcal{H} \to \mathcal{H}'$ be unitary, and consider the image of T, $UTU^* : U\mathcal{D} \to \mathcal{H}'$. Show that T and UTU^* have the same spectrum.

13. The spectrum depends very much on the domain of definition: the larger the domain is, the larger the spectrum is. This is easy to see from the definition of the inverse.

Exercise 31 (Bonus)**.** Consider the operator T defined on $\mathcal{S}(\mathbb{R})$ by $(T\varphi)(x) = x\varphi'(x)$. Show that T is closable.

6 Integration and measures on Banach spaces

In the following Ω is a topological space, Σ is the Borel σ-algebra over Ω, $\mathcal{B}$ is a Banach space, μ is a signed measure on Ω. Integration can be defined on functions from Ω to $\mathcal{B}$, as in usual measure theory, starting with simple functions.

1. A simple function is a sum of indicator functions of measurable mutually disjoint sets with values in $\mathcal{B}$:
$$f(\omega) = \sum_{j\in J} x_j \chi_{A_j}(\omega); \quad \operatorname{card}(J) < \infty \tag{56}$$
where $x_j \in \mathcal{B}$ and $\cup_j A_j = \Omega$.

2. We denote by $\mathcal{L}_s(\Omega, \mathcal{B})$ the linear space of simple functions from Ω to $\mathcal{B}$.

In the sequel, we will define a norm on $\mathcal{L}_s(\Omega, \mathcal{B})$ and find its completion $B(\Omega, \mathcal{B})$ as a Banach space. We then define an integral on $\mathcal{L}_s(\Omega, \mathcal{B})$, and show it is norm continuous. At that stage, the integral on B is defined by continuity. We will then identify the space $B(\Omega, \mathcal{B})$ and find the properties of the integral.

Exercise 32.

1. Let $\mathcal{B} = L^\infty(\mathbb{R})$ and $\mathcal{D} = C_c(\mathbb{R})$, the dense subset consisting of compactly supported functions which are continuous almost everywhere, whose support is a finite union of intervals. Let $B : \mathcal{D} \to \mathbb{C}$ be the Riemann integral (recall Theorem 2.28 in Folland.) Check that $\|f\|_1 = B(|f|)$ is a norm on the vector space $\mathcal{D}$. Show that the completion S of $(\mathcal{D}, \|\|_1)$ can be identified with $L^1(\mathbb{R})$. Clearly, B is bounded and densely defined. Show that the extension of B to $L^1(\mathbb{R})$ is the Lebesgue integral.

2. Perform a similar construction on $\mathbb{R}^n$.

   Note: the point of this exercise is to define the Lebesgue integral as a continuous extension of the Riemann integral. You don't have to follow the steps above if you prefer an approach of your own.

1. $\mathcal{L}_s(\Omega, \mathcal{B})$ is a normed linear space, under the sup norm

$$\|f\| = \|f\|_\infty := \sup_{\omega\in\Omega} \|f(\omega)\| \tag{57}$$

2. We define $B(\Omega, \mathcal{B})$ the completion of $\mathcal{L}_s(\Omega, \mathcal{B})$ in $\|f\|_\infty$.

3. We write $\|f\|_A = \sup_{\omega\in A} \|f(\omega)\|$. Check that, for a disjoint partition $\{A_j\}_{j=1,\dots,n}$ we have

$$\|f\| = \max_{j\in J} \sup_{x\in A_j} \|f(x)\| =: \max_{j\in J} \|f\|_{A_j} \tag{58}$$

4. *Refinements*. Assume $\{A_j\}_{j=1,\dots,n}$ is partition and $\{A'_j\}_{j=1,\dots,n'}$ is a subpartition, in the sense that for any A_j there exist $A'_{j1}, \dots, A'_{jm}$ so that $A_j = \cup_{i=1}^m A'_{ji}$

5. An integral is defined on $\mathcal{L}_s(\Omega, \mathcal{B})$ as in the scalar case by

$$\int f d\mu = \sum_{j\in J} \mu(A_j) x_j \tag{59}$$

   and in a natural way, the integral over a subset of $A \in \Sigma(\Omega)$ is defined

$$\int_A f d\mu = \int \chi_A f d\mu \tag{60}$$

   Check that, if we choose $x'_j = x_j$ for each $A'_j \subset A_j$ then

$$\sum_j x_j \chi_{A_j} = \sum_j x'_j \chi_{A'_j} \tag{61}$$

and

$$\int_\Omega \sum_j x_j \chi_{A_j} d\mu = \int_\Omega \sum_j x'_j \chi_{A'_j} d\mu \tag{62}$$

6. Note that if A, B are disjoint sets in $\Sigma(\Omega)$, then

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu \tag{63}$$

Lemma 6.0.1. If $\{A'_i\}_{i=1,\ldots,n'}$ is a subpartition of $\{A_i\}_{i=1,\ldots,n}$ in the sense that $A'_{i'} \subset A_i$ for any i' and some i, and if $x_{i'} = x_i$ whenever $A'_{i'} \subset A_i$, then $\sum_{i'=1}^{n'} x_{i'}\chi_{A_{i'}} = \sum_{i=1}^{n} x_i \chi_{A_i}$.

7. *Proof.* Since $\chi_{A+B} = \chi_A + \chi_B$, this is immediate. □

Lemma 6.0.2. If $f \in B(\Omega, \mathcal{B})$, then for any $\varepsilon > 0$ there is a partition $\{A_i\}_{i=1,\ldots,n}$ of Ω such that for *any* $\omega_j \in A_j$ we have

$$\|f - \sum_j f(\omega_j)\chi_{A_j}\| \leqslant \varepsilon \tag{64}$$

Proof. Choose a partition $\{A_i\}_{i=1,\ldots,n}$ of Ω and x_j so that

$$\|f - \sum_j x_j \chi_{A_j}\| \leqslant \varepsilon/2$$

This implies by 3 above that

$$\|x_j - f(\omega_j)\| \leqslant \varepsilon/2$$

for *all* $\omega_j \in A_j$. The rest follows from the triangle inequality. □

Lemma 6.0.3. If $f_1, f_2 \in B(\Omega, \mathcal{B})$, then for any $\varepsilon > 0$ there is a (disjoint) partition $\{A_i\}_{i=1,\ldots,n}$ of Ω such that for any $\omega_j \in A_j$ we have

$$\|f_i - \sum_{j=1}^{n} f(\omega_j)\chi_{A_j}\|_{\mathcal{B}} \leqslant \varepsilon, \ i = 1, 2 \tag{65}$$

Proof. Taking as a partition a common refinement of the partitions for f_1 and f_2 which agree with f_1 and f_2 resp. within $\varepsilon/2$ this is an immediate consequence of the previous two lemmas and the triangle inequality. □

Assume $|\mu|(\Omega) < \infty$. If not, we restrict our analysis to subsets $\Omega' \in \Omega$ such that $|\mu|(\Omega') < \infty$.

Lemma 6.0.4. The map $f \to \int f d\mu$ is well defined on simple functions, linear and bounded in the sense

$$\left\|\int_A f d\mu\right\| \leqslant \int_A \|f\| d|\mu| \leqslant \|f\|_{\infty,A}|\mu|(A) \tag{66}$$

where $|\mu|$ is the total variation of the signed measure μ, $|\mu| = \mu^+ + \mu^-$, where $\mu = \mu^+ - \mu^-$ is the Hahn-Jordan decomposition of μ.

Proof. All properties are immediate, except perhaps boundedness. We have

$$\left\|\int_A f d\mu\right\| \leqslant \sum_{j\in J} |\mu|(A_j)\|x_j\| = \int_A \|f\| d|\mu| \leqslant \|f\|_{\infty,A}|\mu|(A) \tag{67}$$

□

8. Thus $\int_A$ is a linear bounded operator from $\mathcal{L}_s(\Omega, \mathcal{B})$ to $\mathcal{B}$ and it extends to a bounded linear operator on from $B(\Omega, \mathcal{B})$ to $\mathcal{B}$. We keep the notation $\int$ for this extension.

   Let $\mathcal{D}$ is a linear space in the Banach space $\mathcal{B}$ and $T : \mathcal{D} \to \mathcal{B}$ a linear operator. If the range of $f \in B(\Omega, \mathcal{B})$ is contained in $\mathcal{D}$, then we naturally define Tf by $(Tf)(\omega) = Tf(\omega)$, $\omega \in \Omega$.

Theorem 6.0.5 (Commutation of closed operators with integration)**.** Let $\mathcal{D} \subset \mathcal{B}$ be a linear space and $T : \mathcal{D} \to \mathcal{B}$ closed. If $f \in B(\Omega, \mathcal{B})$ is such that $f(\Omega) \subset \mathcal{D}(T)$ and moreover $Tf \in B(\Omega, \mathcal{B})$ as well, then

$$T\int_A f d\mu = \int_A Tf d\mu \tag{68}$$

Proof. By definition, there is a sequence of refinements of A, $\{A_{j,n}\}_{j\leqslant n, n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|f - \sum_{j=1}^{n} f(\omega_j)\chi_{A_{n,j}}\| = 0$$

Similarly, there is a sequence of refinements of A, which we can choose to be $\{A_{j,n}\}_{j\leqslant n, n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|Tf - \sum_{j=1}^{n} y_{j,n}\chi_{A_{nj}}\| = 0$$

and in particular, $\sum_{j=1}^{n} y_{j,n}\chi_{A_{nj}}$ converges.

For $\omega_{j,n} \in A_{j,n}$ we have

$$\lim_{n\to\infty} \|(Tf)(\omega_{j,n}) - y_{j,n}\| = 0 = \lim_{n\to\infty} \|T(f(\omega_{j,n})) - y_{j,n}\|$$

This implies

$$\lim_{n\to\infty} \|T\left(\sum_{j\leqslant n} f(\omega_{j,n})\chi_{a_{j,n}}\right) - \sum_{j\leqslant n} y_{j,n}\chi_{a_{j,n}}\| = 0$$

Hence, the sequence $T\left(\sum_{j\leqslant n} f(\omega_{j,n})\chi_{a_{j,n}}\right)$ converges, while $\sum_{j\leqslant n} f(\omega_{j,n})\chi_{a_{j,n}}$ converges to f. Since T is closed, this means

$$\lim_{n\to\infty} T\left(\sum_{j\leqslant n} f(\omega_{j,n})\chi_{a_{j,n}}\right) = Tf$$

Let $f_n = (\sum_{j\leqslant n} f(\omega_{j,n})\chi_{a_{j,n}}$. For any simple function f_s,

$$\int_A Tf_s = T\int_A f_s \tag{69}$$

For any n we have $\int_A Tf_n = T\int_A f_n$. The left side converges, by the continuity of the integral, thus $T\int_A f_n$ converges and $\int_A f_n$ converges to $\int_A f$. Using again the fact that T is closed, the result follows. □

Note 6.0.6. Check, from the definition of the integral, that if $B(\omega)$ is a bounded operator and $T : D(T) \to \mathcal{B}$ is any operator and $u \in D(T)$, then

$$\int_A [B(\omega)Tu]d\mu(\omega) = \left[\int_A B(\omega)d\mu(\omega)\right] Tu \tag{70}$$

Exercise 33. Formulate and prove a theorem allowing to differentiate under the integral sign in the way

$$\frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{\partial}{\partial x} f(x,y)dy$$

Which functions are in $B(\Omega,\mathcal{B})$?
In a metric space a set is *totally bounded* if, for any $\varepsilon > 0$, it can be covered by finitely many balls of radius ε. This is the case iff it is precompact, that is, its closure is a compact set.

Theorem 6.0.7. The function f is in $B(\Omega,\mathcal{B})$ **iff** f is measurable and $f(\Omega)$ is totally bounded.

Proof. Let $f \in B(\Omega,\mathcal{B})$ and $\varepsilon > 0$. By definition, there is a simple function f_ε such that $\|f - f_\varepsilon\| < \varepsilon$. Writing $f_\varepsilon = \sum_{j=1}^m x_j\chi_{A_j}$, we see that $\mathrm{dist}(f(\Omega), \{x_1, ..., x_m\}) < \varepsilon$. This easily implies that $f(\Omega)$ is totally bounded.

Now, assume $f(\Omega)$ is totally bounded. Let $\varepsilon > 0$ and $\mathbb{B}_\varepsilon(x_1), ..., \mathbb{B}_\varepsilon(x_n)$ be a cover of $f(\Omega)$ with balls of radius ε. We can construct out of it a disjoint cover as usual: $A_1 = \mathbb{B}_\varepsilon(x_1), A_2 = \mathbb{B}_\varepsilon(x_2) \setminus \mathbb{B}_\varepsilon(x_1), ..., A_n = \mathbb{B}_\varepsilon(x_n) \setminus \cup_{j=1}^{n-1}\mathbb{B}_\varepsilon(x_j)$. You can check that $\|f - \sum_{j=1}^n x_j\chi_{A_j}\| < \varepsilon$. □

The space $B(\Omega, \mathcal{B})$ looks quite small. If however Ω is compact, as is the important case of the spectrum of a bounded operator, then B contains all continuous functions:

Proposition 6.0.8. If $\Omega' \subset \Omega$ is compact, then $C(\Omega', \mathcal{B})$, the continuous $\mathcal{B}-$valued functions supported on Ω' is a closed subspace of $B(\Omega, \mathcal{B})$.

Proof. The continuous image of a compact set is compact. □

Corollary 6.0.9. In Theorem 6.0.5, if T is bounded we can drop the requirement that $Tf \in B$. (An important such case are functionals on $\mathcal{B}$, that is, $T \in \mathcal{B}^*$.)

Proof. The continuous image of a precompact set is precompact. □

Exercise 34. Go over the properties of the integral thus defined on $B(\omega, \mathcal{B})$ and check that it extends the integral you are familiar with from $\mathbb{R}^n$, while preserving its basic properties.

Exercise 35.

1. Check that, if $\Omega = \mathbb{R}^n$ and $\mathcal{B} = \mathbb{C}$, the construction in this section provides an integral of continuous functions defined on compact sets.

2. Take the completion of $B(\Omega, \mathcal{B})$ under the norm

$$\|f\|_1 = \int_\Omega \|f\| d\mu$$

and proceed as in Exercise 32 to define an integral on this space. The result is the Bochner integral, which we build from first principles in the next section.

7 Bochner integration

Most of the proofs of the statements in this section are quite straightforward, and are left as an exercise. A clear and detailed exposition of Bochner integration is in Yosida's Functional Analysis book [15] available electronically at OSU.

Let (Ω, Σ, μ) be a measure space and $\mathcal{B}$ a Banach space. The Bochner integral is defined in close analogy with the construction of the Lebesgue integral. The integral of a simple function is defined as in the previous section.

Definition 7.0.1 (Bochner integrability). Let (Ω, Σ, μ) be a measure space and $\mathcal{B}$ be a Banach space. A measurable function $f : \Omega \to \mathcal{B}$ is Bochner integrable if there exists a sequence of integrable simple functions S_n such that

$$\lim_{n\to\infty} \int_\Omega \|f - S_n\| \, d\mu = 0$$

where the integral on the left side is the usual Lebesgue integral. If f is Bochner integrable, then the Bochner integral is defined by

$$\int_\Omega f \, d\mu = \lim_{n\to\infty} \int_\Omega S_n \, d\mu$$

You need to check that this definition is consistent.

Proposition 7.0.2 (Bochner's criterion of integrability). Let (Ω, Σ, μ) be a measure space and $\mathcal{B}$ be a Banach space. A measurable function $f : \Omega \to \mathcal{B}$ is Bochner integrable iff

$$\int_\Omega \|f\| \, d\mu < \infty$$

Proposition 7.0.3. Let (Ω, Σ, μ) be a measure space, $\mathcal{B}$ a Banach space, $S \in \Sigma$ and $f : \Omega \to \mathcal{B}$ a Bochner integrable function. Then,

$$\left\| \int_S f \, d\mu \right\|_B \leq \int_S \|f\| \, d\mu$$

Proposition 7.0.4 (Dominated convergence theorem)**.** Let (Ω, Σ, μ) be a measure space, $\mathcal{B}$ a Banach space, $\{f_n\}_{n\in\mathbb{N}}$ a sequence of Bochner integrable functions. Assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges a.e. to f and that, for some $g \in L^1(\Omega, \mu)$ we have $\|f_n(x)\| \leqslant g(x)$ for all n. Then f is Bochner integrable and

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\| d\mu = 0$$

and, finally, for any $S \in \Sigma$, we have

$$\int_S f d\mu = \lim_{n\to\infty} \int_S f_n d\mu$$

Corollary 7.0.5. Let (Ω, Σ, μ) be a measure space, $\mathcal{B}$ a Banach space, $S \in \Sigma$ and $f : \Omega \to \mathcal{B}$ a Bochner integrable function. Then, the set function

$$S \mapsto \int_S f \, d\mu$$

defines a countably-additive $\mathcal{B}$-valued vector measure on Ω which is absolutely continuous with respect to μ.

Exercise 36. Revisit spectral measures with the notions and tools of Bochner integration.

Proposition 7.0.6 (Commutation of closed operators with integration)**.** Let $\mathcal{D} \subset \mathcal{B}$ be a linear space and $T : \mathcal{D} \to \mathcal{B}$ closed. If f is Bochner integrable and Tf is also Bochner integrable, then

$$T \int_A f d\mu = \int_A T f d\mu \tag{71}$$

8 Banach-space valued analytic functions

Definition 8.0.1. Let $\mathcal{B}$ be a Banach space, $\mathcal{O}$ be a domain in $\mathbb{C}$ (a connected open set) and $f : \mathcal{O} \to \mathcal{B}$. Then f is strongly analytic in $\mathcal{O}$ if for any $z_0 \in \mathcal{O}$ there exists an element of $\mathcal{B}$ which we denote by $f'(z_0)$ with the property

$$\lim_{h \to 0} \left\| h^{-1}[f(z_0 + h) - f(z_0)] - f'(z_0) \right\| = 0 \tag{72}$$

Note 8.0.2.

1. You can check that differentiability implies continuity.
2. The definition implies that for any $\varphi \in \mathcal{B}^*$, the complex valued function φf is analytic in $\mathcal{O}$.
3. If f is such that φf is analytic in $\mathcal{O}$ for any $\varphi \in \mathcal{B}^*$, then f is called **weakly analytic**. The notions are however equivalent, as seen in the next proposition.

Proposition 8.0.3. Let $\mathcal{B}$ be a Banach space, $\mathcal{O}$ be a domain in $\mathbb{C}$ (a connected open set) and $f : \mathcal{O} \to \mathcal{B}$ be weakly analytic in $\mathcal{O}$. Then f is strongly analytic in $\mathcal{O}$.

Proof. Let $z_0 \in \mathcal{O}$ and $\varepsilon > 0$ be s.t. $\overline{\mathbb{D}_\varepsilon(z_0)} \subset \mathcal{O}$. Then, for each $\varphi \in \mathcal{B}^*$ and any $z \in \mathbb{D}_\varepsilon(z_0)$ we have

$$(\varphi f)(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z_0)} (s-z)^{-1} (\varphi f)(s) ds = \varphi \left(\frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z_0)} (s-z)^{-1} f(s) ds \right)$$

where we applied Proposition 7.0.6 and the last integral is the Bochner integral (or, by Corollary 6.0.9 and Note 8.0.2, the integral defined in §6), and ds is the arclength measure, $ds = i\varepsilon e^{it} dt$. It follows that

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z_0)} (s-z)^{-1} f(s) ds$$

We now check differentiability of f. We have

$$(z-z_0)^{-1}(f(z) - f(z_0)) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z_0)} (s-z)^{-1}(s-z_0)^{-1} f(s) ds \to \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z_0)} (s-z_0)^{-2} f(s) ds$$

in norm as $z \to z_0$ as you can easily check. $\square$

Note 8.0.4. By using interchangeably weak and strong analyticity, most properties of usual analyticity transfer to the Banach space valued case. We will need some of these properties and convince ourselves that this is the case for them.

9 Analytic functional calculus

9.1 Bounded case: Analytic functional calculus in a Banach algebra

Definition 9.1.1. Let $\mathfrak{B}$ by a Banach algebra, and $T \in \mathfrak{B}$. Let $\mathcal{O}$ be a simply connected domain (i.e., a simply connected open set in $\mathbb{C}$) containing $\sigma(T)$ and $\mathcal{C}$ be a differentiable simple closed contour in $\mathcal{O} \setminus \sigma(T)$. If f is analytic in $\mathcal{O}$, we define $f(T)$ by

$$f(T) := \frac{1}{2\pi i} \oint_{\mathcal{C}} f(s)(s-T)^{-1} ds \tag{73}$$

Where the integral is the one defined in §6, or, with the same effect, the Bochner integral. If f is an analytic function on $\mathcal{O}$ and $\mathcal{D} = \{T \in \mathfrak{B} : \sigma(T) \subset \mathcal{O}\}$, then we say that $T \mapsto f(T)$ defined on $\mathcal{D}$ with values in $\mathfrak{B}$ is an analytic Banach space-valued function.

Note 9.1.2.

1. Here ds is the complex arclength measure: if $\mathcal{C}$ is parametrized by $\gamma : [0,1] \to \mathbb{C}$, then $ds = \gamma'(t)dt$ where dt is the Lebesgue measure on $[0,1]$.

2. We will naturally write $(s-T)^{-1} = \dfrac{1}{s-T}$.

Exercise 37.

1. Check that the definition is consistent, and that $f(T) \in \mathfrak{B}$.

2. Reflect upon the reason we chose to require analyticity of f on an open set containing *all of the spectrum* of T. See also 4. below.

3. Show that the definition above gives the expected result when T is a normal operator on a separable Hilbert space.

4. The spectrum of T may be disconnected. In this case, $\mathcal{O}$ does not have to be a connected set. Formulate and prove a generalization of Definition 9.1.1 suitable for such a setting.

5. Similarly, the connected components of $\sigma(T)$ may not be simply connected. Formulate and prove an extension of Definition 9.1.1 that would allow for this.

Proposition 9.1.3. Let $\mathfrak{B}$ by a Banach algebra, and $T \in \mathfrak{B}$. Let $\mathcal{O}$ be a simply connected domain containing $\sigma(T)$ and $\mathcal{C}$ be a differentiable simple closed contour in $\mathcal{O} \setminus \sigma(T)$. If f and g are analytic in $\mathcal{O}$ and $a, b \in \mathbb{C}$, then

1. $(af + bg)(T) = af(T) + bg(T)$

2. $(fg)(T) = f(T)g(T)$

Proof. Linearity is immediate. For 2., take a simple, closed differentiable contour $\mathcal{C}'$ which contains $\mathcal{C}$ in its interior. You can easily justify the following calculation using Fubini.

$$(fg)(T) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{(fg)(u)}{u-T}du = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(u)g(u)}{u-T}du = \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}'}\oint_{\mathcal{C}} \frac{f(s)}{s-u}\frac{g(u)}{u-T}duds$$
$$= \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}'}\oint_{\mathcal{C}} f(s)g(u)\left(\frac{1}{(s-T)(s-u)} + \frac{1}{(s-T)(u-T)}\right)duds = f(T)g(T) \quad (74)$$

where we used the fact that s is outside $\mathcal{C}$ and theus

$$\oint_{\mathcal{C}} \frac{g(u)}{s-u} du = 0$$

$\square$

Definition 9.1.4. Let $\mathfrak{B}$ by a Banach algebra and $\mathcal{O}$ a domain $\mathbb{C}$. We denote by $H(\mathcal{O}, \mathfrak{B})$ the set of $\mathfrak{B}$-valued analytic functions on $\mathcal{O}$.

Exercise 38. Find conditions on f and g so that $(f \circ g)(T) = f(g(T))$ and prove your result.

9.2 Commutation of analytic functions with the spectrum

Proposition 9.2.1. Let $\mathfrak{B}$ by a Banach algebra, and $T \in \mathfrak{B}$. Let $\mathcal{O}$ be a simply connected domain containing $\sigma(T)$ and $f : \mathcal{O} \to \mathbb{C}$ an analytic function. Then, $\sigma(f(T)) = f(\sigma(T))$.

Proof. The proof essentially stems from the fact that $f \mapsto f(T)$ is linear and multiplicative.

In one direction assume let $f_0 \notin f(\sigma(T))$. We want to show that $f_0 \notin \sigma(f(T))$. Thus, $f(z) - f_0 \neq 0$ for any z in the compact set $\sigma(T)$, and therefore there is some $\mathcal{O}' \supset \sigma(T)$ s.t. $f(z) - f_0 \neq 0$ on $\mathcal{O}'$. This implies that

$$g = \frac{1}{f_0 - f}$$

is analytic in $\mathcal{O}'$ and we have $g(f_0 - f) = (f_0 - f)g = 1$ on $\mathcal{O}'$ which implies $g(T)(f_0 - f(T)) = (f_0 - f(T))g(T) = 1$ and thus $f_0 \notin \sigma(f(T))$.

In the opposite direction we let $f_0 \notin \sigma(f(T))$ and want to show $f_0 \notin f(\sigma(T))$. Assume the contrary, that for some $z_0 \in \sigma(T)$ we had $f(z_0) = f_0$. We will show that $T - z_0$ is invertible, a contradiction.

Consider the function defined by

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \text{ for } z \neq z_0 \text{ and } g(z_0) = f'(z_0)$$

This function is analytic in $\mathcal{O}$, and we have

$$g(z)(z - z_0) = (z - z_0)g(z) = f(z) - f(z_0)$$

implying

$$g(T)(T - z_0) = (T - z_0)g(T) = f(T) - f(z_0)$$

Since $f_0 \notin \sigma(f(T))$ we let $h = [f(T) - f(z_0)]^{-1}$. Then,

$$h(T)g(T)(T - z_0) = (T - z_0)h(T)g(T) = 1$$

the contradiction we mentioned.

□

9.3 Functions analytic at infinity

Definition 9.3.1. f is analytic at infinity if $z \mapsto f(1/z)$ extends to an analytic function in a neighborhood of zero. Equivalently, f is analytic at infinity if there is a compact set $K \subset \mathbb{C}$ such that f is analytic in $\mathbb{C} \setminus K$ and bounded at infinity. In this case, $\lim_{z\to\infty} =: f(\infty)$ exists.

Exercise Check the various assertions above.

Definition 9.3.2. An oriented simple closed curve $\mathcal{C}$ is positively oriented about infinity if infinity is to the left of the curve when it is traversed. Equivalently, $\mathcal{C}$ is negatively oriented relative to points in its interior.

Proposition 9.3.3 (Cauchy formula at infinity)**.** Let f be analytic at infinity and $\mathcal{C}$ simple smooth positively oriented about infinity. Then

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(s)}{s-z}ds = -f(\infty) + \chi_{\mathrm{Ext}(\mathcal{C})}(z) f(z) \tag{75}$$

Proof. This follows as in the scalar case: we can assume without loss of generality that $0 \notin \mathcal{C}$ (otherwise we deform the contour suitably). Let the parametrization be given by $\gamma : [0,1] \to \mathcal{C}$. We denote by $1/\mathcal{C}$ the curve given by the parametrization $\{1/\gamma(1-t) : t \in [0,1]\}$, which makes it positively oriented about its interior. Changing variables, we have

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(s)}{s-z}ds = -\frac{1}{2\pi i}\oint_{1/\mathcal{C}} \frac{f(1/t)}{t^2(1/t-z)}dt \tag{76}$$

Note that

$$\frac{1}{t-zt^2} = \frac{1}{t} + \frac{z}{1-zt} = \frac{1}{t} + \frac{1}{1/z-t}$$

This gives

$$\frac{1}{2\pi i}\oint_{1/\mathcal{C}} \frac{f(t^{-1})dt}{t^2(t^{-1}-z)} = \frac{1}{2\pi i}\oint_{1/\mathcal{C}} \frac{f(t^{-1})dt}{t} - \frac{1}{2\pi i}\oint_{1/\mathcal{C}} \frac{f(t^{-1})dt}{t-z^{-1}} = f(\infty) - \chi_{\mathrm{Ext}(\mathcal{C})}(z) f(z) \tag{77}$$

□

Note 9.3.4 (Functions of unbounded operators)**.** A rich functional calculus is possible for densely defined normal operators in Hilbert spaces. More generally, the existence of nontrivial functions of unbounded operators requires further assumptions; one of them is that the resolvent set is nonempty.

Definition 9.3.5 (Analytic functions of unbounded operators)**.** Let $\mathcal{B}$ be a Banach space, $\mathcal{D}$ a dense subspace of $\mathcal{B}$, and $T : \mathcal{D} \to \mathcal{B}$. Assume that $\rho(T)$ is nonempty and let $\mathcal{C}$ be a differentiable simple closed curve in $\rho(T)$, positively oriented about ∞. Let f be analytic in $\mathrm{Ext}(\mathcal{C})$. Then we define

$$f(T) = -f(\infty) + \frac{1}{2\pi i}\oint_{\mathcal{C}} f(s)(s-T)^{-1}ds$$

We note that these functions of unbounded operators are defined in terms of their resolvents, which are by definition bounded operators. Functional calculus with self-adjoint and normal operators will similarly rely on related bounded operators.

Exercise***.** What do we get if we attempt a weak-* closure of analytic functions of an operator?

10 Compact operators

Operators of the form

$$(\mathcal{K}f)(x) = \int_0^1 K(x,y)f(y)dy \tag{78}$$

are frequently encountered in analysis, for instance in the solution of differential and partial differential equations, in which case $K(x,y)$ is called the Green's function of the equation.

Assuming first that $K \in C([0,1]^2)$, $\mathcal{K}$is clearly bounded from $C([0,1])$ into itself, with norm $M = \max_{[0,1]^2} |K|$.

Fredholm noticed that there is something more going on, and of substantial importance. Since K is is uniformly continuous, for any $\varepsilon > 0$ there is a $\delta > 0$ s.t. $|x - x'| + |y - y'| < \delta$ implies $|K(x,y) - K(x',y')| < \varepsilon$ in $[0,1]^2$. This implies

$$|(Kf)(x) - (Kf)(x')| = \left|\int_0^1 [K(x,y) - K(x',y)]f(y)dy\right| \leqslant \varepsilon\|f\|_u \tag{79}$$

It follows that the family $\{Kf : \|f\|_u \leqslant 1\}$ is equibouunded and equicontinuous and thus, by Ascoli-Arzelà, it is (sequentially) compact. Fredholm showed that such operators have especially good features, including what we now know as the Fredholm alternative property. According to [16] p. 215, Fredholm's work "produced considerable interest among Hilbert and his school, and led to the abstraction of many notions we now associate with Hilbert space theory".

Definition 10.0.1. Let X and Y be Banach spaces. A linear operator $K : X \to Y$ is compact if it maps any bounded subset of X to a relatively compact subset of Y.

Note that compact operators are automatically bounded. An important special case of compact operators are the finite-rank operators.

Definition 10.0.2. Let X and Y be Banach spaces. A linear operator $K : X \to Y$ is finite-rank if $\operatorname{ran}(K)$ is a finite-dimensional subspace of Y.

Exercise 39. Verify that finite-rank operators are compact.

The following result was taken by Hilbert as the definition of compact operators:

Theorem 10.0.3. If X and Y are Banach spaces and $K : X \to Y$ is a compact operator, then K maps any weakly convergent sequence in X to a norm convergent sequence in Y.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in X. If $\varphi \in Y^*$, then $\varphi \circ K \in X^*$. This implies that the weak limit, call it y, of $\{y_n\}_{n\in\mathbb{N}}$ exists. Assume that $\|y - y_n\| \not\to 0$ as $n \to \infty$. Then, there is an $\varepsilon > 0$ and a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ such that $\|y - y_{n_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. But then the sequence $\{Kx_{n_k}\}_{k\in\mathbb{N}}$ contains no norm-convergent subsequence, a contradiction. □

Proposition 10.0.4. Let X be a reflexive Banach space. Then the closed unit ball in X is weakly compact.

Note 10.0.5. In fact, more is true: If X is a Banach space, then X is reflexive iff the closed unit ball of X is weakly compact. This is known as **Kakutani's theorem**.

Proof. We use the fact that the natural embedding J of X in X^{**} is a topological homeomorphism between X with the weak topology and $J(X)$ with the weak-* topology (see Folland).

Let $B_X = \{x \in X : \|x\| \leqslant 1\}$ and $B_{X^{**}} = \{\varphi \in X^{**} : \|\varphi\| \leqslant 1\}$. Since X is reflexive, the embedding J is an isomorphism between X and X^{**}. Since X^* is a normed linear space, by Alaoglu's theorem we have that $B_{X^{**}}$ is weak-* compact. Since J is a homeomorphism, this means that B_X is weakly compact. □

Proposition 10.0.6. Let X and Y be Banach spaces, $K : X \to Y$ a compact operator. If X is reflexive and B is the unit ball in X, then KB is norm-compact.

Proof. Let $\{y_n\}_{n\in\mathbb{N}} \in KB$. Then, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in B s.t. $y_n = Kx_n$. By Proposition 10.0.5, $\{x_n\}_{n\in\mathbb{N}}$ has a a weakly convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, and by compactness, $\{y_{n_k}\}_{k\in\mathbb{N}}$ is norm-convergent. □

Recall the following definition: If X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator, then the adjoint of T is the linear operator $T^* : Y^* \to X^*$ defined by $T^*\varphi = \varphi \circ T$ for all $\varphi \in Y^*$.

Theorem 10.0.7. Let X and Y are Banach spaces. Then,

1. The norm limit of compact operators is compact.
2. An operator is compact iff its adjoint is compact.
3. Let X, Y, Z be Banach spaces, $T : X \to Y$ and $S : Y \to Z$ be linear bounded operators. If one of the operators T and S is compact, then TS is compact. In particular, if $X = Y = Z$, compact operators form an ideal in the space of linear bounded operators.

Proof. 1. Let$\{K_n\}_{n\in\mathbb{N}}$ be a norm-convergent sequence of compact operators from X to Y. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in the closed unit ball B of X. For each n there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\{K_n x_{n_k}\}_{k\in\mathbb{N}}$ converges to a y_n in norm as $n \to \infty$. By the usual diagonal trick, we can choose a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ such that all $\{K_n x_{n_k}\}_{k\in\mathbb{N}}, n \in \mathbb{N}$ converge. We assume without loss of generality that $\{x_n\}_{n\in\mathbb{N}}$ itself is such a sequence. We have

$$\|y_n - y_m\| = \lim_{j\to\infty} \|y_n - K_n x_j + K_n x_j - K_m x_j + K_m x_j - y_m\| \leqslant \|K_n - K_m\|$$

showing that y_n is norm-convergent.

2. Assume that K is compact and let B_X be the unit ball in X and B_{X^*} the unit ball in X^*. Then, $S = \overline{KB}$ is norm-sequentially compact. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in B_{X^*}. Note that $\{\varphi_n|_S\}_{n\in\mathbb{N}}$ is an equibounded equicontinous sequence in $C(S)$ and by Ascoli-Arzelà it has a convergent subsequence, which, without loss of generality we assume is $\{\varphi_n|_S\}_{n\in\mathbb{N}}$ itself. Now

$$\|K^*\varphi_n - K^*\varphi_m\| = \sup_{u\in B_X} \|\varphi_n(Ku) - \varphi_m(Ku)\| \leqslant \sup_{s\in S} \|\varphi_n(s) - \varphi_m(s)\| \to 0$$

uniformly as $n, m \to \infty$, proving the result. □

3. is straightforward.

Theorem 10.0.8. Let $\mathcal{H}$ be a separable Hilbert space and $K : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then K is compact iff it is the norm-limit of finite-rank operators.

Proof. In one direction, the result follows easily from Exercise 39 and Theorem 10.0.7. Assume then that K is compact. Let $\{e_k\}_{k\in\mathbb{N}}$ be a Hilbert basis in $\mathcal{H}$, let $\mathcal{H}_n$ be the subspace generated by $e_1, ..., e_n$, denote $P_n = P_{\mathcal{H}_n}$ and define $K_n = P_n K, n \in \mathbb{N}$. Clearly the K_n are finite-rank. We show that $\|K - K_n\| \to 0$ as $n \to \infty$. Define

$$\lambda_n = \sup\{\|Ku\| : P_n u = 0 \text{ and } \|u\| = 1\}$$

Clearly the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is nonnegative and nonincreasing. Let $\lambda = \lim_{n\to\infty} \lambda_n$. For each n let u_n be s.t. $P_n u_n = 0$ and $\|Ku_n\| \geqslant \lambda/2$. Clearly u_n converge weakly to zero, hence Ku_n converges in norm to zero, hence $\lambda = 0$ and the result follows.

Exercise 40. Let $\mathcal{H}$ be a separable Hilbert space with Hilbert basis $\{e_n\}_{n\in\mathbb{N}}$, and let T be a diagonal operator, $Te_j = t_j e_j$ for all $j \in \mathbb{N}$. Find a necessary and sufficient condition on the set $\{t_j\}_{j\in\mathbb{N}}$ so that T is compact.

□

10.1 Finite-rank operators

Let X, Y be Hilbert spaces and $F : X \to Y$ a linear finite-rank operator. Let $\{e_1, ..., e_n\}$ be orthonormal basis fot $\operatorname{ran} F$. For each $j \leqslant n$ let $f_j = F^* e_j$. The vectors f_j must be linearly independent. Think of the relation between the range of an operator and the kernel of its adjoint. Directly, assume $\sum_m a_m f_m = 0$ and let $e = \sum_m a_m e_m$. Then, for any $x \in X$ we have $0 = (x, F^* e) = (Fx, e)$ which implies $e = 0$ (since $e \in \operatorname{ran} F$) and hence $a_1 = ... = a_m = 0$. For any $x \in X$,

$$Fx = \sum_{j=1}^{n} (Fx, e_j) e_j = \sum_{j=1}^{n} (x, f_j) e_j \tag{80}$$

which is the normal form of a finite-rank operator. The space U generated by the f_j is called the initial space of F.

Note 10.1.1. 1. We see from (80) that, if $\dim Y > n$ we must have $0 \in \sigma(F)$.

From this point on we take $X = Y$.

Note 10.1.2. 1. Note that $Fx = 0$ iff $x \perp \mathcal{U}$, hence $\ker F = \mathcal{U}^\perp$.

2. If F is finite-rank, then F^* has the same finite rank, since $\operatorname{ran} F^* = (\ker F)^\perp = U^{\perp\perp} = U$.

3. finite-rank operators form an ideal in $L(\mathcal{H}, \mathcal{H})$.

4. If P_U be the projection on U, then $1 - P_U$ is the projection on $U^\perp$. From (80) we have $F(1 - P_U) = 0$.

5. Let $E = \operatorname{ran} FP_U = \operatorname{ran} F$. Clearly, $\dim(E) = \dim(U) = n$.

6. Let V be the space spanned by $e_1, \ldots, e_n, f_1, \ldots, f_n$. Then $\dim V \le 2n$ and F is zero on $V^\perp$. Hence essentially F is the square matrix $\tilde{F} = F\big|_V$. Having in mind that $\mathcal{H} = V \oplus V^\perp$ we write $F = \tilde{F} \oplus 0$.

7. Note that
$$\lambda I - F = (\lambda I_V - \tilde{F}) \oplus \lambda I_{V^\perp} \tag{81}$$
Thus we can identify finite-rank operators with a square matrix on some finite dimensional space direct sum with the identity on its complement.

10.2 The Fredholm alternative in Hilbert spaces

Theorem 10.2.1 (The Fredholm alternative)**.** Let $\mathcal{H}$ be a Hilbert space, $K : \mathcal{H} \to \mathcal{H}$ a compact operator and $\lambda \neq 0$. Then $\lambda - K$ is invertible iff $Kx = \lambda x$ has no nonzero solution.

Note 10.2.2.

1. This result can be proved in the more general case when $\mathcal{H}$ is a Banach space.

2. The theorem implies that the spectrum of a compact operator is discrete, except maybe for $\lambda = 0$.

3. This theorem follows from the sharper analytic Fredholm alternative theorem which we prove next, but this weaker result illustrates a different approach to the problem.

Proof. It is clear that $\lambda x = Kx$ for nonzero x implies $\lambda - K$ is not injective thus noninvertible.

In the opposite direction, assume $\lambda - K$ is not invertible and let K_n be a sequence of finite rank operators converging to K. We use the second resolvent formula to note that
$$(\lambda - K_n)^{-1}(I - (K - K_n)(\lambda - K_n)^{-1})^{-1} \tag{82}$$
would be an inverse of $\lambda - K$ if the inverses above existed.

Assume, to arrive at a contradiction, there there is an $\varepsilon > 0$ and a subsequence $K_{n'}$ of K_n such that $\|(\lambda - K_{n'})u\| \geqslant \varepsilon u$ for all u. Since $K_{n'} = M_{n'} \oplus 0$ where $M_{n'}$ are matrices, this means that the $\lambda - K_{n'}$ are invertible and by Proposition 2.1.7, $\|(\lambda - K_{n'})^{-1}\| \le \varepsilon^{-1}$. Since $\|K - K_n\| \to 0$ as $n \to \infty$ then for n' large enough the terms $(K - K_{n'})(\lambda - K_{n'})^{-1}$ will have norm less than 1, therefore all the inverses in (82) exist, hence $\lambda - K$ is invertible, which is a contradiction.

Hence the bounds below of $\lambda - K_n$ converge to zero as $n \to \infty$. Then, for a sequence of unit vectors u_n we have

$$\lambda u_n - (K_n - K)u_n - Ku_n \to 0 \text{ as } n \to \infty \tag{83}$$

Since $\lim_{n\to\infty} \|K - K_n\| = 0$, it follows that $\lambda u_n - Ku_n \to 0$ as $n \to \infty$. Since K is compact and $\|u_n\| = 1$, there is a sequence $v_k = u_{n_k}$ indexed by $k \in \mathbb{N}$ s.t. Kv_k converges, to Kv. Passing to the limit, we have $\lambda v = Kv$. □

Setting. Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and K an analytic function with values in the compact linear operators on a Hilbert space $\mathcal{H}$.

Theorem 10.2.3 (The analytic Fredholm alternative)**.** In the setting above, either

1. $(I - K(z))^{-1}$ exists for no $z \in \mathcal{D}$ or
2. $(I - K(z))^{-1}$ is a meromorphic operator-valued function in $\mathcal{D}$, and if z_p is a pole of order n in $\mathcal{D}$, then the residue $F_p = \lim_{z\to z_p}(z - z_p)^n(I - K(z))^{-1}$ is a finite rank operator.

Note 10.2.4. By rescaling K we can replace I by λI for any nonzero λ.

Proof. Let $\mathfrak{K} \subset \mathcal{D}$ be compact, $\varepsilon \in (0, 1)$ and cover $\mathfrak{K}$ by finitely many open balls B_j such that $\|K(z) - K(z')\| < \varepsilon/2$ for all $z, z' \in B_j$. Let z_0 be any point in any of the B_j for which $I - K(z_0)$ is invertible. Let F a finite rank operator such that $\hat{\varepsilon} := K(z_0) - F$ has norm $< \varepsilon/2$. Then $K - F = \hat{\varepsilon}$ has norm $< \varepsilon$ throughout B_j. We write F in the form $M \oplus 0$ for some finite-dimensional matrix M. Still relying on the second resolvent idea, and noting that $I - \hat{\varepsilon}$ is invertible, we write

$$I - K(z) = (I - F(I - \hat{\varepsilon}(z))^{-1})(I - \hat{\varepsilon}(z)) \tag{84}$$

Note also that $F(I - \hat{\varepsilon}(z))^{-1}$ is finite-rank, and hence $F(I - \hat{\varepsilon}(z))^{-1} = M(z) \oplus 0$ where $M(z)$ is a matrix on a finite-dimensional space $V(z)$. By continuity, there is a finite-dimensional V such that $V(z) \subset V$ for all $z \in B_j$, and $M(z)$ are matrices on V. Thus, (84) has the form

$$I - K(z) = [(I_V - M(z)) \oplus I_{V^\perp}](I - \hat{\varepsilon}) \tag{85}$$

which means

$$(I - K(z))^{-1} = (I - \hat{\varepsilon}(z))^{-1}[(I_V - M(z))^{-1} \oplus I_{V^\perp}] \tag{86}$$

Using the determinant formula for $(I_V - M(z))^{-1}$, the result follows. □

Corollary 10.2.5. If K is a compact operator on a Hilbert space, then $I - K$ has closed range.

Exercise 41. Fill in all the details of the proof of the theorem above, and prove the corollary.

Exercise 42. Let $\mathcal{H} = L^2([0,1])$ and let $K : \mathcal{H} \to \mathcal{H}$ be defined by

$$Kf(x) = \int_0^x f$$

Show that K is compact and that K^* is given by

$$K^* f(x) = \int_x^1 f$$

Define now the compact, self-adjoint operator $A = \frac{i}{2}(K^* - K)$. Show that ran $A \subset AC([0,1])$ $\sigma(A) \setminus \{0\}$ is discrete, and if$0 \neq \lambda \in \sigma(A)$, then

$$-ih = \lambda h'; \quad h(0) = -h(1) = \int_0^1 h$$

whose eigenvalues are $\{\pi^{-1}(2k+1)^{-1} : k \in \mathbb{Z}\}$, with eigenfunctions $\{e^{(2k+1)i\pi x}\}_{k\in\mathbb{Z}}$.

Use the spectral measure theorem(s) (the projector-valued version should be useful here) to show that $\{e^{(2k+1)i\pi x}\}_{k\in\mathbb{Z}}$ is an orthonormal basis is $L^2([0,1])$, and hence so is $\{e^{2k\pi i x}\}_{k\in\mathbb{Z}}$.

Consider now the operator $\tilde{A} = \frac{1}{2}(K + K^*)$ and compare its spectral properties with those of A.

11 Unbounded operators on Hilbert spaces

We first take the general case when $\mathcal{H}, \mathcal{K}$ are Hilbert spaces and T is a linear operator from $\mathcal{H}$ to $\mathcal{K}$, although later on we will be mainly interested in the case $\mathcal{H} = \mathcal{K}$.

Note 11.0.1 (Review of some basic notions)**.**

1. $T : D(T) \subset \mathcal{H} \to \mathcal{K}$ is densely defined if $\overline{D(T)} = \mathcal{H}$.

2. For a densely defined operator T we can define its adjoint T^*. The domain of T^*, $D(T^*)$, consists of those $y \in \mathcal{K}$ with the property that there is a $v \in \mathcal{H}$ such that, for all $x \in D(T)$ we have
$$(y, Tx)_{\mathcal{K}} = (v, x)_{\mathcal{H}}$$

   Note that the density of $D(T)$ implies that if such a v exists it is unique; hence $T^* y = v$ defines a function on $\mathcal{K}$; it is easy to see that $D(T^*)$ is a linear space, and T^* is a linear operator. The domain of T^* can be as small as $\{0\}$.

Exercise 43. Consider again the operator in Exercise 29. Show that the domain of the adjoint is $\{0\}$.

See also Example 4 on p. 252 in [16].

3. The graph of an operator $T : D(T) \to \mathcal{K}$ is defined as the set $\mathbb{G}(T)$ of pairs $[x, Tx]$ in $\mathcal{H} \times \mathcal{K}$ with $x \in D(T)$, and is simply the usual definition of T as a function.

4. $\mathcal{H} \times \mathcal{K}$ is a Hilbert space with the inner product

$$([v,w],[v'w'])_{\mathcal{H}\times\mathcal{K}} = (v,v')_{\mathcal{H}} + (w,w')_{\mathcal{K}}$$

5. A linear operator T is closable if the closure $\mathbb{G}(T)$ in $\mathcal{H} \times \mathcal{K}$ is the graph of an operator, and T is closed if $\mathbb{G}(T)$ is a closed set in $\mathcal{H} \times \mathcal{K}$.

6. Let $\mathcal{K} = \mathcal{H}$ and $A : D(A) \to \mathcal{H}$ a densely defined linear operator. Then, A is self-adjoint if $A = A^*$. This means that the domain of A^* is exactly equal to the domain of A and A is symmetric on its domain $(Ax, y) = (x, Ay)$ for all $x, y \in D(A)$.

11.1 Some remarkable properties of the graph

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. On the Hilbert space $\mathcal{H} \times \mathcal{K}$ define V by

$$V([x,y]) = [-y, x]$$

for all $[x, y] \in \mathcal{H} \times \mathcal{K}$. Note that V is **unitary.** and $V^2 = -I$. Recall that a unitary map (such as V) maps closed subspaces to closed subspaces.

Lemma 11.1.1. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and $T : D(T) \to \mathcal{K}$ a linear operator where $\overline{D(T)} = \mathcal{H}$. Then

$$\mathbb{G}(T^*) = V(\mathbb{G}(T)^{\perp})$$

Proof. We have $[u, v] \in \mathbb{G}(T)^{\perp}$ iff $(u, x)_{\mathcal{H}} + (v, Tx)_{\mathcal{K}} = 0$ for all $x \in D(T)$ iff $(u, x)_{\mathcal{H}} = (-v, Tx)_{\mathcal{K}}$ iff $-v \in D(T^*)$ and $u = T^*(-v)$ iff $[-v, u] \in \mathbb{G}(T^*)$. $\square$

We note the following consequence.

Corollary 11.1.2. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ a densely defined linear operator. Then,

1. T^* is closed.

2. T is closable iff $D(T^*)$ is dense, in which case $\overline{T} = T^{**}$.

3. If T is closable, then $T^* = \overline{T}^*$.

4. $\ker T^* = (\operatorname{ran} T)^\perp$ and $(\ker T)^\perp = \overline{\operatorname{ran} T^*}$.

Proof. 1. and 3. follow from an immediate calculation using the graph properties above.

For 2., note that $D(T^*)$ dense implies T^{**} is well defined. Now,

$$\mathbb{G}(T^{**}) = V(\mathbb{G}(T^*))^\perp = V(V(\mathbb{G}(T))^\perp)^\perp = (V^2\mathbb{G}(T))^{\perp\perp} = \overline{\mathbb{G}(T)}$$

implying that $\overline{\mathbb{G}(T)}$ is the graph of an operator. Conversely, assume that $D(T^*)$ is not dense and let $y \in D(T^*)^\perp$, which implies that $[y,0] \in \mathbb{G}(T^*)^\perp$. This in turn means that $V([y,0]) = [0,y] \in V(\mathbb{G}(T^*)^\perp) = \overline{\mathbb{G}(T)}$ which means $\overline{\mathbb{G}(T)}$ is not the graph of an operator.

For 4. note that $x \in \ker T^*$ iff $[x,0] \in \mathbb{G}(T^*)$ iff $[0,x] = V([x,0]) \perp \mathbb{G}(T)$ iff $x \in (\operatorname{ran} T)^\perp$. The second part is similar. □

Proposition 11.1.3. Let $\mathcal{H}$ be a Hilbert space T be a closed operator from $D(T) \subset \mathcal{H}$ to $\mathcal{H}$. Then, $\mathbb{C} \ni \lambda \in \rho(T)$ iff $(T - \lambda)$ is a bijection between $D(T)$ and $\mathcal{H}$ and $(\lambda - T)^{-1}$ is bounded.

Proof. It is clear that, if the conditions above are satisfied, then $\lambda \in \rho(T)$. Conversely, if $\lambda \in \rho(T)$, then $(\lambda - T)^{-1}$ extends as a bounded operator on $\mathcal{H}$, which we still denote by $(\lambda - T)^{-1}$. By the closed graph theorem, $\mathbb{G}((\lambda - T)^{-1})$ is closed, and it is the closure of $\mathbb{G}((\lambda - T)^{-1}|_{\operatorname{ran}(\lambda - T)})$. If J is the unitary involution $J([x,y]) = [y,x]$ for all $x, y \in \mathcal{H}$, then $J(\mathbb{G}(\lambda - T)^{-1}) = \mathbb{G}(\lambda - T)$. The rest follows from the fact that T (thus $\lambda - T$) is closed. □

11.2 Symmetric and self-adjoint operators

Definition 11.2.1 (Symmetric operators)**.** Let $\mathcal{H}$ be a Hilbert space, and $T : D(T) \to \mathcal{H}$ be a densely defined operator on $\mathbb{H}$. Then, T is symmetric if for all $x, y \in D(T)$ we have $(Tx, y) = (x, Ty)$.

Definition 11.2.2 (Essentially self-adjoint operators)**.** Let $\mathcal{H}$ be a Hilbert space.

1. If $T : D(T) \to \mathcal{H}$ is a closed operator, a **core** for T is a set $D_1 \subset D$ s.t. the closure of $T|_{D_1}$ is T.

2. Let T be a densely defined symmetric operator. Then, T is essentially self-adjoint if its closure is self-adjoint. Note that a self-adjoint operator A is uniquely defined by $A|_{D_1}$ for any core D_1 of A.

3. If T_1 and T_2 are operators on $D(T_1), D(T_2)$ resp., we write $T_1 \subset T_2$ if $\mathbb{G}(T_1) \subset \mathbb{G}(T_2)$.

In this case, we say that T_2 is an extension of T_1.

Exercise 44 (Symmetric is not the same as self-adjoint).

(1). Let $\mathcal{H} = L^2([0,1])$ and let $T = -i\frac{d}{dx}$ be defined on

$$D(T) = \{f \in AC([0,1]) : f(0) = f(1) = 0, f' \in L^2([0,1])\} \tag{87}$$

Check that T is symmetric.

Show that the domain of T^* is

$$D(T^*) = \{f \in AC([0,1]) : f' \in L^2([0,1])\} \tag{88}$$

that is, no boundary condition is required. One way is the following. We need to characterize all the $g \in \mathcal{H}$ with the property that, for any $f \in D(\mathcal{H})$ there is an $u \in \mathcal{H}$ s.t.

$$\int_0^1 f'g = \int_0^1 fu = \int_0^1 f(x)\left(\int_0^x u(s)ds\right)' dx = -\int_0^1 f'(x)\left(\int_0^x u(s)ds\right) du$$

where we used the fact that $L^2([0,1]) \subset L^1([0,1])$ and the antiderivative of an $L^1([0,1])$ function is in $AC([0,1])$ and integrated by parts. [a]

Note that for any φ with compact support in $(0,1)$ and of zero average, $\int_0^1 \varphi = 0$, the equation $Tf = \varphi$ has a (unique) solution in $D(T)$. It follows that $g + \int_0^x u$ is orthogonal on all φ with compact support in $(0,1)$ and of zero average, and is thus a constant. Hence, $g \in AC([0,1])$ and $T^*g = -ig'$.

Note that $\sigma(T^*) = \mathbb{C}$. Indeed, for any $\lambda \in \mathbb{C}$,

$$\ker(T^* - \lambda) = \{ce^{i\lambda x} : c \in \mathbb{C}\}$$

Show that the spectrum of T is also the whole of $\mathbb{C}$.

It will turn out that there are infinitely many domains $D_\alpha(T) \supset D(T)$ such that $-i\frac{d}{dx}$ is self-adjoint on $D_\alpha(T)$. In fact, they are given by

$$D_\alpha(T) = \{f \in AC([0,1]) : f(0) = \alpha f(1), f' \in L^2([0,1])\} \tag{89}$$

for any $\alpha \in \mathbb{T}$.

(2). Let $\mathcal{H} = L^2([0,\infty))$ and let $T = -i\frac{d}{dx}$ be defined on

$$D(T) = \{f \in AC : f(0) = 0, f' \in \mathcal{H}\} \tag{90}$$

where AC denotes the functions which are in $AC([0,b))$ for any $b > 0$. Check that T is symmetric, and

$$D(T^*) = \{f \in AC : f' \in \mathcal{H}\} \tag{91}$$

We will see that, for T as in (2), there is no domain $D \supset D(T)$ such that $-i\frac{d}{dx}$ is self-adjoint on D.

[a] We should have taken $\overline{g}$, but g is general anyway. Also we did not carry i with us in this calculation, it can be reinserted at the end and it changes the sign of integration by parts.

Proposition 11.2.3. Let $\mathcal{H}$ be a Hilbert space, and $T : D(T) \to \mathcal{H}$ be a densely defined **symmetric** operator on $\mathbb{H}$. Then, $T^* \supset T$. Hence T^* is densely defined and, by Corollary 11.1.2, it follows that T is closable.

1. $W(T) \subset \mathbb{R}$
2. $\|(T \pm i)x\|^2 = \|Tx\|^2 + \|x\|^2$
3. if T is closed, then T is self-adjoint iff T^* is symmetric.
4. If T is closed, then $\operatorname{ran}(T \pm i)$ is closed and $T \pm i$ is injective, $\ker(T \pm i) = \{0\}$.
5. If T is self-adjoint, then $\ker(T \pm i) = \{0\}$ and $\operatorname{ran}(T \pm i) = \mathcal{H}$.

Proof.

1. We have $(Tx, x) = (x, Tx) = \overline{(Tx, x)}$ (here $\bar{z}$ is the complex conjugate of z).
2. follows from 1 by a straightforward calculation.
3. By 1. above, $T \subset T^*$, and also $T^* \subset T^{**} = \overline{T} = T$, hence $T = T^*$.
4. Injectivity follows from 2. Let now $\{y_n\}_{n\in\mathbb{N}} \subset \operatorname{ran}(T \pm i)$ be a Cauchy sequence, and write $y_n = (T \pm i)x_n$. Part 2 above implies that both $\{x_n\}_{n\in\mathbb{N}}$ and $\{Tx_n\}_{n\in\mathbb{N}}$ are Cauchy. If $x = \lim_{n\to\infty} x_n$, then, since T is closed, $\lim_{n\to\infty} Tx_n = Tx$ ensuring that $y \in \operatorname{ran}(T \pm i)$. .
5. The fact that $\ker(T \pm i) = \{0\}$ follows from 4. Also from 4., we see that $\operatorname{ran}(T \pm i)$ is closed. But $\operatorname{ran}(T \pm i) = \overline{\operatorname{ran}(T \pm i)} = \ker(T \mp i)^{\perp} = \mathcal{H}$.

□

Theorem 11.2.4 (Basic criteria of self-adjointness)**.** Let $\mathcal{H}$ be a Hilbert space, and $T : D(T) \to \mathcal{H}$ be a densely defined **symmetric** operator on $\mathbb{H}$. Then, the following statements are equivalent:

1. T is self-adjoint.
2. T is closed and $\ker(T^* \pm i) = \{0\}$.
3. $\operatorname{ran}(T \pm i) = \mathcal{H}$.
4. T is closed and $\sigma(T) \subset \mathbb{R}$.

Note. In the above, $\pm i$ can be replaced by $a \pm ib$, for some $a \in \mathbb{R}, b \in \mathbb{R}^+$. Also, it is understood that $D(T \pm i) = D(T)$.

Proof. 1 implies 2 and 3 by Proposition 11.2.3.

$2 \Rightarrow 3$: since T is closed, $\operatorname{ran}(T \pm i)$ are also closed by Proposition 11.2.3, hence $\operatorname{ran}(T \pm i) = \ker(T^* \mp i)^\perp = \mathcal{H}$.

$3 \Rightarrow 1$: recall first that $D(T^*) \supset D(T)$. If we show that $D(T^*) \subset D(T)$, then $T = T^*$ since, by symmetry, $T = T^*$ on $D(T)$. So, let $y \in D(T^*)$. We have,

$$(y, (T + i)x) = ((T^* - i)y, x)$$

Since $\operatorname{ran}(T - i) = \mathcal{H}$ we have $(T^* - i)y = (T - i)y'$ for some $y' \in D(T)$, implying

$$(y, (T + i)x) = ((T^* - i)y, x) = ((T - i)y', x) = (y', (T + i)x)$$

where the last equality uses the symmetry of T. Since $\operatorname{ran}(T + i) = \mathcal{H}$ we have $y = y'$.

At this stage, 1,2,3 are equivalent. Clearly, 4 follows from 1,2,3. If $\sigma(T) \subset \mathbb{R}$, then $\operatorname{ran}(T \pm i) = \mathcal{H} = \ker(T^* \mp i)^\perp$ implying 2. □

11.3 Self-adjoint operators, extensions of symmetric operators and the Cayley transform

Exercise 45 (Self-adjoint vs. unitary)**.** Note that the map $z \mapsto (z - i)(z + i)^{-1}$ maps conformally the upper half plane to the unit disk and the real line to the unit circle. If T is a bounded self-adjoint operator, then $\sigma(T) \subset \mathbb{R}$. Use these facts and the spectral mapping theorem to show that $(T - i)(T + i)^{-1}$ is unitary. Conversely, assume that U is unitary **and** $1 \notin \sigma(U)$. Then $T = i(1 + U)(1 - U)^{-1}$, the **Cayley transform** of T is a bounded self-adjoint operator.

Remarkably perhaps, if we replace $1 \notin \sigma(U)$ in the last part of the exercise above by the weaker condition $\ker(1 - U) = \{0\}$, then it turns out that we obtain a complete characterization of self-adjoint operators on Hilbert spaces, bounded or not. We note that $\ker(1 - U) = \{0\}$ is necessary for the operator $i(1 + U)(1 - U)^{-1}$ to be densely defined.

More generally, symmetric operators arise as Cayley transforms of partial isometries. Recall the notion of partial isometry, Definition 2.3.21 and Note 3.1.15. For our present purposes we can characterize partial isometries in the following way. If $\mathcal{H}$ is a Hilbert space, a partial isometry is a unitary map between two Hilbert subspaces, $\mathcal{H}_+$ and $\mathcal{H}_-$, of $\mathcal{H}$.

Definition 11.3.1. Let $\mathcal{H}$ be a Hilbert space and $(\mathcal{H}_+, \mathcal{H}_-)$ a pair of Hilbert subspaces of $\mathcal{H}$. The pair of dimensions $(\dim(\mathcal{H}_+^\perp), \dim(\mathcal{H}_-^\perp))$, finite or infinite, are called deficiency indices of $(\mathcal{H}_+, \mathcal{H}_-)$.

Proposition 11.3.2. Let $\mathcal{H}$ be a separable Hilbert space and W a partial isometry with initial space $\mathcal{H}_+$ and final space $\mathcal{H}_-$. Then W extends to a unitary transformation of $\mathcal{H}$ iff the deficiency indices of $\mathcal{H}_\pm$ are equal.

Proof. Note that there is a unitary map U_{+-} from $\mathcal{H}_+^\perp$ onto $\mathcal{H}_-^\perp$ iff the deficiency indices are equal. Assuming U_{+-} exists, write $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_+^\perp$ and define $U = W \oplus U_{+-}$, clearly a unitary map from $\mathcal{H}$ onto $\mathcal{H}$. In the opposite direction, if W extends to a unitary map U, then you can check that $U|_{\mathcal{H}_+^\perp}$ is a unitary map of $\mathcal{H}_+^\perp$ onto $\mathcal{H}_-^\perp$. □

Lemma 11.3.3. Let $\mathcal{H}$ be a separable Hilbert space and T a closed, densely defined symmetric operator. Then, the Cayley transform of T, $W = (T-i)(T+i)^{-1}$ is a partial isometry from $\mathcal{H}_+ = \operatorname{ran}(T+i)$ onto $\mathcal{H}^- = \operatorname{ran}(T-i)$. Furthermore, $D(T) = \operatorname{ran}(I-W)$.

Proof. Clearly, $W : \mathcal{H}_+ \to \mathcal{H}_-$. By Proposition 11.2.3, the spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are closed and $(T+i)^{-1}$ is one-to-one from $\mathcal{H}_+$ onto $D(T)$ while $T-i$ is one-to one from $D(T)$ onto $\operatorname{ran}(T-i)$. Hence W is bijective between $\mathcal{H}_+$ and $\mathcal{H}_-$. We now check that W is unitary between $\mathcal{H}_+$ and $\mathcal{H}_-$. Let $u_+ \in \mathcal{H}_+$ and $u_- = Wu_+$. Then $u_+ = (T+i)f$ for a unique $f \in D(T)$, and, by Proposition 11.2.3, $\|u_+\|^2 = \|f\|^2 + \|Tf\|^2$. Then, $u_- = (T-i)f$ and, again by Proposition 11.2.3, $\|u_-\|^2 = \|f\|^2 + \|Tf\|^2$.

For the last statement note that $(T-i)(T+i)^{-1} = 1 - 2i(T+i)^{-1}$, hence $1 - W = 2i(T+i)^{-1}$ and the rest follows from Proposition 11.2.3. □

Lemma 11.3.4 (Duality between self-adjoint operators and unitary ones)**.** Let $\mathcal{H}$ be a separable Hilbert space, let T be a closed, densely defined symmetric operator, and $U : \mathcal{H} \to \mathcal{H}$ a unitary operator. Then,

1. T is self-adjoint iff its Cayley transform is unitary.

2. If $\ker(1-U) = \{0\}$, then there is a unique self-adjoint operator T such that U is the Cayley transform of T, and $T : \operatorname{ran}(1-U) \to \mathcal{H}$.

Proof. 1. Assume first T is self-adjoint. Then, by Theorem 11.3.6, $\operatorname{ran}(T \pm i) = \mathcal{H}$ and $\ker(T \pm i) = \{0\}$. Lemma 11.3.3 implies that $W = (T-i)(T+i)^{-1}$ is a partial isometry from $\mathcal{H}$ onto $\mathcal{H}$, that is, it is unitary. Conversely, if W is unitary, then $(T+i)^{-1}$ is defined everywhere, it has range $D(T)$, hence $\operatorname{ran}(T+i) = \mathcal{H}$, and $\operatorname{ran} W = \operatorname{ran}(T-i) = \mathcal{H}$, and Theorem 11.3.6 implies the result.

2. If U is unitary, then $\ker(1-U) = \ker(1-U^*)$ since $x = Ux$ is equivalent to $U^*x = x$. Since $\ker(1-U) = \{0\}$, we have $\operatorname{ran}(1-U)^\perp = \ker(1-U^*) = \{0\}$. We define $T = i(1+U)(1-U)^{-1}$ on $D(T) = \operatorname{ran}(1-U)$, a dense set in $\mathcal{H}$ by the above. To check that T is symmetric, we calculate $(x, i(1+U)(1-U)^{-1}x)$ for $x \in D(T)$, that is for $x = (1-U)v$ for some v. We have

$$(x, i(1+U)(1-U)^{-1}x) = ((1-U)v, i(1+U)v) = (-i(2-U+U^*)v, v) = (i(1+U)(1-U)^{-1}x, x)$$

as it can be checked. We see that $\mathbb{G}(T)$ is the set

$$\Big\{[-i(2-U+U^*)v,v]:v\in\mathcal{H}\Big\}=J\Big(\Big\{[v,i(2-U^*+U)v]:v\in\mathcal{H}\Big\}\Big)=J\Big(\mathbb{G}(i(2-U^*+U))\Big)$$

were $\mathbb{G}(i(2-U^*+U))$ is the graph of a bounded operator, hence closed, and $J([x,y])=[y,x]$ is unitary. This implies T is closed and symmetric.,

To calculate $\operatorname{ran}(T+i)$ we take $x\in D(T)$, that is $x=(1-U)v$. Then, $(T+i)x=i(1+U)v+i(1-U)v=2iv$. Similarly, $(T-i)x=2iUv$. Since v is any element of $\mathcal{H}$, we have $\operatorname{ran}(T\pm i)=\mathcal{H}$ and Theorem 11.3.6 implies that T is self-adjoint. Uniqueness is obvious. □

Definition 11.3.5. Let $\mathcal{H}$ be a separable Hilbert space, T a closed, densely defined symmetric operator and $W:\mathcal{H}_+\to\mathcal{H}_-$ its Cayley transform. The deficiency indices of T are by definition the deficiency indices of the pair $(\mathcal{H}_+,\mathcal{H}_-)$.

Theorem 11.3.6 (Von Neumann's theorem on self-adjoint extensions)**.** Let $\mathcal{H}$ be a Hilbert space and T a closed, densely defined symmetric operator. Then, T admits a self-adjoint extension iff its deficiency indices are equal.

Any self-adjoint extension of T is of the form $i(1+U)(1-U)^{-1}$ where U is a unitary map extending W.

Proof. Assume that A is a self-adjoint extension of T and $U=(A-i)(A+i)^{-1}$ is its Cayley transform, a unitary operator. Let $x\in D(T)=\operatorname{ran}(1-W)$. By definition, $Ax=Tx$ which implies $Ux=(T-i)(T+i)^{-1}x=Wx$ and U is a unitary extending W, and this implies that the deficiency indices are indeed equal.

If the deficiency indices of T are equal, then there exist unitary operators U extending W, and they are of the form $U=W\oplus\tilde{U}$ where $\tilde{U}$ is unitary from $\mathcal{H}_+^\perp$ onto $\mathcal{H}_-^\perp$. Then, as in the proof of Lemma 11.3.4, $\ker(1-U)=\ker(1-U^*)=\operatorname{ran}(1-U)^\perp\subset\operatorname{ran}(1-W)=\{0\}$. Hence, for any unitary extension U of W, $A=i(1+U)(1-U)^{-1}$ is self-adjoint.

Since U extends W, $A=i(1+U)(1-U)^{-1}$, extends $T=i(1+W)(1-W)^{-1}$ (think graphs!) and the result follows. The self-adjoint extensions of T are parametrized by the unitaries $\tilde{U}$ in this proof.

□

Exercise 46. Let $\mathcal{H}$ be a Hilbert space and T a closed, densely defined symmetric operator. Check that any symmetric extension of T is a restriction of T^*. Indeed, if S is symmetric and $T\subset S$, then $T^*\supset S^*\supset S$, where we used symmetry of S. Using the definition of the adjoint using the graph, and the ideas in this section, prove the following result.

Theorem 11.3.7 (Von Neumann's theorem on self-adjoint extensions)**.** Let $\mathcal{H}$ be a Hilbert space and T a closed, densely defined symmetric operator. Let $N_\pm = \operatorname{ran}(T \pm i)^\perp = \mathcal{H}_\pm^\perp$. Then, $N_\pm = \ker(T^* \mp i)$ and

$$\operatorname{dom}(T^*) = \operatorname{dom}(T) \oplus N_+ \oplus N_- \tag{92}$$

where the decomposition is orthogonal relative to the graph inner product.

A fundamental result of Von Neumann is the following.

Theorem 11.3.8 (Von Neumann's theorem on T^*T)**.** Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces and $T : D(T) \subset \mathcal{H} \to \mathcal{H}'$ be closed and densely defined. Then T^*T is self-adjoint in $\mathcal{H}$ and $D(T^*T)$ is a core of T.

At first sight, it is remarkable even that T^*T is necessarily densely defined for any T closed and densely defined!

Proof. By Lemma 11.1.1, we have $\mathbb{G}(T) \oplus V(\mathbb{G}(T^*)) = \mathcal{H} \times \mathcal{H}'$. Hence, any vector $[u, u']$ in $\mathcal{H} \times \mathcal{H}'$ can be written in the form $[v, Tv] + [-T^*v', v']$ for some $v \in D(T)$ and $v' \in D(T^*)$, In particular, for $u' = 0$ we get $u = v - T^*v'$ and $0 = Tv + v'$. Hence, $Tv = -v' \in D(T^*)$ and $u = (1 + T^*T)v$. Since $u \in \mathcal{H}$ was arbitrary, we see that $S = 1 + T^*T$ has range $\mathcal{H}$. It can be easily checked that S is symmetric and $\|S^{-1}\| \leqslant 1$. Hence, S^{-1} is symmetric and bounded, therefore self-adjoint. Since $\ker S^{-1} = \{0\}$, $\overline{\operatorname{ran} S^{-1}} = \mathcal{H}$. This implies that S is self-adjoint with domain $\operatorname{ran} S^{-1}$, as you should easily check.

To show that $D(T^*T)$ is a core of T, we check that the elements of the form $[v, Tv]$ with $v \in D(T^*T)$ are dense in $\mathbb{G}(T)$, or, that if $u \in D(T)$ and $[u, Tu] \perp [v, Tv]$ for all $v \in D(T^*T)$, then $u = 0$. But, this orthogonality implies $0 = (u, v) + (Tu, Tv) = (u, (1 + T^*T)v)$. Since $\operatorname{ran}(1 + T^*T) = \mathcal{H}$, the result follows. □

Definition 11.3.9 (Normal operators)**.** Let $\mathcal{H}$ be a Hilbert space, and T a closed densely defined operator. Then, T is normal if $T^*T = TT^*$.

Definition 11.3.10. A closed, symmetric operator is said to be **maximal** if at least one of the deficiency indices is zero.

We see that a closed, symmetric operator is self-adjoint iff it is maximal with deficiency indices $(0, 0)$. We also see that a closed, symmetric operator with just one deficiency index equal to zero has no self-adjoint extension.

Exercise 47 (Maximal operators have no proper symmetric extensions). Show that a maximal symmetric operator T has no proper symmetric extension.

Solution: Say the deficiency indices are $(0,n)$. Then, by definition $\operatorname{ran}(T+i)=\mathcal{H}$. Let $T'\supset T$ be symmetric. We may assume T' is closed (otherwise, take the closure of T', and check that this closure is a symmetric extension as well). We clearly have $\operatorname{ran}(T'+i)=\mathcal{H}$ as well. Hence, for any $u\in D(T')$ there is a $v\in D(T)$ such that $(T+i)v=(T'+i)u$ and since $T'\supset T$, we have $(T+i)v=(T'+i)v$, and therefore $(T'+i)(u-v)=0$. Proposition 11.2.3 implies $u=v, D(T')\subset D(T)$ and thus $T=T'$.

11.3.1 Examples

1. Let $\mathcal{H}=L^2(\mathbb{R})$ and T the multiplicative operator defined by $(T\varphi)(x)=x\varphi(x)$ on the domain $D(T)=\{\varphi\in\mathcal{H}:T\varphi\in\mathcal{H}\}$. Equivalently, $D(T)=\{\psi:(1+|x|)\psi\in L^2(\mathbb{R})\}$. The deficiency indices of T are $(0,0)$: Clearly, if $\psi\in L^2(\mathbb{R})$, then $(x\pm i)^{-1}\psi\in D(T)$ and, since $(x\pm i)[(x\pm i)^{-1}\psi]=\psi$ we have $\operatorname{ran}(T\pm i)=\mathcal{H}$. Hence T is self-adjoint. It is a good exercise to obtain this result from first principles, that is, by finding the domain and expression of the adjoint.

2. More generally, if F is a real-valued measurable function on $\mathcal{H}=L^2(\mathbb{R},\mu)$, then the operator T defined by $Tf=Ff$ on $D(T)=\{f\in\mathcal{H}:Ff\in\mathcal{H}\}$ is self-adjoint. This result clearly extends to the case when F is real-valued measurable on $\oplus_n L^2(\mathbb{R},\mu_n)$.

3. We now reexamine $T=-i\frac{d}{dx}$ on various domains of symmetry in $L^2([0,1])$ (part 1 of Exercise 44). We have

$$\mathcal{H}_\pm^\perp=\operatorname{ran}(T\pm i)^\perp=\ker(T^*\mp i)=\{ce^{\mp x}:c\in\mathbb{C}\}\tag{93}$$

Both subspaces $\mathcal{H}_\pm$ are one dimensional: the deficiency indices of T are $(1,1)$, and thus there exists a one-parameter family of self-adjoint extensions of T, parametrized by the unitary maps from $\mathcal{H}_+^\perp$ to $\mathcal{H}_-^\perp$. Any such unitary map is of the form $x\mapsto\alpha x$ where $\alpha\in\mathbb{T}$.

Let's check that the operator $T_\alpha=-i\frac{d}{dx}$ on the domain D_α in (89) is self-adjoint for any $\alpha\in\mathbb{T}$. Let $g\in L^2([0,1])$. Solving the equation $-if'+if=g$ we get

$$f(x)=ie^x\int_0^x e^{-s}g(s)ds+ce^x\tag{94}$$

The condition $f(0)=\alpha f(1)$ translates to

$$c=\alpha ie\int_0^1 e^{-s}g(s)ds+\alpha ce \text{ hence } c=i(1-\alpha e)^{-1}e\alpha\int_0^1 e^{-s}g(s)ds\tag{95}$$

Thus a c can always be chosen so that $f\in D_\alpha$ and we get $\operatorname{ran}(T_\alpha+i)=L^2([0,1])$. Similarly, $\operatorname{ran}(T_\alpha-i)=L^2([0,1])$, and T_α is self-adjoint by Theorem 11.3.6.

Exercise 48. Show that the T_α above are all the self-adjoint extensions of T.

Solution. Let T' be a self-adjoint extension of T. Then, $T \subset T' \subset T^*$. In particular, if $\varphi \in D(T')$, then $\varphi \in AC[0,1]$ and $T'\varphi = -i\varphi'$. Hence,

$$(T'\varphi, \psi) = \int_0^1 -i\varphi'\overline{\psi} = \varphi\overline{\psi}|_0^1 + \int_0^1 \varphi\overline{(-i\psi')} = \varphi\overline{\psi}|_0^1 + (\varphi, T'\psi) \tag{96}$$

Hence the bilinear form $\varphi(1)\overline{\psi(1)} - \varphi(0)\overline{\psi(0)}$ must be zero. This implies $|\varphi(0)| = |\varphi(1)|$ for all $\varphi \in D(T')$ and since the domain is a linear space $\supsetneq D(T)$ this implies that $\varphi(0) = \alpha\varphi(1)$ for some $\alpha \in \mathbb{T}$ and all $\varphi \in D(T')$.

4. Here we reexamine $T = -i\frac{d}{dx}$ on $L^2([0,\infty)$ (part 2 of Exercise 44). We have

$$\mathcal{H}_+^\perp = \ker(T^* - i) = \{ce^{-x} : c \in \mathbb{C}\} \text{ and } \mathcal{H}_-^\perp = \ker(T^* + i) = \{0\}$$

The deficiency indices are $(1,0)$, hence T is maximal, but it has no self-adjoint extensions.

Exercise 49. Let now $T = -i\frac{d}{dx}$ in $\mathcal{H} = L^2(\mathbb{R})$. Take $D(T) = \{\psi \in AC : \psi \in L^2(\mathbb{R}), \psi' \in L^2(\mathbb{R})\}$. Check that $D(T)$ is the Sobolev space $H^1(\mathbb{R})$ and that T is self-adjoint on $D(T)$. Check that the Schwarz space $\mathcal{S}(\mathbb{R})$ is a core for T.

Note that T is maximal, and by Exercise 51 it is impossible to extend differentiation on a subspace of L^2 strictly larger than H^1 as a symmetric operator (meaning, if we want to keep the Leibniz rule, $(fg)' = f'g + fg'$–check this).

6. $-\Delta$ in $\mathbb{R}^n$. We define the Laplacian $T = -\Delta$ on $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, which is a core for T: its closure is self-adjoint. Indeed, the Fourier transform $\mathcal{F}$ is a unitary automorphism of L^2 mapping $\mathcal{S}$ onto $\mathcal{S}$. The image of $-\Delta$, $\tilde{T} = \mathcal{F}T\mathcal{F}^{-1}$ is the multiplication operator $(\tilde{T}\varphi)(\xi) = \xi^2\varphi(\xi)$. The extension of this operator to $D(\tilde{T}) = \{\varphi \in L^2(\mathbb{R}^n) : \xi^2\varphi \in L^2(\mathbb{R}^n)\}$ is self-adjoint, by virtually the same argument as in 1. It is an easy exercise to show that that the closure of the restriction of $\tilde{T}$ to $\mathcal{S}$ is indeed $\tilde{T}$ on $D(\tilde{T})$. This proves the claim, and shows that $D(-\Delta) = H^2(\mathbb{R}^n)$. We note that, by the Sobolev embedding theorem, the functions in $D(-\Delta)$ are only continuous if $2 - n/2 \geqslant 0$, and the derivatives are in general taken in the sense of distributions.

11.3.2 Spectral teorems for unbounded self-adjoint operators

Using the correspondence between self-adjoint operators and unitary ones, one can essentially translate the various versions of spectral theorems to corresponding ones for self-adjoint operators. Let us prove te following.

Theorem 11.3.11 (Spectral theorem, multiplication operator form)**.** Let T be a self-adjoint operator on the separable Hilbert space $\mathcal{H}$.

Then,

1. There exists $J \subset \mathbb{N}$, measures $\{\mu_n\}_{n\in J}$ and a unitary operator U from $\mathcal{H}$ onto $\bigoplus_{n\in J} L^2(\mathbb{R}, \mu_n)$, were μ_n are supported on $\sigma(T) \subset \mathbb{R}$. Furthermore, $f \in D(T)$ iff the function $x(Uf)(x) \in L^2(\mathbb{R}, \mu_n)$ for all n.

2. The map U gives the **spectral representation** of T: for any

$$\langle \varphi_1, \cdots, \varphi_n, \cdots \rangle \in \bigoplus_{n\in J} L^2(\mathbb{R}, \mu_n)$$

and any $x \in \mathbb{R}$ and $n \in J$, we have

$$(UTU^*\varphi)_n(x) = x\varphi_n(x)$$

Proof. Let W be the Cayley transform of T. W is a unitary operator, in particular it is normal and theorem 4.0.6 applies to it. We have $\sigma(W) \subset \mathbb{T}$, and there is a unitary operator from $\mathcal{H}$ onto $\mathcal{H}' = \bigoplus_{n\in J} L^2(\mathbb{T}, \lambda_n)$ were λ_n are supported on $\sigma(W)$ such that for any

$$\langle \varphi_1, \cdots, \varphi_n, \cdots \rangle \in \bigoplus_{n\in J} L^2(\mathbb{R}, \lambda_n)$$

and any $\lambda \in \mathbb{R}$ and $n \in J$, we have

$$(UWU^*\varphi)_n(z) = z\varphi_n(z); \;\; z \in \mathbb{T}$$

Note that the function $F(z) = i(1+z)(1-z)^{-1}$ is real-valued for $z \in \mathbb{T}$ (indeed, if $z = e^{i\varphi}$, then $F(z) = -\cot(t/2)$). Hence, we see by Example 2 above that F is self-adjoint on $\mathcal{H}'$. Now, $f \in D(T)$ iff $f = (1-W)g$ for some $g \in \mathcal{H}$ iff for all n $Uf = (1-z)g'$ for some $g' \in \oplus_n L^2(\lambda_n)$, that is, iff $Uf \in D(F)$. With $g = (1-z)g'$ we have $T(U^*g) = T(1-W)U^*g' = i(1+W)U^*g'$ which means that $UTU^* = i(1+z)(1-z)^{-1}$ on $D[(1-z)^{-1}]$. Since $F(t) = -\cot(t/2)$ is measurable, for each n it induces a positive Borel measure $\mu_n'(\mathbb{R})$ by $\mu_n'(A') = \mu_n(A)$ where $A = \{t : -\cot(t/2) \in A\})$ and the result follows, since $\int_{A'} \lambda d\mu_n' = \int_A -\cot(t/2)d\mu_n$. □

Another form of the spectral measure which can be obtained by an easy adaptation from the bounded case (see [16]), or from the spectral theorem above and Bochner integration is the following. Recall Definition 4.2.1. Here, $X = \mathbb{R}$.

Theorem 11.3.12 (Spectral theorem, –projection valued measure form)**.** Let $\mathcal{H}$ be a separable Hilbert space. There is a one-to-one correspondence between self-adjoint operators A

and projection-valued measures $\{P\}$ on $\mathcal{H}$. The correspondence is

$$A = \int_{\mathbb{R}} \lambda dP_\lambda$$

If g is a real-valued measurable function on $\mathbb{R}$, then $g(A)$ defined on

$$D_g = \{\varphi \in \mathcal{H} : \int_{\mathbb{R}} |g(\lambda)|^2 dP_\lambda < \infty\}$$

by

$$g(A) = \int_{\mathbb{R}} g(\lambda) dP_\lambda$$

is self-adjoint.

Spectral theorems are particularly important in the unbounded case. Indeed, one cannot define, for unbounded self-adjoint A, the important unitary operator e^{iA} by the Taylor series of the exponential since the series may not converge.

11.4 Quantum mechanics and operators

A physical system is generally described by three basic ingredients: states; observables; and dynamics (or law of time evolution) or, more generally, a group of physical symmetries. In classical mechanics, a system is described by a phase space model of mechanics: states are points in a symplectic phase space, observables are real-valued functions on it, time evolution is given by a one-parameter group of symplectic transformations of the phase space, and physical symmetries are realized by symplectic transformations. A quantum description normally consists of a Hilbert space of states, observables are self adjoint operators on the space of states, time evolution is given by a one-parameter group of unitary transformations on the Hilbert space of states, and physical symmetries are realized by unitary transformations. (It is possible to map this Hilbert-space picture to a phase space formulation, invertibly.)

The following summary of the mathematical framework of quantum mechanics can be partly traced back to the Dirac-von Neumann axioms.

Each physical system is associated with a separable Hilbert space $\mathcal{H}$. Rays (that is, subspaces of complex dimension 1) in $\mathcal{H}$ are associated with quantum states of the system. Each element ψ of a ray is called a *wave function*. In other words, quantum states can be identified with equivalence classes of vectors of length 1 in $\mathcal{H}$. Separability is a mathematically convenient hypothesis, with the physical interpretation that countably many observations are enough to uniquely determine the state.

Physical observables are represented by self-adjoint operators on $\mathcal{H}$. The spectral measures give the expectation value of a physical quantity. Let A be a self-adjoint operator representing a physical quantity, and assume that the state of the system is represented by the wave function ψ. Then, the measured expected value of A is given by $(\psi, A\psi)$. Note that the expectation value is necessarily real-valued, as it should, and the expectation value only depends on the ray of ψ. Under a unitary transformation using the spectral theorem, A becomes a multiplication operator (say, with the variable itself) on $L^2(\mathbb{R}, \mu)$. Then, the probability that the physical quantity is in the set A is $\int_A d\mu$. Hence, the only possible values of A lie in the spectrum of A. In the special case when A has only discrete spectrum, the possible outcomes of measuring A are its eigenvalues.

The momentum operator is given, in normalized units, by $p = -i\nabla$ and then the kinetic energy operator is $p^2/2 = -\frac{1}{2}\Delta$. If the evolution is on the whole space, we know that these are self-adjoint on a corresponding Sobolev space. An important physical quantity is the Hamiltonian of a system $H = H(x,t)$. In the simple case of one particle, as in classical mechanics $H = K + V$ the kinetic plus potential energy, viz.

$$H(x,t) = -\frac{1}{2}\Delta + V(x,t)$$

For physically reasonable potentials, it can be shown that H is self-adjoint.

The Schrödinger picture. An important physical quantity is the Hamiltonian of a system $H = H(x,t)$. If the system is initially in the state $\psi_0(x)$, then at time t its state is given by $\psi(x,t)$, the solution of the Schrödinger equation

$$i\psi_t = H\psi; \;\; \psi(0,x) = \psi_0(x)$$

The Dirac picture looks at the evolution of operators themselves. It follows that, if $A(t)$ is a time-dependent observable, then

$$i\frac{d}{dt}A = [H,A]$$

where $[B,C]$ is the commutator of the operators B, C. Note that A is conserved if it commutes with H. Such is the case of the momentum p on a line or in the whole space. On a half-line, $-i\nabla$ cannot be an observable, since its commutation with Δ on a half-line, with boundary condition zero, would mean p is conserved when $V = 0$. But this is impossible, since a particle coming towards the origin would eventually hit the origin, where the momentum would necessarily change regardless of the boundary conditions there.

12 Elements of the theory of resurgent functions

The phenomenon of resurgence (the terminology will become clear in the sequel) was discovered by J. Écalle in the late 70's. What he realized is that functions of "natural origin" in analysis, as he called them, have a rich set of special analytic properties, which are invariant under common algebraic and analytic operations, conferring resurgence a high degree of universality. By now it is known that these "natural problems" in analysis include systems of linear or nonlinear ODEs, difference equations, wide classes of evolution PDEs, integral equations, multidimensional integrals with parameters, all classical "named" special functions and more.

An analytic function at a point of analyticity or a pole, or at a branch point where there is a local convergent expansion in ramified powers of the variable and logs, is resurgent at that point in a rather trivial way. Much more interesting are essential singularities. Typically, solutions of classical equations have essential an singularity at infinity, and that is taken conventionally to be the point of analysis. We recall the basic definitions and notation of asymptotic expansions.

Definition 12.0.1 (Poincaré asymptotics)**.**

1. Let f be defined for all large $x \in \mathbb{R}$. We say that f has an asymptotic power series in integer powers of $1/x$ as $x \to +\infty$, written $f \sim \sum_{k\geqslant 0} c_k x^{-k-1}$ if for all $n \in \mathbb{N}$

$$f - \sum_{k=1}^{n} \frac{c_k}{x^k} = o(x^{-n}) \quad \text{as } x \to +\infty \tag{97}$$

In (97), g is $o(x^{-n})$ as $x \to \infty$ if by definition $x^n g(x) \to 0$ as $x \to \infty$. Similarly, g is $O(x^{-n})$ as $x \to \infty$ if $\limsup_{x\to\infty} |x^n g(x)| < \infty$. For the purpose of the definition (97), $o(x^{-n})$ is equivalent to $O(x^{-n-1})$.

2. These notions extend naturally to series in fractional powers of x, or complex powers $\beta_1, \beta_2, ...$ provided $\Re\beta_1 > \Re\beta_2 > \cdots$ and $\Re\beta_j \to -\infty$ fast enough (for instance linearly in j, which is most often the case).

3. Other natural extensions are asymptotic series in some sector in the complex domain, where we replace $x \to +\infty$ to $|x| \to \infty$ for all x with $\arg x \in (a, b)$. Powers of x can be replaced with combinations of powers, exponentials and logs, such as in $g = O(e^{-x} x^a \ln x)$ as $x \to \infty$. More generally, we write $f = O(g)$, or $f = o(g)$ with a similar definition, making sure that g does not vanish in a neighborhood of infinity. The notation $f \ll g$ stands for $f = o(g)$.

4. More generally, one can take an asymptotically ordered family of functions, $f_1(x) \gg f_2(x) \gg f_3(x) \gg \cdots$ and define the formal asymptotic series $\sum_{k\in\mathbb{N}} f_k(x)$. A simple example would be $f_k(x) = e^{-x} x^{-k}$. Then, $g(x) \sim \sum_{k\in\mathbb{N}} f_k(x)$ is defined using obvious extensions of the above.

Note also that, if we take $x = 1/z$ and if $f \in C^\infty[(0, a)]$, then (97) states that f has a one-sided Maclaurin series at 0^+, with coefficients $c_1, c_2, ...$.

It is easy to check that the asymptotic series of a function, when one exists, is unique.

One can use the solutions of Euler's equation

$$y' - y = 1/x \tag{98}$$

for large x to illustrate some of the most basic concepts and methods in resurgence theory. We choose not to explicitly solve (98), since for few equations do we have this privilege. A commonly used method of solving ODEs is by power series. Inserting $\tilde{y} = \sum_{k=0}^{\infty} c_k x^{-k-1}$ and identifying the coefficients, we see that $c_k = (-1)^{k+1} k!$, and the series solution has empty domain of convergence. This $\tilde{y}$,

$$\tilde{y} = \sum_{k\geqslant 0} \frac{k!}{(-x)^{k+1}} \tag{99}$$

is called a formal solution of (98)–it is a solution in the space of formal power series, the term "formal" contrasting these with convergent ones.

We will see that there is a unique solution y_0 of (98) such that $y_0 \sim \tilde{y}$ as $x \to +\infty$, and in fact this asymptotic behavior holds in a sector of opening 3π on the Riemann surface of the log. Uniqueness can be shown from the fact that if y_0 is a solution of (98), then the general solution is $y = y_0 + Ce^x, C \in \mathbb{C}$.

However, if we change one sign in (98), to get $y' + y = 1/x$, then *all* solutions are asymptotic to the formal solution $\sum_{k=0}^{\infty} k! x^{-k-1}$. Indeed, in this case, given one solution y_0, the general solution is $y_0 + Ce^{-x}, C \in \mathbb{C}$, and certainly $e^{-x} = o(x^{-n})$ as $x \to +\infty$, for any $n \in \mathbb{N}$.

In general, divergence of a formal asymptotic series is associated with loss of information about the underlying function *if this information is to be obtained from Poincaré asymptotics*. It is also important to remark that the notions of divergence and convergence *are not absolute, but topology-dependent*.

The formal solution $\tilde{y}_0$ can be obtained in a way that sheds some light into the reasons behind its divergence. We note that if f is a power series at infinity, then $f' \ll f$ as $x \to \infty$. Then we can define an iteration scheme for the formal solution of (98),

$$f_{n+1} = x^{-1} + f_n; \quad f_0 = 0 \tag{100}$$

and we get

$$f_1 = \frac{1}{x}, f_2 = \frac{1}{x} - \frac{1}{x^2}, ..., f_n = \sum_{k=1}^{n} \frac{k!}{(-x)^{k+1}}, \cdots \tag{101}$$

and it can be shown that f_n converges to $\tilde{y}$ as $n \to \infty$ in the space of formal series endowed with a natural topology.

In a space of actual functions, the iteration above amounts to the following. Write (98) in the form

$$(1 - D)y = -x^{-1}; \quad D := \frac{d}{dx} \tag{102}$$

and solve it by inverting $(1 - D)$ as a power series:

$$y = \left(\sum_{k=0}^{\infty} D^k \right) x^{-1} \tag{103}$$

Of course, since D is an unbounded operator, the right side cannot be expected to converge, and it doesn't. But this calculation also suggests an approach to address the divergence issue, that can be generalized. We can apply the spectral theorem to (102) or (103), where the unitary map is a Fourier transform. We should choose to take it along $i\mathbb{R} + 0$ to make D self-adjoint (and in fact for a good number of better reasons), $Uf = \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} e^{px} f(x) dx$–in fact, this is the inverse Laplace transform $\mathcal{L}^{-1}$. Noting that $\mathcal{L}^{-1}(1/x) = 1$ we get

$$\mathcal{U}y = \mathcal{L}^{-1} y = -\sum_{k=0}^{\infty} (-p)^k \tag{104}$$

We see that the transformation $\mathcal{U}$ maps a series with empty domain of convergence to one which is convergent near zero. For obvious reasons the series in (105) cannot converge on $\mathbb{R} = \sigma(-iD)$), but we may hope that *the analytic continuation* of the geometric sum, namely

$$\mathcal{U}y = -\frac{1}{1+p} \tag{105}$$

should lead to actual solutions of (98). Indeed, you can easily check that by returning to the

$x-$space through $\mathcal{L}$, $(\mathcal{L}F)(x) = \int_0^\infty F(p)e^{-xp}dp$ where $F = (1+p)^{-1}$ for the case at hand we get

$$y_0(x) = -\int_0^\infty e^{-px}(1+p)^{-1}dp \tag{106}$$

is a solution of (98). You should check that y_0 has $\tilde{y}$ as an asymptotic series, for instance by integrating by parts. In fact, in this case we know from spectral theory that (106) must be a solution, since $(1+p)^{-1}$ is the spectral image of $(1-D)^{-1}$.

More work is needed to make such a procedure general enough.

There are two other conceptually important approaches to addressing the issue of divergence. The first one goes back to Euler who indeed took (99) to be a prototypical divergent series and investigated what "value" should one assign to it. His approach, in modern language, relies on the notion of isomorphism.

Take as a model the correspondence between convergent power series at some point, say zero, as a subspace of all formal power series, and their sums, the germs of analytic functions at zero.

Remark. Summation of convergent series is an operator from the subspace of convergent power series to the space of analytic germs at zero, and it is an **extended algebra isomorphism**: summation commutes with the usual algebra operations, and moreover it commutes with series composition $f \circ g$ where $g(0) = 0$, with differentiation and integration, and a number of other operations such as series inversion f^{-1} when $f(0) = 0, f'(0) \neq 0$.

Again, in modern language Euler proposes to extend the summation operator to a wider class of series, in a property-preserving fashion. That is, the extended summation operator should still be an isomorphism between formal series and a space of functions. In "De Seriebus divergentibus" (1760) Euler shows that *if* such an extension existed, it would necessarily assign to $\tilde{y}$ the value (106).

Emile Borel went further, and constructed such an extended summation procedure, now known as Borel summation. Borel summation gives a rigorous meaning to the following formal manipulation. Note that $k! = \int_0^\infty e^{-t}t^k dt$ to write

$$\sum_{k\geqslant 0}\frac{k!}{(-x)^{k+1}} = \sum_{k\geqslant 0}\frac{1}{(-x)^{k+1}}\int_0^\infty e^{-t}t^k dt = \int_0^\infty \sum_{k\geqslant 0}\frac{t^k}{(-x)^{-k-1}}e^{-t}dt = \int_0^\infty \frac{e^{-xp}}{1+p}dp = -e^x\mathrm{Ei}(-x) \tag{107}$$

by summing the geometric series and changing variables to $p = xt$. Here Ei is the exponential integral, a special function

$$\mathrm{Ei}(-x) = \int_x^\infty \frac{e^{-s}}{s}ds \tag{108}$$

Of course, we cannot even meaningfully attempt to prove commutation of summation and integration since we started with a formal series that diverges, and ended up with an actual function. The mathematical way to proceed is to transform all this in a definition, with appropriate conditions to ensure the existence of the final object.

Definition 12.0.2 (The Borel transform)**.** Let $\tilde{f} = \sum_{k\geqslant 0} c_k x^{-k-1}$ be a formal power series. The Borel transform $\mathcal{B}$ is defined by $\mathcal{B}\tilde{f} = \sum_{k\geqslant 0}\frac{c_k}{k!}p^k$. We note that $\mathcal{B}$ is simply the inverse Laplace transform, defined on formal power series, by applying it term-by-term to the series.

Definition 12.0.3 (Borel summation)**.** The *domain D* of Borel summation is the subspace of formal power series $\tilde{f} = \sum_{k\geqslant 0} c_k x^{-k-1}$ with the properties:

1. $\mathcal{B}\tilde{f}$ has a nonzero radius of convergence. Let F be the analytic function to which $\mathcal{B}\tilde{f}$ converges.

2. F is real-analytic on $\mathbb{R}^+$.

3. $\mathcal{L}F$ is well defined. For this, we impose exponential bounds on F along $\mathbb{R}^+$. Two spaces are particularly useful here. For $\geqslant 0$,

$$L^\infty_\nu = \{f : \mathbb{R}^+ \to \mathbb{C} : \|e^{-\nu p} f\|_\infty < \infty\}$$

and

$$L^1_\nu := \{f : \mathbb{R}^+ \to \mathbb{C} : \|f e^{-\nu x}\|_1 < \infty\} \tag{109}$$

The Laplace transform exists on both of them, with values in a space of analytic functions for $\Re z > \nu$.

If $\tilde{f} \in D$, then the **Borel sum** of $\tilde{f}$ is defined to be $\mathcal{L}\mathcal{B}\tilde{f} = \mathcal{L}F$.

Note 12.0.4.

1. The operator $\mathcal{L}\mathcal{B}$ is the composition of four operators: first is inverse Laplace transform $\mathcal{L}^{-1}$, which applied to formal series is denoted by $\mathcal{B}$. Its domain is a space of formal power series in x^{-1} and the range is another space of formal series, in powers of p. The second operator, convergent summation $\mathcal{S}$, is defined on all formal series in p with nonzero radius of convergence, with range the family of germs of analytic functions at zero. The third is analytic continuation $\mathcal{A}_c$, here on $[0, \infty)$ and the last one is the Laplace transform. The two operations in the middle, convergent summation and analytic continuation, are extended structural isomorphisms. Borel summation conjugates this pair through $\mathcal{L}\mathcal{L}^{-1}$ and it is thus also an extended structural isomorphism, as discussed in the note below.

2. The operator $\mathcal{A}_c\mathcal{S}\mathcal{B}$ associates to a possibly divergent series $\tilde{f} \in D$ an analytic function $F = \mathcal{A}_c\mathcal{S}\mathcal{B}\tilde{f}$. The structure of singularities of this function provides a rich set of analytic invariants and extensive information about the origin of $\tilde{f}$. In particular if $\tilde{f}$ is a formal power series solution of some system of linear or nonlinear ODEs with meromorphic coefficients, the system is completely, and constructively, determined from this special F.

Exercise 50. One can define a form of composition of series in powers of $1/x$, where if $S_1(x)$ and $S_2(x)$ are formal series one replaces x in S_1 by $x + C + S_2$, for some $C \in \mathbb{C}$, formally reexpands each term, and collects the powers of x. Check that this is well defined on formal series at infinity. Check that function inversion is well defined, in the same sense: $S_1(y) = x + C + S_2$ can be inverted to obtain y as a power series.

Note 12.0.5.

1. It is not difficult to show that the domain D of Borel summation is a differential algebra of power series. This differential algebra is also closed under series composition and series inversion, as in Exercise 50 above.

2. However, analytic equations also involve operations which are undefined on D. Hence, D is insufficient to asymptotically represent important functions in analysis.

   (a) The first missing operation is $\tilde{f} \mapsto 1/\tilde{f}$; to remedy this, one would extend D to the field of fractions generated by D.

   (b) The second one however, integration, maps $1/x$ to $\log x$, which, as an asymptotic object has to be treated as a new primitive symbol (it is not expressible in terms of powers of x). Function inversion results in yet another primitive symbol, e^x with reciprocal e^{-x}. Further closures introduce no other new object, a remarkable mathematical fact.

   Closure of asymptotic power series under all these operations and more leads to the space of *transseries*. Remarkably, transseries have a fairly simple structure in terms of which one can express the general formal solution to analytic equations or the physical quantities obtained by various formal perturbative expansions.

3. Borel summation also requires a highly nontrivial extension. Indeed, in our model equation with one changed sign, $y' + y = 1/x$, $\mathcal{A}_c \mathcal{S}\mathcal{B}\tilde{y} = (1-p)^{-1}$ does not have analytic continuation along $\mathbb{R}$. $(1-p)^{-1}$ does continue uniquely on $\mathbb{R}^+$ as a meromorphic function, but this is a peculiarity of this very simple equation. In general, one encounters arrays of branch points along $\mathbb{R}^+$, in which case analytic continuation on the real line has no obvious natural definition.

4. Écalle-Borel summation is a composition of the form $\mathcal{L}\mathcal{A}\mathcal{S}\mathcal{B}$ where $\mathcal{A}$ is now an average of analytic continuations along paths in $\mathbb{C}$ avoiding the singularities. It is also remarkable that such an average exists, with universal coefficients, so that $\mathcal{L}\mathcal{A}\mathcal{S}\mathcal{B}$ commutes with "all" operations.

5. In view of 4., Laplace transforms of functions analytic at zero and on some Riemann surface play a key role in resurgence theory.

6. The concept of transseries and well-behaved averages appears in the pioneering work of Écalle [11].

A typical form of a transseries expansion is

$$\tilde{y} = \sum_{k \geqslant 0} e^{-kx} \left(\sum_{m \geqslant 0} \frac{c_{km}}{x^m} \right); \quad x \to +\infty \tag{110}$$

You can regard (110) as the formal Taylor series of a function of two variables, $z_1 = x^{-1}, z_2 = e^{-x}$, the **generators** of this transseries. We note that, in general, an expression such as (110) is not a classical asymptotic expansion: the terms cannot be defined by asymptotic limits. Indeed, any term with $k = 0$ is much larger than any of the terms with $k > 0$.

Such simple examples of transseries were known in asymptotic ODE theory already at the end of the 19th century and are part of classical asymptotic ODE literature, but before resurgence theory they were just purely formal solutions. A multisum of the form (110) but with $n+1$ generators, $x^{-1}, x^{\alpha_1} e^{-\lambda_1 x}, ..., x^{\alpha_n} e^{-\lambda_n x}$, $\Re\lambda_i > 0, \alpha_i \in \mathbb{C}$ is the most general formal solution of a generic nonlinear meromorphic nth order ODE as $x \to +\infty$. More precisely, the general formal solution of these equations has the shape

$$\tilde{y} = \sum_{\mathbf{k} \geqslant 0} e^{-\boldsymbol{\lambda} \mathbf{k} x} \mathbf{C}^{\mathbf{k}} y_{\mathbf{k}}(x) \tag{111}$$

where $\mathbf{C}^{\mathbf{k}} = C_1^{k_1} \cdots C_n^{k_n}$, $\mathbf{k} \geqslant 0$ means every component is a nonnegative integer, and $y_{\mathbf{k}}(x)$ are formal power series in powers of $1/x$. For a given ODE, the multi-constant $\mathbf{C}$ parametrizes the formal solutions, and all other ingredients, $\boldsymbol{\lambda}$ and the $\{y_{\mathbf{k}}\}_{\mathbf{k} \geqslant 0}$ are uniquely determined.

Example 12.0.6 (Solving differential equations in transseries)**.** Consider the equation

$$y' + y + y^3 = x^{-1} \tag{112}$$

This is an Abel-type equation with no explicit solutions (it is "nonintegrable"). It is a simple exercise however to show that (112) has a particular formal solution $\tilde{y}_0$ as an integer asymptotic power series. You can obtain it by inserting in the equation a series with unknown coefficients and finding a recurrence relation for them. Or, we can obtain it by an iteration which converges in the space of formal series,

$$y_{n+1} = x^{-1} - y_n' - y_n^3; \quad y_0 = 0 \tag{113}$$

The topology on the linear space of formal power series is: the sequence $\left\{ \sum_{k \geqslant 0} c_k^{[n]} x^{-k} \right\}_{n \in \mathbb{N}}$ of formal power series converges to zero as $n \to \infty$ if for any k there is an $n(k)$ such that, for all $n > n(k)$, $c_k^{[n]} = 0$. That is, eventually, each coefficient vanishes exactly. Check that the iteration (113) converges in this topology. Note also that a sequence $\{x_n\}$ of real numbers converges to zero if all the binary digits of $\{x_n\}$ become zero eventually.

Further solutions: hands-on calculation of a transseries. Let y_0 be this power series.

By uniqueness, if further formal solutions $y = y_0 + \delta$ exist, then $\delta = o(x^{-n})$ for any n. Taking $y = y_0 + \delta$ in (112) we get

$$\delta' + \delta + 3y_0^2\delta + 3y_0\delta^2 + \delta^3 = 0 \tag{114}$$

Since $\delta = o(1)$, we have $\delta^2 = o(\delta)$. Hence, to leading order, (115) gives

$$\delta' + \delta + 3y_0^2\delta = 0 \text{ hence } \frac{\delta'}{\delta} = -1 - 3y_0^2 \tag{115}$$

which implies

$$\delta = Ce^{-x}e^{-3y_0^2} = Ce^{-x}y_1, \tag{116}$$

where $y_1 = 1 + \sum_{k\geqslant 1} y_{1k}x^{-k}$ is the asymptotic expansion of $e^{-3y_0^2}$ as $x \to +\infty$. (Writing $x = 1/z$, the coefficients y_{1k} are the one-sided Maclaurin coefficients of $\exp(-3y_0^2(z))$.) For any given C, $\delta^2 = O(e^{-2x})$ and $\delta^3 = O(e^{-3x})$, both $o(e^{-x}x^{-n})$ for all $n \in \mathbb{N}$ and cannot affect the calculation up to $O(e^{-2x})$. Next, we take $y = y_0 + Ce^{-x}y_1 + \delta_1$ in (112) we similarly get

$$\delta_1' + \delta_1 + 3y_0^2\delta_1 = C^2R_1e^{-2x} + O(e^{-3x}) \tag{117}$$

where R_1 is some specific formal power series, a quadratic expression of y_0 and y_1. (The only dependence on C is in the prefactor C^2.) Formally, (117) implies δ_1 is of order e^{-2x}. Inserting $\delta_1 = e^{-2x}y_2$ we get

$$f_1' - f_1 + 3y_0^2f_1 = C^2R_1 + O(e^{-x}) \tag{118}$$

which has a unique unique power series solution of the form $f_1 = C^2y_2$ where the coefficients of y_2 are determined from those of y_0 and y_1. Now we write $y = y_1 + Ce^{-x}y_1 + C^2e^{-2x}y_2 + \delta_2$ and so on. Inductively, at step n we get (compare with (117))

$$\delta_n' + \delta_n + 3y_0^2\delta_n = C^nR_ne^{-nx} + O(e^{-(n+1)x}) \tag{119}$$

where R_n has a polynomial expression in terms of $y_0, ..., y_{n-1}$.

Note that, for any n we get a linear non-homogeneous equation for δ_n and that the linear part is independent of n. This is typical of very general *perturbative expansions*– expansions in some small parameter, which here is simply $1/x$.

The solution of (119) is of the form $C^ne^{-nx}y_n$ where y_n is a uniquely determined power series. The limiting result, in a topology of transseries where a sequence $\left\{\sum_{k,l} c_{kl}^{[n]}e^{-kx}x^{-l}\right\}_{n\in\mathbb{N}}$ converges if for any fixed (k,l) the coefficient $c_{kl}^{[n]}$ vanishes identically beyond some $n(k,l)$, we get an expression

$$\tilde{y} = \sum_{n=0}^{\infty} C^ne^{-nx}y_n \tag{120}$$

which is a formal solution of (112). This is the general solution of (112) in the space of all transseries. It can be checked that the coefficients of series $y_n(x)$ grow at a common rate, roughly $y_{nk} \sim k!$.

Note the presence of an arbitrary constant C : as it turns out, C parametrizes the family of all **actual** solutions that exist in a region of the form $\{x : |x| > x_0, |\arg x| < \pi/2\}$. This also indicates that we should not expect meaningful further formal solutions of (112), beyond (120).

Transseries were independently discovered in quantum field theory (also in the 70's) by t'Hooft and Polyakov who called them multi-instanton expansions. There, the exponentials are of the form $e^{-n\gamma^{-1}A(x)}$ where A is the classical action and γ is some small coupling constant.

See [13] for a really simple and nice introduction to **transseries** and their rigorous construction. In a nutshell, transseries are formal multiseries in small variables $z_1, ..., z_n$ called **generators** or **transmonomials**, where n and the z_i may depend on the transseries. Each transmonomial is a combination of powers and logs, such as $x^{-3/2}e^{-x-x^2}e^{-e^x}\log x$ or $e^{-x}/\log\log\log x$ (In practice, very rarely do transmonomials do involve iterated exponentials or logs.). The multiseries may start with a fixed negative power of the transmonomial, such as in $\sum_{k\geqslant -4} c_k x^{-k}$. They are closed under a general class of operations.

Closure of transseries under all operations we could think of, confers them a high degree of **universality** in the class of formal solutions to various problems. Indeed, to get formal solutions of some new type, we need first to exit the class of operations under which transseries are closed. All solutions of analysis problems we are aware of are Écalle-Borel sums of transseries, and all the(numerical) evidence coming from the formal expansions in string theory, quantum field theory and other areas of physics indicate that that is the case with those expansions as well.

Écalle-Borel summation extends to transseries. In fact, once well defined on series, the difficult part of the problem, the extension is quite straightforward: The summation of an exponential is the exponential itself. For some deep reasons, after Écalle-Borel summing all the $y_{\mathbf{k}}$, the resulting **function series** converges uniformly.

12.1 A useful asymptotic tool: Watson's lemma

Heuristics. Consider the following asymptotic problem: Determine the behavior of

$$\int_{-a}^{a} e^{xG(p)}h(p)dp, \quad x \to +\infty \tag{121}$$

where $a > 0$, G is real valued, smooth, and has a unique maximum at some $m \in [-a, a]$. Intuitively, it is clear that, as $x \to \infty$ most the contribution to the integral will come from an increasingly narrow region around m, since for fixed $p \neq m$, as $x \to \infty$, $e^{xG(p)} \ll e^{xG(m)}$.

Watson's lemma is a very useful tool to transform this intuition into proofs, as well as for dealing with the asymptotics of integrals arising in applications, which because of the universality of resurgence, can be reduced to Laplace transforms.

Lemma 12.1.1 (Watson's lemma)**.** (i) Assume that $\|F\|_{L^1_\nu} < \infty$ (cf. (109)) and

$$F(p) = p^{\alpha-1}\sum_{k=0}^{m} c_k p^{k\beta} + o(p^{\alpha-1+m\beta}) \text{ as } p \to 0^+ \text{ for all } m \leqslant m_0 \in \mathbb{N} \cup \infty \tag{122}$$

for some α and β, with $\Re\alpha, \Re\beta > 0$. Then as $|x| \to \infty$ along an arbitrary ray in the right half plane, we have

$$f(x) := (\mathcal{L}F)(x) = \int_0^\infty e^{-xp} F(p)dp = \sum_{k=0}^{m} c_k \Gamma(k\beta + \alpha) x^{-\alpha - k\beta} + o\left(x^{-\alpha - m\beta}\right). \tag{123}$$

for any $m \leqslant m_0$.

(ii) If $F \in L^1(0,a)$ and (122) holds, then $f(x) = \int_0^a F(p)e^{-px}dp$ has the same asymptotic expansion (123).

Note that a convergent Taylor series at zero in p results in a divergent asymptotic expansion as $x \to \infty$. This asymptotic series is gotten by simply Laplace transforming termwise the Taylor series at the origin, and this results in each coefficient getting multiplied by a factorial.

The maximum of e^{-xp} on $[0,\infty)$ is at $p = 0$, and part (ii) shows that indeed the asymptotic behavior of the integral is obtained from the germ behavior of F at zero: for any $\varepsilon > 0$, the behavior of F on (ε, ∞) is irrelevant.

For the proof, we rely on the intermediate result below.

Lemma 12.1.2. Let $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\varphi}$, $\rho > 0$, $\varphi \in (-\pi/2, \pi/2)$ and assume

$$F(p) \sim cp^\beta \quad \text{as } p \to 0^+$$

with $\Re(\beta) > -1$. Then

$$\int_0^\infty F(p)e^{-px}dp \sim c\Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. By definition $F(p) \sim p^\beta$ means $p^{-\beta}F(p) \to 1$ as $p \to 0$. We have

$$x^{\beta+1}\int_0^\infty e^{-xp}F(p)dp = \int_0^\infty \frac{F(t/|x|)}{(t/|x|)^\beta} e^{-te^{i\varphi}} e^{i(\beta+1)\varphi} t^\beta dt \to c\Gamma(\beta+1) \text{ as } |x| \to \infty \tag{124}$$

by dominated convergence, since $F(t/|x|)/(t/|x|)^\beta$ converges pointwise to 1. □

Proof of Watson's lemma. The proof of part (i) is straightforward induction from Lemma 12.1.2 applied to $F(p) - p^{\alpha-1}\sum_{k=0}^{m-1} c_k p^{k\beta}$, and Laplace transforming explicitly the finite sum of powers. Part (ii) follows from (i), replacing F by $F\chi_{[0,a)}$. □

Exercise 51 (Stirling's formula)**.** Start with the integral representation of the Gamma function,

$$\Gamma(x+1) = \int_0^\infty e^{-t}t^x dt = \int_0^\infty e^{-t + x\ln t}dt \tag{125}$$

To bring it to the form (121), we simply change variable to $t = xs$,

$$\Gamma(x+1) = x^{x+1}\int_0^\infty e^{-x(t-\ln t)}dt = x^{x+1}e^{-x}\int_{-1}^\infty e^{-x(u-\ln(1+u))}dt \tag{126}$$

where we noted that $G(t-1) = -(t-\ln t)$ has a maximum at $t=1$, and we re-centered the integral around that maximum. G' has a unique simple zero at 0. Hence we can change variable $G(t) = p$ separately on $(-1,0)$ and $(0,\infty)$ [17]. Let t_- and t_+ be the inverse functions of G on $(-1,0)$ and $(0,\infty)$ resp. and $F = t'_+ - t'_-$. We then have

$$\Gamma(x+1) = x^{x+1}e^{-x}\int_0^\infty e^{-xp}F(p)dp \tag{127}$$

Note that $t - \ln(1+t) = \sum_{k\geqslant 2}\frac{(-t)^k}{k} = \frac{t^2}{2}(1+H(t))$ where $H(0)=0, H'(0)\neq 0$ means that the function inverse $(1+H)^{-1}$ exists and is analytic at zero. Use this to show that F has a convergent power series of the form $F(p) = \sum_{k\geqslant 0} c_k p^{k/2}$, it is analytic in a neighborhood of $\mathbb{R}^+$ and polynomially bounded. By Watson's formula, the asymptotic expansion of the integral in (127) is the formal Laplace transform of this series, and putting everything together, we get Stirling's formula. It reflects the Borel summed transseries of $\Gamma(x+1)$, (127). To write it in canonical form, the transseries is of the form

$$e^{(x+1)\ln x - x}S(x)$$

where S is a power series in powers of $x^{-1/2}$. There is one transmonomial, $x^{-1/2}e^{(x+1)\ln x - x}$

Calculate a few terms c_k and compare to Stirling's formula. One gets a much streamlined calculation with more explicit coefficients by examining $\ln\Gamma$ first.

12.2 Example: $\log\Gamma$

Using the basic relation of the Γ function, $\Gamma(x+1) = x\Gamma(x)$ and taking the log, it is a relatively simple exercise [6] to show that

$$\log\Gamma(x) = x(\log x - 1) - \frac{1}{2}\log x + \frac{1}{2}\log(2\pi)\int_0^\infty \frac{\frac{p}{2}\coth\left(\frac{p}{2}\right)-1}{p^2}e^{-xp}dp \tag{128}$$

1. Let $G(p) = p^{-2}\left(\frac{p}{2}\coth\left(\frac{p}{2}\right)-1\right)$ and $g = \mathcal{L}G$, the integral above. You can check that, by extending G by $G(0) = \frac{1}{12}$, G is mermomorphic in $\mathbb{C}$, analytic in $\mathbb{C}\setminus\{2\pi i n : n\in\mathbb{Z}\setminus\{0\}\}$ and its simple poles have residues $\{1/(2n\pi i) : n\in\mathbb{Z}\setminus\{0\}\}$.

2. We note that, for $x>0$, by deformation of contour we can equally write

$$g(x) = \int_0^{\infty e^{i\varphi}} \frac{\frac{p}{2}\coth\left(\frac{p}{2}\right)-1}{p^2}e^{-xp}dp \tag{129}$$

for any $\varphi\in(-\pi/2,\pi/2)$. Taking $\varphi\in(-\pi/2,0)$, g exists, is analytic for all x with $\arg x \in$

[17] We note here that $G(p) = -W(-\exp(-1-p)) - 1$ where W is the Lambert function, defined as the inverse function of xe^x.

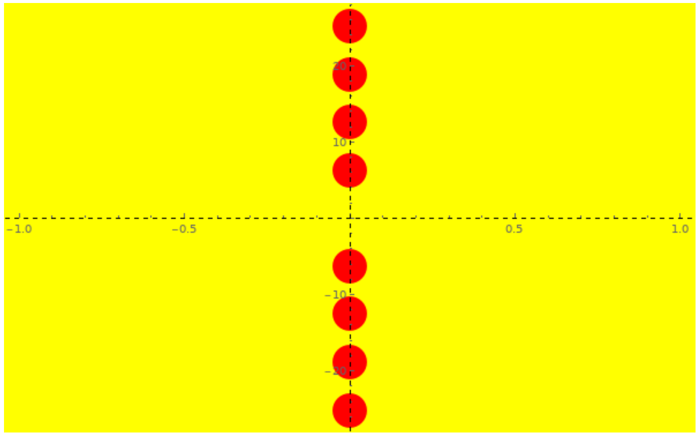

Figure 1: The poles of G.

$(-\pi/2+\varphi, \pi/2+\varphi)$, and has an asymptotic series independent of φ, $\tilde{g}$ – the Stirling series for $\ln\Gamma$ – that you can get from Watson's Lemma.

3. All classical special functions which have essential singularities at infinity are Laplace transforms of functions with only regular singularities, and *properly normalized as discussed in the sequel*, they are Laplace transforms of *elementary* functions.

4. By deformation of contour, we can always ensure that $xp \in \mathbb{R}^+$ along the ray of integration.

Definition 12.2.1. The condition $\arg(xp) = 0$, linking the direction of integration in p to the argument of x, **defines the directional Borel summation** in the direction of $x \in \mathbb{C}$.

5. According to the definition above, the series $\tilde{g}$ is not classically Borel summable along $i\mathbb{R}^+$ (where the contour of integration crosses poles). Analytic continuation in x past $\arg x = \pi/2$ is possible by collecting the residues of the integrand and deforming the contour past $-i\pi/2$.

   This gives

$$g(x) = \int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp + 2\pi i \sum_{j=1}^{\infty} \mathrm{Res} G(p)e^{-xp}|_{p=2j\pi i}$$
$$= \int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp + \sum_{j=1}^{\infty} \frac{1}{je^{2xj\pi i}} = \int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp - \ln(1-\exp(-2x\pi i)), \quad (130)$$

6. We note that both the integral and the sum are convergent when $\arg x = -\theta = -\pi/2 - \varepsilon$ for $\varepsilon \in \left(0, \frac{\pi}{2}\right]$ if x is not an integer. We see that the middle term in (130) is a Laplace transform of an analytic function *along the direction* $e^{i\theta}$, which is the directional Borel sum of $\tilde{g}$ in the direction $-\theta$, and a convergent sum of exponentials. The whole expression is the Borel summed transseries of the function g in the second quadrant.

7. Note that G is analytic in the left half plane, and hence so is g.

The reflection formula. From (128) and (130) we get

$$\Gamma(x) = \frac{\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}e^{g(x)}}{1-\exp(-2x\pi i)} \tag{131}$$

1. It follows that $\Gamma(x)$ is meromorphic, with poles at $x \in -\mathbb{N}$. Note how these poles are dual to those in the plane of G, where everything is explicit.

2. At this stage, we have obtained a representation of Γ in its whole domain of analyticity, and we have determined its singularities on the boundary of this domain. This is part of the information we can get from general transseries of functions.

3. Furthermore, taking $x = -y$ with $y \notin \mathbb{N}$ and then setting $\theta = \pi$ in the expression for $\Gamma(-x)$, we get, by changes of variables and straightforward algebra, $\Gamma(-x)\Gamma(x) = -\dfrac{\pi}{x\sin(\pi x)}$ from which it follows that
$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)} \tag{132}$$

Further discussion.

1. Similar conclusions hold for general resurgent functions discussed in the next section.

2. We also note that the exponentials in the transseries result from collecting residues (more generally, collecting branch cut jumps), and they can **only** be of the form $e^{-\lambda x}$ for some $\lambda \in \mathbb{C}$. Indeed, a pole at $p = -\lambda$ produces an exponential $e^{-\lambda x}$. In particular, a transseries with exponentials e^{-x^2} are not obtainable from a Laplace transform, **unless we take x^2 to be the variable**. This is an important step in the initial normalization of the function y, or series $\tilde{y}$, or underlying equation: choosing that independent variable with respect to which the *exponents* in the transseries are linear in x.

3. The variable with respect to which the exponents in the transseries are linear in x is called **Écalle critical time**, and it ensures that the Écalle-Borel summability process succeeds. Should we use a power of this variable < 1 , $\mathcal{B}\tilde{y}$ will still be divergent, while if the power is > 1 then $\mathcal{B}\tilde{y}$ will grow faster than exponentially in $\mathbb{C}$ and then $\mathcal{L}\mathcal{B}\tilde{y}$ will not make sense.

12.3 Resurgent functions

Definition 12.3.1 (Resurgent functions)**.** A formal power series $\tilde{f}$ belonging to the class of Écalle-Borel summable series is called a resurgent series. Assuming it is properly normalized, the function $F = \mathcal{A}_c\mathcal{S}\mathcal{B}\tilde{f}$ is called a Borel-plane (or convolutive model) resurgent function, and its Écalle-Borel sum is called a resurgent function in the physical domain (or in the geometric model). The Borel plane of $\tilde{f}$ is the complex domain of F.

Following usual convention in which the ordinary summation operator and $\mathcal{A}_c$ are omitted, we write $F = \mathcal{B}\tilde{f}$ and $\mathcal{LB}$ for the (Écalle-)Borel summation operator.

12.4 Structure of the Borel plane

Here is a typical Borel plane of a solution of a nonlinear ODE, whose *appropriately normalized* normalized asymptotic series is $\tilde{f}$, and, in fact, of much more general functions. If $F = \mathcal{B}f$, then F is analytic at zero, and on the universal covering of $\mathbb{C} \setminus \cup_j \lambda_j \mathbb{Z}$ where λ_j are the eigenvalues of the linearized equation. For clarity, we specialize further to the Borel plane of the solutions of the Painlevé equation P_I, for which there is only one λ, $\lambda = 1$.

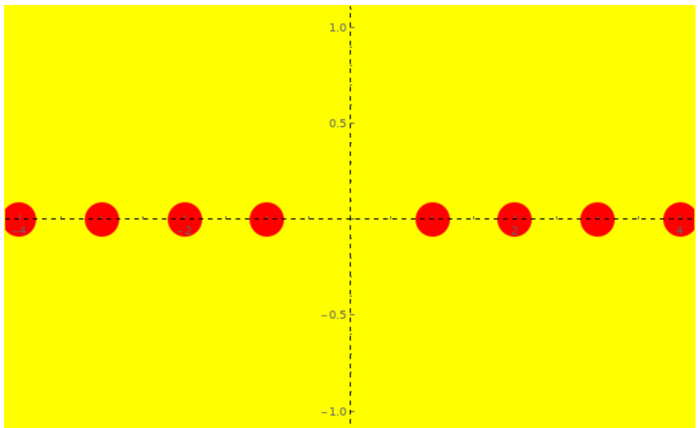

Figure 2: A typical Borel plane: the Borel plane of the Painlevé equations P_I–P_V, that is the plane of $\mathcal{B}\tilde{y}$, where $\tilde{y}$ is a typical asymptotic series solution.

1. All the singularities of F on its Riemann surface are branch points. On the first Riemann sheet, they are of the form

$$(k-p)^{|k|/2-3/2}A_k(p) + B_k(p),\ (k = 2m+1);\ \text{ and }\ (k-p)^{|k|/2-1}\ln(k-p)A_k(p) + B_k(p);\ (k = 2m) \tag{133}$$

   where A_k, B_k are analytic at $p = k$.

2. There are deep relations connecting the various A_k, B_k at all singular points on the universal covering of $\mathbb{C} \setminus \mathbb{Z}$, and hence the fundamental group of the Riemann surface of F is highly nontrivial. In a nutshell, the A_k, B_k can be calculated in terms of each-other. These are resurgence relations. (The first singularity "resurges" later on in various incarnations, infinitely often.)

3. The transseries of the general solution of a system is intimately related to these singularities. For ODEs, the general solution with sectorial existence is of the form

$$y = \mathcal{LB}\tilde{y} = \sum_{\mathbf{k}\geqslant 0} e^{-\boldsymbol{\lambda}\mathbf{k}x}\mathbf{C}^{\mathbf{k}}\mathcal{LB}\tilde{y}_{\mathbf{k}}(x) \tag{134}$$

   For ODEs, this representation is where $\mathbf{C} \in \mathbb{C}^n$. Here $\mathcal{B}\tilde{y}_k = A_k$, the branch jumps of $\mathcal{B}\tilde{y}_0$ at the kth singularity and (134) converges at a geometric rate [1].

4. The transseries representation (134) converges throughout the sector of existence of the associated solution y, which can be determined from it, [2] and down to actual singularities of y, whose position and type can be calculated from it [3]. The global Riemann structure of y can also be obtained, by combining the transseries representation in the region of existence with KAM techniques in the singular ones [4].

5. These representations are of particularly important interest in PDEs, when often even existence and uniqueness of solutions is otherwise unknown [5].

12.5 The Borel transform as a regularizing tool

The series $y(x) \sum_{k=0}^{\infty} k!(-x)^{k+1}$ satisfies our model equation (98): $y' - y = 1/x$. Its Borel transform is a geometric series. It must be that the transformed *equation* is more regular.

The equation satisfied by this geometric series, clearly, is the inverse Laplace transform of (98). Let $Y = \mathcal{L}^{-1}y$. Then,

1. $\mathcal{L}^{-1}y' = -pY$. Indeed, if $y(x) = \int_0^\infty Y(p)e^{-px}dp$, then $y'(x) = -\int_0^\infty pY(p)e^{-px}dp$.
2. $\mathcal{L}^{-1}(1/x) = p$ Since $\int_0^\infty e^{-px}dp = 1/x$, we have .

Hence, $\mathcal{L}^{-1}y$ satisfies the equation

$$-(p+1)Y(p) = 1 \tag{135}$$

We note that while the solutions of (98) have an essential singularity at infinity the transformed equation has meromorphic solutions, and in fact elementary ones.

Example 12.5.1 (The Airy equation)**.**

$$y'' - xy = 0 \tag{136}$$

There are no formal power series solutions, as you can check. The reason is that all solutions have nontrivial transseries, starting with an exponential prefactor. To see what the prefactor should be, insert $y(x) = e^{ax^b}$ in (136): you will find that $b = 3/2$ and $a = \pm 2/3$. Inserting $y(x) = e^{-2/3x^{3/2}}h(x)$ in (136) we get an equation for h which has (noninteger) formal power solutions. An inductive argument however shows that the rate of convergence of this series

$$y \sim Cx^{-\frac{1}{4}}e^{-\frac{2}{3}x^{3/2}} \tag{137}$$

We use the transformation $y(x) = g(\frac{2}{3}\,x^{\frac{3}{2}})$ to achieve normalization and get

$$g'' + \frac{1}{3t}g' - g = 0 \tag{138}$$

In view of (137) we have

$$g(t) \sim Ct^{-\frac{1}{6}}e^{\pm t} \tag{139}$$

To eliminate the exponential behavior of one solution, say of the decaying one, we substitute $g = he^{-t}$, and get

$$h'' - \left(2 - \frac{1}{3t}\right) h' - \frac{1}{3t} h = 0 \tag{140}$$

To obtain a second solution, we can resort to the substitution $g = he^{t}$, or we can rely on the Stokes phenomenon to obtain it from the one above. We get

$$p(2+p)H' + \frac{5}{3}(1+p)H = 0 \tag{141}$$

with the solution

$$H = Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} \tag{142}$$

and thus

$$h(t) = \mathcal{L}\left(Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}\right) \tag{143}$$

which is, up to constants and returning to the original variable $t = -2/3x^{3/2}$, the Airy function.

$$\mathrm{Ai}(x) = \frac{3^{-\frac{1}{6}} \exp(-\frac{2}{3}x^{3/2})}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{6})} \int_0^\infty e^{-\frac{2}{3}x^{\frac{3}{2}}p} p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} dp \tag{144}$$

To get the asymptotic expansion of Airy, we go back to t and apply Watson's lemma to the Laplace transform above.

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