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1 Introduction

We start by looking at various simple examples. Some properties carry over to more general settings, and many don't. It is useful to look into this, as it gives us some idea as to what to expect. Some intuition we have on operators comes from linear algebra. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be linear. Then A can be represented by a matrix, which we will also denote by A. Certainly, since A is linear on a finite dimensional space, A is continuous. We use the standard scalar product on $\mathbb{C}^n$,

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

with the usual norm $\|x\|^2 = \langle x, x\rangle$. The operator norm of A is defined as

$$\|A\| = \sup_{x\in\mathbb{C}^n} \frac{\|Ax\|}{\|x\|} = \sup_{x\in\mathbb{C}^n} \left\| A\frac{x}{\|x\|}\right\| = \sup_{u\in\mathbb{C}^n:\|u\|=1} \|Au\| \tag{1}$$

Clearly, since A is continuous, the last sup (on a compact set) is in fact a max, and $\|A\|$ is bounded. Then, we say, A is bounded.

The spectrum of A is defined as

$$\sigma(A) = \{\lambda \mid (A - \lambda) \text{ is not invertible}\} \tag{2}$$

This means $\det(A - \lambda) = 0$, which happens iff $\ker\,(A - \lambda) \neq \{0\}$ that is

$$\sigma(A) = \{\lambda \mid (Ax = \lambda x) \text{ has nontrivial solutions}\} \tag{3}$$

For these operators, the spectrum consists exactly of the eigenvalues of A. This will not extend to infinite dimensional cases.

• **Self-adjointness** A is symmetric (self-adjoint, it turns out) iff

$$\langle Ax, y\rangle = \langle x, Ay\rangle \tag{4}$$

for all x and y. As an exercise, you can show that this is the case iff $(A)_{ij} = \overline{(A)_{ji}}$.

We can immediately check that all eigenvalues are real, using (4).

We can also check that eigenvectors x_1, x_2 corresponding to distinct eigenvalues λ_1, λ_2 are orthogonal, since

$$\langle Ax_1, x_2\rangle = \lambda_1\langle x_1, x_2\rangle = \langle x_1, Ax_2\rangle = \lambda_2\langle x_1, x_2\rangle \tag{5}$$

More generally, we can choose an orthonormal basis consisting of eigenvectors u_n of A. We write these vectors in matrix form,

$$U = \begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \cdots & \cdots & \cdots & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{pmatrix} \tag{6}$$

and note that

$$UU^* = U^*U = I \tag{7}$$

where I is the identity matrix. Equivalently,

$$U^* = U^{-1} \tag{8}$$

We have

$$AU = A \begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \cdots & \cdots & \cdots & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} Au_1 & Au_2 & \cdots & Au_n \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{pmatrix} = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} =: UD \tag{9}$$

where D is a diagonal matrix. In particular

$$U^* AU = D \tag{10}$$

which is a form of the spectral theorem for A. It means the following. If we pass to the basis $\{u_j\}$, that is we write

$$x = \sum_{k=1}^{n} c_k u_k \tag{11}$$

we have

$$c_k = \langle x, u_k \rangle \tag{12}$$

that is, since $\tilde{x} = (c_k)_k$ is the new representation of x, we have

$$\tilde{x} = U^* x \tag{13}$$

We also have

$$Ax = \sum_{k=1}^{n} c_k A u_k = \sum_{k=1}^{n} c_k \lambda_k u_k = D\tilde{x} =: \tilde{A}\tilde{x} \tag{14}$$

another form of (10). This means that after applying U^* to $\mathbb{C}^n$, the new A, $\tilde{A}$ is diagonal (D), and thus is *it acts multiplicatively.*

• **A few infinite-dimensional examples.** Let us look at $L^2[0,1]$; here, as we know,

$$\langle f, g \rangle = \int_0^1 f(s)\overline{g(s)} ds$$

We can check that X, defined by

$$(Xf)(x) = xf(x)$$

is symmetric. It is also bounded, with norm ≤ 1 (exactly 1, it turns out), since

$$\int_0^1 s^2 |f(s)|^2 ds \le \int_0^1 |f(s)|^2 ds \tag{15}$$

What is the spectrum of X? We have to see for which λ $X-\lambda$ is not invertible, that is the equation

$$(x-\lambda)f = g \tag{16}$$

does not have L^2 solutions for all g. This is clearly the case iff $\lambda \in [0,1]$.

But we note that now $\sigma(X)$ has no eigenvalues! Indeed,

$$(x-\lambda)f = 0 \Rightarrow f = 0 \forall x \neq \lambda \Rightarrow f = 0 \ a.e, \Rightarrow f = 0 \text{ in the sense of } L^2 \tag{17}$$

Finally, let us look at X on $L^2(\mathbb{R})$. The operator stays symmetric, wherever defined. Note that now X is unbounded, since, with χ the characteristic function,

$$x\chi_{[n,n+1]} \geq n\chi_{[n,n+1]} \tag{18}$$

and thus $\|X\| \geq n$ for any n. Likewise, X is not everywhere defined. Indeed, $f = (|x|+1)^{-1} \in L^2(\mathbb{R})$ whereas $|x|(|x|+1)^{-1} \to 1$ as $x \to \pm\infty$, and thus Xf is not in L^2. What is the domain of definition of X (domain of X in short)? It consists of all f so that

$$f \in L^2 \ \text{ and } \ xf \in L^2 \tag{19}$$

This is not a subspace of L^2. Rather, it is a dense set in L^2 since C_0^∞ is contained in the domain of X and it is dense in L^2. X is said to be densely defined.

2 Bounded and unbounded operators

1. Let X, Y be Banach spaces and $D \subset X$ a *linear* space, not necessarily closed.

2. A linear operator is any linear map $T : D \to Y$.

3. D is the domain of T, sometimes written Dom (T), or $\mathcal{D}(T)$.

4. $T(D)$ is the *Range* of T, Ran(T).

5. The *graph of* T is

$$\Gamma(T) = \{(x, Tx) | x \in \mathcal{D}(T)\}$$

6. The kernel of T is

$$\text{Ker}(T) = \{x \in \mathcal{D}(T) : Tx = 0\}$$

2.1 Operations

1. $aT_1 + bT_2$ is defined on $\mathcal{D}(T_1) \cap \mathcal{D}(T_2)$.

2. if $T_1 : \mathcal{D}(T_1) \subset X \to Y$ and $T_2 : \mathcal{D}(T_2) \subset Y \to Z$ then $T_2T_1 : \{x \in \mathcal{D}(T_1) : T_1(x) \in \mathcal{D}(T_2)\}$.

   In particular, if $\mathcal{D}(T)$ and $\mathrm{Ran}(T)$ are both in the space X, then, inductively, $\mathcal{D}(T^n) = \{x \in \mathcal{D}(T^{n-1}) : T(x) \in \mathcal{D}(T)\}$. The domain may become trivial.

3. Inverse. The inverse is defined iff $\mathrm{Ker}(T) = \{0\}$. This condition implies T is bijective. Then $T^{-1} : \mathrm{Ran}(T) \to \mathcal{D}(T)$ is defined as the usual function inverse, and is clearly linear. ∂ is not invertible on $C^\infty[0,1]$: $\mathrm{Ker}\partial = \mathbb{C}$. How about ∂ on $C_0^\infty((0,1))$ (the set of C^∞ functions with compact support contained in $(0,1)$)?

4. Closable operators. It is natural to extend functions by continuity, when possible. If $x_n \to x$ and $Tx_n \to y$ we want to see whether we can define $Tx = y$. Clearly, we must have

$$x_n \to 0 \text{ and } Tx_n \to y \Rightarrow y = 0, \tag{20}$$

   since $T(0) = 0 = y$. Conversely, (20) implies the extension $Tx := y$ whenever $x_n \to x$ and $Tx_n \to y$ is consistent and defines a linear operator.

   An operator satisfying (20) is called *closable*. This condition is the same as requiring

$$\overline{\Gamma(T)} \text{ is the graph of an operator} \tag{21}$$

   where the closure is taken in $X \oplus Y$. Indeed, if $(x_n, Tx_n) \to (x, y)$, then, by the definition of the graph of an operator, $Tx = y$, and in particular $x_n \to 0$ implies $Tx_n \to 0$. An operator is closed if its graph is closed. It will turn out that symmetric operators are closable.

5. Many common operators are closable. E.g., ∂ defined on a subset of *continuous, everywhere differentiable functions* is closable. Assume $f_n \xrightarrow{L^2} 0$ (in the sense of L^2) and $f_n' \xrightarrow{L^2} g$. Since f_n' exist everywhere, see [4],

$$f_n(x) - f_n(0) = \int_0^x f_n'(s)ds = \langle f_n', \chi_{[0,x]}\rangle \to \langle g, \chi_{[0,x]}\rangle = \int_0^x g(s)ds \tag{22}$$

   Thus

$$\lim_{n\to\infty} (f_n(x) - f_n(0)) = 0 = \int_0^x g(s)ds \tag{23}$$

   implying $g = 0$.

6. As an example of non-closable operator, consider, say $L^2[0,1]$ (or any separable Hilbert space) with an orthonormal basis e_n. Define $Ne_n = ne_1$, extended by linearity, whenever it makes sense (it is an unbounded operator). Then $x_n = e_n/n \to 0$, we have $Nx_n = e_1 \neq 0$. Thus N is not closable.

   Every infinite-dimensional normed space admits a nonclosable linear operator. The proof requires the axiom of choice and so it is in general nonconstructive,

   The closure through the graph of T is called the *canonical closure* of T.

   Note: if $\mathcal{D}(T) = X$ and T is closed, then T is continuous, and conversely (see §2.3, 5).

7. As we know, T is continuous iff $\|T\| < \infty$. Indeed, by linearity only continuity at zero needs to be checked. For the latter, simply note that $\|Tx_n\| \leq \|T\|\|x_n\| \to 0$ if $\|x_n\| \to 0$.

8. The space $\mathcal{L}(X,Y)$ of *bounded operators* from X to Y is a Banach space too, with the norm $T \to \|T\|$.

9. We see that $T \in \mathcal{L}(X,Y)$ takes bounded sets in X into bounded sets in Y.

2.2 A brief review of bounded operators

1. $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y.

2. We have the following topologies on $\mathcal{L}(X,Y)$ in increasing order of weakness:

   (a) The **uniform operator topology** or **norm** topology is the one given by $\|T\| = \sup_{\|u\|=1} \|Tu\|$. Under this norm, $\mathcal{L}(X,Y)$ is a Banach space.

   (b) The **strong operator topology** is the one defined by the convergence condition $T_n \to T \in \mathcal{L}(X,Y)$ iff $T_n x \to Tx$ for all $x \in X$. We note that if $T_n x$ is Cauchy for every x then, in this topology, T_n is convergent to a $T \in \mathcal{L}(X,Y)$. Indeed, it follows that $T_n x$ is convergent for every x. Now note that $\|T_n x\| \leq C(x)$ for every x because of convergence. Then, $\|T_n\| \leq C$ for some C by the uniform boundedness principle. But $\|Tx\| = \|(T - T_{n_0} + T_{n_0})x\| \leq \|(T - T_{n_0})x\| + C\|x\| \to C\|x\|$ as $n \to \infty$ thus $\|T\| \leq C$.
   We write $T = s - \lim T_n$ in the case of strong convergence.

   Note also that the *strong operator topology* is a pointwise convergence topology while the **uniform operator topology** is the "L^∞" version of it.

(c) The **weak operator topology** is the one defined by $T_n \to T$ if $\ell(T_n(x)) \to \ell(Tx)$ for all linear functionals on X, i.e. $\forall \ell \in X^*$. In the Hilbert case, $X = Y = \mathcal{H}$, this is the same as requiring that $\langle T_n x, y\rangle \to \langle Tx, y\rangle$ $\forall x, y$ that is the "matrix elements" of T converge. It can be shown that if $\langle T_n x, y\rangle$ converges $\forall x, y$ then there is a T so that $T_n \to^w T$.

3. Reminder: The Riesz Lemma states that if $T \in \mathcal{H}^*$, then there is a unique $y_T \in \mathcal{H}$ so that $Tx = \langle x, y_T\rangle$ for all x.

4. Adjoints. Let first $X = Y = \mathcal{H}$ be separable Hilbert spaces and let $T \in \mathcal{L}(\mathcal{H})$. Let's look at $\langle Tx, y\rangle$ as a linear functional $\ell_T(x)$. It is clearly bounded, since by Cauchy-Schwarz we have

$$|\langle Tx, y\rangle| \leq (\|y\|\|T\|)\|x\| \tag{24}$$

$|\langle Tx, y\rangle| \leq (\|y\|\|T\|)\|x\|$. Thus $\langle Tx, y\rangle = \langle x, y_T\rangle$ for all x. Define T^* by $\langle x, T^*\rangle = \langle x, y_T\rangle$. Using (24) we get

$$|\langle x, T^*\rangle| \leq \|y\|\|T\|\|x\| \tag{25}$$

This is clearly linear, well defined by linearity by $\langle u, T^*\rangle = \langle u, y_T\rangle$ when $\|u\| = 1$. Furthermore

$$\|T^*x\|^2 = |\langle T^*x, T^*x\rangle| \leq \|T^*y\|\|T\|\|x\|$$

by definition , and thus $\|T^*\| \leq \|T\|$. Since $(T^*)^* = T$ we have $\|T\| = \|T^*\|$.

5. In a general Banach space, we mimic the definition above, and write $T'(\ell)(x) =: \ell(T(x))$. It is still true that $\|T\| = \|T'\|$, see [2].

2.3 Review of some results

1. **Uniform boundedness theorem**. If $\{T_j\}_{j\in\mathbb{N}} \subset \mathcal{L}(X, Y)$ and $\|T_j x\| < C(x) < \infty$ for any x, then for some $C \in \mathbb{R}$, $\|T_j\| < C\ \ \forall\ j$.

2. In the following, X and Y are Banach spaces and T is a linear operator.

3. **Open mapping theorem**. Assume $T : X \to Y$ is *onto*. Then $A \subset X$ open implies $T(A) \subset Y$ open.

4. **Inverse mapping theorem**. If $T : X \to Y$ is *one to one*, then T^{-1} is continuous.

   Proof. T is open so $(T^{-1})^{-1} = T$ takes open sets into open sets.

5. **Closed graph theorem** $T : X \to Y$ (note: T is defined *everywhere*) is bounded *iff* $\Gamma(T))$ is closed.

   Proof. It is easy to see that T bounded implies $\Gamma(T)$ closed.

Conversely, we first show that $Z = \Gamma(T) \subset X \oplus Y$ is a Banach space, in the norm

$$\|(x, Tx)\|_Z = \|x\|_X + \|Tx\|_Y$$

It is easy to check that this is a norm, and that $(x_n, Tx_n)_{n\in\mathbb{N}}$ is Cauchy *iff* x_n is Cauchy and Tx_n is Cauchy. Since X, Y are already Banach spaces, then $x_n \to x$ for some x and $Tx_n \to y = Tx$. But then, by the definition of the norm, $(x_n, Tx_n) \to (x, Tx)$, and Z is complete, under this norm, thus it is a Banach space.

Next, consider the *projections* $P_1 : z = (x, Tx) \to x$ and $P_2 : z = (x, Tx) \to Tx$. Since both $\|x\|$ and $\|Tx\|$ are bounded above by $\|z\|$, then P_1 and P_2 are continuous.

Furthermore, P_1 is *one-to one* between Z and X (for any x there is a unique Tx, thus a unique (x, Tx), and $\{(x, Tx) : x \in X\} = \Gamma(T)$, by definition. By the open mapping theorem, thus P_1^{-1} is continuous. But $Tx = P_2P_1^{-1}x$ and T is also continuous.

6. **Hellinger-Toeplitz theorem**. Assume $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. That is, T is everywhere defined. Furthermore, assume T is symmetric, *i.e.* $\langle x, Tx'\rangle = \langle Tx, x'\rangle$ for all x, x'. Then T is bounded.

   Proof. We show that $\Gamma(T)$ is closed. Fix x' and assume $x_n \to x$ and $Tx_n \to y$. Let x' be arbitrary. Then $\lim_{n\to\infty}\langle x_n, Tx'\rangle = \lim_{n\to\infty}\langle Tx_n, x'\rangle = \langle x, Tx'\rangle = \langle Tx, x'\rangle = \langle y, x'\rangle$, for all x', thus $Tx = y$, and the graph is closed.

7. Consequence: The differentiation operator $i\partial$, say, with domain is C_0^∞ (and many other *unbounded* symmetric operators in applications), which are symmetric on certain domains *cannot be extended to the whole space.* We cannot "invent" a derivative for general L^2 functions in a linear, symmetric way! General L^2 functions are fundamentally nondifferentiable.

   Unbounded symmetric operators come with a nontrivial domain $\mathcal{D}(T) \subsetneq X$, and addition, composition etc are to be done carefully.

3 Closed operators, examples of unbounded operators, spectrum

1. We let X, Y be Banach space. Y' is a subset of Y.

2. We recall that bounded operators are closed. Note that T is closed iff $T + \lambda I$ is closed for some/any λ.

3.

   Proposition 1. *If $T : \mathcal{D}(T) \subset X \to Y' \subset Y$ is closed and injective, then T^{-1} is also closed.*

Proof. Indeed, the graph of T and T^{-1} are the same, modulo switching the order. Directly: let $y_n \to y$ and $T^{-1}y_n = x_n \to x$. This means that $Tx_n \to y$ and $x_n \to x$, and thus $Tx = y$ which implies $y = T^{-1}x$. ☐

4.

Proposition 2. *If T is closed and $T : \mathcal{D}(T) \to X$ is bijective, then T^{-1} is bounded.*

Proof. We see that T^{-1} is defined everywhere and it is closed, thus bounded. ☐

Exercise*

5. Show that if T is not closed but bijective between $\mathrm{Dom}\, T$ and Y, there exist sequences $x_n \to x \neq 0$ such that $Tx_n \to 0$. (One of the "pathologies" of non-closed, and more generally, non-closable operators.)

Spectrum

6. The spectrum of an operator plays a major role in characterizing it and working with it. Of course in a more sophisticated way, we can, in "good" cases, find unitary transformations that essentially transform an operator to the multiplication operator on the spectrum, an infinite dimensional analog of diagonalization of matrices.

The spectrum of T, as we noted, equals

$$\sigma(T) =: \{\lambda \in \mathbb{C} : (T-\lambda)^{-1} \text{does not exist as a bounded operator} : X \to \mathrm{Dom}\, T\}$$

By the above and the closed graph theorem, **if T is closed**, we have $\sigma(T) = \{z : (T - z) \text{ is not bijective}\}$. That is, the possibility $(T - z)^{-1} : Y \to \mathrm{Dom}\,(T - z)$ is unbounded is ruled out *if T is closed*.

For closed operators, there are thus two possibilities: (a) $T : \mathcal{D}(T) \to Y$ is not injective. That means that $(T - z)x = (T - z)y$ for some $x \neq y$, which is equivalent to $(T - \lambda)u = 0$, $u = x - y \neq 0$, or $\mathrm{Ker}(T - z) \neq \{0\}$ or, which is the same $Tu = \lambda u$ for some $u \neq 0$. This u is said to belong to the *point spectrum* of T. (b) $\mathrm{Ran}T \neq Y$. There is a *subcase*, $\mathrm{Ran}T$ *is not dense*. This subcase is called the *residual spectrum*.

Examples

7. (a) Consider the operator X on $L^2([0, 1])$. We noted already that $\sigma(X) = [0, 1]$. You can show that there is no point spectrum and residual spectrum for this operator.

   (b) Consider now operator X on $\{f \in L^2(\mathbb{R}) \mid xf \in L^2(\mathbb{R})\}$. Show that $\sigma(X) = \mathbb{R}$.

Exercise

   (c) Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces, and let $U : \mathcal{H} \to \mathcal{H}'$ be unitary. Let $T\mathcal{H} \to \mathcal{H}$ and consider its image UTU^*. Show that T and UTU^* have the same spectrum.

   (d) Show that $-i\partial$ densely defined on the functions in $L^2(\mathbb{R})$ so that f' exists and is in $L^2(\mathbb{R})$, and that it has as spectrum $\mathbb{R}$. For this, it is useful to use item 7c above and the fact that $\mathcal{F}$, the Fourier transform is unitary between $L^2(\mathbb{R})$ and $L^2(\mathbb{R})$.

(e) The spectrum of unbounded operators, even closed ones, can be any closed set, including $\emptyset$ and $\mathbb{C}$. The domain of definition plays an important role. In general, the larger the domain is, the larger the spectrum is. This is easy to see from the definition of the inverse.

(f) Let $T_1 = \partial$ be defined on $\mathcal{D}(T_1) = \{f \in C^1[0,1] : f(0) = 0\}$ [(1)] with values in the Banach space $C[0,1]$ (with the sup norm). (Note also that $\mathrm{Dom}\, T$ is dense in $C[0,1]$.) Then the spectrum of T_0 is empty. In particular T_1 is closed.

Indeed, to show that the spectrum is empty, note that by assumption $(\partial - z)\mathcal{D}(T_1) \subset C[0,1]$. Now, $(\partial - z)f = g$, $f(0) = 0$ is a linear differential equation with a unique solution

$$f(x) = e^{xz} \int_0^x e^{-zs} g(s) ds$$

We can therefore check that f defined above is an inverse for $(\partial - z)$, by checking that $f \in C^1[0,1]$, and indeed it satisfies the differential equation. Clearly $\|f\| \le const(z)\|g\|$.

(g) At the "opposite extreme", $T_0 = \partial$ defined on $\mathcal{D}(T_0) = C^1[0,1]$ has as spectrum $\mathbb{C}$.

Indeed, if $f(x;z) = e^{zx}$, then $T_0 f - zf = 0$.

We note that T_0 is closed too, since if $f_n \to 0$ then $f_n - f_n(0) \to 0$ as well, so we can use 6 and 7f above.

Exercise

8. Operators which are not closable are ill-behaved in many ways. Show that the spectrum of such an operator must be the whole of $\mathbb{C}$.

9. An interesting example is S defined by $(S\psi)(x) = \psi(x+1)$. This is well defined and bounded (unitary) on $L^2(\mathbb{R})$. The "same" operator can be defined on the polynomials on $[0,1]$, an L^∞ dense subset of $C[0,1]$. Note that now S is unbounded.

10. $S : P[0,1] \to SP[0,1]$ is bijective and thus invertible in a function sense. But the inverse is unbounded as seen in a moment.

    (a) It is also not closable. Indeed, since $P[0,1] \subset P[0,2]$ and $P[0,2]$ is dense in $C[0,2]$, it is sufficient to take a sequence of polynomials P_n converging to a continuous nonzero function which vanishes on $[0,1]$. Then $P_n \to 0$ as restricted to $[0,1]$ while $P_n(x+1)$ converges to a nonzero function, and closure fails [(2)]. In fact, P_n can be chosen so that $P_n(x+1)$ converges to any function that vanishes at $x = 1$.

    (b) Check that $S - z$ is not injective if $z \ne 0$. So $S - z$ is bijective iff $z = 0$.

[(1)] $f(0) = 0$ can be replaced by $f(a) = 0$ for some fixed $a \in [0,1]$.
[(2)] Proof due to Min Huang.

(c) However, S restricted to $\{P \in P[0,1] : P(0) = 0\}$ (a dense subset of $\{f \in C[0,1] : f(0) = 0\}$ it is injective for all z. Again, $(S-z)^{-1}$ (understood in the function sense), is unbounded.

Exercise 1. *Show that $T_2 = \partial$ defined on $\mathcal{D}(T_2) = \{f \in C^1[0,1] : f(0) = f(1)\}$ has spectrum exactly $2\pi i\mathbb{Z}$.*

11. It is also useful to look at the extended spectrum, on the $\mathbb{C}_\infty$ We say that $\infty \in \sigma_\infty(T)$ if $(T-z)^{-1}$ is not analytic in a neighborhood of infinity.

4 Integration and measures on Banach spaces

In the following Ω is a topological space, $\mathcal{B}$ is the Borel σ-algebra over Ω, X is a Banach space, μ is a signed measure on Ω. Integration can be defined on functions from Ω to X, as in standard measure theory, starting with simple functions.

(a) A simple function is a sum of indicator functions of measurable mutually disjoint sets with values in X:

$$f(\omega) = \sum_{j\in J} x_j \chi_{A_j}(\omega); \quad \text{card}(J) < \infty \tag{26}$$

where $x_j \in X$ and $\cup_j A_j = \Omega$.

(b) We denote by $B_s(\Omega, X)$ the linear space of simple functions from Ω to X.

(c) We will define a norm on $B_s(\Omega, X)$ and find its completion $B(\Omega, X)$ as a Banach space. We define an integral on $B_s(\Omega, X)$, and show it is norm continuous. Then the integral on B is defined by continuity. We will then identify the space $B(\Omega, X)$ and find the properties of the integral.

(d) $B_s(\Omega, X)$ is a normed linear space, under the sup norm

$$\|f\|_\infty = \sup_{\omega\in\Omega} \|f(\omega)\| \tag{27}$$

(e) We define $B(\Omega, X)$ the completion of $B_s(\Omega, X)$ in $\|f\|_\infty$.

(f) Check that, for a partition $\{A_j\}_{j=1,\dots,n}$ we have

$$\|f\|_\Omega = \max_{j\in J} \sup_{x\in A_j} \|f(x)\| =: \max_{j\in J} \|f\|_{A_j} \tag{28}$$

(g) *Refinements*. Assume $\{A_j\}_{j=1,\dots,n}$ is partition and $\{A'_j\}_{j=1,\dots,n'}$ is a subpartition, in the sense that for any A_j there exists $A'_{j1}, \dots, A'_{jm}$ so that $A_j = \cup_{i=1}^m A'_{ji}$

(h) The integral is defined on $B_s(\Omega, X)$ as in the scalar case by

$$\int f d\mu = \sum_{j \in J} \mu(A_j) x_j \tag{29}$$

and likewise, the integral over a subset of $A \in \mathcal{B}(\Omega)$ by

$$\int_A f d\mu = \int \chi_A f d\mu \tag{30}$$

which is the natural definition since A is also a topological space with a Borel σ-algebra (the induced one) and with the same measure μ. Check that, is we choose $x'_j = x_j$ for each $A'_j \subset A_j$ then

$$\sum_j x_j \chi_{A_j} = \sum_j x'_j \chi_{A'_j} \tag{31}$$

and

$$\int_\Omega \sum_j x_j \chi_{A_j} d\mu = \int_\Omega \sum_j x'_j \chi_{A'_j} d\mu \tag{32}$$

(i) Note that if A, B are disjoint sets in $\mathcal{B}(\Omega)$, then

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu \tag{33}$$

(j)

Lemma 3. *If $f \in B(\Omega, X)$, then for any ϵ there is a (disjoint) partition $\{A_i\}_{i=1,\ldots,n}$ of Ω so that for any $\omega_j \in A_j$ we have*

$$\|f - \sum_{i=1}^n f(\omega_j) \chi_{A_j}\|_X \leqslant \epsilon \tag{34}$$

Proof. Since $f \in B(\Omega, X)$ there is a $g = \sum_{i=1}^n x_j \chi_{A_j}$, so that $\|g - f\| \leqslant \epsilon/2$. This means in particular that for any j, $\|\sum_{\omega \in A_j} f(\omega) - x_j\| \leqslant \epsilon/2$. Choosing any $\omega_j \in A_j$, it follows that $\|f(\omega_j) - x_j\| \leqslant \epsilon/2$. The rest follows from the triangle inequality. ☐

Lemma 4. *If $\{A'_i\}_{i=1,\ldots,n'}$ is a subpartition of $\{A_i\}_{i=1,\ldots,n}$ in the sense that $A'_{i'} \subset A_i$ for any i' and some i, and if $x_{i'} = x_i$ whenever $A'_{i'} \subset A_i$, then $\sum_{i'=1}^{n'} x_{i'} \chi_{A_{i'}} = \sum_{i=1}^n x_i \chi_{A_i}$.*

Proof. Since $\chi_{A+B} = \chi_A + \chi_B$, this is immediate. ☐

Lemma 5. *If $f_1, f_2 \in B(\Omega, X)$, then for any ϵ there is a (disjoint) partition $\{A_i\}_{i=1,\ldots,n}$ of Ω so that for any $\omega_j \in A_j$ we have*

$$\|f_i - \sum_{i=1}^n f(\omega_j) \chi_{A_j}\|_X \leqslant \epsilon, \; i = 1, 2 \tag{35}$$

Proof. Taking as a partition a common refinement of the partitions for f_1 and f_2 which agree with f_1 and f_2 resp. within $\epsilon/2$ this is an immediate consequence of the previous two lemmas and the triangle inequality. □

Lemma 6. *Assume* $|\mu|(\Omega) < \infty$ *(otherwise choose* $\Omega' \in \Omega$ *so that* $|\mu|(\Omega') < \infty$*). The map* $f \to \int f d\mu$ *is well defined, linear and bounded in the sense*

$$\left\| \int_A f d\mu \right\| \leq \int_A \|f\| d|\mu| \leq \|f\|_{\infty,A} |\mu|(A) \tag{36}$$

where $|\mu|$ is the total variation of the signed measure μ, $|\mu| = \mu^+ + \mu^-$, where $\mu = \mu^+ - \mu^-$ is the Hahn-Jordan decomposition of μ.

Proof. All properties are immediate, except perhaps boundedness. We have

$$\left\| \int_A f d\mu \right\| \leq \sum_{j \in J} |\mu|(A_j) \|x_j\| = \int_A \|f\| d|\mu| \leq \|f\|_{\infty,A} |\mu|(A) \tag{37}$$

□

(k) Thus $\int_A$ is a linear bounded operator from $B_s(\Omega, X)$ to X and it extends to a bounded linear operator on from $B(\Omega, X)$ to X.

Lemma 7. $f \in B(\Omega, X) \to \forall \epsilon$ *there is a partition* $\{A_i\}_{i=1,\dots,n}$ *of* Ω *so that for* any $\omega_j \in A_j$ *we have*

$$\|f - \sum_j f(\omega_j) \chi_{A_j}\| \leq \epsilon \tag{38}$$

Proof. Choose a partition $\{A_i\}_{i=1,\dots,n}$ of Ω and x_j so that

$$\|f - \sum_j x_j \chi_{A_j}\| \leq \epsilon/2$$

This implies by the 11f above that

$$\|x_j - f(\omega_j)\| \leq \epsilon/2$$

for *all* $\omega_j \in A_j$. The rest follows from the triangle inequality. □

Exercise

Show that the same holds for a pair of functions f_1, f_2, namely there is a *common* partition $\{A_i\}_{i=1,\dots,n}$ of Ω so that

$$\|f_i - \sum_j f_i(\omega_j) \chi_{A_j}\| \leq \epsilon, i = 1, 2$$

(l) Let T be a closed operator and $f \in B(\Omega, X)$ be such that $f(\Omega) \subset \mathcal{D}(T)$. Assume further that f is such that $Tf \in B(\Omega, X)$.

Theorem 1 (Commutation of closed operators with integration)**.** *Under the assumptions above we have*

$$T \int_A f d\mu = \int_A T f d\mu \tag{39}$$

Proof.

- For any m we can $f_m \in B_s(\Omega, X)$, so that $\|f - f_m\| \le 1/m$, where
$$f_m = \sum_{j=1}^{n_m} \chi_{A_j} x_j$$
and $g_m \in B_s(\Omega, X)$, so that $\|Tf - g_m\| \le 1/m$,
$$g_m = \sum_{j=1}^{n_m} \chi_{A_j} y_j$$
where we have assumed N_m, A_j are the same, since this can be arranged by a subpartition of the A_js.
- Furthermore, we can arrange that $x_j = f(\omega_j)$ for some $\omega_j \in A_j$. Indeed, we have, throughout A_j, $\|f(\omega) - f(\omega_j)\| \le \|f(\omega) - x_j\| + \|x_j - f(\omega_j)\| \le 2/m$ since the estimate $\|f(\omega) - x_j\| < 1/m$ is uniform in A_j.
- Then T applies to f_m, and we have, on A_j, $Tf_m - g_m = Tf(\omega_j) - y_j$ which is estimated in norm by $1/m$ since $\|Tf(\omega) - y_j\| < 1/m$ throughout A_j.
- On the other hand, $\int_A Tf_m = \sum_{j=1}^{N_m} \mu(A_j) T x_j = T_A \int f_m$.
- But g_m converges in the sup norm, by assumption, to Tf. Thus Tf_m converges in norm to Tf.
- Furthermore, $\int_A Tf_m$ converges, since g_m converge uniformly, thus $\int_A g_m$ converge, and $\|Tf_m - g_m\| < 1/m$. Since T is closed, and $T \int_A f_m$ converges, and $\int_A f_m \to \int_A f$, then $T \int_A f_m \to T \int_A f$. □

Exercise 1. *Formulate and prove a theorem allowing to differentiate under the integral sign in the way*

$$\frac{d}{dx} \int_a^b f(x, y) dy = \int_a^b \frac{\partial}{\partial x} f(x, y) dy$$

Corollary 8. *In the setting of Theorem 1, if T is bounded we can drop the requirement that $Tf \in B$.*

Proof. Tf_m is a simple function, and it converges to Tf. The rest is immediate. □

(m) An important case of Corollary 8 is that for any $\phi \in X^*$, we have

$$\phi \int_A f d\mu = \int_A \phi f d\mu \tag{40}$$

(n) We recall a corollary of the Hahn-Banach theorem:

Proposition 9. Let X be a normed linear space, $Z \subset X$ a subspace of it and $y \in X$ such that $\text{dist}(y, Z) = d$. Then there exists a $\phi \in X^*$ such that $\|\phi\| \le 1$, $\phi(y) = d$ and $\phi(Z) = \{0\}$.

(see [2], p.77).

Corollary 10. *If $\phi(x) = \phi(y)$ $\forall \phi \in X^*$, then $x = y$.*

(o)

Definition 11. *The last results allow us to transfer many properties of the usual integral to the vector setting.*
For instance, if A and B are disjoint, then

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu \tag{41}$$

Proof. By Corollary 10 , (42) is true iff for any ϕ we have

$$\int_{A\cup B} \phi f d\mu = \int_A \phi f d\mu + \int_B \phi f d\mu \tag{42}$$

which clearly holds.

□

f is measurable between two topological spaces if the preimage of a measurable set is measurable.

Proposition 12. *If $f \in B(\Omega, X)$ then f is measurable.*

Proof. Usual proof, f is a uniform limit of measurable functions, f_m. □

Remark 1. *We recall that in a complete metric space M, a set S is precompact* **iff** *it is totally bounded, that is for any $\epsilon > 0$ there is an $N(\epsilon)$ and a set of points $J_N = \{x_1, ..., x_N\} \subset M$ so that $\sup_{x\in S} dist(x, J_N) < \epsilon$.*

Theorem 2. *The function f is in $B(\Omega, X)$* **iff** *f is measurable and $f(\Omega)$ is relatively compact.*

Proof. We first show that f is in $B(\Omega, X)$ implies $f(\Omega)$ is relatively compact. We have $f = f_m + O(1/m)$ and $f_m = \sum_{j=1}^{n_m} \chi_{A_j} x_j$. Thus the whole range of f is within $1/m$ of the finite set $x_1, ..., x_{N_m}$. Thus $f(\Omega)$ is totally bounded, thus precompact.
Now, if $f(\Omega)$ is precompact, then it is totally bounded and $f(\omega)$ is within ϵ of a set $\{x_1, ..., x_N\}$. Out of it, it is easy to construct a simple function approximating f within ϵ. □

Corollary 13. *Let Ω be compact and denote by $C(\Omega, X)$ the continuous functions from Ω to X. Then $C(\Omega, X)$ is a closed subspace of $B(\Omega, X)$.*

Proof. If $f \in C(\Omega, X)$, then $f(\Omega)$ is compact, since Ω is compact and f is continuous. Measurability follows immediately, since the preimage of open sets is open. $\square$

(p) If $f_n \in B(\Omega, X)$, $f_n \to f$ in the sup norm, then $\int f_n d\mu \to \int f d\mu$. This is clear, since $\int_A$ is a continuous functional.

5 Extension

A theory of integration similar to that of Lebesgue integration can be defined on the measurable functions from Ω, X to Y ([1])

The starting point are still simple functions. Convergence can be understood however in the sense of L^1. We endow simple functions with the norm

$$\|f\|_1 = \int_A \|f(\omega)\| d|\mu|$$

and take the completion of this space. Convergence means: $f_n \to f a.e.$ and f_n are Cauchy in $L_1(A)$. Then $\lim \int f_n d\mu$ is by definition $\int_A f d\mu$.

Integration is continuous, and then the final result is $L^1(A)$.

Then, the usual results about dominated convergence, Fubini, etc. hold.

6 Basic Banach algebra notions

We need to work in a systematic way with sums, products, and more generally functions of operators.

Definition 14. *A Banach algebra is a Banach space which is also an associative algebra, in which multiplication is continuous:*

$$\|ab\| \leq \|a\| \|b\| \tag{43}$$

Examples: (1) $C(\Omega)$;

(2) $L^1(\mathbb{R})$ with the product given by $fg := f * g$ where

$$(f * g)(p) = \int_0^p f(s) g(p - s) ds$$

Exercise 1. *Show that (2) is indeed a Banach algebra.*

A Banach algebra may or may not have an identity element, for which $\forall\, a,\ 1a = a1 = a$. (1) has an identity, the function $f(x) = 1$, while (2) does not.

Note. Often the condition of existence of an identity is included in the definition of a Banach algebra.

(a)

Remark 2. *If $\mathcal{A}$ has an identity, then $\|1\| \geq 1$. It is clear that $\|1\| \neq 0$. Then, we must have $\|1\| = \|1^2\| \leq \|1\|^2$. We can arrange that $\|1\| = 1$ by changing to an equivalent norm. Indeed, let*

$$\|x\|_\sim = \sup_{a\in\mathcal{A}:\|a\|=1} \|ax\|$$

Then clearly, by the continuity of multiplication, we have $\|x\|_\sim \leq \|x\|$. On the other hand, $\|a\|_\sim \geq \|1a\|/\|1\|$, so the two norms are equivalent*.*

Clearly, $\|1\|_\sim = 1$ essentially by definition.

(b) There is a good notion of spectrum on Banach algebras:

Definition 15. Let $\mathcal{A}$ be a Banach space with an identity and let $a \in \mathcal{A}$. The spectrum of a, denoted by $\sigma_b(a)$ is defined by

$$\sigma_b(a) = \{z \in \mathbb{C} : z - a \text{ has no inverse in } \mathcal{A}\} \tag{44}$$

Note 3. If T is a bounded operator, then T is called invertible if T is one-to-one with bounded inverse. Then $TT^{-1} = T^{-1}T = 1$. In general, invertible means *two sided inverses exist.* Note that the shift operator on l^2, $S(a_1, a_2, ..., a_n, \ldots) = (0, a_1, a_2, ..., a_n, \ldots)$ has a left inverse, $S'(a_1, a_2, ..., a_n, \ldots) = (a_2, a_3, , ..., a_n, \ldots)$ but not a right inverse, since the image $S(l^2)$ has codimension one (all vectors in the image have 0 as the first component). *Show that if the right inverse and left inverse of an element in a group exist, then they coincide.*

(c) Spectral radius. This is defined by

$$r(a) = \limsup_{n\to\infty} \|a^n\|^{1/n} \tag{45}$$

Clearly, $r(a) \leq \|a\|(< \infty)$. But it can be smaller. Think of the algebra generated by a nilpotent matrix.

In fact we will show that $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$. This explains the name and also shows that $\sigma(a) \neq \mathbb{C}$.

7 Analytic vector valued functions

Let X be a Banach space. Analytic functions $: \mathbb{C} \to X$ are functions which are, locally, given by convergent power series, with coefficients in X.

(a) More precisely, let $\{x_k\} \subset X$ be such that $\limsup_{n\to\infty} \|x_n\|^{1/n} = \rho < \infty$. Then, for $z \in \mathbb{C}$ $|z| < R = 1/\rho$ the series

$$S(z, x_k) = \sum_{n=0}^{\infty} x_k z^k \tag{46}$$

converges in X (because it is *absolutely convergent* [3]). (There is an interchange of interpretation here. We look at (46) also as a series over X, with coefficients z^k.)

(b) **Abel's theorem**. Assume $S(z)$ converges for $z = z_0$. Then the series converges uniformly, together with all formal derivatives on D_r where $D_r = \{z : |z| \le r\}$, if $r < |z_0|$.

Proof. This follows from the usual Abel theorem, since the series $\sum_{n=0}^{\infty} \|x_k\| |z|^k =: S(|z|, \|x_k\|_k)$ converges (uniformly) in D_r. □

7.1 Functions analytic in an open set $\mathcal{O} \in \mathbb{C}$

(a) Let $\mathcal{O} \in \mathbb{C}$ be an open set. The space of X-valued analytic functions $H(\mathcal{O}, X)$ is the space of functions defined on $\mathcal{O}$ with values in X such that for any $z_0 \in \mathcal{O}$, there is an $R(z_0) \neq 0$ and a power series $S(z; z_0)$ with radius of convergence $R(z_0)$ such that

$$f(z) = S(z; z_0, x_k) =: \sum_{k=0}^{\infty} x_k(z_0)(z - z_0)^k \quad \forall z, |z - z_0| < R(z_0) \tag{47}$$

(b) If f is analytic in $\mathbb{C}$, then we call it entire.

Proposition 16. *An analytic function is continuous.*

Proof. We are dealing with a uniform limit on compact sets of continuous functions, $\sum_{k=0}^{N} x_k(z_0)(z - z_0)^k$. □

Corollary 17. *Let $\mathcal{O}$ be precompact. Then $H(\mathcal{O}, X) \subset B(\Omega, X)$.*

Proof. This follows from Proposition 16 and Corollary 13. □

(c) Let now X be a Banach algebra with identity.

Lemma 18. *The sum an product of two series $s(z; z_0, x_k)$ and $S(z; z_0, y_k)$ with radia of convergence r and R respectively, is convergent in a disk D of radius at least* $\min\{r, R\}$.

[3] That is, $\sum_{n=0}^{\infty} \|x_k\| |z|^k$ converges.

Proof. By general complex analysis arguments, the real series $s(\|z\|; z_0, \|\|x_k\|)$ and $S(\|z\|; z_0, \|\|x_k\|)$ converge in D and then so does $s(\|z\|; z_0, \|\|x_k\|)+ S(\|z\|; z_0, \|\|x_k\|)$ etc. □

Corollary 19. *The sum and product of analytic functions, whenever the spaces permit these operations, is analytic.*

(d) Let $\mathcal{O}$ a relatively compact open subset of $\mathbb{C}$. We can introduce a norm on $H(\mathcal{O}, X)$ by $\|f\| = \sup_{z\in\mathcal{O}} \|f(z)\|_X$.

(e) Let us recall what a rectifiable Jordan curve is: This is a set of the form $\Gamma = \gamma([0, 1])$ where $\gamma : [0, 1] \to \mathbb{C}$ is in CBV (continuous functions of bounded variation), such that $x \le y$ and $\gamma(x) = \gamma(y) \Rightarrow (x = y$ or $x = 0, y = 1)$ (that is, there are no nontrivial self-intersections; if $\gamma(0) = \gamma(1)$ then the curve is closed). (Of course, we can replace $[0, 1]$ by any $[a, b]$, if it is convenient.) Then γ' exists a.e., and it is in L^1. Thus $d\gamma(s) = \gamma'(s)ds$ is a measure absolutely continuous w.r.t ds. As usual, we define positively oriented contours, the interior and exterior of a curve etc.

(f) We can define complex contour integrals now. Note that if f is analytic, then it is continuous, and thus $f(\gamma) : [0, 1] \mapsto \mathbb{C}$ is continuous, and thus in $B(\mathcal{O}, X)$. Then, by definition,

$$\int_\Gamma f(z)dz := \int_0^1 f(\gamma(s))\gamma'(s)ds =: \int_0^1 f(\gamma(s))d\gamma(s) \tag{48}$$

(g)

Proposition 20. *If $f \in B(S^1, X)$ (S^1 =the unit circle) and $z \in \mathbb{D}^1$ (the open unit disk), then*

$$F(z) = \oint_{S^1} f(s)(s - z)^{-1}ds \tag{49}$$

is analytic in $\mathbb{D}_1$.

Proof. As usual, we pick $z_0 \in \mathbb{D}^1$, let d =dist(d, S^1) and take the disk $\mathbb{D}^{d/2}(z_0)$. We write $(s - z) = (s - z_0)^{-1}/(1 - (z - z_0)/(s - z_0)) =: (s - z_0)^{-1}/(1 - x)$. Using

$$1/(1 - x) = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

we get

$$\oint_{S^1} f(s)(s - z)^{-1}ds - \oint_{S^1} f(s)(s - z_0)^{-1}ds$$
$$= \sum_{k=0}^{n}(z-z_0)^k \oint_{S^1} \frac{f(s)ds}{(s - z_0)^{k+1}} + (z-z_0)^{n+1} \oint_{S^1} \frac{f(s)ds}{(s - z)(s - z_0)^{n+1}} ds \tag{50}$$

We now check that the last integral has norm $\leq 2^{-n}C$ where C is independent of n, and the result follows. □

(h)

Proposition 21. Let Y be a Banach space, $T : X \to Y$ continuous (i.e. bounded), and $f \in H(\mathcal{O}, X)$. Then $Tf \in H(\mathcal{O}, Y)$.

Proof. Let $z_0 \in \mathcal{O}$. Then there is a disk $\mathbb{D}(z_0, \epsilon)$ such that for all $z \in \mathbb{D}(z_0, \epsilon)$ we have $\|f(z) - \sum_{k=0}^{N} x_k(z - z_0)^k\| \to 0$, as $n \to \infty$. then,

$$\|Tf(z) - \sum_{k=0}^{N} Tx_k(z - z_0)^k\| \leq \|T\|\|f(z) - \sum_{k=0}^{N} x_k(z - z_0)^k\| \to 0$$

and the result follows. □

(i) In particular, f analytic implies ϕf analytic for any $\phi \in X^*$.

(j) As a result of (11m) on p. 14 we have, for any $\phi \in X^*$

$$\phi \int_\Gamma f(z)dz = \int_\Gamma \phi f(z)dz \tag{51}$$

(k) The last few results allow us to transfer the results that we know from usual complex analysis to virtually identical results on strongly analytic vector valued functions.

(l)

Proposition 22. *The function f is analytic* **iff** *it is* weakly analytic*, that is ϕf is analytic for any $\phi \in X^*$.*

Proof 1. Let $\phi \in X^*$ be arbitrary. Then $\phi f(s)$ is a scalar valued analytic function, and then

$$\phi f(z) = \oint_{\bigcirc} \phi f(s)(s - z)^{-1}ds = \phi \oint_{\bigcirc} f(s)(s - z)^{-1}ds \tag{52}$$

if the circle around z is small enough. Since this is true for all ϕ, we thus we conclude that

$$f(z) = \oint_{\bigcirc} f(s)(s - z)^{-1}ds \tag{53}$$

and by Proposition 20, f is analytic. □

Proof 2. Consider the family of operators $f_{zz'} := (f(z) - f(z'))/(z - z')$ on $\mathcal{O}^2 \setminus D$ where D is the diagonal $(z, z) : z \in \mathcal{O}$. Then $|\phi f_{zz'}| < C_\phi$ for all $\phi \in X^*$. Now we interpret $f_{zz'}$ as a family of functionals on X^{**}, indexed by z, z'. By the uniform boundedness principle,

$\|f_{zz'}\|_{X^{**}} \leq B < \infty$ is bounded with the bound independent of z, z', and by standard functional analysis $\|f_{zz'}\|_{X^{**}} = \|f_{z,z'}\|_X \leq B$. Then, $f(z)$ is continuous, and thus integrable. But then, since $\oint_\Delta \phi f(s)ds = 0$ it follows that $\oint_\Delta f(s)ds = 0$ for all $\Delta \in \mathcal{O}$, and thus f is analytic. $\square$

For instance:

i. $\int_\Gamma$ does not depend on the parametrization of Γ, but on Γ alone.

ii. If we take a partition of Γ, $\Gamma = \cup_{i=1}^N \Gamma_i$ then

$$\int_\Gamma f(z)dz = \sum_{i=1}^N \int_{\Gamma_i} f(z)dz \tag{54}$$

iii. We have Cauchy's formula: Assume $\mathcal{O}$ is a precompact open set in $\mathbb{C}$, Γ a a closed, positively oriented contour in $\mathcal{O}$ and if $z \notin \Gamma$, where Γ is a closed, positively oriented contour, then

$$\oint_\Gamma f(s)(s-z)^{-1}ds = 2\pi i f(z)\chi_{\mathrm{Int}\Gamma}(z) \tag{55}$$

Proof. Note that $f(s)(s-z)^{-1}$ is analytic in z, for any s. Thus it is continuous, integrable, etc. We have for any $\phi \in X^*$,

$$\oint_\Gamma \phi f(s)(s-z)^{-1}ds = 2\pi i \phi f(z)\chi_{\mathrm{Int}\Gamma}(z) \tag{56}$$

$\square$

iv. Liouville's theorem: If f is analytic in $\mathbb{C}$ and bounded, in the sense that $\|f(\mathbb{C}, X)\| \subset K \in \mathbb{R}$ where K is compact, then f is a constant.

Proof. Indeed, it follows that $\phi(f)(z) = \phi(f(z)$ is entire and bounded, thus constant. Hence, $\phi(f)(z) - (\phi f)(0) = 0 = \phi(f(z) - f(0))$. Since ϕ is arbitrary, we have $f(z) = f(0)\ \forall z$. $\square$

v. Morera's theorem. Let $f : (\mathcal{O}, X) \to Y$ be continuous (it means it is single-valued, in particular), and assume that $\int_\Delta f(s)ds = 0$ for every triangle in Ω. Then, f is analytic.

Proof. Indeed, ϕf is continuous on $\mathcal{O}$ and we have $\oint_\Delta \phi f ds = 0$ for every triangle in Ω. But then ϕf is a usual analytic function, and thus

$$\phi f(z) = \frac{1}{2\pi i} \oint_{\bigcirc} \phi f(s)(s-z)^{-1}ds$$

By Proposition 20, the right side is analytic. $\square$

vi.

Corollary 23. *f is analytic iff f is strongly differentiable in z iff it is weakly differentiable in z.*

Proof. Assume that f is weakly differentiable. This implies that for any $\phi \in X^*$ we have ϕf is analytic, which in turn implies that for a small enough circle around z we have

$$\phi f = \oint_{\bigcirc} \frac{\phi f(s)}{s-z} ds = \phi \oint_{\bigcirc} \frac{ds f(s)}{s-z} \tag{57}$$

which means

$$\oint_{\bigcirc} \frac{f(s)}{s-z} ds = \oint_{\bigcirc} \frac{f(s)}{s-z} ds \tag{58}$$

and, again by Proposition 20, f is analytic. The rest is left as a simple exercise. □

vii. (Removable singularities) If f is analytic in $\mathcal{O} \setminus a$ and $(z-a)f = o(z-a)$ then f extends analytically to $\mathcal{O}$.

Proof. This is true for ϕf. □

(m) If $f_n \to f$ in norm, *uniformly on compact sets*, then furthermore for any $\phi \in X^*$ we have $\phi f_n \to \phi f$ uniformly on compact sets. Then f is weakly analytic thus strongly analytic.

Corollary 24. *$H(\mathcal{O}, X)$ is a linear space; if X is a Banach algebra then $H(\mathcal{O}, X)$ is a Banach algebra.*

Proof. Straightforward verification. □

We can likewise define double integrals, as integrals with respect to the product measure. If $f \in B(\Omega, X_1 \times X_2)$, then $f(\cdot, x_2)$ and $f(x_1, \cdot)$ are in $B(\Omega, X_2)$ and $B(\Omega, X_1)$ respectively, and Fubini's theorem applies (since it applies for every functional). Check this.

7.2 Functions analytic at infinity

(a) f is analytic at infinity if f is analytic in $\mathbb{C} \setminus K$ for some compact set K (possibly empty) and f is bounded at infinity. Equivalently, $f(1/z)$ is analytic in a punctured neighborhood of zero and bounded at zero. Then $f(1/z)$ extends analytically uniquely by $f(\infty)$.

Cauchy formula at ∞ 12. Let f be analytic at infinity and Γ positively oriented about infinity, which by definition means that the neighborhood of infinity is to the left of the curve as we traverse it. (That is, Γ is **negatively oriented** if seen as a curve around 0). Then

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(s)}{s-z} ds = -f(\infty) + \chi_{Ext\Gamma}(z) f(z) \tag{59}$$

Proof. This follows from the scalar case, which we recall. Let f be analytic in $\mathbb{C}_\infty \setminus K$ We have

$$J = \frac{1}{2\pi i}\oint_\Gamma \frac{f(s)}{s-z}ds = -\frac{1}{2\pi i}\oint_{1/\Gamma}\frac{f(1/t)}{t^2(1/t-z)}dt \tag{60}$$

where $1/\Gamma$ is the positively oriented closed curve $\{1/\gamma(1-t) : t \in [0,1]\}$ (where we assumed a standard parametrization of γ, and we can assume that $\gamma \neq 0$, since otherwise f is analytic at zero, and the contour can be homotopically moved away from zero). There is a changes in sign: $z \to 1/z$ changes the orientation of the curve, so it becomes positively oriented in τ as claimed; the second, from $ds = -dt/t^2$.

Then, with $g(t) = f(1/t)$ and writing

$$\frac{1}{t-zt^2} = \frac{1}{t} + \frac{z}{1-zt} = \frac{1}{t} + \frac{1}{1/z-t}$$

$$J = -\frac{1}{2\pi i}\oint_{1/\Gamma}\frac{g(t)}{t} + \frac{1}{2\pi i}\oint_{1/\Gamma}\frac{g(t)}{t-1/z} = -g(0) + \chi_{Ext\Gamma}(z)f(z) \tag{61}$$

□

8 Analytic functional calculus

Let $\mathcal{B}$ be a Banach algebra with identity, *not necessarily commutative* and let $a \in \mathcal{B}$, with spectrum $\sigma(a)$. Then $\mathbb{C} \setminus \sigma(a)$ is the resolvent set of a, $\rho(a)$. Let $z \in \rho(a)$.

The resolvent. Then $(a-z)$ is invertible by definition, and thus $(z-a)^{-1} : \rho(a) \mapsto \mathcal{B}$ is well defined and it is denoted by $R_z(a)$, the resolvent of a. See again Note 3

Note 4. *This may be confusing, but the definitions $R_z(a) = (a-zI)^{-1}$ [3] and $R_z(a) = (zI-a)^{-1}$ [2] are almost evenly distributed in the literature. Of course, they only differ by a sign and the theory is the same, but we have to pay attention to which definition is adopted. I would prefer the one in [3], but we are more closely following [2].*

The domain of $R_z(a)$, $\rho(a)$ is nonempty since we have

Proposition 25. $\rho(a) \supset \{z : |z| > r(a)$

Note. We have, formally,

$$\frac{1}{z-a} = \frac{1}{z}\,\frac{1}{1-a/z} = \frac{1}{z}\sum a^k z^{-k}$$

Proof. The series

$$h(z) = \sum a^k z^{-k} \tag{62}$$

is norm convergent, thus convergent and analytic for $|z| > r(a)$, since $\limsup \|a^n\|^{1/n} = r(a)$. We can check that $z^{-1}h(z)(z-a) = 1$. Indeed, we can, as usual, truncate the series and estimate the difference:

$$\left(\frac{1}{z}\sum_{j=0}^{n} \frac{a^j}{z^j}\right)(z-a) = 1 - z^{-n}a^{-n-1} \tag{63}$$

□

Proposition 26 (First resolvent formula)**.** *For $s, t \in \rho(a)$ we have*

$$R_s(a) - R_t(a) = (t-s)R_t(a)R_s(a) \tag{64}$$

and in particular R_s and R_t commute.

Note. Formally, we would write

$$\frac{1}{s-a} - \frac{1}{t-a} = \frac{(t-a)-(s-a)}{(s-a)(t-a)} \tag{65}$$

Of course, this is nonsense as such, especially if the algebra is noncommutative, but it can serve as a template to work out a correct formula.

Proof. If $a, b \in \mathcal{B}$ and b is invertible, then $ab = cb$ is equivalent to $a = c$. This is seen by direct multiplication. Now, by assumption, $(s-a)$, $(t-a)$ are invertible. Note that, if b is invertible, then $x = y$ is **equivalent to** $bx = by$. Then (64) is equivalent to

$$(t-a)(R_s(a) - R_t(a))(s-a) = (t-a)(t-s)R_t(a)R_s(a)(s-a) \tag{66}$$

which means

$$R_s(a)(s-a)(t-a) - R_t(a)(s-a)(t-a) = t-s \tag{67}$$

$$(t-a) - (s-a) = t-s \tag{68}$$

□

Proposition 27. *The resolvent set is open. The resolvent is analytic on the resolvent set.*

Proof. We use the resolvent formula, first formally. Suppose R_t exists, $\|R_t\| = m$ and $\epsilon < 1/m$. Then we *define* R_s from the first resolvent equation, written as

$$R_s(a)(1 + (t-s)R_t(a)) = R_t(a) \tag{69}$$

More precisely, let $z_0 \in \rho(a)$ and let s be such that $(s - z_0)^{-1}\|R_{z_0}| < 1$. Define

$$T_s = (1 + (z_0 - s)R_{z_0}(a))^{-1}R_{z_0}(a) \tag{70}$$

This is clearly well defined since $1/(z_0 - s)$ is *outside* the disk of radius $\|R_{z_0}\|$. Clearly, the right side of (70) is analytic in s near z_0, since we can write the inverse as a norm-convergent power series, see (62). Then, $(s - a)T_s = (s - z_0 + z_0 - a)T_s = 1$. Check that $T_s(s - a) = 1$ as well. □

Theorem 3. *For every $a \in \mathcal{B}$, the set $\sigma(a)$ is nonempty and compact.*

Proof. We have already shown that diam $\sigma(a) \le r(a)$ and that the complement is open. It remains to show that $\sigma(a)$ is nonempty. We'll use (64). Let $z_0 \in \rho(a)$ be fixed and ν be any large enough complex number. Let $N = \|R_{z_0+\nu}\|$, $A = \|R_{z_0}\|$ and $B = \|R_{z_0}^{-1}\| = \|a - z_0\|$. Then,

$$\nu R_{z_0+\nu} = (R_{z_0} - R_{z_0+\nu})(R_{z_0})^{-1} \Rightarrow$$

$$|\nu|N \le (A + N)B \Rightarrow N(|\nu| - B) \le AB \Rightarrow N \le \frac{AB}{|\nu| - B} = O(|\nu|^{-1} \tag{71}$$

and thus $\phi R_{z_0+\nu}$ is entire and $\to 0$ as $|z| \to \infty$, implying $\phi R_{z_0+\nu} \equiv 0 \Rightarrow R_{z_0+\nu} = 0$ which is impossible, since 0 is not the inverse of anything. □

The spectrum of 1 is just 1 (why?)

Exercise 1. ** Let K be a compact set in $\mathbb{C}$. Find a Banach algebra and an element a so that its spectrum is exactly K. Hint: look at $f(z) = z$ restricted to a set.*

9 Functions defined on $\mathcal{B}$

Clearly, right from the definition of an algebra, for any polynomial P and any element $a \in \mathcal{B}$, $P(a)$ is well defined, and it is an element of the algebra. A rational function $R(a) = P(a)/Q(a)$ can be also defined provided that the zeros of Q are in $\rho(a)$, with obvious notations, by

$$R(a) = (p_1 - a)\cdots(p_m - 1)(q_1 - a)^{-1}\cdots(q_n - a)^{-1}$$

(since we have already shown the resolvents are commutative, the definition is unambiguous and it has the expected properties.

What functions can we define on Banach algebras? Certainly analytic ones (and many more, in fact).

A natural way is to start with Cauchy's formula,

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(s)ds}{s - z} \tag{72}$$

and replace z by a. We have to ensure that (a) The integral makes sense and it is useful. For that we choose the contour so that a "is inside" that is, Γ should be outside the spectrum.

(b) on the other hand, f should be analytic in the interior of Γ, $\sigma(a)$ included. We can see that by looking at Exercise 1, where a would be z. If we don't assume analyticity of f on the spectrum of z, we don't get an analytic function.

So, let's make this precise. Let a be given, with spectrum $\sigma(a)$. Consider the set of functions analytic in an open set $\mathcal{O}$ *containing* $\sigma(a)$. For a generalization see 5 below.

Proposition 28. *Let Γ be a Jordan curve in $\mathcal{O} \setminus \sigma(a)$. Then $f(s)R_s(a)$ is continuous thus integrable, and the function*

$$f(a) := \frac{1}{2\pi i} \oint_\Gamma f(s)(s-a)^{-1} ds \tag{73}$$

is well defined and it is an element of $\mathcal{B}$.

Exercise 1. ** Prove Proposition 28 above.*

We thus **define** $f(a)$ by (73).

1. Show that the integral only depends on the homotopy class of Γ in $\mathcal{O}\setminus\sigma(a)$. So in this sense, $f(a)$ is canonically defined.

2. $P(a)$ defined through (72) coincides with the direct definition. By linearity it suffices to check this on monomials z^n Then we can choose a disk $\mathbb{D}_r$ around zero with $r > r(a)$ Then

$$a^n = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_r} s^n s^{-1} (1 - a/s)^{-1} ds \tag{74}$$

   Indeed, $(1 - a/s)^{-1}$ has a convergent expansion in $1/s$ where $|s| > r(a)$. Then the integral can be expanded convergently, as usual, and the result coincides with a^n.

3.

   Proposition 29. *(i) $(fg)(a) = f(a)g(a)$ (ii) in particular $f(a)g(a) = g(a)f(a)$. The set of functions of a forms a commutative algebra.*

   Proof. (i) Indeed let $\mathrm{Int}\Gamma \supset \Gamma'$. Then (by Fubini, etc.) and the first resolvent formula,

$$f(a)g(a) = \frac{1}{-4\pi^2} \oint_\Gamma \oint_{\Gamma'} f(s)(s-a)^{-1} g(s')(s'-a)^{-1} ds' ds$$
$$= \frac{1}{-4\pi^2} \oint f(s)(s-a)^{-1} \oint f(s)(s-a)^{-1} g(s')(s'-a)^{-1} ds'$$
$$= \frac{1}{2\pi i} \oint f(s) g(s)(s-a)^{-1} ds = (fg)(a) \tag{75}$$

   (ii) is immediate. □

4. We can now of course re-prove in a simpler way 2 above.

   Exercise 2. More generally than in item 2 above, let $E(x) = \sum_{k=0}^{\infty} e_k x^k$ be an entire function. Then $E(a)$ as defined by (73) coincides with

$$\sum_{k=0}^{\infty} e_k a^k \tag{76}$$

5. Note also that $\mathcal{O}$ need not be connected. We have defined analyticity in terms of local Taylor series. More generally, we consider an open set $\mathcal{O}_1 \supset \sigma(a)$, connected or not. If $\Gamma = \Gamma_1 = \partial\mathcal{O}_1$ consists of a finite number of rectifiable Jordan curves, then the definition (73) is still meaningful. Clearly, a function analytic on a disconnected open set is simply any collection of analytic functions, one for each connected component, and no relation needs to exist between two functions belonging to different components. Cauchy's formula still applies on Γ_1. This will allow us to define projectors, and some analytic functions of unbounded operators.

6. We are technically dealing with classes of equivalence of functions of a, where we identify two elements f_1 and f_2 if they are analytic on a common subdomain and coincide there. But this just means that f_1 and f_2 are analytic continuations of each-other, and we make the choice of not distinguishing between a function and its analytic continuation. So we'll write $f(a)$ and not $[f](a)$ where $[f]$ would be the equivalence class of f.

   Exercise 3. Let $\Omega \supset \sigma(a)$ be open in $\mathbb{C}$.

   (i) Show that if $f_n \to f$ in the sup norm on Ω, then $f_n(a) \to f(a)$

   (ii) Show that the map $f \to f(a)$ is continuous from $H(\Omega)$, with the sup norm, into X.

   Remark 5. Let $\rho \notin \sigma(a)$ and $f(z) = 1/(\rho - z)$. Then $f(a) = (\rho - a)^{-1}$. Indeed $f(x)(\rho - x) = (\rho - x)f(x) = 1$ and thus, by 2, we have $f(a)(\rho - a) = (\rho - a)f(a) = 1$, that is $f(a) = (\rho - a)^{-1}$.

   More generally, if P and Q are polynomials and the roots $q_1, ..., q_n$ of Q are outside $\sigma(a)$, then $(P/Q)(a) = P(a)(a - q_1)^{-1} \cdots (a - q_k)^{-1}$. [4].

7. Thus we can write

$$(s - a)^{-1} = \frac{1}{s - a}$$

Exercise 4. Show that $f \to f(a)$, for fixed a, defines an algebra homomorphism between $H(\Omega)$ and its image.

[4] Note however that there is no immediate extension in general of the *local* Taylor theorem $f(x) = \sum f^{(k)}(x_0)(x - x_0)^k$, since $\|x_0 - a\|$ would be required to be arbitrarily small to apply this formula for every analytic f, which in turn wold imply $a = x_0$.

Exercise 5. Let f be analytic in a neighborhood of the spectrum of a and at infinity. Let Γ be a simple closed curve outside $\sigma(a)$, positively oriented about ∞. Then we can define

$$f(a) = -f(\infty) + \frac{1}{2\pi i} \oint_{\Gamma} f(s)(s-a)^{-1} ds$$

This allows for defining functions (analytic at infinity) of unbounded operators having a nonempty resolvent set. The properties are very similar to the case where f is analytic on $\sigma(a)$.

10 Further properties of analytic functional calculus

10.1 Spectrum of $f(a)$

The spectrum of an operator, or of an element of a Banach Algebra is very robust, in that it "commutes" with many operations.

Proposition 30. *Let f be analytic on $\sigma(a)$. Then $\sigma(f(a)) = f(\sigma(a))$.*

Proof. We first prove that $\sigma(f(a)) \subset f(\sigma(a))$. Indeed, let $z \notin f(\sigma(a))$. That is, $f(z) - z'$ does not vanish for z' on the compact set $\sigma(a)$, and therefore it does not vanish on some open set $\mathcal{O} \supset \sigma(a)$.

Then $g = 1/(z-f)$ is analytic in $\mathcal{O} \supset \sigma(a)$. Thus there is an analytic g ($g = 1/(f-z')$), so that $g(z')(z-f(z')) = (z-f(z'))g(z') = 1$ for $z' \in \mathcal{O}$, and thus $1 = [g(z')(z-f(z'))](a) = [(z-f(z'))g(z')] = g(a)(z-f(a)) = (z-f(a))g(a) = 1$ and thus $z \notin \sigma(f(a))$.

In the opposite direction, we show that $z \notin \sigma(f(a)) \Rightarrow z \notin f(\sigma(a))$. We show that [$\sigma_0 \in \sigma(a)$ and $(f(\sigma_0) - f(a))^{-1} := h(a) \ \exists$] leads to a contradiction. Note first that $[f(z) - f(\sigma_0)]/[z - \sigma_0] =: g(z)$ is analytic on $\sigma(a)$ (as it has only one –removable– singularity at σ_0. We would have

$$[f(a) - f(\sigma_0)] = (a - \sigma_0)g(a) = g(a)(a - \sigma_0)$$

and using the invertibility of $f(a) - f(\sigma_0)$ we have

$$\begin{aligned} 1 = h(a)[f(a) - f(\sigma_0)] = h(a)(a-\sigma_0)g(a) = (a-\sigma_0)h(a)g(a) \\ = h(a)g(a)(a-\sigma_0) \end{aligned} \quad (77)$$

which means $(a - \sigma_0)$ is invertible (with inverse $h(a)g(a)$), and thus $\sigma_0 \in \rho(a)$, contradiction. ☐

Exercise 1. *Prove the following corollary.*

Corollary 31. *Let $\mathcal{B}$ be a Banach algebra, $a \in \mathcal{B}$ and $\sigma(a) = K$. Assume further that f, g are analytic in the open set $\mathcal{O} \supset K$.*

Then, $\Big[f(a) = g(a)\Big] \Leftrightarrow \Big[f(z) = g(z)\ \forall z \in K\Big]$. In particular, if card$(K) = \infty$, then $f \equiv g$.

10.2 Behavior with respect to algebra homeomorphisms

Proposition 32. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Banach algebras with identity and H : $\mathcal{A}_1 \to \mathcal{A}_2$ a Banach Algebra homeomorphism.*

(i) Then $\sigma_{\mathcal{A}_2}(H(a)) := \sigma(H(a)) \subset \sigma(a)$.

In particular, if H is an isomorphism, then the spectrum is conserved: $\sigma(H(a)) = \sigma(a)$.

(ii) $H(f(a)) = f(H(a))$.

Proof. (i) This is because $(a-\alpha)g(a) = g(a)(a-\alpha) = 1$ implies $H((a-\alpha)g(a)) = H(g(a)(a-\alpha)) = 1$, which, in view of the assumptions on H means $H(a) - \alpha)H(g(a)) = (H(g(a))(H(a) - \alpha)) = 1$.

(ii) We know that H is continuous and it thus commutes with integration. If we write

$$f(a) = \frac{1}{2\pi i} \oint f(s)(s-a)^{-1}$$

we simply have to show that $H(s-a)^{-1} = (s - H(a))^{-1}$ which holds by (i). □

10.3 Composition of operator functions

Proposition 33. *Let f be analytic in an open set $\Omega \supset \sigma(a)$ and g be analytic in $\Omega' \supset f(\sigma(a))$. Then $g(f(a))$ exists and it equals $(g \circ f)(a)$.*

Proof. That $g(f(a))$ exists follows from Proposition 30 and the definition of f and g. We have, for suitably chosen contours (find the conditions!)

$$g(f(a)) = \oint_{\Gamma_1} \frac{g(s)}{s - f(a)} ds \tag{78}$$

while on the other hand

$$\frac{1}{s - f(a)} = \oint_{\Gamma_2} \frac{1}{s - f(t)} \frac{1}{t-a} dt \tag{79}$$

and thus

$$g(f(a)) = \oint_{\Gamma_1} g(u) du \oint_{\Gamma_2} \frac{1}{u - f(s)} \frac{1}{s-a} ds \tag{80}$$

We also have

$$(g \circ f)(a) = \oint_{\Gamma_1} \frac{g(f(s))}{s-a} dt = \oint_{\Gamma_1} \frac{1}{s-a} \oint_{\Gamma_3} \frac{g(u)}{u - f(s)} du \tag{81}$$

The rest follows easily from the uniform bounds on the integrand, which allow interchange of orders of integration. □

Remark 6. *In particular, this proves again, in a different way that $(fg)(a) = f(a)g(a)$. Indeed, since $2fg = (f+g)^2 - f^2 - g^2$, it is enough to show that $f^2(a) = f(a)^2$. But this follows from Proposition 33 with $g(x) = x^2$. (Note that $f(\sigma(a))$ is a bounded set, $f(a)$ is a bounded operator, and the contours in Proposition 33 can be chosen so that no use of Proposition 30 is needed).*

10.4 Spectral radius

We now want to show that

$$R(a) := \limsup_{n\to\infty} \|a^n\|^{1/n} = \sup\{|\lambda| : \lambda \in \sigma(a)\} =: R_\sigma(a) \tag{82}$$

Indeed, $(z-a)^{-1}$ exists and is analytic for all $|z| > R(a)$ (simply write $(z-a)^{-1} = z^{-1}(1-a/z)^{-1}$ and expand in powers of $1/z$). Furthermore, the very existence and convergence of this series means $z^{-1}(1-a/z)^{-1}$ is analytic at ∞. On the other hand, by Proposition 27, $(z-a)^{-1} = z^{-1}(1-a/z)^{-1}$ is analytic in $\sigma(a)^c$, in particular outside the disk of radius $R_\sigma(a)$. That, is, the function $\zeta(1-a\zeta)^{-1}$ is analytic inside the disk of radius $1/R_\sigma(a)$, and in no larger disk, since otherwise $(z-a)^{-1}$ would be analytic on a part of the spectrum of a. Thus $1/R_\sigma(a)$ is the maximal disk of analyticity of $\zeta(1-a\zeta)^{-1}$, which implies $R_\sigma(a) = \limsup \|a^n\|^{1/n}$.

10.5 Extended spectrum

We say that ∞ is in the extended spectrum, σ_∞, of an operator if $(z-T)^{-1}$ is not analytic at infinity.

Proposition 34. *If T is closed, then ∞ is not an essential singularity of $(z-T)^{-1}$ iff T is bounded.*

Proof. We have already shown that T bounded implies $(z-T)^{-1}$ is analytic for large z, thus, by definition ∞ is not a singularity of R_z.

In the opposite direction, assume that $R_z := (z-T)^{-1}$ has at most a pole at infinity where the series is convergent and k is the largest power of z so that the Laurent coefficient A is nonzero. We have

$$R_z = z^k A + z^{k-1}B + \cdots$$

On the other hand,

$$1 = (z-T)R_z = zR_z - TR_z$$
$$TR_z = zR_z - 1 \tag{83}$$

We first show that $k \le -1$. Indeed, assume $k \ge 0$. On the one hand, as have $z^{-k-1}R_z = A/z + \cdots \to 0$. On the other hand

$$T(z^{-k-1}R_z) = z^{-k-1}TR_z = z^{-k-1}(zR_z - 1) = A + O(1/z)$$

Since

$$z^{-k-1}R_z = A/z + \cdots$$

we have $z^{-k-1}R_z \to 0$. This together with $T(z^{-k-1}R_z)$ convergent (to A) and the closure of T implies that $T(z^{-k-1}R_z) \to T0 = 0$ and thus $A = 0$. Here we used $k \ge 0$ since otherwise $1z^{-k-1} \ne (1/z)$. Thus $k \le -1$ and $zR_z \to A$. We have

$$TR_z = zR_z - 1 \to A - 1$$

Since $k \le -1$ $R_z \to 0$ and, as before, we must have $A - 1 = 0$ that is $A = 1$. We see that $zR_z \to 1$. Now, for any vector u,

$$TzR_zu = z^2(1/z + B/z^2 + \cdots) - z = Bu + O(1/z)$$

Since T is closed, $zR_zu \to u$ is convergent and TzR_zu converges to Bu. Since T is closed, the limits have to coincide, $u = \lim zR_zu \in \text{Dom}\,(T)$ and $Tu = Bu$, thus T is bounded.

It is clear that for any closed operator, the extended spectrum is nonempty.

□

10.6 Further remarks on Banach algebras

1. Let $\mathcal{A}$ be a Banach algebra with identity, and $x \in \mathcal{A}$, not necessarily commutative. Then $x_l(a) = xa$ and $x_r(a) = ax$ define linear bounded operators from $\mathcal{A}$ into $\mathcal{A}$.

2. Assume a and b commute, $ab = ba(*)$. If a^{-1} exists, then $a^{-1}b = ba^{-1}$. Indeed, from (*) we have $a^{-1}(ab) = a^{-1}(ba)$ and thus $b = a^{-1}(ba)$. Multiplying with a^{-1} on the right, we have $ba^{-1} = a^{-1}b$. Likewise, if b^{-1} exists as well, then b^{-1} commutes with a^{-1}.

3. If a and b are in $\mathcal{A}$ and commute with each other and $f(a)$ and $g(b)$ are analytic functions of a and b, then $f(a)$ and $g(b)$ commute as well.

   Proof. Clearly, it is enough to show that $f(a)$ and b commute. For that we use 1 above

$$\begin{aligned} bf(a) = b_l(f(a)) = b_l\left(\frac{1}{2\pi i}\oint f(s)(s-a)^{-1}ds\right) \\ = \frac{1}{2\pi i}\oint b_l\left(f(s)(s-a)^{-1}\right)ds = \frac{1}{2\pi i}\oint f(s)b_l(s-a)^{-1}ds \\ = \frac{1}{2\pi i}\oint f(s)b(s-a)^{-1}bds = \frac{1}{2\pi i}\oint f(s)(s-a)^{-1}bds \\ = \frac{1}{2\pi i}\oint f(s)b_r(s-a)^{-1}ds = \cdots = b_r(f(a)) = f(a)b \end{aligned} \tag{84}$$

   □

4. The notion of spectrum in a Banach algebra and spectrum of operators are closely related. Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. As above, we can define x as an operator on $\mathcal{A}$ (either left or right multiplication). If $x - z$ is invertible in $\mathcal{A}$ then the operator $x - z$ is invertible, and vice-versa. Conversely, the space B of all bounded operators on X is a Banach algebra and the spectrum of T *as an operator on* X is clearly the same as the spectrum of T as an element of B.

11 Projections, spectral projections

This is an important ingredient in understanding operators and in spectral representations.

Note. The spectrum K does not have to be connected for the spectral theorem to hold. Indeed, if $\sigma(K) \in K_1 + K_2$ (disjoint union) it means there exist $\mathcal{O}_1 \supset K_1$ and $\mathcal{O}_2 \supset K_1$ open sets, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. A function analytic on $\mathcal{O}_1 \cup \mathcal{O}_2$ is any pair of functions $(f_1, f_2) =: f$ so that f_i is analytic in $\mathcal{O}_i$, $i = 1, 2$. Check this. Check that, if $\Gamma_i \in \mathcal{O}_i$, then $f(z) = \oint_{\Gamma_1+\Gamma_2} f(s)/(s-z)ds$ and thus the definition of $f(a)$ in this case is the same,

$$f(a) = \oint_{\Gamma_1+\Gamma_2} f(s)(s-a)^{-1}ds$$

Let us first look at operators. Assume for now the operator is bounded, though we could allow unbounded operators too.

Definition 35. *Let $\mathcal{A}$ be a Banach algebra, and $P \in \mathcal{A}$, P is a projector if $P^2 = 1$. Perhaps the simplest example, and a relevant one as we shall see is a characteristic function χ_A in L^∞.*

General properties.

1. Assume $P \in \mathcal{L}(X)$ is a projector. Then PX is a closed subspace of X and P is the identity on PX.

   Proof. Assume $Px_k \to z$. Continuity of P implies P^2x_k converges to Pz, thus $z = Pz$, and thus $z \in PX$. We have $x = Py \Rightarrow Px = P^2y = Py = x$ and thus $Px = x$. □

2. If T and P commute, and $X_P = PX$ then $TX_P \subset X_P$, that is T can be restricted to X_P. Indeed, $TX_P = TPX_P = PTX_P \subset X_P$.

Let us first look at operators. Assume for now the operator is bounded, though we could allow unbounded operators too.

As we have seen in 6 on p. 8, $\lambda \in \sigma(T)$ iff $T - \lambda$ is not bijective.

Assume now that the spectrum of T is not connected. Then $\sigma(T) = K_1 + K_2$ where K_1 and K_2 are compact, nonempty, and disjoint (of course, K_1 and K_2 could be further decomposable).

Theorem 4 (Elementary spectral decomposition)**.** *Let $T \in \mathcal{L}(X)$ an operator such that $\sigma(T) = K_1 + K_2$, $K_{1,2}$ as above.*

Then there exist nonzero closed subspaces of X, X_1 and X_2 so that
(i) $X = X_1 + X_2$, $X_1 \cap X_2 = \{0\}$ (X is isomorphic to $X = X_1 \oplus X_2$).
(ii) $TX_i \subset X_i$ and $\sigma_{X_i}(T) \subset K_i$.

Proof. 1. Let $\mathcal{O}_{1,2} \supset K_{1,2}$ be two open disjoint sets in $\mathbb{C}$, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, and let $f_i(z) = \chi_{\mathcal{O}_i}(z)$. Note that $f_i(z)$ are *analytic* in $\mathcal{O}$ (the parts are disjoint; f is differentiable in each piece).

2. Let $P_i = \chi_{\mathcal{O}_i}(T)$. Since $\chi^2_{\mathcal{O}_i}(z) = \chi_{\mathcal{O}_i}(z)$, we have $P_i^2 = P_i$, that is, P_i are projectors.

3. Similarly, we have $P_1P_2 = \chi_1(T)\chi_2(T) = (\chi_1\chi_2)(T) = 0$.

4. We have $P_1 + P_2 = 1$ (the identity). Indeed $\chi_{\mathcal{O}_1} + \chi_{\mathcal{O}_2} = \chi_{\mathcal{O}}$. Then,

$$\oint_{\Gamma_1\cup\Gamma_2} \frac{\chi_{\mathcal{O}_1}(s) + \chi_{\mathcal{O}_2}(s)}{s-a} ds = \oint_{\Gamma_1\cup\Gamma_2} \frac{\chi_{\mathcal{O}}(s)}{s-a} ds = \oint_{\Gamma_1\cup\Gamma_2} \frac{1}{s-a} ds = 1$$

   (recall the calculation (74)).

5. Let $X_i = P_iX$. These are closed subspaces of X by 1.

6. If $x \in P_1X$ and $x \in P_2X$, then $x = P_1x_1 = P_2x_2$. We multiply the latter equality by P_1. We get $P_1x = P_1^2x_1 = P_1x_1 = P_1P_2x_2 = 0$. Similarly, $P_2x = 0$. Thus $x = P_1x + P_2x = 0$ and $X_1 \cap X_2 = \{0\}$.

7. Since $1 = P_1 + P_2$, any $x \in X$ can be written as $P_1x + P_2x = x_1 + x_2$, $x_i \in X_i$.

8. We have $\sigma(P_iT) = \sigma(\chi_i(T)Ide(T))$ (where, here, $Ide(z) = z$) $= (z\chi_i(z))(\sigma(T)) = K_i$.

□

12 Analytic functional calculus for unbounded, closed operators with nonempty resolvent set

As we have seen, the spectrum of an unbounded closed operator can be any closed set (that it is necessarily closed we will see shortly), including $\mathbb{C}$ and $\emptyset$. (The extended spectrum is never empty though, as we have seen). If the spectrum of T is the whole of $\mathbb{C}$, then it is closed, and this is about all we will say for now in this case, and will shall assume from this point on that $\rho(T) \neq \emptyset$.

Also, calculus with better behaved operators (normal, self-adjoint) is richer, and we will later focus on that.

If an operator is closed, then it is invertible iff it is bijective, 6, on p. 8.

Evidently, the domain of $T - z$ is the same as the domain of T, so $T - z; z \in \mathbb{C}$ share a common domain. So $z \in \sigma(T)$ iff $T - z$ is not bijective from $D(T)$ to Y. With this remark, the proof in the first resolvent formula goes through essentially without change.

Assume in the following that $X = Y$.

Proposition 36 (First resolvent formula, closed unbounded case)**.** *(i) $\mathbb{C} \cap \sigma_\infty(T) = \rho(T)$ is open (possibly empty).*

(ii) Assume $\rho(T) \neq \emptyset$ and $(s,t) \in \rho(T)$. Then,

$$R_s(T) - R_t(T) = (t-s)R_t(T)R_s(T) \tag{85}$$

In particular R_s and R_t commute.

Proof. We start with (85), assuming for the moment that there are at least *two* elements, $s, t \in \rho(T)$. Of course, this will follow once we have proved that $\rho(T)$ is open. The proof is not circular, check this.

Obviously (85) holds iff it holds for any $y \in Y$. Once more, since T is closed, $s \in \rho(T)$ **iff** $(s - T)$ is bijective between $D(T) \subset X$ and X. In particular, $x \in D(T)$ is zero iff $(s - T)x = 0$. We have by the def. of the resolvent, $R_s(T)$ is a bounded bijection between Y and $D(T)$. So is $R_t(T)$. In particular, $[R_s(T) - R_t(T)]x$ and $R_t(T)R_s(T)x \in D(T)$ If we apply $(t - T)$ to both sides we get

$$\{s-T+(t-s)\}R_s-(t-T)R_t(T) = (t-s)R_s \;\; (s-t)(t-T)R_tR_s = (s-t)R_s \tag{86}$$

so the equality checks $\forall x \in X$.

Suppose R_t exists, $\|R_t\| = m$, $\epsilon < 1/m$, and $|s - t| < \epsilon$. *Define* R_s from the first resolvent equation (as on pp. 23), written as by

$$R_s(T) = (1 + (t - s)R_t(T))^{-1}R_t(T) \tag{87}$$

Then clearly $R_s : X \to D(T)$ and

$$(s - T)R_s = \frac{(t - T + (s - T))R_t}{1 + (t - s)R_t(T)} = \frac{1 + (s - t)R_t}{1 + (t - s)R_t(T)} = 1$$

$\forall x \in X$. Similarly, $R_s(s - T) = 1$. ☐

12.1 Analytic functions of unbounded, closed operators

Since in this section we specifically deal with unbounded operators, we shall will always have $\infty \in \sigma_\infty(T)$, see Proposition 34.

In the following, we **assume that** $\sigma_\infty(T) \neq \mathbb{C}_\infty$, to be able to define non-trivial analytic functions on $\sigma_\infty(T)$.

Note 7. Note that a function analytic on $\sigma_\infty(T)$ is therefore analytic in an open set $\mathcal{O}_\infty$ containing ∞. Clearly, the complement in $\mathbb{C}$ of $\mathcal{O}_\infty$ is compact. So f is analytic in $\text{Ext}(K)$ for some K and on the *rest of the spectrum* of T, necessarily contained in a compact set. We can break $\sigma_{\mathbb{C}}(T)$ in a (possibly infinite, possibly consisting of just one set) disjoint union of connected compact sets. Each connected component K_α is contained in a connected $\mathcal{O}_\alpha$, where all $\mathcal{O}_\alpha$ are disjoint. By the finite covering theorem, f is analytic in $\text{Ext}(K)$ and in $\cup_{i=1}^n \mathcal{O}_i$ where $\mathcal{O}_i$ are open and connected. Now, for each of these, we can assume that the boundary is a polygonal arc (check!). In fact, by a similar construction, we can take the boundary to be an analytic curve. How?

Recall that, from (59), if $z \in Ext\ \Gamma$ we have

$$f(z) = f(\infty) + \frac{1}{2\pi i}\oint_\Gamma \frac{f(s)}{s - z}ds \tag{88}$$

1. An identity: If f is analytic at infinity, and $0 \in \mathrm{Ext}_\infty(\Gamma)$ that is, to the right of Γ traversed clockwise, where Γ is a curve or system of curves so that f is analytic in $\mathrm{Int}_\infty \Gamma$ then we have

$$\frac{1}{2\pi i} \oint s^{-1} f(s) ds = -f(\infty) \tag{89}$$

(The left side is *not* necessarily $f(0)$ which could be undefined. Even if defined, there are typically other singularities of f in $\mathrm{Ext}_\infty(\Gamma)$.) Indeed,

$$f(z)/z = 0 + \frac{1}{2\pi i} \oint \frac{f(s)}{s(s-z)} ds = -\frac{1}{2\pi i z} \oint \frac{f(s)}{s} ds + \frac{1}{2\pi i z} \oint \frac{f(s)}{s-z} ds$$
$$= -\frac{1}{2\pi i z} \oint \frac{f(s)}{s} ds + \frac{1}{z}\left(-f(\infty) + f(z)\right) \tag{90}$$

and the conclusion follows after multiplication by z.

2. Let f be analytic on $\mathcal{O} \supset K = \sigma_\infty(T)$. We take $\Delta \subset \mathcal{O}$ be an open set between $\mathcal{O}$ and K , such that its boundary consists of a finite number of nonintersecting simple Jordan curves, positively oriented with respect to infinity. We can always reduce to this case as explained at the beginning of the section.

3. Analytic functions on the spectrum are now functions analytic at infinity as well. Let f be such a function.

4. Therefore, we define for such an f and a (multi) contour Γ such that the spectrum of T lies in $Ext\, \Gamma$, or in the language of item 2 above the spectrum F is in Δ and $\Gamma = \partial\Delta$,

$$f(T) = f(\infty) + \frac{1}{2\pi i} \oint_\Gamma f(s)(s-T)^{-1} ds \tag{91}$$

Note that this is a bounded operator, by the definition of the spectrum and the properties of integration. Note also that now we don't have the luxury to define polynomials first, etc.

Note 8 (Independence of contour)**.** *Assume* Γ_1 *is a contour homotopic to* Γ_2 *in* $\mathcal{D}_1$*, inFig. 2. Then*

$$\int_{\Gamma_1} f(s)(s-T)^{-1} ds = \int_{\Gamma_2} f(s)(s-T)^{-1} ds \tag{92}$$

(As usual, this can be checked using functionals.)

If ϕ is a linear functional in X^*, since $(s-T)^{-1}$ is analytic outside the spectrum of T, Γ_i, and Γ_1 and Γ_2 are as above, we have

$$\int_{\Gamma_1} f(s)\phi(s-T)^{-1} ds = \int_{\Gamma_2} f(s)\phi(s-T)^{-1} ds \tag{93}$$

That is,

$$\phi \int_{\Gamma_1} f(s)(s-T)^{-1} ds = \phi \int_{\Gamma_2} f(s)(s-T)^{-1} ds \tag{94}$$

and therefore, the contour of integration is immaterial in the definition of $f(T)$, modulo homotopies.

Proposition 37. *(i) Assume f is as in item 3, and satisfies $f(\infty) = 0$. Then $f(T)X \in D(T)$.*

(ii) f be as above and let $g = zf(z)$. Then clearly g is analytic at infinity. We have $g(T) = Tf(T) = f(T)T$. That is, also, $(zf(z))(T) = Tf(T)$.

Recall the fundamental result of commutation of closed operators with integration, Theorem 1.

For convenience, we repeat it here: **Theorem (1, p. 13 above)** Let T be a closed operator and $a \in B(\Omega, X)$ be such that $a(\Omega) \subset \mathcal{D}(T)$. Assume further that T is measurable, in the sense that $Ta \in B(\Omega, X)$. Under the assumptions above we have

$$T \int_A a(\omega) d\mu(\omega) = \int_A T a(\omega) d\mu(\omega) \tag{95}$$

(In particular $\int_A a(\omega) d\mu(\omega) \in D(T)$ if $a(\omega) \in D(T)$ for all ω.) Check that the following satisfies all requirements w.r.t. T.

$$a(s) = (s-T)^{-1}$$

Proof. We can assume that the contour of integration does not pass through 0; in case it did, then 0 would be in the domain of analyticity of f and we can deform the contour around zero.

By definition we have

$$\begin{aligned} g(T) = g(\infty) + \frac{1}{2\pi i} \oint_\Gamma s f(s)(s-T)^{-1} ds \\ = g(\infty) + \frac{1}{2\pi i} \oint_\Gamma (s - T + T) f(s)(s-T)^{-1} ds \\ = g(\infty) + \frac{1}{2\pi i} \oint_\Gamma s^{-1} g(s) ds + \frac{1}{2\pi i} \oint_\Gamma T f(s)(s-T)^{-1} ds \\ = \frac{1}{2\pi i} \oint_\Gamma T f(s)(s-T)^{-1} ds = T \frac{1}{2\pi i} \oint_\Gamma f(s)(s-T)^{-1} ds \end{aligned} \tag{96}$$

where we used (89) and Theorem 1. □

Corollary 38. *If $\rho \in \rho(T)$ and $f(z) = 1/(\rho - z)$, then $(\rho - T)^{-1} = f(T)$, and we can again write*

$$(\rho - T)^{-1} = \frac{1}{\rho - T}$$

Proof. Evidently, by a change of T, we can assume that $\rho = 0$. Take $f(z) = 1/z$. We see that $f(\infty) = 0$, Proposition 37 applies and thus

$$1 = (zf(z))(T) = Tf(T) = f(T)T$$

□

Proposition 39. *Let* $T \in \mathcal{C}(X)$ *and* $f \in H(\sigma_\infty(T), \mathbb{C})$. *Then,* $\sigma_\infty(f(T)) = f(\sigma_\infty(T))$.

Proof. The proof is similar to the one in the bounded case. If $\rho \notin f(\sigma(T))$ then $\rho - f(z)$ is invertible on $\sigma(T)$, and let the inverse be g. Then (note that g is always a bounded operator),

$$(\rho - f(z))g(z) = g(z)(\rho - f(z)) = 1 \Rightarrow g(T)(\rho - f(T)) = (\rho - f(T))g(T) = 1$$

and thus $\rho \notin \sigma(f(T))$. Conversely, assume that $f(\sigma_0) \in f(\sigma_\infty(T))$, but that $f(T) - f(\sigma_0)$ was invertible. Without loss of generality, by shifting T and f we can assume $\sigma_0 = 0 \in \sigma_\infty(T)$ and $f(0) = 0$. Then, $f(z) = zg(z)$ with g analytic at zero (and on $\sigma_\infty(T)$)). Then $g(\infty) = 0$ and thus, since $f(T)$ was assumed invertible, we have

$$f(T) = Tg(T) = g(T)T, \text{ or} 1 = T[g(T)f^{-1}(T)] = [g(T)f^{-1}(T)]T \tag{97}$$

so that T is invertible and thus $0 \notin \sigma_\infty(T)$, a contradiction. □

12.2 *Analyticity at zero and at infinity: a discussion*

Consider an open set $\mathcal{D} \subsetneq \mathbb{C}_\infty$, including a neighborhood of infinity and let g be analytic in $\mathcal{D}$. (Functions analytic in $\mathbb{C}_\infty$ are trivial.) We then take a point in $\mathbb{C}_\infty \setminus \mathcal{D}$, say the point is zero, and make an inversion: define $f(z) = g(1/z)$ defined in $D_1 = 1/\mathcal{D} := \{1/z : z \in \mathcal{D}\}$. Then g is analytic in $\mathcal{D}_1$. If $\mathcal{D}_2 \subset \mathcal{D}_1$ is a multiply connected domain whose boundary is a finite union of disjoint, simple Jordan curves, then Cauchy's formula still applies, and we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \mathcal{D}_2} \frac{f(s)}{s - z} ds \tag{98}$$

where the orientation of the contour is as depicted. Green regions are regions of analyticity, red ones are excluded regions. The first region is an example of a relatively compact $\mathcal{D}$. Cauchy's formula (98) applies on the boundary of the domain, the integral gives $f(z)$ at all $z \in \mathcal{D}_2$. The orientation of the curves must be as depicted.

The second domain is $\mathcal{D} = \{1/z : z \in \mathcal{D}_2\}$. It is the domain of analyticity of $g(z) = f(1/z)$ and it includes ∞. Here we can apply Cauchy's formula at infinity to find g in the green region.

The orientation of the contours is the image under $z \to 1/z$ of the original orientations.

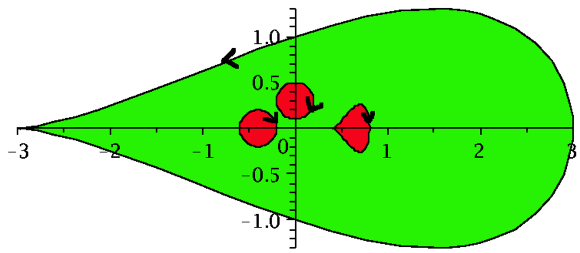


Figure 1: Analyticity near zero

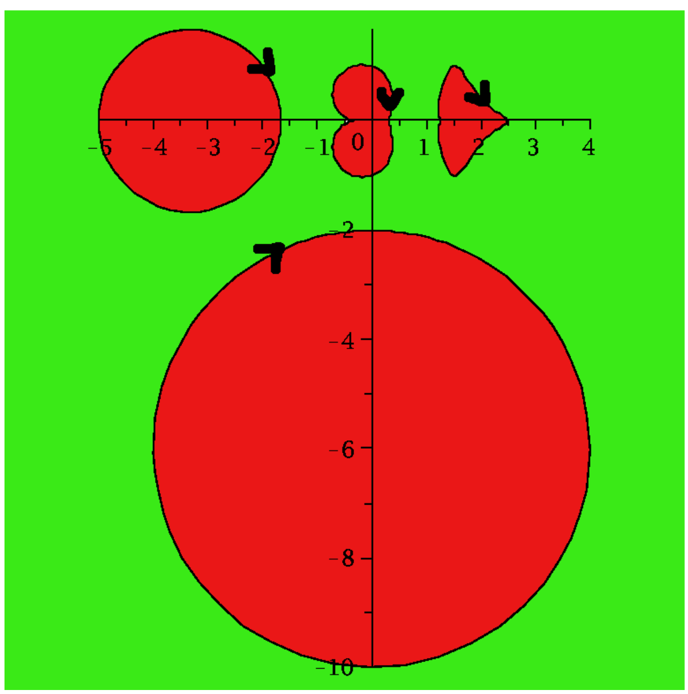


Figure 2: Analyticity near ∞

If the spectrum of an operator is contained in the green region in the second figure (infinity included, clearly) then the contours should be taken as depicted.

Assume as before that $\sigma_\infty(T) \neq \mathbb{C}_\infty$. Then there is a $z_0 \in \mathbb{C} \setminus \sigma_\infty(T)$ and we assume without loss of generality that $z_0 = 0$. We can define, as before, the set $\sigma_1 = 1/\sigma_\infty(T) = \{1/z : z \in \sigma_\infty(T)\}$.

Then a function f is analytic on $\mathcal{D} \supset \sigma_\infty(T)$ iff $f(1/z)$ is analytic on $\sigma_1(T)$. The spectrum $\sigma_\infty(T)$ is contained in $\mathcal{D}$ iff $1/\mathcal{D} \supset 1/\sigma_\infty(T)$. A curve, or set of curves, gives the value of $f(T)$ iff the curves are so chosen that $1/\sigma_\infty(T)$ is contained in the domain defined by the curves.

If f is nontrivial and analytic on $\sigma_\infty(T)$, then

Proposition 40. *The Banach algebra of analytic functions on $\mathcal{O}$ with the sup norm is isomorphic to the algebra of bounded operators $f[T]$, in the operator norm.*

Proof. Linearity, continuity etc are proved as before. Multiplicativity could also be proved by density, taking say polynomials in $1/(z - z_0)$, $z_0 \notin \sigma(T)$ as a dense set.

Alternatively, it could be proved directly from the definition (91) by Fubini.

In any case, the analysis is rather straightforward and we leave all details to the reader. $\square$

Finally,

Proposition 41. *Assume $\sigma_\infty(T) = K_1 + K_2$ (disjoint union) where K_1 is compact in $\mathbb{C}$. Let $\mathcal{O} \supset K_1$ be open, relatively compact and disjoint from K_2, and (w.l.o.g.) with rectifiable boundary Γ. Then*

$$P_{K_1} = \frac{1}{2\pi i} \oint_\Gamma \frac{1}{z - T} dz \tag{99}$$

defines a projector such that

(i) $P_{K_1}(X) \subset D(T)$, $T(P_{K_1}(X)) \subset P_{K_1}(X)$.

(ii) $\sigma(T|_{P_{K_1}(X)}) = K_1$.

(iii) T restricted to $P_{K_1}(X)$ is bounded.

Proof. Let $P = P_{K_1}, \chi = \chi_{K_1}$, First, we note that $P = \chi(T)$ and χ is analytic on $\sigma_\infty(T)$, $\chi(\infty) = 0$ and thus, by Proposition 37 (i), PX is in $D(T)$. $P^2 = P$ since $\chi^2 = \chi$. Thus P is a projector, as in §11. By Prop. 37 (ii) we have $TP = (z\chi)(T) = (\chi z)(T) = PT$. Thus $TP = TP^2 = PTP$, and $TPX \subset PX$, and thus $TP|_{PX} \subset PX$. We further have $\sigma_\infty(TP) = \chi(\sigma_{\infty(T)}) = K_1$. Since $\infty \notin \sigma_\infty(TP)$, TP is bounded.

$\square$

12.3 Bounded self-adjoint and normal operators on a Hilbert space $\mathcal{H}$

Note 9. Assume A is bounded on the Hilbert space $\mathcal{H}$ and self-adjoint, that is $\langle Ax, y\rangle = \langle x, Ay\rangle$ $\forall x, y \in \mathcal{H}$ (this condition is not enough if A is unbounded). Then $\sigma(A) = K \subset \mathbb{R}$.

Indeed, if $z \in \sigma(A)$ then $A - z$ is either not injective or not surjective. If it is not injective, then $(A-z)x = 0$ for some x, which implies $\langle Ax, x\rangle = \langle zx, x\rangle = z\langle x, x\rangle = \langle x, Ax\rangle = \langle x, zx\rangle = \overline{z}\langle x, x\rangle$, thus $(z - \overline{z})\|x\| = 0$, or, $z \in \mathbb{R}$. If the range of $A - z$ is not dense in $\mathcal{H}$, then there is a vector y such that $\langle (A-z)x, y\rangle = 0 = \langle x, (A-\overline{z})y\rangle$ for all $x \in \mathcal{H}$, that is $(A-\overline{z})y = 0$ which, as above, implies $z \in \mathbb{R}$. We are left with $\mathrm{Ran}(A-z)$ dense in $\mathcal{H}$ but $\neq \mathcal{H}$. Then, the densely defined $(A-z)^{-1}$ is not bounded, or, which is the same , there is a sequence u_n, $\|u_n\| = 1$ and $\|(A-z)u_n\| \to 0$. In particular

$$\mathrm{Im}(\langle (A-z)u_n, u_n\rangle) = \mathrm{Im}(\langle Au_n, u_n\rangle - \langle zu_n, u_n\rangle) = (\mathrm{Im}z)\langle u_n, u_n\rangle \to 0$$

which is not possible unless $\mathrm{Im}z = 0$.

For a bounded operator to be self-adjoint, it suffices that it is symmetric, that is $\langle Ax, y\rangle = \langle x, Ay\rangle$. Note that if A is self-adjoint, then $\|A^2\| = \|A\|^2$. Indeed, with $\|u\| = 1$, we have

$$\sup_{\|u\|=1} \|A^2\| \geq \sup_{\|u\|=1} \|\langle A^2u, u\rangle\| = \sup_{\|u\|=1} \|\langle Au, Au\rangle\| = \sup_{\|u\|=1} \|Au\|^2 = \|A\|^2 \tag{100}$$

From our excursion in Banach algebras we know that $\|A^2\| \leq \|A\|^2$; thus $\|A^2\| = \|A\|^2$.

Exercise 1. ** Show that this holds for all powers of A, that is $\|A^n\| = \|A\|^n$.*

The conjugate f^* of a function which is analytic in a region containing an interval of $J \subset \mathbb{R}$ is given by $f^*(z) = \sum_k \overline{c_k}(z - z_0)^k$, where $z, z_0 \in J$.

Exercise 2. ** Show that if A is self-adjoint, then $f^*(A) = (f(A))^*$ where $(f(A))^*$ denotes the (Hilbert-space) adjoint of A.*

Exercise 3. *Show that the analysis above shows that $\|f(A)\| = \sup_{\lambda\in\sigma(N)} |f(\lambda)| =: \sup |f(\sigma(A)|$. In particular $\|A\| = \sup |\sigma(A)| = R(A)$.*

Exercise 4. *More generally, if N is a bounded* normal *operator (i.e., $NN^* = N^*N$) and f is as above, then*

$$\|f(N)\| = \sup_{\lambda\in\sigma(N)} |f(\lambda)| \tag{101}$$

13 Unbounded operators: adjoints, self-adjoint operators etc.

In this section we work with operators in Hilbert spaces.

1. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces (we will most often be interested in the case $\mathcal{H} = \mathcal{K}$), with scalar products $\langle , \rangle_{\mathcal{H}}$ and $\langle , \rangle_{\mathcal{K}}$.

2. $T : D(T) \subset \mathcal{H} \to \mathcal{K}$ is **densely defined** if $\overline{D(T)} = \mathcal{H}$.

3. Assume T is densely defined.

4. The adjoint of T is defined as follows. We look for those y for which

$$\exists\ v = v(y) \in \mathcal{H} \text{ s.t.} \forall\ x \in D(T),\ \langle y, Tx \rangle_{\mathcal{K}} = \langle v, x \rangle_{\mathcal{H}} \tag{102}$$

Since $D(T)$ is dense, such a $v = v(y)$ is unique.

5. We define $D(T^*)$ to be the set of y for which $v(x)$ as in (102) exists, and define $T^*(y) = v$. Note that $Tx \in \mathcal{K}$, $y \in \mathcal{K}$, $T^*y \in \mathcal{H}$.

6. Check that $D(T^*)$ is a linear space and $T^*(a_1 y_1 + a_2 y_2) = a_1 T^*(y_1) + a_2 T^*(y_2)$ that is, T^* is linear.

7.

Exercise 1. *Show that $y \in D(T^*)$* **iff** *$\langle y, Tx \rangle_{\mathcal{K}}$ extends to a bounded (linear) functional on $\mathcal{H}$. Does this mean that Tx exists for all x?*

Exercise 2. *Define the operator $T(P(x)) = P(x+1)$ where $\mathcal{H} = L^2[0,1]$ and $D(T)$ consists of the polynomials on $[0,1]$. What is $D(T^*)$?*

8. **Definition.** We write $T_1 \subset T_2$ iff $\Gamma(T_1) \subset \Gamma(T_2)$.

Exercise 3. *Show that $T_1 \subset T_2 \Rightarrow T_2^* \subset T_1^*$. (A short proof is given in Corollary 46.)*

9.

Proposition 42. *In the setting of 1 and 3, T^* is closed.*

Proof. Let $y_n \to y$ and $T^* y_n \to v$. Then, for any $x \in D(T)$ we have

$$\langle y_n, Tx \rangle =: \langle T^* y_n, x \rangle \to \langle v, x \rangle = \lim \langle y_n, Tx \rangle = \langle y, Tx \rangle$$

Thus, for any $x \in D(T)$ we have $\langle y, Tx \rangle = \langle v, x \rangle$ and thus, by definition, $y \in D(T^*)$ and $T^* y = v$. □

10.

Proposition 43. *Let T and $D(T)$ be as in 1 and 3. Then,*

(i) For any $\alpha \in \mathbb{C}$, we have $(\alpha T)^* = \overline{\alpha} T^*$.

(ii) If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, then T^* is the usual adjoint, everywhere defined, and for two such operators, we have $(T_1 + T_2)^* = T_1^* + T_2^*$.

Proof. Exercise. □

Proposition 44. *(i) Let T_i and $D(T_i)$ be as in 1 and 3. Assume furthermore that $D(T_1) \cap D(T_2)$ is dense. Then $(T_1+T_2)^* \supset T_1^* + T_2^*$ (this means that the domain is larger, and wherever all adjoints make sense, we have $(T_1+T_2)^* = T_1^* + T_2^*$).*

(ii) Let $T_1 : \mathcal{H} \to \mathcal{K}_1$, $T_2 : \mathcal{H} \to \mathcal{K}_2$ and assume $D(T_2)$ as well as $D(T_2T_1)$ are dense. Then, $(T_2T_1)^* \supset T_1^* T_2^*$.

Proof. We define $T_1 + T_2$ on $D(T_1) \cap D(T_2)$. Let $x \in D(T_1) \cap D(T_2)$ and assume that $y \in D(T_1)^* \cap D(T_2)^*$. This is, by definition, the domain of $T_1^* + T_2^*$. Then

$$\begin{aligned}\langle y, (T_1+T_2)x\rangle &= \langle y, T_1x + T_2x\rangle = \langle y, T_1x\rangle + \langle y, T_2x\rangle \\ &= \langle T_1^* y, x\rangle + \langle T_2^* y, x\rangle = \langle T_1^* y + T_2^* y, x\rangle =: \langle (T_1+T_2)^* y, x\rangle \end{aligned} \quad (103)$$

and thus $y \in D(T_1+T_2)^*$, and $(T_1+T_2)^* = T_1^* + T_2^*$.

(ii) Let $x \in D(T_2T_1)$ (note that, by definition, $D(T_2T_1) \subset D(T_1)$) and $w \in D(T_1^*T_2^*)$. Then,

$$\langle T_1^*(T_2^* w), x\rangle = \langle (T_2^* w), T_1 x\rangle = \langle w, T_2T_1x\rangle \quad (104)$$

and thus $w \in D(T_2T_1)^*$ etc. □

13.1 Example: An adjoint of d/dx

1. Let $T = d/dx$ be defined on $C^1[0,1]$. We need to see for which y do we have $\langle y, Tf\rangle = \langle w, f\rangle \ \forall f \in D(T)$, that is, $\forall f \in D(T)$

$$\int_0^1 y(s)\frac{df(s)}{ds}ds = \int_0^1 w(s)f(s)ds$$

Let $h(s) = \int_0^s w(t)dt$. Then, $h \in AC[0,1]$ and we can integrate by parts

$$\int_0^1 y(s)\frac{df(s)}{ds}ds = \int_0^1 \frac{dh(s)}{ds}f(s)ds = f(1)h(1) - \int_0^1 h(s)\frac{df(s)}{ds}ds \quad (105)$$

hence

$$\int_0^1 (y(s)+h(s))\frac{df(s)}{ds}ds = f(1)h(1) \quad (106)$$

Let $v \in C[0,1]$ be arbitrary and take $f = \int_1^x v$. Then $f \in C^1[0,1]$, $f' = v$ and $f(1) = 0$. Then,

$$\int_0^1 (y(s)+h(s))\, v(s)ds = 0 \quad (107)$$

Note that the set of such v is dense. We can also set, for each $v \in C[0,1]$ $f = \int_1^x v + 1$. As before, $f \in C^1[0,1]$, $f' = v$ but now $f(1) = 1$. Thus, also on a dense set,

$$\int_0^1 (y(s) + h(s))\, v(s)ds = h(1) \tag{108}$$

Thus $h(1) = 0$. Also, by density, say from (107), $y(s) + h(s) = 0$ Thus, $D(T^*) = AC[0,1]_{01}$ where the subscript 01 indicates that the function vanishes at both ends. Clearly, $(d/dx)^* = -d/dx$.

13.2 Symmetric does not mean self-adjoint

One property we certainly want to preserve is that $\sigma(A) \subset \mathbb{R}$ for a self-adjoint operator. Note that $Ti = d/dx$ defined on $\{f \in C^1[0,\infty]_0 \cap L^2(\mathbb{R}^+) : f' \in L^2(\mathbb{R}^+)\}$, the subscript 0 indicating that the functions vanish at zero, is symmetric on its domain, just by integration by parts. But note that, for $\lambda > 0$, $i\lambda \in \sigma(T)$, since $f' - \lambda f = g$, $f(0) = 0$ has the unique solution $f(x) = e^{\lambda x} \int_0^x e^{-\lambda s} g(s)ds$, which is not everywhere defined (apply it to $e^{-\lambda s}$. So $i\lambda \in \sigma(T)$, and in fact, all λ with $\operatorname{Im} \lambda > 0$ are in the spectrum.

13.3 Operators on graphs and graph properties

From now on, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Recall that $\mathcal{H} \otimes \mathcal{K}$ is a Hilbert space under the scalar product

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle \tag{109}$$

1. Recall that the graph of an operator T is the set of pairs

   $$\Gamma(T) = \{(x, Tx) : x \in D(T)\} \subset \mathcal{H} \otimes \mathcal{K}$$

   Remember also that an operator T is closed iff $\Gamma(T)$ is a closed set, and it is closable iff the closure of $\Gamma(T)$ is the graph of an operator.

   Exercise 4. *If T is closable, then* $\Gamma(\overline{T}) = \overline{\Gamma(T)}$

13.3.1 A formula for the adjoint

1. Let us note something simple but very important. Assume $y \in D(T^*)$. Then

   $$\langle w, x \rangle_{\mathcal{H}} - \langle y, Tx \rangle_{\mathcal{K}} = 0 \Rightarrow (-T^*y, y) = (-w, y) \perp (x, Tx) = V(y, T^*y) \tag{110}$$

   where $V : \mathcal{K} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ is defined by

   $$V\ (x, y) = (-y, x) \tag{111}$$

Exercise 5. *Compute the adjoint of V (in $\mathcal{K}\otimes\mathcal{H}$) and show that $VV^* = I$, that is, V is unitary. Then $V(E^\perp) = (V(E))^\perp$ for any subspace of $\mathcal{K}\otimes\mathcal{H}$. Clearly also, $V^2 = -1$.*

Thus there is a simple link between $\Gamma(T)$ and $\Gamma(T^*)$:

Lemma 45 (A unitary operator, and action on graphs)**.** *We have*

$$\Gamma(T^*) = V[\Gamma(T)]^\perp$$

Proof. Of course, $(y, w) \perp \Gamma(T)$ **iff** $(y, w) \perp (x, Tx)\ \forall x$, that is

$$\forall x \in D(T): \quad (y, w) \perp (x, Tx) \Leftrightarrow \langle y, x\rangle = \langle -w, Tx\rangle$$

that is, by definition, **iff** $w \in D(T^*)$ and $w = -T^*y$. That is,

$$\Gamma(T)^\perp = \{(-T^*w, w) : w \in D(T^*)\}$$

and

$$V(\Gamma(T))^\perp = \{(w, T^*w) : w \in D(T^*)\} = \Gamma(T^*) \tag{112}$$

□

Corollary 46. $T_1 \subset T_2 \Rightarrow T_2^* \subset T_1^*$.

2.

Theorem 5. *Let T be densely defined. Then,*

(i) T^* is closed.

(ii) T is closable **iff** *$D(T^*)$ is dense, and then $\overline{T} = T^{**}$.*

(iii) If T is closable, then the adjoints of T and of $\overline{T}$ coincide.

Proof. (i) By (112), T^* is closed (since its graph is closed).

(ii) Assume $D(T^*)$ is dense. Then T^{**} is well defined. By (112) its graph is given by

$$(V\Gamma(T^*))^\perp = (V(V(\Gamma(T)))^\perp)^\perp = (V^2\Gamma(T)^\perp)^\perp = (\Gamma(T)^\perp)^\perp = \overline{\Gamma(T)} \tag{113}$$

and thus $\overline{\Gamma(T)}$ is the graph of an operator. Since T^* is densely defined, T^{**} exists, and by Lemma 45 the left side of (113) is $\Gamma(T^{**})$. Conversely, assume $D(T^*)$ is not dense. Let $y \in D(T^*)^\perp$ ($y \neq 0$); then $(y, 0)$ is orthogonal on all vectors in $\Gamma(T^*)$, so that $(y, 0) \in (\Gamma(T^*))^\perp$. Then, $V(y, 0) = (0, y) \in V(\Gamma(T^*))^\perp = \overline{\Gamma(T)}$ by (113), and thus $\overline{\Gamma(T)}$ is not the graph of an operator.

(iii) We have by definition $\Gamma(\overline{T}) = \overline{\Gamma(T)}$ and thus

$$\Gamma(\overline{T}^*) = V[(\overline{\Gamma(T)})^\perp] = V(\Gamma(T)^\perp) = \Gamma(T^*) \tag{114}$$

since $E^\perp = \overline{E}^\perp$ for any E.

□

13.4 Self-adjoint operators

1. Assume now that $\mathcal{K} = \mathcal{H}$.

2. **Definition: Symmetric (or Hermitian) operators.** Let T be densely defined on Hilbert space. T is symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $\{x, y\} \subset D(T)$. This means that $T \subset T^*$.

3. A symmetric operator is always closable, by Theorem 5 since T^* is densely defined, since $T^* \supset T$, by 2.

4. If T is closed, then $\Gamma(T) = \Gamma(\overline{T})$ and thus by Theorem 5,
$$T = T^{**} \tag{115}$$

5. For symmetric operators, T is closable (since $D(T^*)$ is dense) and then by Theorem5 (ii) we have
$$\overline{T} = T^{**}$$

6. Note that, if T is symmetric, then $\langle Tx, x\rangle \in \mathbb{R}$, since
$$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$
and also, for any $x \in D(T)$, $\|(T \pm i)x\|^2 = \|Tx\|^2 + \|x\|^2$, since
$$\|(T \pm i)x\|^2 = \langle Tx, Tx\rangle + \langle x, x\rangle \pm \langle Tx, ix\rangle \pm \langle ix, Tx\rangle = \langle Tx, Tx\rangle + \langle x, x\rangle \tag{116}$$

7.

Lemma 47. *If T is* **symmetric and** *(assume already)* **closed***, then (i)* $\mathrm{Ran}(T \pm i)$ *is closed and (ii) $(T \pm i)$ are injective.*

Proof. (ii) if $Tx = \pm ix$, then
$$\langle x, Tx\rangle = \langle Tx, x\rangle = \langle \pm ix, x\rangle = \langle x, \pm ix\rangle = - \pm i\langle x, x\rangle = 0 \tag{117}$$

(i) Let now $(T \pm i)x_n$ be a convergent sequence, or, which is the same, a Cauchy sequence. But then, by (116) Tx_n and x_n are both Cauchy sequences. Then x_n converges to x; since T is closed, $x \in D(T)$, and $Tx_n \to Tx$.

Exercise 6. *Note that $\|(T \pm i)x\|$ and the norm on the (closed) graph $\Gamma(T)$ coincide and that the space $(0, Ran(T \pm i))$ is an orthogonal projection ($P = P^*$) of $\Gamma(T \pm i)$. Provide an alternative proof of (i) based on this.*

Definition: Self-adjoint operators. T is self-adjoint if $T = T^*$. It means $D(T) = D(T^*)$, and T is symmetric.

□

8. It follows that **self-adjoint operators are closed.**

9. For self-adjoint operators we have from 4 that $T = T^* = T^{**}$.

   Lemma 48. *If T is* **closed and symmetric** *and T^** **is symmetric**. *then T is self-adjoint.*

   Proof. By 2, $T \subset T^*$. Since T^* is symmetric, $T^* \subset T^{**}$ But $T^{**} = T$ since T is closed, thus $T^* \subset T$, hence $T^* = T$. □

13.5 Symmetric versus self-adjoint: an example

Define

$$T_0 = id/dx \text{ on } D(T_0) = AC[0,1]_{01} := \{f \in AC[0,1] : f(0) = f(1) = 0\} \quad (118)$$

If $\{x, y\} \subset D(T_0)$, then a simple integration by parts shows that $\langle y, T_0 x\rangle = \langle T_0 y, x\rangle$, and thus $T_0 \subset T_0^*$.

The range of T_0 consists of the functions of average zero, since $\int_0^1 Ty = y|_0^1 = 0$ and conversely $y = g'$ has a solution g if $\int_0^1 y = 0$. Another way to phrase it is

$$\text{Ran}T_0 = \mathbb{C}^\perp \quad (119)$$

Note 10. *Similarly,* $\text{Ran}(T_0 \pm i) = (e^{\mp ix}\mathbb{C})^\perp$.

What is T_0^*?

Lemma 49. *We have*

$$T_0^* = id/dx \text{ on } D(T_0^*) = AC[0,1] \quad (120)$$

Proof. (First proof) Let τ be the operator in §13.1; we showed there that $\tau^* = T_0$. Thus T_0 is closed. We also proved before that τ is closed. (See also the interesting discussions in the chapter "The Fundamental Theorem of Calculus" in Rudin, Real and Complex analysis.) Then $\tau = \tau^{**} = T_0^*$.

□

Proof. (Direct proof)

Assume that $y \in D(T_0^*)$. Then, there exists a $v \in L^2[0,1]$ such that for all $x \in D(T_0)$, we have

$$\langle y, T_0 x\rangle = \langle v, x\rangle \quad (121)$$

Since $v \in L^2[0,1]$, we have $v \in L^1[0,1]$, and thus $h(x) = \int_0^x v(s)ds \in AC[0,1]$, and $h(0) = 0$. With $u = \overline{x}$, it follows from (121) by integration by parts that

$$\int_0^1 y(s)\frac{du(s)}{ds} = -\int_0^1 \frac{dh(s)}{ds}u(s)ds = \int_0^1 h(s)\frac{du(s)}{ds}$$

Let $\phi = y - h$. By (119), we have

$$\int_0^1 \phi(s)w(s)ds = 0 \quad \forall w \in \mathbb{C}^\perp$$

This means that $\phi \in \mathbb{C}^{\perp\perp} = \mathbb{C}$. Thus y is an element of $AC[0,1]_0$ plus an arbitrary constant, which simply means $y \in AC[0,1] = D(T_0^*)$. □

By essentially the same argument as in §13.1, we have $T_0^{**} = id/dx$ on $D(T_0)$. Thus T_0 is closed. We note that T_0^* is not symmetric. In some sense, T_0 is too small, and then T_0^* is too large.

Definition: Normal operators. The definition of an unbounded normal operator is essentially the same as in the bounded case, namely, operators which commute with their adjoints.

More precisely: an operator T which is **closed and densely defined** is normal if $TT^* = T^*T$. The questions of domain certainly become very important. The domain if T^*T is $\{x \in D(T) : Tx \in D(T^*)\}$. (Similarly for TT^*.) Thus, the operator T in (118) is not normal. Indeed, $D(T^*T)$ consists of functions in $AC[0,1]$ vanishing at the endpoints, with derivative in $AC[0,1]$, while $D(TT^*)$ consists of functions in $AC[0,1]$, with the derivative, in $AC[0,1]$, vanishing at the endpoints.

Definition: Essentially self-adjoint operators. A **symmetric** operator is essentially self-adjoint if its closure is self-adjoint. "Conversely", if T is closed a **core** for T is a set $D_1 \subset D(T)$ such that $\overline{T|_{D_1}} = T$.

Note 11. We will show that T^*T and TT^* are selfadjoint, for any **closed and densely defined** T. If an operator T is normal, then in particular $D = D(T^*T) = D(TT^*)$, and since D is dense and T is closed, we see that D is a core for both T and T^*.

Lemma 50. *(i) If T is essentially self-adjoint, then there is a unique self-adjoint extension, $\overline{T}$. (Phrased differently, if $S \supset T$ is self-adjoint, then $S = T^{**}$.)*

(ii) (Not proved now). Conversely, if T has only one self-adjoint extension, then it is essentially self-adjoint.

Proof. (i) By 9, T is closable and $\overline{T} = T^{**}$. But, by assumption, $\overline{T}$ is self-adjoint. Thus

$$T^{***} = T^{**}$$

Let $S \supset T$ be self-adjoint. Then, firstly, S is closed, by 8. Then, $T^* \supset S^* = S$ and $T^{**} \subset S$, while $T^{***} = T^{**} \supset S$. Therefore, $T^{**} = S$. □

Corollary 51. *A self-adjoint operator is* **uniquely specified by giving it on a core.**

Exercise 7. ** Consider the operator T in §13.2. Is T closed? What is the adjoint of T? What is the range of $T \pm i$?*

1. A symmetric operator is essentially self-adjoint if and only if its adjoint is self-adjoint.

   Indeed, assume T is e.s.a. and symmetric. Since T^{**} is the closure of T, then $T \subset T^{**}$. But this means $T^* \supset T^{***} = T^{**}$ since T^{**} is self-adjoint, and thus $T^* = T^{**}$ is self-adjoint.

   Conversely, if T is symmetric and T^* is self-adjoint, then $T^{**} = T^*$ is also self-adjoint, and thus T is e.s.a.

Lemma 52. *If T is self-adjoint, then* $\operatorname{Ker}(T \pm i) = \{0\}$.

Theorem 6 (Basic criterion of self-adjointness)**.** *Assume that T is* **symmetric**.

Then the following three statements are equivalent:

(a) *T is self-adjoint*

(b) *T is closed and* $\operatorname{Ker}(T^* \pm i) = \{0\}$.

(c) $\operatorname{Ran}(T \pm i) = \mathcal{H}$. *(Since T is closed, this is equivalent to $\pm i \notin \sigma(T)$.)*

(d) $\sigma(T) \subset \mathbb{R}$

Of course (we can take) $D(T \pm i) = D(T)$. Remember the operator T in §13.2: it follows from (c) that T is not s.a.

Proof. (a)$\Rightarrow$ (b) is simply Lemma 52.

(b)$\Rightarrow$ (c). By Lemma 47 it is enough to show that $\operatorname{Ran}(T \pm i)$ are dense.

Assume $y \in \operatorname{Ran}(T + i)^{\perp}$. Then, $\langle y, (T + i)x\rangle = 0$ for all $x \in D(T)$. In particular, there is a $v(= 0)$ such that $\langle y, (T + i)x\rangle = \langle v, x\rangle$, and thus $y \in D(T^*)$ and $(T^* - i)y = v = 0$, thus $y \in \operatorname{Ker}(T^* - i) = \{0\}$. The other sign is treated similarly.

(c) $\Rightarrow$ (a) Let $y \in D(T^*)$. We want to show that $y \in D(T)$. By definition, there is a v such that for any $x \in D(T)$ we have

$$\langle y, (T + i)x\rangle = \langle v, x\rangle$$

Since $\operatorname{Ran}(T - i) = \mathcal{H}$, we have

$$v = (T - i)s$$

for some $s \in D(T)$. Thus, since $x \in D(T)$ and since T is symmetric, we have

$$\begin{aligned}\langle y, (T + i)x\rangle &= \langle v, x\rangle = \langle (T - i)s, x\rangle \\ &= \langle Ts, x\rangle - i\langle s, x\rangle = \langle s, Tx\rangle + \langle s, ix\rangle = \langle s, (T + i)x\rangle \end{aligned} \tag{122}$$

Now we use the fact that $\operatorname{Ran}(T + i) = \mathcal{H}$ to conclude that $s = y$. But s was in $D(T)$ and the proof is complete. $\square$

Corollary 53. *Let T be symmetric. Then, the following are equivalent:*

1. *T is essentially self-adjoint.*

2. $\operatorname{Ker}(T^* \pm i) = \{0\}$.

   Note 12. If $T_1 \subset T_2$ and T_i are self-adjoint, then $T_1 = T_2$. Indeed, $T_1 \subset T_2 \Rightarrow T_1 = T_1^* \supset T_2^* = T_2$.

3. $\operatorname{Ran}(T \pm i)$ *are dense.*

Corollary 54. *Let A be a self-adjoint operator. Then $\sigma(A) \subset \mathbb{R}$.*

Proof. Let $\lambda = a + ib, b \neq 0$. Then $A - a = A'$ and $A'' = A'/b$ are is self-adjoint as well. Then $\operatorname{Ker}(A'' + i) = \{0\}$ and $\operatorname{Ran}(A'' + i) = \mathcal{H}$. ☐

Note 13. Let again $T = id/dx$ on $AC[0,1]_{01}$. The range of T (see (119)) and Corollary 53 show that T is symmetric (thus closed since $D(T)$ is dense), but **not** essentially self-adjoint. Also, we note that $\operatorname{Ker}(T^* \pm i) \neq \{0\}$. Indeed,

$$i\frac{d\phi}{dx} \pm i\phi = 0 \Rightarrow \phi = Ce^{\mp x}$$

13.5.1 All self-adjoint extensions of id/dx on $[0,1]$

See once more the discussions in the chapter "The Fundamental Theorem of Calculus" in Rudin, Real and Complex analysis. The natural, "maximal" domain of d/dx in which the fundamental theorem of calculus holds w.r.t. Lebesgue integration is $AC[0,1]$.

Lemma 55. *If $T = id/dx$ defined on a dense domain in $AC[0,1]$ then T is selfadjoint* **iff** *$D(T) = AC[0,1]_\phi$ where $AC[0,1]_\phi = \{f \in AC[0,1] : f(0) = e^{i\phi} f(1)\}$.*

Since $\langle f, Tg\rangle = \langle Tf, g\rangle$ then $f(0)\overline{g(0)} = f(1)\overline{g(1)}$. In particular this holds for $f = g$ and thus $|f(0)|^2 = |f(1)|^2$. If there is no $f \in D(T)$ so that $|f(0)|^2 + |f(1)|^2 > 0$, then we are in the case $D(T) \subset AC[0,1]_{01}$. $D(T^*) \supset\subset AC[0,1]$ This case has been settled: T is symmetric, but not self-adjoint. Otherwise $f(0) = e^{i\phi_f} f(1)$, $\phi_f \in \mathbb{R}$. Now taking f and g so that $f(0) \neq 0$ and $g(0) \neq 0$, we have $f(0)\overline{g(0)} = e^{i\phi_f - \phi_g} f(1)\overline{g(1)} = f(1)\overline{g(1)}$ meaning that ϕ does not depend on f. We let $AC[0,1]_\phi = \{f : f(0) = e^{i\phi} f(1)\}$. T is clearly symmetric as well as densely defined. To see whether it is self-adjoint we check $\operatorname{Ran}(T \pm i)$.

$$if' \pm if = h \Rightarrow if(x) = e^{\mp x} \int_0^x = e^{\pm s} h(s)ds + Ce^{\mp x}$$

and we have

$$f(0) = C = f(1)e^{-i\phi} = e^{\mp 1 - i\phi} \left(C + \int_0^1 e^{\mp s} h(s)ds \right)$$

This can be always solved for C. Thus $\mathrm{Ran}(T \pm i) = \mathcal{H}$ implying that T is self-adjoint. This shows that $T_\phi := id/dx$ with $D(T) = AC[0,1]_\phi$ are all the self-adjoint extensions of id/dx on $[0,1]$.

T_0 is called the minimal operator associated to id/dx, T_0^* is the maximal operator. We see that T_0^* has as spectrum $\mathbb{C}_\infty$ (since e^{as} is an eigenvalue for any a); T_0^* is already a bit "too large").

13.6 Further remarks on closed operators

Proposition 56. *Let T be closed, densely defined and bounded below (that is, $\inf_{\|x\|=1} \|Tx\| = a > 0$). Then $\mathcal{H}_T = \mathrm{Ran}(T) = \mathrm{Ker}(T^*)^\perp$ is closed, and $T : D(T) \mapsto \mathcal{H}_T$ is invertible.*

Note that under these assumptions, $\mathcal{H}_T$ can still be a nontrivial closed subspace of $\mathcal{H}$. An example is provided by $T = (x_1, x_2, ...) \mapsto (0, x_1, x_2, ...)$.

Proof. If Tx_n is Cauchy, then x_n is Cauchy, by the lower bound, $x_n \to x$ and thus, since T is closed, $Tx_n \to Tx$. Clearly, T is injective, and also surjective. Then T^{-1} is well defined, and bounded by $1/a$. The rest follows from Proposition 56.

□

14 Extensions of symmetric operators; Cayley transforms

Overview: The Cayley transform of a self-adjoint operator, $U = \frac{T-i}{T+i}$ is unitary (by functional calculus). Conversely, we will show that unitary operators such that $1-U$ is injective generate self-adjoint operators. Thus self-adjoint operators can be fully analyzed if we simply understand unitary operators! Of course, more careful proofs are needed, and they follow below.

Partial isometries are used to measure self-adjointness, find whether self-adjoint extensions exist, and find these extensions.

1. Let $H^\pm$ be closed subspaces of a Hilbert space $\mathcal{H}$.

2. **Definitions**

   (a) A unitary transformation U from H^+ to H^- is called a **partial isometry**.

   (b) The dimensions of the spaces $(H^\pm)^\perp$, finite or infinite, are called deficiency indices.

3.

   Lemma 57. *Let U_p by a partial isometry. Then U_p extends to a unitary operator on $\mathcal{H}$ iff the deficiency indices of U_p are equal.*

Proof. This is straightforward: assume the deficiency indices coincide. Two separable Hilbert spaces are isomorphic **iff** they have the same dimension. Let $U_\perp$ be any unitary between $(H^\pm)^\perp$. Let $U = U_p$. We write $\mathcal{H} = H^+ \oplus (H^+)^\perp$ and $\mathcal{H} = H^- \oplus (H^-)^\perp$ and define $U = U_p \oplus U_\perp$. Clearly this is a unitary operator.

Conversely, let U is a unitary operator that extends U_p. Then $U(H^{+\perp}) = (U(H^+))^\perp = (U_p(H^+))^\perp = H^{-\perp}$, and in particular $H^{+\perp}$ and $H^{-\perp}$ have the same dimension.

□

4. In the steps below, T **is closed and symmetric.**

   Proposition 58. *Assume T is as in 4. Then (i)* $\mathrm{Ker}(T^*) = \mathrm{Ran}(T)^\perp$ *and (ii)* $\mathrm{Ker}(T)^\perp = \overline{\mathrm{Ran}(T^*)}$.

   Proof. The proof is straightforward; it consists of interpreting the equality $\langle T^* y, x\rangle = 0 = \langle y, Tx\rangle$ where y spans $\mathrm{Ker}(T^*)$ and x spans $D(T)$, or x spans $\mathrm{Ker}(T)$ and y spans $D(T^*)$. For instance

   $$y \in \mathrm{Ker}(T^*) \ iff \ (y \in D(T^*) \text{ and } T^* y = 0) \\ \Leftrightarrow \langle T^* y, x\rangle = 0 = \langle y, Tx\rangle \ \forall x \in D(T) \quad (123)$$

   □

5. By Lemma 47 $H^\pm = \mathrm{Ran}(T \pm i)$ are closed, and $T \pm i$ are one-to-one onto from $D(T)$ to $H^\pm$.

   Lemma 59. *The operator $(T - i)(T + i)^{-1}$ is well defined and a partial isometry between* $\mathrm{Ran}(T + i)$ *and* $\mathrm{Ran}(T - i)$.

   Note 14. *The partial isometry $(T - i)(T + i)^{-1}$ is known as* **the Cayley transform** *of T.*

   By 5, $U_p = (T - i)(T + i)^{-1}$ is well defined on H^+ with values in H^-, and also, for any $u^- \in H^-$ there is a unique $f \in D(T)$ such that $(T - i)f = u^-$. With $u^+ \in H^+$, we have $U_p u^+ = u^-$. Since $\|u_-\| = \|(T - i)f\|$ and $(T + i)^{-1} u^+ = f$, that is $u^+ = (T + i)f$, we have $\|u_\pm\|^2 = \|f\|^2 + \|Tf\|^2$.

   Lemma 60. *$x \in D(T)$* **iff** *$x = y - U_p y$ for some $y \in H^+$, i.e., $D(T) =$* $\mathrm{Ran}(1 - U_p)$.

   Proof. Let $y \in H^+$ and $(T + i)^{-1} y = f$. Evidently, $f \in D(T + i) = D(T)$ (by the definition of $T + i$). Then $(T + i)f = y$ and $(T - i)(T + i)^{-1} y = (T - i)f$, which means that $x = (T + i)f - (T - i)f = 2if \in D(T)$. Conversely, if $f \in D(T)$, then $2if = (T + i)f - (T - i)f = y - (T - i)(T + i)^{-1} y$. □

14.1 Duality between self-adjoint operators and unitary ones

Lemma 61. *(i) If U is unitary and* $\mathrm{Ker}(1-U)=\{0\}$ [5]*, then there exists a self-adjoint T such that $U=(T-i)(T+i)^{-1}$.*

(ii) Conversely, if T is symmetric, then T is self-adjoint **iff** *its Cayley transform is unitary on $\mathcal{H}$.*

Proof. 1. We start with (ii). Since T is s.a., $T\pm i$ are invertible, and in particular $\mathrm{Ran}(T\pm i)=\mathcal{H}$, and $\sigma(T)\subset\mathbb{R}$. The map $(t-i)/(t+i)=u$ is conformal between the upper half plane and the unit disk, and in particular it maps the real line bijectively on the unit circle. Its inverse is $t=i(1+u)/(1-u)$. We could proceed by functional calculus, which we leave as an exercise. Direct proof: By the above, $(T-i)(T+i)^{-1}\mathcal{H}\to\mathcal{H}$, since $D(T-i)^{-1}=\mathrm{Ran}(T+i)=\mathcal{H}$. We have already shown that $(T-i)(T+i)^{-1}$ is an isometry wherever defined, thus in this case it is unitary. Conversely, $U=(T-i)(T+i)^{-1}$ is defined on $\mathcal{H}$, thus $\mathrm{Ran}(T+i)=\mathcal{H}$ and it since it is an isometry, it follows that $(T-i)\mathcal{D}(T)=(T-i)(T+i)^{-1}\mathcal{H}=\mathcal{H}$ thus $\mathrm{Ran}(T\pm i)=\mathcal{H}$.

2. We now prove (i). Note that $\mathrm{Ker}(1-U)=\mathrm{Ker}(1-U^*)$. Indeed, $x=Ux\Leftrightarrow U^*x=x$. Let $T=i(1+U)(1-U)^{-1}$. Then T is defined on $\mathrm{Ran}(1-U)$, which is dense in $\mathcal{H}$, since $\mathrm{Ran}(1-U)^\perp=\mathrm{Ker}(1-U^*)=\{0\}$.

   Now we want to see that for all $\{x,y\}\subset D(T)$ we have $\langle x,Ty\rangle=\langle Tx,y\rangle$.

3. Note first that $\mathrm{Ran}(1-U)=(1-U)\mathcal{H}=-U(U^*-1)\mathcal{H}=U\mathrm{Ran}(1-U^*)$.

4. Let $x\in D(T)$. Symmetry: $x\in D(T)$ is, by the above, the same as

$$x=U(U^*-1)z=(1-U)z;\quad(\text{and } z=(1-U)^{-1}x)\tag{124}$$

   Thus, to determine the adjoint, by (124) we analyze the expression $\langle x,Ty\rangle$ where we first show T is symmetric:

$$\langle x,Ty\rangle:=\langle U(U^*-1)z,i(1+U)(1-U)^{-1}y\rangle$$
$$\langle Uz,i(U-1)(1+U)(1-U)^{-1}y\rangle=\langle Uz,-i(1+U)y\rangle=\langle i(1+U^*)Uz,y\rangle$$
$$=\langle i(U+1)z,y\rangle=\langle i(U+1)(1-U)^{-1}x,y\rangle=\langle Tx,y\rangle\tag{125}$$

5. Let's check the range of $T+i$. We have $D(T)=\mathrm{Ran}(1-U)$, and thus

$$(T+i)D(T)=(T+i)(1-U)\mathcal{H}=i(1+U)(1-U)^{-1}(1-U)\mathcal{H}+i(1-U)\mathcal{H}$$
$$=i(1+U)\mathcal{H}+i(1-U)\mathcal{H}=2i\mathcal{H}=\mathcal{H}\tag{126}$$

   Likewise, $\mathrm{Ran}(T-i)=\mathcal{H}$.

□

[5] Or, equivalently, $\mathrm{Ran}(1-U)$ is dense. This condition is needed to ensure $D(T)$ is dense, see Lemma 60.

14.2 Von Neumann's Theorem on self-adjoint extensions

Theorem 7. Let T be a closed, densely defined, symmetric operator. Then T has a self-adjoint extension *iff* the deficiency indices of its Cayley transform are equal.

Proof. (i) Assume T_e is a self-adjoint extension of T. Let $U = (T - i)(T + i)^{-1}$ and $U = (T_e - i)(T_e + i)^{-1}$. By Lemma 61, U is unitary. We want to show that U_e is an extension of U. Remember $U : H^+ \to H^-$. Let $x \in H^+$, then $x = (T+i)f = (T_e+i)f$ and $U_e x = (T-i)f = (T_e-i)f = (T_e-i)(T_e+i)^{-1}x = Ux$. Since U_e admits a unitary extension, then the deficiency indices are equal, see Lemma 57.

(ii) Conversely, assume that $H_\pm$ have the same dimension. Then there is a unitary U_e extending U. We first need to show that $1 - U_e$ is injective. If this was not the case, and $z \in \mathrm{Ker}(1 - U_e)$, thus, as before, $z \in \mathrm{Ker}(1 - U_e^*)$ thus $z \in \mathrm{Ran}(1 - U_e)^\perp \subset \mathrm{Ran}(1 - U)^\perp = D(T)^\perp = \{0\}$.

The rest of the proof is quite similar to that of (i). Let $T_e = i(1 + U)(1 - U)^{-1}$, a self-adjoint operator. We want to show that $T_e \supset T$. U_e is defined on $\mathrm{Ran}(T+i)$ and $D(T) = \mathrm{Ran}(1-U_e)$. If $x \in D(T)$, then $x = (1-U_e)y = (1-U)y$ for some y and thus $Tx = i(1 + U_e)(1 - U_e)^{-1}x = i(1 + U)(1 - U_e)^{-1}x = i(1 + U)(1 - U)^{-1}x = T_e x$ ☐

Corollary 62. Let T be symmetric and closed. Then T has a self-adjoint extension iff $\mathrm{Ker}(T^* - i)$ and $\mathrm{Ker}(T^* + i)$ have the same dimension, that is, iff $\mathrm{Ran}(T \pm i)$ have the same dimension.

Exercise 1. *Show that the symmetric operator in §13.2 has* **no** *self-adjoint extension.*

15 Spectral theorem: various forms

We first formulate the various forms of this theorem, then apply it on a number of examples, and then prove the theorem.

15.1 Bounded operators

Theorem 8 (Functional calculus form (I))**.** *Let A be a bounded self-adjoint operator on $\mathcal{H}$, a separable Hilbert space. Then there exists a finite measure space $\{M, \mu\}$, a unitary operator from $\mathcal{H}$ onto $L^2(M, d\mu)$ and a* **bounded** *function F so that the image of A under U is the operator $F_\times$ of multiplication by F. That is $A = U^{-1}F_\times U$, where $(F_\times f)(\omega) =: F(\omega)f(\omega)$.*

Equivalently (the equivalence is simple, we'll show it later).

Theorem 9 (Functional calculus form (II))**.** *Let A be a bounded self-adjoint operator on $\mathcal{H}$, a separable Hilbert space. Then there exists a decomposition*

$$\mathcal{H} = \oplus_{n=1}^{N} \mathcal{H}_n, \quad N \le \infty$$

so that, on each H_n, A is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu_n)$ for some finite measure μ_n (meaning $A = U^{-1}X_\times U$ where $X(f(x)) = xf(x)$). That is, A is unitarily equivalent to multiplication by X. The measures depend on A, of course. Furthermore, they are however non-unique, thus non-canonical. Also, for any analytic f, $f(A) = U^{-1}f_\times U$.

The measures $d\mu_n$ are called spectral measures.

15.2 Unbounded operators

Theorem 10. *Let A be a self-adjoint operator on $\mathcal{H}$, a separable Hilbert space. Then there exists a finite measure space $\{M, \mu\}$, a unitary operator from $\mathcal{H}$ onto $L^2(M, d\mu)$ and a function F* **finite a.e.** *so that so that the image of A under U is the operator of multiplication by F. That is $A = U^{-1}F_\times U$. Also, $\psi \in D(A)$* iff *$F(\omega)(U\psi)(\omega) \in L^2(M, d\mu)$).*

Furthermore, the measure space can be arranged $F \in L^p(d\mu)$ for any $p \in [0, \infty)$.

15.2.1 Spectral projection form

Let A be selfadjoint. There is a family of orthogonal projections P associated to A, with the following properties. For every measurable set S in $\mathbb{R}$ there is a projection P_S (the projection on the part S of the spectrum of A). They have the following properties.

Theorem 11 (Spectral projection form. This form is the same for bounded or unbounded operators)**.** *We have (i) $P_\emptyset = 0, P_\mathbb{R} = I$.*

(ii) if $S = \cup_1^\infty S_n$ (all sets being measurable) and $S_n \cap S_m = \emptyset$ for $n \neq m$, then $P_S = s - \lim \sum_1^\infty P_n$ where $s - \lim$ is the strong limit.

(iii) $P_{S_1}P_{S_2} = P_{S_1 \cap S_2}$.

(iv)

$$A = \int_\mathbb{R} \lambda dP_\lambda$$

For now, we understand (iv) in the sense

$$\langle f, Ag\rangle = \int_\mathbb{R} \lambda d\langle f, P_\lambda g\rangle$$

where $d\langle f, P_\lambda g\rangle$ is a usual measure, as it is straightforward to check using the properties of the family P.

Note 15. *It is important to emphasize that, unlike the other forms of the spectral theorem, this is canonical: given A there is a unique family of projections with the properties above.*

Returning to our functional calculus, if g is analytic on the spectrum of A (contained in $\mathbb{R}$, of course), the image of $g(A)$ is the function $g(F)$. But note that now we can define $e^{itA} = \int_\mathbb{R} e^{it\lambda} dP_\lambda$.

16 Proof of the functional calculus form

16.1 Cyclic vectors

Consider a bounded self-adjoint operator A, a vector ψ in $\mathcal{H}$ and the vectors $\{A^n\psi\}_{n\in\mathbb{N}}$. If we take all linear combinations of $A^n\psi$ and then its closure, the "span" of $A^n\psi$ denoted by $\bigvee_{n=1}^{\infty} A^n\psi$, is a Hilbert space $\mathcal{H}_\psi$. If ψ is such that $\mathcal{H}_\psi = \mathcal{H}$, then ψ is a cyclic vector. Not all operators A have cyclic vectors.

Exercise 1. *Do self-adjoint matrices have cyclic vectors?*

Note that we can take any vector, form $\mathcal{H}_\psi$ and then pick a vector in $\mathcal{H}_\psi^\perp$. The construction can go on indefinitely, if $\oplus_{n=1}^{N}\mathcal{H}_{\psi_N} \neq \mathcal{H}$ for some finite N.

Exercise 2. *Show (for instance using Zorn's Lemma) that every separable Hilbert space can be written as*

$$\mathcal{H} = \oplus_{n=1}^{N}\mathcal{H}_{\psi_N}, \quad N \le \infty \tag{127}$$

Now, to prove Theorem 9 it suffices thus to prove it in each $\mathcal{H}_{\psi_N}$, or, w.l.o.g., assume that ψ is already cyclic for $\mathcal{H}$. Of course, we can assume $\|\psi\| = 1$.

We can define a functional L, first on analytic functions f on $\sigma(A)$ by

$$L(f) = \langle f(A)\psi, \psi\rangle \tag{128}$$

L is bounded since

$$|L(f)| \le \|f(A)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\} = \|f\|_\infty \tag{129}$$

(see Prop. 30 above, p. 27, (100), and Exercise 4 on p. 39.

On the other hand, this is a positive functional, since if f is positive on $\sigma(A)$ then $f = g^2$ with g analytic too and real valued, and thus

$$\langle f(A)\psi, \psi\rangle = \langle g(A)^2\psi, \psi\rangle = \langle g(A)\psi, g(A)\psi\rangle = \|g(A)\psi\|^2 \ge 0 \tag{130}$$

Since analytic functions on a compact set in $\mathbb{R}$, $\sigma(A)$, are dense in $C(\sigma(A))$ (polynomials are already dense) and the functional $f \mapsto \langle f(A)\psi, \psi\rangle$ is positive and of norm one on a dense subset, it extends to $C(\sigma(A))$ and, by the Riesz-Markov theorem, there is a measure $d\mu_\psi$ so that

$$\langle f(A)\psi, \psi\rangle = \int_{\sigma(A)} f d\mu_\psi(x) \tag{131}$$

For simplicity, we drop the subscript $\psi : \mu = \mu_\psi$. We can now consider $\mathcal{H}_A = L^2(\sigma(A), d\mu_\psi)$. We want to define a unitary transformation between $\mathcal{H}$ and $\mathcal{H}_A$. (Note that up to now we showed bounds between $\mathcal{H}$ and $L^\infty(\sigma(A))$.)

It is more convenient to define U on $\mathcal{H}_A$ with values in $\mathcal{H}$. The vectors in $\mathcal{H}_A$ are of course functions on $\sigma(A)$. From (131) we expect φ to be mapped into $\varphi(A)\psi$. Define thus for f analytic (not necessarily real-valued) on $\sigma(A)$,

$$Uf = f(A)\psi \tag{132}$$

Remember (see p. 39) that

$$f(A)^* = f^*(A) \text{ and for } x \in \mathbb{R} \text{ we have } f^*(x) = \overline{f(x)} \tag{133}$$

Therefore

$$\int |f|^2 d\mu \overset{(131)}{=} \langle f(A)(f(A))^*\psi, \psi\rangle = \|f(A)\psi\|^2 = \|Uf\|^2 \tag{134}$$

Thus the transformation U, defined on a dense subset of $L^2(\sigma(A), d\mu)$ extends to an isometry from $L^2(\sigma(A), d\mu)$ into $\mathcal{H}$.

We only have to check that indeed the image $U^{-1}AU$ of A is the operator of multiplication by the variable. We have for analytic f,

$$\begin{aligned} U^{-1}A\underline{Uf} = U^{-1}A\underline{f(A)\psi} = U^{-1}\underline{Af(A)\psi} \\ = U^{-1}\underline{(X_\times f_\times)(A)\xi} = U^{-1}UX_\times f_\times = X_\times f_\times \end{aligned} \tag{135}$$

Since holds on a dense set, the proof is complete.

Note 16. Whether ψ is cyclic or not, formulas (132) and (134) above allow us to define any L^2 functions of A. Indeed, analytic functions (or even polynomials) are dense in L^2. If $f_n \to h$ in the sense of L^2, then $f_n(A)\psi$ converges to (by notation) $h(A)\psi$. Thus we can define, given ψ arbitrary, $h(A)\psi$ for any $h \in L^2$. It is easy to check that h is bounded and linear: Indeed, if ψ is cyclic for the whole of $\mathcal{H}$ there is essentially nothing to show. If it is not, then it is correctly defined and bounded on each $\mathcal{H}_n$ with values in $\mathcal{H}_n$ while $\mathcal{H}_n$ are mutually orthogonal. Simply define $h(a)x = \sum h(A)x_n$ where $(x_n)_{n\le N}$ is the orthogonal decomposition of x. Now, if $h = \chi_S$, the characteristic function of a measurable set, then $\chi_S(A)$, is, of course, a spectral projection.

Note 17. Note that (129) shows that if we take the closure in *the operator norm* of $\mathcal{A}(A) := \{f(A) : f \text{ analytic on } \sigma(A)\}$ then we get $C(A)$ the continuous functions on $\sigma(A)$. In Note 16 we have taken the *strong-limit closure* of $\mathcal{A}(A)$, since convergence of f_n is only used pointwise– convergence of $f_n\psi$ for any ψ. This gives us in particular $M(A)$, the bounded measurable functions on $\sigma(A)$, applied to A. Compare also with what we have done in §4. The apparent similarity can be misleading.

Note 18. Note also that we have made substantial use of the fact that A was self-adjoint to define $M(A)$ (for instance, in using (133)).

17 Examples: The Laplacian in $\mathbb{R}^3$

Most of this section is based on [3], where more results, details, and examples can be found.

In a number of cases, the unitary transformation mapping an unbounded self-adjoint operator T to a multiplication operator is explicit. Then, we can use the exercise below to find all information about T.

Exercise 1. *Show that, if U is unitary between $\mathcal{H}_1$ and $\mathcal{H}_2$, then T is self-adjoint on $D(T) \subset \mathcal{H}_1$ iff $UTU^{-1} = UTU^*$ is self-adjoint in $UD(T)$.*

As an example, consider defining $-\Delta$ in $\mathbb{R}^3$. Define it first on $D_0(\Delta) = C_0^\infty$ where it is symmetric. Let U be the Fourier transform, a unitary operator. It is in fact easier to work with another set of functions, still dense in $L^2(\mathbb{R}^3)$ as can be easily checked, which admit an explicit Fourier image:

$$D_1 = \left\{ e^{-\mathbf{x}^2/2} P(\mathbf{x}) : P \text{ polynomial} \right\} \tag{136}$$

This is Fourier transformed to

$$UD_1 = \left\{ e^{-\mathbf{k}^2/2} P(\mathbf{k}) : P \text{ polynomial} \right\} \tag{137}$$

Exercise 2. *Check that D_1 is dense in $L^2(\mathbb{R}^3)$. Check that on D_1 T is symmetric and that $UT(f) = |\mathbf{k}^2| Uf$, a multiplication operator.*

We will denote $|\mathbf{k}^2|$ by k^2 and Uf by $\hat{f}$.

Exercise 3. *Check that k^2 is self-adjoint on*

$$D(k^2) = \{ f \in L^2(\mathbb{R}^3) : k^2 f \in L^2(\mathbb{R}^3) \} = \{ f : (k^2+1) f \in L^2(\mathbb{R}^3) \} \tag{138}$$

Thus, we have

Proposition 63. *$-\Delta$ is self-adjoint on $U^{-1}D(k^2)$. Call this operator H_0.*

The characterization of $D(H_0)$ is simplest through Fourier transform. This gives another dimension to the need for **Sobolev spaces** etc.

Let $u \in D(H_0)$ and $\hat{u}$ be its Fourier transform. A direct space characterization is more difficult, though we might simply say that u together with all second order partial derivatives exist *as weak derivatives, in distributions* (equivalently, defined in L^2 as as inverse Fourier transforms of $k_i k_j \hat{u}$), and the derivatives are in L^2.

We can see some classical properties of elements of $u \in D(H_0)$. We can show that u is bounded and uniformly continuous as follows. Noting that both $\hat{u}$ and $k^2\hat{u}$ are in L^2 it follows immediately that $(\alpha^2 + |k|^2)\hat{u} \in L^2$ for any α. Let $k = |\mathbf{k}|$ and dk be the Lebesgue measure on $\mathbb{R}^3$.

For boundedness, we use Cauchy-Schwarz:

$$\left(\int_{\mathbb{R}^3} |\hat{u}| dk \right)^2 \le \int_{\mathbb{R}^3} \frac{dk}{(k^2+\alpha)^2} \int_{\mathbb{R}^3} [(k^2+\alpha)|\hat{u}|]^2 dk = \frac{\pi^2}{\alpha} \|(H_0+\alpha)u\|^2 < \infty \tag{139}$$

Note also that

$$|e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{y}}| \le \max\{2, k|\mathbf{x}-\mathbf{y}|\} \tag{140}$$

Similarly, we can calculate that, if $\beta < 1/2$, then

$$|u(\mathbf{x}) - u(\mathbf{y})| \le C_\beta |\mathbf{x}-\mathbf{y}|^\beta \left(\alpha^{-(1/2-\beta)} \|H_0 u\| + \alpha^{3/2+\beta} \|u\| \right) \tag{141}$$

which easily implies that u is Hölder continuous with any exponent $< 1/2$. However, u is not even once classically differentiable!

The inequalities (139) and (141) are special cases of Sobolev inequalities.

For more details and examples, see [3], pp. 299.

Here is a very useful and general criterion of self-adjointness, in the unbounded case. First, a definition:

Definition 64. *(i) An operator A is relatively bounded with respect to T ($T-$bounded) if $D(A) \supset D(T)$ and for some a, b and all $u \in D(T)$ we have*

$$\|Au\| \leq a\|u\| + b\|Tu\| \tag{142}$$

An equivalent condition is that for some (different) a', b' we have

$$\|Au\|^2 \leq (a')^2\|u\|^2 + (b')^2\|Tu\|^2 \tag{143}$$

The equivalence of (142) *and* (143) *is left as an exercise (details are given in [3], p. 287, but the exercise is simple enough). (ii) The $T-$bound is defined as the greatest lower bound of the b for which there is an a so that* (142) *holds, or equivalently the greatest lower bound of the b' for which there is an a' so that* (143) *holds.*

Theorem 12 (Kato-Rellich)**.** *Let T be self-adjoint. If A is symmetric and $T-$bounded with $T-$bound $b' < 1$ then $T + A$ is* **self-adjoint***. This is the case in particular if A is bounded.*

Proof. First, it is clear that $D(T + A) = D(T)$ and $T + A$ is symmetric. We will show that (c) in Theorem 6 holds. Without loss of generality we take also $a' > 0$. Recall (116). This implies immediately that

$$\|Ax\| \leq \|(b'T \mp ia')x\|, \quad (x \in D(T)) \tag{144}$$

Denote $c' = a'/b'$ and $(T \mp ic')x = y$, and recall Theorem 6 (b) (c). This implies that $(T \mp ic') = \mathcal{R}(\mp ic')$ exist and are bounded. Thus

$$\|A\mathcal{R}(\mp ic')y\| \leq b'\|y\|, \quad (x \in D(T)) \tag{145}$$

In particular (again by Theorem 6 (b) (c)) this means that

$$B_{\pm} = -A\mathcal{R}(\mp ic') \text{ are bounded and } \|B_{\pm}\| < b' < 1 \tag{146}$$

Therfore, by the standard Neumann series argument, $(1 - B_{\pm})^{-1}$ exist and are bounded (with norm $\leq 1/(1 - b')$) and thus $(1 - B_{\pm})$ are bounded and one-to-one. Note now that $T \mp ic'$ is one to one, so is $(1 - B_{\pm})$ and thus their product is one-to-one.

$$(1 - B_{\pm})(T \mp ic') = T \mp ic' + A\mathcal{R}(\mp ic')(T \mp ic') = T \mp ic' + A \tag{147}$$

Thus $\mathrm{Ran}(T+A\mp ic') = \mathrm{Ran}\frac{1}{c'}(T+A)\mp i = \mathcal{H}$ and thus $T+A$ is self-adjoint. $\square$

Example The operator $-\Delta + V(\mathbf{x})$ is self-adjoint on $D(-\Delta)$ for any bounded real function V. This follows trivially from Theorem 12.

In one dimension, say on $L^2[0,1]$, for any bounded, measurable, real f, $id/dx + f(x)$ (or $-d^2/dx^2 + f(x)$) are self-adjoint on any domain on which id/dx (or $-d^2/dx^2$) is self-adjoint. Find self-adjointness domains for $-d^2/dx^2$.

More generally, one can show (see [3]) the following.

Proposition 65 ([3], p. 302)**.** *Consider functions of the form $q = q_0 + q_1$ where $q_0 \in L^\infty(\mathbb{R}^3)$ and $q_1 \in L^2(\mathbb{R}^3)$. Then*

$$-\Delta + q \tag{148}$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ and self-adjoint on $D(-\Delta)$.

Example $q = 1/r$, $r = |\mathbf{x}|$ satisfies the assumptions of Proposition 65. Thus the Hamiltonian of the Coulomb atom,

$$-\Delta + e/r \tag{149}$$

is self-adjoint on $D(H_0)$.

17.0.1 Second resolvent formula

This is a simple but very important identity. It holds in more generality than described below, but this is all we need for now.

Theorem 13. *Let $A, B, A+B$ be defined common domain D, and all of them invertible. Then*

$$A^{-1} - (A+B)^{-1} = A^{-1}B(A+B)^{-1} \tag{150}$$

Proof. Since all operators involved are invertible, (150) holds iff it holds after multiplication by $(A+B)$. This gives:

$$A^{-1}(A+B) - 1 = A^{-1}B \tag{151}$$

which clearly holds. □

18 Compact operators: a summary

We present some of the main results (some without proofs). See [2] for a compact yet complete and clear presentation.

Definition 66. *A (bounded) operator T between two Banach spaces X, Y is compact if it takes bounded sets into relatively compact ones (that is, every sequence has a convergent subsequence).*

Example On $C[0,1]$ or $L^2[0,1]$ taking $Tf = \int_0^x h(s)f(s)ds$ (or $Tf = \int_0^1 h(s)f(s)$) where $h \in L^\infty$, then the image of T consists of functions which are in $AC[0,1]$ with derivative uniformly bounded by $\|f\|$. Then Arzela-Ascoli applies and T is compact.

Theorem 14. *Let X, Y be Banach spaces and $T, T_n \in \mathcal{L}(X,Y)$ (recall that $\mathcal{L}(X,Y)$ denotes the bounded operators from X to Y). Then:*

(a) If $\{T_n\}_{n\in\mathbb{N}}$ are compact and $T_n \to T$ in **norm**, *then T is compact.*

(b) T is compact iff T^* is compact.

(c) If $S \in \mathcal{L}(X,Y)$ and S, or T, is compact, then ST is compact. (d) The sum of compact operators is compact.

Corollary 67. *The resolvent of $id/dx + z$ on $AC[0,1]$ with periodic boundary conditions, where $z \notin \sigma(id/dx)$ is compact. Indeed the resolvent is*

$$(\mathcal{R}(z)g)(x) = -ie^{izx}\int_0^x g(s)e^{-izs}ds + \frac{ie^{iz(x+1)}}{e^{iz-1}}\int_0^1 g(s)e^{-izs}ds \qquad (152)$$

a combination of compact and bounded operators as in Theorem 14.

Corollary 68. *If A, B are as in Theorem 13 and A^{-1} is compact and B is bounded, then $(A+B)^{-1}$ is compact.*

This is immediate from the second resolvent formula and Theorem 14.

Corollary 69. *Let $T = id/dx + V(x)$ on $AC[0,1]$ with periodic boundary conditions, where $V(x)$ is real valued and bounded. Then T is self-adjoint with compact resolvent.*

Proof. This follows from Theorem 12, Corollary 67 and the second resolvent formula, Theorem 13. □

Theorem 15 (The Fredholm alternative)**.** *If A is a compact operator on $\mathcal{H}$ then either $(I-A)^{-1}$ exists or $A\psi = \psi$.*

Theorem 16 (Riesz-Schauder)**.** *Let A be a compact operator on $\mathcal{H}$. Then $\sigma(A)$ is a discrete set of eigenvalues λ_n of finite multiplicity, except* 0 *which might be an accumulation of the λ_n.*

Theorem 17 (The Hilbert-Schmidt theorem)**.** *Let A be compact and self-adjoint. Then there is a complete orthonormal basis φ_n for $\mathcal{H}$ so that $A\varphi_n = \lambda_n\varphi_n$ and $\lambda_n \to 0$ as $n \to \infty$.*

In view of Theorem 15 this is essentially the spectral theorem (functional form I).

Corollary 70. *In the setting of Corollary 69, the spectrum of T is discrete, with finite degeneracy, and the eigenfunctions form a basis in $L^2[0,1]$.*

That follows from Corollary 70 and the fact that $\sigma(1/A) = 1/\sigma(A)$.

Theorem 18. *Let $\mathcal{H}$ be a* **Hilbert space**. *Then T is compact* **iff** *it is the norm limit of operators of finite rank (meaning that* $\dim \operatorname{Ran}(T_n) < \infty$.

18.0.2 Equivalence with Theorem 8

For that we simply take N copies of $\mathbb{R}$, take their disjoint union $M = \cup_{n=1}^{N}\mathbb{R}$ and on each copy we take a vector of norm 2^{-n}. Then we take the measure $\mu = \oplus\mu_n$ on M, and clearly $\mu(M) < \infty$. The rest is immediate.

18.0.3 Extension to Borel functions of A

Since A corresponds to multiplication by x, if h is any bounded Borel measurable function on $\sigma(A)$ we define $h(A) = Uh(x)U^{-1}$. In particular, if h is chosen to be χ_S, the characteristic function of a subset S, then $P_S = U\chi_S U^{-1}$ is an orthogonal projection, in fact a spectral measure. We want to see that

$$A = \int x dP(x)$$

Let S be an arbitrary measurable subset of $\sigma(A) \subset \mathbb{R}$ and define, for each pair $x, y \in \mathcal{H}$,

$$\nu(S) = \langle P_s x, y\rangle$$

Then,

18.1 Defining self-adjoint operators

Suppose A is symmetric, and that we can find a unitary transformation U that maps $\mathcal{H}$ into $L^2(\mathbb{R}, d\mu)$ in such a way that A is mapped onto $a\cdot$ for all $\psi \in D(A)$, and that means that $a\cdot$ is initially defined on $U(D(A))$ which is also dense, since U is unitary. Then, clearly, $(A\pm i)^{-1}$ are mapped to $(a\pm i)^{-1}$, since $(A\pm i)(A\pm i)^{-1} = \mathbf{1}$ on $D(A)$. It is clear that $U_a = (a\cdot -i)(\underline{a\cdot +i})^{-1}$, the image of $U_A = (A-i)(A+i)^{-1}$ is extends to a unitary operator on $\overline{UD(A)} = U\mathcal{H}$. But this does not mean that it was unitary to start with, since it was only defined on $\{(a+i)f : f \in UD(A)\}$ which may not be dense in $L^2(\mathbb{R}, d\mu)$.

Nevertheless, U_a extends to a unitary operator, and thus U_A extends to a unitary operator, and thus A has *some* self-adjoint extension A_1, (canonical wrt this particular construction...). What is the domain of A_1? This can be written in terms of U_A: $D(A) = \text{Ran}(U_A - 1)$, which is simply $U^{-1}\text{Ran}(u_a - 1) = D(a\cdot)$. On the other hand, this is simple to calculate, it consists of the functions g such that

$$D(a) = \left\{ g \in L^2(\mathbb{R}, d\mu) : \int_{\mathbb{R}} |g|^2 a^2 da < \infty \right\} \tag{153}$$

Exercise 1. Apply these arguments to $A = id/dx$ defined originally on C_0^∞, using as U the discrete Fourier transform and, as usual k instead of a as a discrete variable. What is $UD(A)$? On $UD(A)$ $k\cdot$ is simply multiplication by k. The domain of the adjoint is *larger* than the set of sequences $\{c_k\}_{k\in\mathbb{Z}}$ with $kc_k \in L^2$. What is it? Find a self-adjoint extensions of A. Can you find more than one?

Thus,

$$D(A) = U^{-1}\left\{g \in L^2(\mathbb{R}, d\mu) : \int_{\mathbb{R}} |g|^2 a^2 da < \infty\right\} \tag{154}$$

19 Spectral measures and integration

We have seen that finitely many isolated parts of the spectrum of an operator yield a "spectral decomposition": The operator is a direct sum of operators having the isolated parts as their spectrum. Can we allow for infinitely many parts?

We also saw that, if the decomposition of the spectrum is $K_1 + K_2$, where $\chi_1\chi_2 = 0$, then the decomposition is obtained in terms of the spectral projections $P_i = \chi_i(T)$ where $P_1P_2 = 0$ and $P_1 + P_2 = I$. We can of course reinterpret (probability-)measure theory in terms of characteristic functions. Then a measure becomes a functional on a set of characteristic functions such that $\chi_i\chi_j = 0$ (the sets are disjoint) then

$$\chi(\bigcup_{i=1}^{\infty} A_i) = \sum_{k=1}^{\infty} \chi_i$$

and

$$\chi(A_1 \cap A_2) = \chi_1\chi_2$$

In terms of measure, we then have

$$\mu(\sum_{k=1}^{\infty} \chi_k) = \sum_{k=1}^{\infty} \mu(\chi_k)$$

$$\mu(1) = 1$$

(continuity of the functional) etc.

In the following, we let $E_i := E(A_i)$.

Note 19. 1. In a Hilbert space it is natural to restrict the analysis of projectors to orthogonal projections. Indeed, if $\mathcal{H}_1 \subsetneq \mathcal{H}$ is a closed subspace of $\mathcal{H}$, itself then a Hilbert space, then any vector x in $\mathcal{H}$ can be written in the form $x_1 + x_2$ where $x_1 \in \mathcal{H}_1$ and $x_2 \perp x_1$. So to a projection we can associate a natural orthogonal projection, meaning exactly the operator defined by $(x_1 + x_2) \to x_1$.

2. An orthogonal projection P is a self-adjoint operator. Indeed,

$$\langle Px, y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle = \langle x_1 + x_2, y_1\rangle = \langle x, Py\rangle$$

3. Conversely, if P is self-adjoint, and $x = x_1 + x_2$, where $Px_1 = x_1$, then

$$\langle x_1, x_2\rangle = \langle Px, (1 - P)x\rangle = \langle x, P(1 - P)x\rangle = 0$$

Let Ω be a topological space, $\mathcal{B}_\Omega$ be the Borel sets on Ω and $\mathcal{A}(\mathcal{H})$ be the *self-adjoint* bounded operators on the Hilbert space $\mathcal{H}$: $\forall(x,y), \langle Tx, y\rangle = \langle x, Ty\rangle$. Thus, with $A_i \in \mathcal{B}_\Omega$, the spectral projections $E := \chi(T)$ should satisfy

1. $$(\forall i \neq j, A_i \cap A_j = \emptyset) \;\Rightarrow\; E(\bigcup_{i=1}^{\infty} A_i)x = \sum_{k=1}^{\infty} E_i x$$

2. $$E((A_1 \cap A_2) = E_1 E_2$$

3. $$E(\Omega) = I; \quad E(\emptyset) = 0$$

Definition 71. *A projector valued spectral measure on $\mathcal{B}_\Omega$ with values in $\mathcal{A}(\mathcal{H})$ is a map $E : \mathcal{B}_\Omega \to \mathcal{H}$ with the properties 1,2,3 above.*

Note 20. *$E(A)$ is always a projector in this case, since $E(A) = E(A \cap A) = E(A)^2$ and $E(A)E(B) = E(B)E(A)$ since they both equal $E(A \cap B)$.*

Proposition 72 (Weak additivity implies strong additivity)**.** *Assume 2. and 3. above, and that for all $(x,y) \in \mathcal{H}^2$ we have*

$$(\forall i \neq j, A_i \cap A_j = \emptyset) \;\Rightarrow\; \langle E(\bigcup_{i=1}^{\infty} A_i)x, y\rangle = \langle \sum_{k=1}^{\infty} E_i x, y\rangle$$

Then 3. above holds.

Remark 21. *Note a pitfall: we see that*

$$E(\bigcup_{i=1}^{\infty} A_i)x = \sum_{k=1}^{\infty} E_i x \tag{155}$$

for all x. But this does not mean that $\sum_{i=1}^{N} E_i x$ converge normwise!

Note 22. If $\langle v_n, y\rangle \to \langle v, y\rangle$ for all y and $\|v_n\| \to \|v\|$ then $v_n \to v$. This is simply since

$$\langle v_n - v, v_n - v\rangle = \langle v_n, v_n\rangle - \langle v_n, v\rangle - \langle v, v_n\rangle + \langle v, v\rangle$$

Proof. Note first that $E(A_i)x$ form an orthogonal system, since

$$\langle E(A_1)x, E(A_2)x\rangle = \langle x, E(A_1)E(A_2)x\rangle = \langle x, E(A_1 \cap A_2)x\rangle = 0 \tag{156}$$

$$\langle \sum_{i=1}^{N} E_i x, \sum_{i=1}^{N} E_i x, \rangle = \sum_{i=1}^{N} \langle E_i x, E_i x, \rangle$$

while

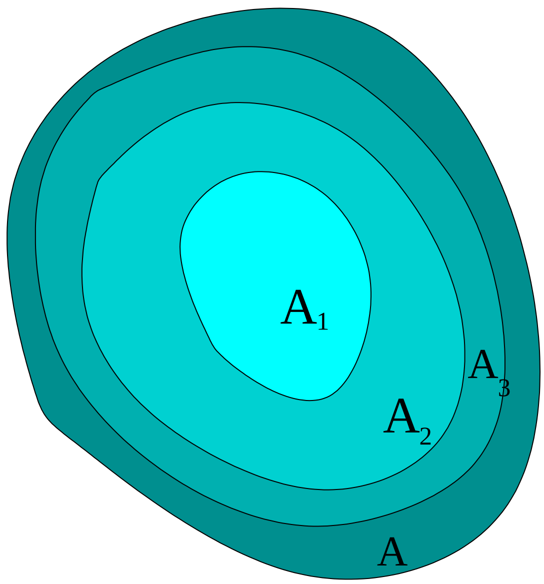
$$\sum_{i=1}^{N} E_i x$$

converges. Thus $\sum_{i=1}^{N} \|E_i x\|$ converges too and thus $\| \sum_{i=1}^{N} E_i x - \sum_{i=1}^{\infty} E_i x\| \to 0$.

□

Lemma 73. *If $A_j \uparrow A$ then $E_j x \to E(A)x$.*

Proof. Same as in measure theory. $A_{j+1} \setminus A_j$ are disjoint, their sum to N is A_{N+1} and their infinite sum is A, see figure. □



20 Integration wrt projector valued measures

Let f be a simple function defined on $S \in \mathcal{B}_\Omega$. This is a dense set in $\mathcal{B}(\Omega, \mathbb{C})$.

$$f(\omega) = \sum_{i=1}^{N} f_j \chi(S_j) \tag{157}$$

As usual, $S_i \cap S_j = \emptyset$ and $\cup S_i = S$. Then we naturally define

$$\int_S f(\omega) dE(\omega) = \sum_{i=1}^{N} f_j E(S_j) \tag{158}$$

This is the usual way one defines integration wrt a scalar measure.

Note 23. We note an important distinction between this setting and that of §4: here we integrate scalar functions wrt operator valued measures, whereas in §4 we integrate operator-valued functions against scalar measures.

Note 24. [(6)] In §4 as well as in the few coming sections, we restrict ourselves to integrating uniform limits of simple functions. The closure in the sup norm of simple functions are the measurable bounded functions, which is not hard to verify, so this is a more limited integration than Lebesgue's (e.g., L^1 is not accessible in this way.)

It is also interesting to note the following theorem:

Theorem 19 (Šikić, 1992)**.** A function f is Riemann integrable iff it is the uniform limit of simple functions, where the supporting sets A_i are Lebesgue measurable with the further property that $\mu(\partial A_i) = 0$.

Without the restriction on $\mu(\partial A_i)$, the closure of simple functions in the sup norm is the space of bounded measurable functions (exercise).

Proposition 74. *The definition (158) is independent of the choice of S_j, for a given f.*

Proof. Straightforward. □

Proposition 75. *The map $f \to \int_S f$ is uniformly continuous on simple functions, and thus extends to their closure. We have*

$$\left\| \int_S f(\omega) dE(\omega) \right\| \le \sup_S |f(\omega)| \tag{159}$$

Proof. We can assume that the sets S_i are disjoint. Then the vectors $E_i x$ are orthogonal and

$$\left\| \int_S f(\omega) dE(\omega) \right\|^2 = \sum_{i=1}^N f_i \|E(S_i)x\|^2 \le \sup_S |f|^2 \sum_{i=1}^N \langle E_i x, x\rangle \le \sup_S |f|^2 \|x\|^2$$

□

Consider then the functions in $B(\Omega, \mathbb{C})$ which are uniform limits of simple functions. The integral extends to $B(\Omega, \mathbb{C})$ by continuity and we write

$$\int_S f(s) dE(s) := \lim_{n\to\infty} \int_S f_n(s) dE(s) \tag{160}$$

Definition 76. *A C^* algebra is a Banach algebra, with an added operation, conjugation, which is antilinear and involutive ($(f^*)^* = f$) and $\|f^* f\| = \|f\|^2$.*

Exercise 1. *Show that $\|f^*\| = \|f\|$.*

Proposition 77. *$f \mapsto \int_S f(\omega) dE(\omega)$ is a C^* algebra homeomorphism.*

[(6)] Pointed out by Sivaguru Ravisankar.

Proof. We only need to show multiplicativity, and that only on characteristic functions, and in fact, only on very simple functions, where $N = 1$, and the coefficients are one. We have

$$\int \chi(S_1)\chi(S_2)dE(S) = E(S_1 \cap S_2) = E(S_1)E(S_2) \tag{161}$$

Exercise 2. *Complete the details.*

□

20.1 Spectral theorem for (unbounded) self-adjoint operators

Theorem 20. *Let A be self-adjoint. There is a spectral measure on the Borel σ-algebra of $\mathbb{R}$ such that*

$$Ax = \int_{\mathbb{R}} tdE(t)\,x := \lim \int_{-k}^{k} tdE(t)\,x \tag{162}$$

for all $x \in D(A)$. Furthermore,

$$D(A) = \{x \in \mathcal{H} : \int_{\mathbb{R}} t^2 d\langle E(t)x, x\rangle < \infty\} \tag{163}$$

We know that self-adjoint operators are in a one-to-one correspondence with unitary operators. Assuming we have the spectral theorem for bounded normal operators and u is a unitary operator, then we have

$$U = \int_C udE(u) \tag{164}$$

where C is the unit circle. Then, with $A = i(1+U)(1-U)^{-1}$ we expect

$$A = \int_C i\frac{1+u}{1-u}dE(u) \tag{165}$$

which we can formally write

$$A = \int_{\mathbb{R}} td(E\circ\phi)(t) = \int_{\mathbb{R}} tdE'(t) \tag{166}$$

where

$$\phi(t) = \frac{t-i}{t+i} \tag{167}$$

and where $E'(S) = E(\phi(S))$ is a new spectral measure.

We have to ensure that the theory of integration holds, and that it is compatible with changes of variable, and to interpret the singular integrals obtained.

21 Spectral representation of self-adjoint and normal operators

We first state the main results, which we will prove in the sequel.

Theorem 21 (Spectral theorem for bounded normal operators)**.** *Let N be bounded and normal (that is, $NN^* = N^*N$). Then there is spectral measure E defined on the Borel sets on $\sigma(N)$, see Definition 71, such that*

$$N = \int_{\sigma(N)} z dE(z) \tag{168}$$

Theorem 22 (Uniqueness of the spectral measure)**.** *Let N be a normal, bounded operator.*

(i) Assume $E^{[1]}$ and $E^{[2]}$ are spectral measures on the Borel sets on $\sigma(N)$ such that

$$N = \int_{\sigma(N)} z dE^{[1]} = \int_{\sigma(N)} z dE^{[2]} \tag{169}$$

Then $E^{[1]} = E^{[2]}$.

(ii) More generally, if $\Omega \in \mathbb{C} \supset \sigma(N)$ is a closed set and $E^{[3]}$ is a spectral measure on Ω such that

$$N = \int_{\sigma(N)} z dE^{[3]} \tag{170}$$

Then $E(\Omega \setminus \sigma(N)) = 0$ and the Borel measure induced by $E^{[3]}$ on $\sigma(N)$ coincides with E, the spectral measure of N.

What can the spectrum of a normal operator be? Let $\mathcal{D}$ be a compact set in $\mathbb{R}^2$, and consider $L^2(\mathcal{D})$. Consider the operator $Z := f(x, y) \mapsto (x + iy)f(x, y)$. Then $Z^* = f(x, y) \mapsto (x - iy)f(x, y)$, and Z is a normal operator. Clearly, if $(x_0, y_0) \notin \mathcal{D}$ and $\lambda = x_0 + iy_0$, then $1/(x + iy - \lambda)$ is bounded on $\mathcal{D}$ and it is the inverse of $(x + iy - \lambda)$; otherwise, if $(x_0, y_0) \in \mathcal{D}$, then $(x + iy - \lambda)$ has no nonzero lower bound, and thus is not invertible. Then $\sigma(Z) = \mathcal{D}$.

21.1 Spectral theorem in multiplicative form

Definition 78. *A vector $\psi \in \mathcal{H}$ is* **cyclic** *for the (bounded) operator A if the closure of the set of linear combinations of vectors of the form $A^n\psi$, $n \in \mathbb{N}$ (denoted by $\bigvee_{n=1}^{\infty} A^n\psi$) is $\mathcal{H}$.*

In this formulation, self-adjoint operators are unitarily equivalent with multiplication operators on a direct sum of $L^2(\sigma(A)), d\mu_n$ with respect to certain measures. If A has a cyclic vector, then the measure is naturally defined in the following way. We know how to define bounded measurable functions of A (for instance taking limits of polynomials). Then, $\langle \psi, f(A)\psi \rangle = \int_{\sigma(A)} f(\omega) d\mu_\psi(\omega)$.

Note 25. In Quantum mechanics, if o (angular momentum, energy, etc.) is an observable, then it is described by a *self-adjoint* operator O. The spectrum of O consists of all possible measured values of the quantity o. An eigenvector ψ_a of O is a state in which the measured observable has the value $o = a$ with probability one.

If ψ is not an eigenvector, then the measured value is not uniquely determined, and for an ensemble of measurements, a number of values are observed, with different probabilities. The average measured value of o or *the expected value of o*, in an ensemble of particles, each described by the wave function ψ is $\langle \psi, O\psi \rangle$.

Imagine that the measured quantity is $\chi_{[a,b]}(o)$. That is, if say o is the energy, then $\chi_{[a,b]}(o)$ is a filter, only letting through particles with energies between a and b (a spectrometer only lets through photons in a certain frequency range, and since $E = h\nu$, certain energies).

Then, $\langle \psi, \chi_{a,b}(E)\psi \rangle$ is the expected number of times the energy falls in $[a, b]$, that is, the density of states. This would be the physical interpretation of $d\nu_\psi$.

Theorem 23. *Assume A is a bounded self-adjoint operator with a cyclic vector ψ. Then there is a unitary transformation U that maps $\mathcal{H}$ onto $\mathcal{H}_A = L^2(\sigma(A), d\mu_\psi)$ and A into the multiplication operator $\Lambda = f(\lambda) \mapsto \lambda f(\lambda)$:*

$$UAU^{-1} = \Lambda \tag{171}$$

Note 26. This unitary transformation is non-canonical. But cyclicity means that the transformation $\phi \mapsto O\psi$ is ergodic, mixing, and the measures are expected to be equivalent, in some sense.

Clearly, uniqueness cannot be expected. $L^2([0, 1])$ with respect to dx and with respect to $f(x)dx$ $\alpha < 0 < f < \beta$ are unitarily equivalent, and the unitary transformation is $g \mapsto \sqrt{f}g$. X, multiplication by x is invariant, since $UXU^{-1} = X$. The measure is determined up to measure equivalence: two measures are equivalent iff they have the same null sets. Instead, the spectral projections are unique.

In case A does not have a cyclic vector, then $\mathcal{H}$ can be written as a finite or countable direct sum of Hilbert spaces $\mathcal{H} = \oplus_{n=1}^N \mathcal{H}_n$, $N \le \infty$ such that $A|_{\mathcal{H}_n}$ has a cyclic vector ψ_n. Then, as before, there is a unitary transformation $U : \mathcal{H} \to \oplus_{n=1}^N L^2(\mathbb{R}, d\mu_n)$ so that

$$(UAU^{-1})_{\mathcal{H}_n} = \Lambda \tag{172}$$

Let F be the operator $g \mapsto fg$ where $f, g \subset L^2(M, d\mu)$. As a corollary,

Proposition 79. *Let A be bounded and self-adjoint. Then, there exists a finite measure space (M, μ) ($\mu(M) < \infty$), a bounded function f on M and a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$ so that*

$$UAU^{-1} = F \tag{173}$$

Like all other measures, μ_ψ can be decomposed uniquely into three disjoint measures: $\mu_{pp} + \mu_{ac} + \mu_{sing}$. Let μ be a finite measure on $\mathbb{R}$.

(i) μ_{pp} (pure point). A measure is pure point, μ_{pp} if, by definition, for any $B \in \Omega$ we have $\mu(B) = \sum_{\omega \in B} \mu(\omega)$. We note that the support of μ_{pp} is a countable set, since the sum of any uncountable family of positive numbers is infinite.

(ii) μ_{ac}. A measure is absolutely continuous with respect to the Lebesgue measure dx if, by definition, $d\mu = f(x)dx$ where $f \in L^1(dx)$.

(iii) A measure is singular with respect to the Lebesgue measure if there is a set S of full Lebesgue measure ($\mathbb{R} = S + S'$ where S' is a set of measure zero, and $\mu(S) = 0$. Thus μ_{sing} is concentrated on a zero Lebesgue measure set.

(iv) Pure point measures are clearly singular. To distinguish further, we say that a measure μ is continuous if $\mu(x) = 0$ for any point x.

(v) μ_{sing}. A measure is singular continuous if μ is continuous and singular with respect to the Lebesgue measure.

Decomposition theorem. Any measure μ on $\mathbb{R}$ has a canonical decomposition

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sing} \tag{174}$$

$$L^2(\sigma(A), d\mu_\psi) = L^2(\sigma(A), d\mu_{pp}) \oplus L^2(\sigma(A), d\mu_{ac}) \oplus L^2(\sigma(A), d\mu_{sing}) \tag{175}$$

where the norm square in the lhs space is the sum of the square norms of the three rhs spaces. We can then decompose $\varphi \in L^2(\sigma(A), d\mu_\psi)$ in $(\varphi_1, \varphi_2, \varphi_3)$, in the usual orthogonal decomposition sense.

Then, through the unitary transformation, we have

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing} \tag{176}$$

where

$$H_{pp} := U^{-1} L^2(\mathbb{R}, d\mu_{pp})$$

etc. How is A acting on $\mathcal{H}_{pp}$? Assume μ_{pp} is concentrated on the points $x_1, ..., x_n, ...$ We see that $UAU^{-1} = F$. Here we take $g_k = \delta(x = x_k)$. Then, $Fg_k = x_k g_k$, and also any $g \in L^2(\mathbb{R}, d\mu_{pp}) = \sum_{n=1}^{\infty} g_k g(x_k)$ and thus $AU^{-1}g_k = x_k U^{-1} g_k$, and for any $\psi \in H_{pp}$ $\psi = \sum c_k \phi_k$ where $\psi_k = U^{-1} g_k$. Thus, H_{pp} is a Hilbert space where the restriction of A has a complete set of eigenvectors (relative to H_{pp}).

In Quantum Mechanics, these are the bound states, or eigenstates. For instance, for the Hydrogen atom, we have infinitely many bound states, or orbitals, of increasing size. Here, the *energy* operator is unbounded though,

$$H = -\Delta + \frac{1}{r}$$

Likewise, H_{ac} consists of *scattering states*. For the Hydrogen atom, these would be energies too large for the electron to be bound, and it travels to infinity.

H_{sing} is in some sense non-physical, and the struggle is to show it does not exist.

21.2 Multiplicity free operators

Definition 80. *A bounded self-adjoint operator A is multiplicity-free if, by definition, A is unitarily equivalent with Λ on $L^2(\mathbb{R})$.*

Proposition 81. *The following three statements are equivalent:*

1. *A is multiplicity-free*
2. *A has a cyclic vector*
3. *$\{B : AB = BA\}$ is an Abelian algebra*

Exercise 1. *Let A be a diagonal matrix. Show by direct calculation that 2. and 3. hold iff the eigenvalues are distinct.*

Exercise 2. *Does the operator X of multiplication by x on $L^2[0,1]$ have cyclic vectors? How about id/dx?*

21.3 Spectral theorem: continuous functional calculus

Assume N is normal and $\mathfrak{S} = \sigma(N)$. Let $C(\mathfrak{S})$ be the C^*-algebra of continuous functions on $\mathfrak{S}$ in the sup norm.

Let on the other hand $C^*(N)$ be the C^* algebra generated by N. This is the norm closure of the set $\{f(N), f(N^*) : f \text{ analytic}\}$. Since it is a closure in "sup" norm, we get continuous functions in this way. When we need measurable functions, we have to take a strong (not norm) closure.

Theorem 24 (Spectral theorem: continuous functional calculus)**.** *There is a unique isomorphism between $C(\mathfrak{S})$ and $C^*(N) \subset \mathcal{L}(\mathcal{H})$ such that $\phi(x \to x) = N$. Furthermore, the spectrum of $\check{g} \in C(\mathfrak{S})$ is the same as the spectrum of $g(N)$ and if $N\psi = \lambda\psi$, then $f(N)\psi = f(\lambda)\psi$.*

We consider $C_{\mathfrak{S}} = C(\mathfrak{S})$ in the sup norm, and the closure C_N of $P(N, N^*)$ in the operator norm. Let f be an analytic function. We write $\phi(f)(N) = f(N)$ and $\phi(\overline{f})(N) = f^*(N^*)$. It is easy to check that this is an algebra homeomorphism. Furthermore, we have

$$\|P(N)\|_{\mathcal{L}(\mathcal{H})} = \sup_{\lambda \in \mathfrak{S}} |P(\lambda)| = \|P\|_\infty \tag{177}$$

so ϕ is an isometry. In particular, ϕ is one-to-one on its image. The domain of ϕ, $D(\phi)$, contains all functions of the form $P(z)$ and $\overline{P}(z)$ where P is a polynomial, and by the Stone-Weierstrass theorem, the closure of D is $C(\mathfrak{S})$. Thus, ϕ extends by continuity to $C(\mathfrak{S})$. On the other hand, since ϕ is one-to-one isometric, $\phi(C(\mathfrak{S})) = C^*(N)$, the closure of polynomials in N, N^* in the operator norm.

21.4 Spectral theorem, measurable functional calculus form

Now that we have defined continuous functions of N, we can extend calculus to measurable functions by taking strong limits, in the following way.

Recall that:

Lemma 82. *T_n is a sequence of operators which converges strongly iff $\langle\psi, T_n\phi\rangle$ converge for all ψ, ϕ. By the polarization identity, T_n converges strongly iff $\langle\psi, T_n\psi\rangle$ converges for all ψ.*

Proof. This is Theorem VI.1 in [2]. □

We then take the closure of $C^*(N)$ in the strong limit. This is the von Neumann, or W^*-algebra. Strong closure of $C^*(N)$ corresponds to essup-type closure of $C(\sigma(N)$, giving Borel functions. This essup is however calculated with respect to an infinite system of measures, one for each pair of vectors in the original Hilbert space. For every $\psi_1, \psi_2 \in \mathcal{H}$ and $g \in C(\mathfrak{S})$, consider the application $g \mapsto \langle\psi_1, g(N)\psi_2\rangle$. This is a linear functional on $C(\mathfrak{S})$, and thus there is a complex Baire measure so that

$$\langle\psi_1, g(N)\psi_2\rangle = \int_{\mathfrak{S}} g(s)d\mu_{\psi_{12}}(s) \tag{178}$$

(cf. Theorem IV.17 in [2]). (alternatively, we could use the polarization identity and work with positive measures generated by $\langle\phi, g(N)\phi\rangle$. The closure contains all pointwise limits of uniformly bounded sequences.

Proposition 83. *If $g_n \to f$ pointwise and $|g_n| < C\ \forall n$, then $g_n(N)$ converges strongly to an element in $W^*(N)$, which we denote by $f(N)$. We have $\|f(N)\| = \|f\|_\infty$.*

Proof. Let $\mathcal{B}(\mathfrak{S})$ be the bounded Borel functions on $\mathfrak{S}$.

We take a sequence g_n converging in the sense of the Theorem to $f \in \mathcal{B}(\mathfrak{S})$. Then the rhs of (180) converges, by dominated convergence, and so does the left side. Then, the sequence of $g_n(T)$ converge strongly, let the limit be T. We then have,

$$\langle\phi, T\psi\rangle = \int_{\mathfrak{S}} f(s)d\mu_{\psi_{12}}(s) \tag{179}$$

and it is natural to denote T by $f(N)$. Likewise, we could have defined $f(N)$ by

$$\langle\psi, f(N)\psi\rangle = \int_{\mathfrak{S}} f(s)d\mu_{\psi}(s) \tag{180}$$

with $\psi_2 = \psi$ and ψ_1 spanning $\mathcal{H}$.

We have an isometric correspondence Φ between operators $g(T)$, and bounded measurable functions. The isometric isomorphism extends to $W^*(N)$.

Note that the collection of spectral measures only depends on N.

Remark 27. *We note that $f(N)$ is self-adjoint iff f is real-valued. Indeed, in this case we have $f(N) \leftrightarrow f(z) = \overline{f(z)} \leftrightarrow f^*(N^*) = f(N^*)$ since f is real.*

□

Now we can define spectral measures! It is enough to take $E(S) = \chi_S(N)$.

By Remark 27, we have that $E(S)$ are self-adjoint. They are projections, since $P^2 = P$, orthogonal if $S_1 \cap S_2 = \emptyset$. In fact, $\{E(S) : S \in \mathrm{Bor}(\mathfrak{S})$ is a spectral family, the only property to be checked is sigma-additivity, which follows from the same property of characteristic functions, and continuity of Φ.

What is the integral wrt dE? We have, for a simple function $f = \sum f_k \chi(S_k)$, by definition,

$$\int f dE = \sum_k f_k E(S_k) = \Phi^{-1} \sum f_k \chi(S_k) = \Phi^{-1}(f) = f(N)$$

We thus get the projection-valued measure form of the spectral theorem

Theorem 25 (Spectral theorem for bounded normal operators)**.** *Let N be bounded and normal (that is, $NN^* = N^*N$). Then there is a spectral measure E defined on the Borel sets on $\sigma(N)$, see Definition 71, such that*

$$N = \int_{\sigma(N)} z dE(z) \tag{181}$$

Furthermore, if $f \in B(\mathfrak{S})$, we have

$$f(N) = \int_{\sigma(N)} f(z) dE(z) \tag{182}$$

21.5 Changes of variables

Let now $u(s)$ be measurable from $\mathfrak{S}$ to the measurable set $\Omega \in \mathbb{C}$, taken as a measure space with the Borel sets. Then, $E(u^{-1}(O))$ is a spectral family, on $Bor(\Omega)$. It is defined in the following way: if $O \in \Omega$ is measurable, then $u^{-1}(O) = S \in \mathfrak{S}$ is measurable, $\chi(S)$ is well defined, and so is $E_1(O) := \Phi^{-1}(\chi(u^{-1}(O)))$. We have, for a simple function f on Ω: $f = f_k$ on O_k, by definition,

$$\int_\Omega f_k dE_1 = \sum f_k E_1(O_k) = \sum f_k E(u^{-1}(O_k)) = \sum f_k E(S_k) = \int_{\mathfrak{S}} \tilde{f} dE \tag{183}$$

where $\tilde{f} = f_k$ if $s \in S_k$ or, which is the same, $u(s) \in O_k$. In other words, $\tilde{f} = f \circ u$. Thus,

$$\int_{u(\mathfrak{S})} f(t) dE_1(t) = \int_{u(\mathfrak{S})} f(t) d(E \circ u^{-1})(t) \int_{\mathfrak{S}} f(u(s)) dE(s) \tag{184}$$

In particular, assuming u is one-to-one, taking f to be the identity, we have

$$\int_{\mathfrak{S}} u(s) dE(s) = \int_{u(\mathfrak{S})} t dE_1(t) = \int_{u(\mathfrak{S})} t d(E \circ u^{-1})(t) \tag{185}$$

This form of the spectral measure theorem is still a form of functional calculus, it allows to define Borel functions of N, in particular characteristic functions of N which are projections, and the Hilbert space is, heuristically, $\bigoplus_{\lambda\in\sigma(N)} dE(\lambda)(\mathcal{H})$, but this not does not yet really relate the action of N on $\mathcal{H}$ to multiplication by n on $\sigma(N)$. And in fact, $\mathcal{H}$ is not, in general, isomorphic to $L^2(\sigma(N))$ in such a way that N becomes multiplication by N, but in fact to a direct sum, maybe infinite, of such spaces.

Remark 28. *Let N be normal. Then $A = \frac{1}{2}(N + N^*)$ and $B = -\frac{i}{2}(N - N^*)$ are self adjoint, commute with each-other and $N = A + iB$. Thus the spectral theorem for normal operators follows from the one on self-adjoint operators, once we deal with families of commuting ones.*

Let first A be a self-adjoint, operator.

Proposition 84. *If A is multiplicity-free, then there is a measure μ so that $\mathcal{H}$ is isomorphic with $L^2(\sigma(A), d\mu)$ in such a way that the unitary equivalence U has the property $UAU^{-1} = a\cdot$ where $a\cdot$ is the operator of multiplication by a.*

Proof. If f is, say, continuous, then $f(A)\psi$ is dense in $\mathcal{H}$. We then define U on this dense set: $U(f(A)\psi) = f$. We have

$$\|f(A)\psi\| = \langle f(A)\psi, f(A)\psi\rangle = \langle f(A)\overline{f}(A)\psi, \psi\rangle = \langle f\overline{f}(A)\psi, \psi\rangle \\ = \int |f|^2 d\mu_\psi = \|f\|_2 \quad (186)$$

so $f(A)\psi = g(A)\psi$ iff $f = g[d\mu_\psi]$. This is a point is where the type of measure μ_ψ, which depends of course on A, is important. The same equality shows that $\|Uf(A)\psi\| =: \|f\|_2 = \|f(A)\psi\|$ so that U is an isometry, on this dense set, so U extends to an isometry on $\mathcal{H}$. Clearly, $UAf(A)\psi = af(a)$, so $UAU^{-1} = a\cdot$.

In general, we can "iterate" the construction: we take any ϕ and look at the closure $\mathcal{H}_1$ of the orbit of $g(A)\phi$; if this is not the whole of $\mathcal{H}$, then we take ϕ' in the orthogonal complement of $\mathcal{H}_1$ and consider the orbit of that. This construction requires transfinite induction (Zorn's lemma). For separable spaces we can be more constructive, but this is not the main point. Then, $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ and on each, $\mathcal{H}_n$, the theorem holds, with a different measure.

□

Let A be selfadjoint, unbounded (the bounded case has been dealt with). Then $N = (A - i)^{-1}$ is a normal operator, and the spectrum of N is $1/(z - i)(\sigma(A)) = \Omega$, a piece of a circle.

We have

$$N = \int_\Omega n dE(n) = \int_{\sigma(A)} (t - i)^{-1} dE_1(t) \quad (187)$$

We define the operator of multiplication by t on $L^2(\sigma(A))$. To it we attach an operator A:

Theorem 26. *Let A be self-adjoint. There is a spectral measure $E : Bor(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$, so that*

(i)

$$D(A) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} t^2 d\langle E(t)x, x\rangle < \infty \right\}$$

(ii) For $x \in D(A)$ we have

$$Ax = \left(\int_{\mathbb{R}} t dE \right) x$$

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