Contents

1. Review: Complex numbers, functions of a complex variable

∘ Complex numbers, $\mathbb{C}$ form a field; addition, multiplication of complex numbers have the same properties as their counterparts in $\mathbb{R}$.

∘ There is no "good" order relation in $\mathbb{C}$. Except for that, we operate with complex numbers in the same way as we operate with real numbers.

∘ A function f of a complex variable is a function defined on some subset of $\mathbb{C}$ with complex values. Alternatively, we can view it as a pair of real valued functions of two real variables. We write $z = x + iy$ with x, y real and $i^2 = -1$ and write $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. We write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

∘ We note that $i^2 = (-i)^2 = -1$. There is no fundamental distinction between i and $-i$, or an intrinsic way to prefer one over the other. This entails a fundamental symmetry of the theory, symmetry with respect to complex conjugation [1].

∘ Based on the basic properties of complex numbers, we can right away define a number of elementary complex functions: z, $1/z$ and more generally for $m \in \mathbb{Z}$ we easily define z^m and in fact any polynomial $\sum_{m=0}^{K} c_m (z - z_0)^m$.

∘ To be able to define and work with more interesting functions we need to define continuity, derivatives and so on. For this we need to define limits. Seen as a pair of real numbers (x, y), the modulus of z, $|z| = \sqrt{x^2 + y^2}$ gives a measure of length and thus of smallness which induces a natural norm which makes $\mathbb{C}$ a complete metric space. Convergence then reduces to one of real numbers:

$$z_n \to z \quad \Leftrightarrow |z - z_n| \to 0 \quad \text{as} \ \ n \to \infty \tag{1.2}$$

The *topology of* $\mathbb{C}$ is the same of that of $\mathbb{R}^2$, if we identify $z = x + iy$ with the point $(x, y) \in \mathbb{R}^2$. Some basic facts in topology are reviewed in Appendix §42.1.

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[1] More precisely, $z = x + iy \to \overline{z} = x - iy$ is an involution and a field isomorphism of $\mathbb{C}$

In the sequel, a *domain in* $\mathbb{C}$ is an open connected set in $\mathbb{C}$. Examples are disk of radius $r \geqslant 0$ centered at some point $z_0 \in \mathbb{C}$:

$$\mathbb{D}(z_0, r) := \{z : |z - z_0| < r\} \tag{1.3}$$

The special cases $r = 0$ (the empty set, $\emptyset$) and $r = \infty$ (the whole of $\mathbb{C}$) are open. Usually, we will assume that an open set is nontrivial, $r \in \mathbb{R}^+$. The unit disk $\mathbb{D}$

$$\mathbb{D} := \mathbb{D}(0, 1)$$

will play a special role as a canonical choice of a disk.

Exercise 1.1. Show that $z_n \to z$ if and only if $\mathrm{Re}\,(z_n) \to \mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z_n) \to \mathrm{Im}\,(z)$. Using completeness of $\mathbb{R}$ show that $\mathbb{C}$ is a complete normed space.

Definition 1.2. *For functions, limits are similarly reduced to the real case:* $\lim_{z\to z_0} f(z) = a$ *if* $|f(z) - a| \to 0$ *as* $z \to z_0$.

2. Convergent power series

2.1. **Series.** A series is written as

$$\sum_{k=0}^{\infty} a_k \tag{2.2}$$

where a_k are complex, and is said to converge if, by definition, the sequence of *partial sums*

$$S_N := \sum_{k=0}^{N} a_k \tag{2.3}$$

converges as $N \to \infty$.

The series is said to converges absolutely if the real-valued series

$$\sum_{k=0}^{\infty} |a_k| \tag{2.4}$$

converges.

Exercise 2.3. Check that a *necessary* condition of convergence is $a_k \to 0$ as $k \to \infty$ and that absolute convergence implies convergence. Verify that the convergence criteria that you know from real analysis: the ratio test, the n-th root test, in fact any test that does not rely on signs carry over to complex series. The proofs over $\mathbb{C}$ require minor, if any, modifications of the standard proofs in $\mathbb{R}$.

2.2. **Power series.** A power series centered at z_0 is a series of the form

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{2.5}$$

where c_k, z, z_0 are complex.

Theorem 2.4 (Abel). *If for some $z_1 \neq z_0$ the series*

$$\sum_{k=0}^{\infty} c_k (z_1 - z_0)^k \tag{2.6}$$

converges, then

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{2.7}$$

converges absolutely and uniformly in any disk $\mathbb{D}(z_0, r)$ if $r < |z_1 - z_0|$.

For a proof, reduce the question to a familiar one about real series and use the completeness of $\mathbb{C}$.

Abel's theorem tells us that the region of convergence of a power series is a disk (perhaps together with parts of its boundary). The largest r for which a series (2.7) converges for all $z \in \mathbb{D}(z_0, r)$ is called the *radius of convergence*. The disk of convergence may be degenerate: in one extreme situation it is a point, $z = z_0$ (zero radius of convergence) in the other, the whole complex domain ("infinite radius of convergence").

2.3. **Differentiability of power series.**

Theorem 2.5. *If the power series*

$$S(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{2.8}$$

converges in the open disk $\mathbb{D}(z_0, r)$, $r > 0$ *(see Theorem 2.4), then* $S(z)$ *has derivatives of all orders in* $\mathbb{D}(z_0, r)\}$. *In particular,*

$$S'(z) = \sum_{k=0}^{\infty} k c_k (z - z_0)^{k-1} \tag{2.9}$$

$$S''(z) = \sum_{k=0}^{\infty} k(k-1) c_k (z - z_0)^{k-2} \tag{2.10}$$

$$\ldots\ldots \tag{2.11}$$

$$S^{(p)}(z) = \sum_{k=0}^{\infty} k(k-1)\cdots(k-p+1) c_k (z - z_0)^{k-p} \tag{2.12}$$

$$\ldots\ldots\ldots \tag{2.13}$$

and all these series converge in $\mathbb{D}(z_0, r)$ *to the corresponding derivative of* S.

Proof. For the proof we only need to show the result for S': for larger p the proof follows by induction. Furthermore, by taking $z' = z - z_0$ we reduce the problem to the case when $z_0 = 0$. Let $|z| < \rho < r$ and choose h small enough so that $|z| + |h| < \rho$. Note that

$$(z+h)^n - z^n = nz^{n-1}h + \frac{n(n-1)}{2} z^{n-2}h^2 + \cdots + h^n$$

and thus

$$\frac{(z+h)^n - z^n}{h} = nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}h + \cdots + h^{n-1}$$

hence

$$\left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| = \left| \frac{n(n-1)}{2} z^{n-2}h + \cdots + h^{n-1} \right| \tag{2.14}$$

$$\leqslant \frac{n(n-1)}{2} |z|^{n-2}|h| + \cdots + |h|^{n-1} \leqslant \frac{n(n-1)}{2} |\rho|^{n-2}|h|$$

The last inequality is obtained by replacing z, h by $|z|, |h|$ respectively, repeating otherwise the calculations above, and applying Taylor's remainder formula to the *real valued* function $(|z| + |h|)^n$ at the end:

$$(|z| + |h|)^n - |z|^n - n|z|^{n-1}|h| = \tfrac{n(n-1)}{2}(|z| + \delta)^{n-2}|h|^2 \quad \text{with } \delta \in (0, |h|).$$

Thus, for the partial sums $S_N(z) = \sum_{k=0}^{N} c_k (z - z_0)^k$ we have

$$\left| \frac{S_N(z+h) - S_N(z)}{h} - S_N'(z) \right| \leqslant K \frac{h}{r'^2} \sum_{k=0}^{N} \frac{k(k-1)}{2} \left(\frac{\rho}{r'}\right)^{k-2}$$

for some $r' \in (\rho, r)$ and $K > 0$ (check!) and taking $N \to \infty$

$$\left|\frac{S(z+h)-S(z)}{h} - S'(z)\right| \leqslant K\frac{h}{r'^2}\sum_{k=0}^{\infty}\frac{k(k-1)}{2}\left(\frac{\rho}{r'}\right)^{k-2} \tag{2.15}$$

The series on the right hand side of (2.15) is convergent, ρ is independent of h and thus the right hand side of (2.15) converges to zero as $h \to 0$. □

Corollary 2.6. *Show that* $S^{(k)}(z_0) = k!c_k$ *and thus* (2.8) *is the convergent Taylor series of* S.

Corollary 2.7. *Assume* $S(z)$ *converges in a disk* $\mathbb{D}(z_0, r)$ *and that there is a sequence* $\{z_n\}_{n\in\mathbb{N}}$ *with an accumulation point at* z_0 *so that* $S(z_n) = 0$ *for all* $n \in \mathbb{N}$. *Then* $S(z)$ *is identically zero.*

Proof. We can assume without loss of generality that the sequence itself converges to z_0 and $z_0 = 0$. We show that this implies that all coefficients of $S(z)$ are zero. We write

$$S(z) = c_0 + zT(z) \tag{2.16}$$

where T converges in $\mathbb{D}(0, r)$. We have, by assumption

$$S(z_n) = 0 = \lim_{n\to\infty}[c_0 + z_nT(z_n)] \tag{2.17}$$

and thus $c_0 = 0$. From here we proceed by induction, as $S(z)/z$ is a power series with the same properties as S etc. (check!) ∎

2.4. **Some basic functions.**

○ *The exponential.* We define

$$e^z = \sum_{k=0}^{\infty}\frac{z^n}{n!} \tag{2.18}$$

This series converges for any $z \in \mathbb{C}$ and thus it is differentiable for any z in $\mathbb{C}$ by Theorem 2.5. We have, by (2.9)

$$(e^z)' = e^z \tag{2.19}$$

Thus,

$$(e^ze^{-z})' = 0 \tag{2.20}$$

and thus e^ze^{-z} does not depend on z, and takes the same value everywhere, the value for $z = 0$. But we see immediately that $e^0 = 1$. Thus

$$e^ze^{-z} = 1 \quad \Leftrightarrow e^{-z} = 1/e^z \tag{2.21}$$

Exercise 2.8. *We tacitly used something more: what? Fill in the missing details.*

In the same way,

$$(e^{z+a}e^{-z})' = (e^{z+a})'e^{-z} + e^{z+a}(e^{-z})' = 0 \\ \Leftrightarrow e^{z+a}e^{-z} = e^a e^0 = e^a \quad \Leftrightarrow e^{z+a} = e^z e^a \tag{2.22}$$

which provides us with the fundamental property of the exponential. Also, we immediately check Euler's formula: for $\phi \in \mathbb{R}$ we have

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{2.23}$$

Exercise 2.9.

$$e^s = 1 \quad \Leftrightarrow s = 2N\pi i, \quad N \in \mathbb{Z} \tag{2.24}$$

∘ *The logarithm.* In the complex domain the log is a trickier function. For the moment we look at a simpler question, that of defining $\log(1+z)$ only for $|z| < 1$. This is done via the convergent Taylor series

$$\log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots \tag{2.25}$$

By (2.9) we get

$$\frac{d}{dz}\log(1+z) = 1 - z + z^2 - z^3 + \cdots = \frac{1}{1+z} \quad \text{if} \quad |z| < 1 \tag{2.26}$$

Exercise 2.10. *Show that if $|s|$ is small we have*

$$\log(e^s) = s; \qquad e^{\log(1+s)} = 1 + s$$

We will return later to the question of defining $\log z$ for more general $z \in \mathbb{C}$, $z \neq 0$ and we will study its properties carefully. It is one of the fundamental "branched" complex functions.

2.5. **Operations with power series.** If S and T are power series convergent at z_0, then so are $S + T, S{\times}T$, also S/T if $T(z_0) \neq 0$ and $S(T)$ if $T(z_0) = z_0$ etc. Formulas for these new series are obtained by working with the series as if they were polynomials. For instance,

$$ST = s_0t_0 + (s_1t_0 + s_0t_1)(z - z_0) + (s_2t_0 + s_1t_1 + s_0t_2)(z - z_0)^2 + \cdots \tag{2.27}$$

Exercise 2.11. *(a) If S and T are two power series with radius of convergence r, then ST has radius of convergence at least r.*

(b) Write three terms of the series S/T if $T(z_0) \neq 0$.

(c) Under the assumptions above, show that S/T has nonzero radius of convergence.*

3. Differentiability

Definition. A complex function is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$.

Exercise 3.12. Show that polynomials are continuous in $\mathbb{C}$.

Likewise, we can now define differentiability.

Definition. A function f is differentiable at z_0 if, by definition, there is a number, call it $f'(z_0)$ such that

$$\frac{f(z) - f(z_0)}{z - z_0} \to f'(z_0) \quad \text{as} \quad z \to z_0$$

Exercise 3.13. Show that differentiation has the properties we are familiar with from real variables: sum rule, product rule, chain rule etc. hold for complex differentiation. (As you will see, proving this amounts to nothing more than mimicking the proofs over the reals.)

3.1. **The Cauchy-Riemann equations.**

Analytic functions can be defined by many equivalent properties, that we will soon explore.

Definition 3.14. *The function f defined in a domain $\mathcal{D}$ is analytic in $\mathcal{D}$ if it is differentiable at all points in $\mathcal{D}$.*

Differentiability in $\mathbb{C}$ is far more demanding than differentiability in $\mathbb{R}$. For the same reason, complex differentiable functions are much more regular and have better properties than real-differentiable ones.

We will see that if f is analytic, then its derivative is also analytic, implying that f has continuous derivatives of all orders.

We will also see later that analyticity in a domain $\mathcal{D}$ is equivalent to the convergence of the Taylor series at all points $z_0 \in \mathcal{D}$.

As a first definition equivalent to differentiability, an analytic function is a function which satisfies the Cauchy-Riemann (C-R) equations:

Theorem 3.15 (C-R)**.** *(1) Assume that $f = u + iv$ is analytic in a domain $\mathcal{D}$ in $\mathbb{C}$. Then the Cauchy-Riemann equations hold:*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{3.2}$$

throughout $\mathcal{D}$ and u, v are continuously differentiable in $\mathcal{D}$ ("belong to $C^1(\mathcal{D})$").

(2) Conversely, if (u, v) are differentiable and satisfy (3.2) in $\mathcal{D}$, then f is differentiable in $\mathcal{D}$.

Proof. (1) Let $f(z) = u(x,y) + iv(x,y)$ and $f'(z_0) = a + ib$. We have

(3.3)
$$f(z)-f(z_0) = u(x,y)-u(x_0,y_0)+iv(x,y)-iv(x_0,y_0) = [f'(z_0)+\varepsilon(z)](z-z_0)$$
$$= (a+ib)[x-x_0+i(y-y_0)] + \varepsilon(z)(z-z_0)$$
$$= a(x-x_0) - b(y-y_0) + ia(y-y_0) + ib(x-x_0) + \varepsilon(z)(z-z_0)$$

where $\varepsilon(z) \to 0$ as $z \to z_0$ implying u_x, u_y, v_x, v_y exist at z_0 and satisfy the C-R equations.

(ii) Differentiability of u and v at (x_0, y_0) implies

(3.4) $$f(z) - f(z_0) = u(x,y) - u(x_0,y_0) + iv(x,y) - iv(x_0,y_0)$$
$$= u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)$$
$$+iv_x(x_0,y_0)(x-x_0)+iv_y(x_0,y_0)(y-y_0)+\varepsilon(x,y)(x-x_0)+\eta(x,y)(y-y_0)$$

where ε and η go to zero as $z \to z_0$. ∎

Exercise 3.16. Show that (3.3) and (3.4) are compatible if and only if (3.2) hold. (The real and imaginary parts must be equal to each other, and $x - x_0$ and $y - y_0$ are independent quantities.)

3.2. **Analyticity at infinity.** As $|z| \to \infty$, $1/z \to 0$. By definition f is *analytic at infinity* if $f(1/z)$ is analytic at zero.

4. Integrals

Integration plays an important role in complex analysis. As we shall see, the derivative of a function can be written as an integral, and many of the nice properties of analytic functions originate in this fact.

If $f(t) = u(t)+iv(t)$ is a complex-valued function of one real variable t then $\int_a^b f(t)dt$ is defined by

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt \tag{4.2}$$

This reduces the questions of complex integration to the familiar real integration.

Note 4.17. *In the following, unless otherwise specified, we assume that the curves we use are piecewise differentiable.*

Note.

Let $\gamma(t) = x(t) + iy(t)$, $t \in [a,b]$ be a parametrized curve. We define using (4.2)

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Note that the sign of the integral depends on the *orientation* of γ, specified by stating that t *goes from* a *to* b, rather than only $t \in [a, b]$. It is natural to say that $\gamma(a)$ is the starting point of γ, and $\gamma(b)$ is its final point. The same geometric curve with opposite orientation is denoted by $-\gamma$; a formula is easily found as $(-\gamma)(t) = x(a+b-t)+iy(a+b-t)$, $t \in [a, b]$. We see that

$$\int_{-\gamma} f(z)dz = \int_b^a f(\gamma(t))\gamma'(t)dt = -\int_\gamma f(z)dz \tag{4.3}$$

If $\gamma(a) = \gamma(b)$ the curve is called *closed*. Positive orientation, the counterclockwise one, is assumed (unless otherwise specified), and we denote $\oint_\gamma := \int_\gamma$.

Exercise 4.18. *Show that*

$$\int_\gamma f(z)dz = \int_\gamma \Big(udx - vdy\Big) + i\int_\gamma \Big(udy + vdx\Big) \tag{4.4}$$

Exercise 4.19. *Show that the integral along a curve (as a set) depends on the parametrization of the curve it only through a sign.*

A curve is called *simple* if it has no self-intersections. For example, the circle is simple, but the figure "8" is not.

A domain $\mathcal{D}$ is called *simply connected* if any simple closed curve γ contained it $\mathcal{D}$ can be deformed to a point continuously through curves completely contained in $\mathcal{D}$. More precisely, there is a continuous function of two variables $F(t, s) = x(t, s) + iy(t, s)$ defined on $[a, b] \times [0, 1]$ with values in $\mathcal{D}$ such that $F(t, 0) = \gamma(t)$ and $F(t, 1) = p$, a point in $\mathcal{D}$. Intuitively, a simply connected domain has no holes. For example a disk is a simply connected domain, but a punctured disk, or an *annulus*: $\{z \in \mathbb{C} | \, r < |z| < R\}$, are not simply connected.

Theorem 4.20 (Cauchy)**.** *Assume $\mathcal{D}$ is a simply connected domain and that f is continuously differentiable in $\mathcal{D}$. If γ is a piecewise differentiable simple closed curve contained in $\mathcal{D}$ then*

$$\oint_\gamma f(z)dz = 0 \tag{4.5}$$

Proof. Start with the decomposition (4.4) and use Green's theorem to write

$$\int_\gamma \Big(udx - vdy\Big) = -\int\int_{\mathrm{Int}(\gamma)} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy = 0 \tag{4.6}$$

which vanishes by (3.2) The second integral in (4.4) is dealt with similarly. □

It is sometimes useful to integrate analytic functions along the boundary of their analyticity domain. This can be done for instance if f is continuous up to this boundary; Cauchy's theorem still holds:

Exercise 4.21. *Assuming that f is analytic on $\mathcal{D}$, continuous on $\overline{\mathcal{D}}$, and that $\overline{\mathcal{D}}$ is a rectifiable curve of winding number one show that* (4.5) *holds if γ is a simple closed curve in $\overline{\mathcal{D}}$.*

5. Cauchy's formula

5.1. **Homotopic curves.** Let $\mathcal{D}$ be domain in $\mathbb{C}$. Two curves in $\mathcal{D}$ are said to be *homotopic in* $\mathcal{D}$ if they can be continuously deformed into each other by a deformation inside $\mathcal{D}$ (see paragraph preceding Theorem 4.20). For example, if $\mathcal{D}$ is simply connected then any simple closed curve is homotopic to a point, see Fig. 1. As another example, if $\mathcal{D}$ is the annulus $\{z\,|1 < |z| < 2\}$ then all circles $|z| = r$ with $1 < r < 2$ are homotopic to each other, but not to a point, while any simple closed curve not going around 0 is homotopic to a point.

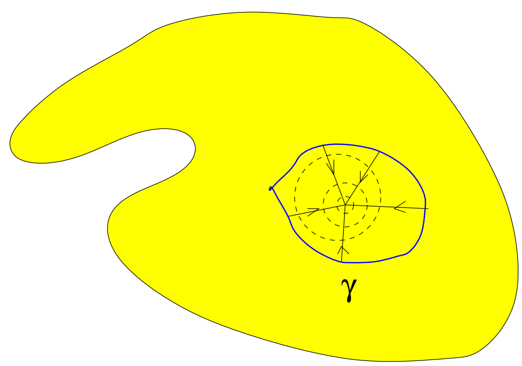


Figure 1. All dotted curves inside γ are homotopic to each other and to the central point.

We will find useful to consider curves $\gamma_{1,2}$ in $\mathcal{D}$, say given by two functions $\gamma_{1,2}(t)$ for $t \in [a, b]$, which have the same endpoints, $\gamma_1(a) = \gamma_2(a)$, and $\gamma_1(b) = \gamma_2(b)$. Two such curves are called *homotopic with fixed endpoints* if they can be continuously deformed into each other through a transformation preserving the endpoints with range within $\mathcal{D}$, see Fig. 2.

5.2. **Independence of the integral on the path.** Line integrals are additive w.r.t. the domain of integration: consider an oriented curve γ_1, and then let γ_2 start at the final point of γ_1, say, $\gamma_1(t)$ for t from

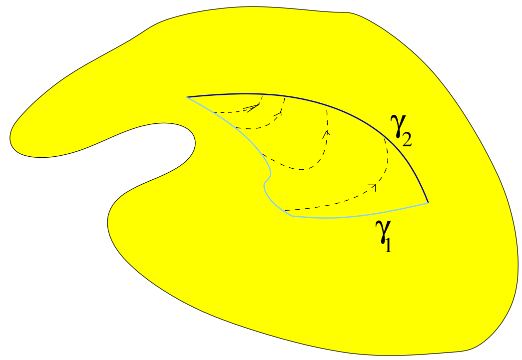


FIGURE 2. Two homotopic curves, γ_1 and γ_2.

a to b, $\gamma_2(t)$ for t from b to c, with $\gamma_1(b) = \gamma_2(b)$. We denote for short $\gamma_1 + \gamma_2$ the concatenated curve from t from a to c and we have by definition

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz \tag{5.2}$$

(i) Suppose f is analytic in a simply connected domain $\mathcal{D}$, and $\gamma_{1,2}$ are two oriented curves in $\mathcal{D}$, having the same endpoints. Then $\gamma = \gamma_1 - \gamma_2$ is a closed curve and by (4.5), (5.2), (4.3) we find

$$0 = \int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$$

and therefore *the integral of an analytic function on a simply connected domain is path independent*:

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz \tag{5.3}$$

(ii) If the domain is not simply connected, formula (5.3) clearly still holds if the domain enclosed by the closed curve $\gamma_1 - \gamma_2$ is simply connected and completely included in $\mathcal{D}$; in other words, (5.3) *holds if if* γ_1 *and* γ_2 *are homotopic with fixed endpoints in* $\mathcal{D}$, see Fig. 3.

(iii) Moreover, (5.3) *holds if* $\gamma_{1,2}$ *are closed curves homotopic in* $\mathcal{D}$.

This is easily seen by writing $\gamma_1 - \gamma_2$ as a sum of closed curves, each homotopic in $\mathcal{D}$ to a point.

5.3. **Cauchy's Formula.** Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and $z_0 \in \mathcal{D}$. The functions $(z-z_0)^{-n}$, $n = 1, 2, \ldots$ are analytic in $\mathcal{D}\setminus\{z_0\}$. Thus, if γ_1 and γ_2 are two closed curves in $\mathcal{D}$ not passing through z_0, and homotopic

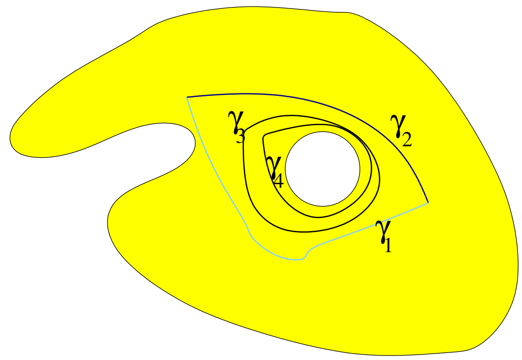


FIGURE 3. γ_1 and γ_2 are not homotopic in the yellow domain while γ_3 and γ_4 are.

to each-other in $\mathcal{D} \setminus \{z_0\}$ then

$$\int_{\gamma_1} (z - z_0)^{-n} dz = \int_{\gamma_2} (z - z_0)^{-n} dz \tag{5.4}$$

Clearly, these integrals are zero if γ_i does not contain z_0 inside. To calculate the integrals on a simple closed curve encircling z_0 it suffices, by (5.4), to do calculation when the curve is a circle, which can be done explicitly. Indeed, a circle centered at z_0 with radius ρ is parametrized by $z = z_0 + \rho(\cos(t) + i\sin(t)) = \rho e^{it}$, $t \in [0, 2\pi]$ (where we used Euler's formula), and we get

$$\oint \frac{dz}{(z - z_0)^n} = \frac{i}{\rho^{n-1}} \int_0^{2\pi} e^{-i(n-1)t} dt = \begin{cases} 2\pi i & \text{if } \; n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

Theorem 5.22 (Cauchy's formula)**.** *If f is analytic in the simply connected domain $\mathcal{D}$ and γ is piecewise a differentiable simple closed curve in $\mathcal{D}$ around z, we have*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{s - z} ds \tag{5.6}$$

Proof. We write

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{s - z} ds = \frac{1}{2\pi i} \oint_\gamma \frac{f(s) - f(z)}{s - z} ds + \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{s - z} ds = f(z) \tag{5.7}$$

where the middle interval vanishes by Cauchy's theorem 4.20.

6. Taylor series of analytic functions

Assume f is analytic in $\mathcal{D}$ and let $z_0 \in \mathcal{D}$. Consider the disk

$$\mathbb{D}(z_0, \rho) = \{s : |s - z_0| < \rho\}$$

with ρ small enough so that its closure $\overline{\mathbb{D}}(z_0, \rho) = \{s : |s - z_0| \le \rho\}$ is contained in $\mathcal{D}$. Let $C(z_0, \rho) = \partial\mathbb{D}(z_0, \rho)$ be its boundary (the circle of radius ρ centered at z_0).

By Theorem 5.22 we have, for $z \in \mathbb{D}(z_0, \rho)$

$$f(z) = \frac{1}{2\pi i} \oint_{C(z_0,\rho)} \frac{f(s)}{s - z} ds \tag{6.2}$$

We write

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{s - z_0 - (z - z_0)} \\ &= \frac{1}{s - z_0}\left[1 + \frac{z - z_0}{s - z_0} + \cdots + \left(\frac{z - z_0}{s - z_0}\right)^n\right] + \frac{1}{s-z}\left(\frac{z - z_0}{s - z_0}\right)^{n+1} \end{aligned} \tag{6.3}$$

and thus

$$\begin{aligned} f(z) = \frac{1}{2\pi i} \sum_{k=0}^{n} \oint_{C(z_0,\rho)} (z - z_0)^k \frac{f(s)}{(s - z_0)^{k+1}} ds \\ + \frac{1}{2\pi i} \oint_{C(z_0,\rho)} \frac{f(s)}{(s - z)} \frac{(z - z_0)^{n+1}}{(s - z_0)^{n+1}} \end{aligned} \tag{6.4}$$

therefore

$$f(z) = \sum_{k=0}^{n} c_k (z - z_0)^k + E(z, z_0, n) \tag{6.5}$$

where

$$c_k = \frac{1}{2\pi i} \oint_{C(z_0,\rho)} \frac{f(s)}{(s - z_0)^{k+1}} ds$$

and

$$E(z, z_0, n) = \frac{1}{2\pi i} \oint_{C(z_0,\rho)} \frac{f(s)}{(s - z)} \frac{(z - z_0)^{n+1}}{(s - z_0)^{n+1}} \tag{6.6}$$

It is not hard to see that the remainder $E(z, z_0, n) \to 0$ as $n \to \infty$, and this implies that series $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ converges to $f(z)$, by the definition of convergent series. But we can do better, we can provide an estimate for the remainder, as follows.

Let R so that $|z-z_0|<R<\rho$. The path $C(z_0,\rho)$ can be replaced by $C(z_0,R)$ in (6.6), and then a direct estimate gives

$$\begin{aligned}(6.7)\quad \big|E(z,z_0,n)\big| &= \Big|\frac{1}{2\pi i}\oint_{C(z_0,R)}\frac{f(s)}{(s-z)}\frac{(z-z_0)^{n+1}}{(s-z_0)^{n+1}}\Big| \\ &\leqslant \frac{1}{2\pi}\max_{\overline{D}(z_0;\rho)}|f|\,\frac{|z-z_0|^{n+1}}{(R-|z-z_0|)\;R^{n+1}}\,2\pi R \\ &= \max_{\overline{D}(z_0;\rho)}|f|\,\frac{|z-z_0|^{n+1}}{(R-|z-z_0|)\;R^{n}} = \frac{\alpha^n}{1-\alpha}\max_{\overline{D}(z_0;\rho)}|f|;\;\; \alpha:=\frac{\rho}{R}\end{aligned}$$

From these considerations it follows that

Theorem 6.23. *If f is analytic in a domain $\mathcal{D}$ and $z_0\in\mathcal{D}$, then f has derivatives of any order at z_0.*

Therefore if f is analytic on $\mathcal{D}$, so are f', f'', etc.

Corollary 6.24. *If f is differentiable in the domain $\mathcal{D}$ then it is continuously differentiable and thus analytic.*

Proof. As we saw, $F(z)=\int_a^z f(s)ds$ is continuously differentiable thus analytic. Therefore $F''=f'$ exists in $\mathcal{D}$ implying f' is continuous in $\mathcal{D}$. ∎

Furthermore:

Theorem 6.25 (Taylor series; Cauchy's formula for higher derivatives)**.** *If $f(z)$ is continuously differentiable in $\mathcal{D}$ and $z_0\in\mathcal{D}$ then there exists ρ such that, for $z\in\mathbb{D}(z_0;\rho)$ we have*

$$(6.8)\quad f(z)=\sum_{k=0}^{\infty}c_k(z-z_0)^k \quad \textit{where}$$

$$c_k=\frac{1}{2\pi i}\oint_{C(z_0;\rho)}\frac{f(s)}{(s-z_0)^{k+1}}ds=\frac{f^{(k)}(z_0)}{k!}$$

Proof. We have already proved everything in this theorem except the last equality, which follows since by Theorem 2.5

$$f^{(n)}(z)=\sum_{k=n}^{\infty}c_k\,\frac{k!}{(k-n)!}\,(z-z_0)^{k-n}$$

which for $z=z_0$ gives $f^{(n)}(z_0)=c_n\,n!$, hence the last equality of (6.8). □

Note 6.26. The expression of $f^{(k)}$ as an integral makes differentiation a "smooth" operation on analytic functions, unlike usual differentiation in real analysis.

Remark 6.27. The disk of convergence of the Taylor series of an analytic function cannot, by the estimate (6.7), be zero. We claim that the radius of convergence of the series exactly equals the radius r of the largest disk centered at z_0 where f is analytic ("the distance to the nearest singularity"), see Fig. 4. Indeed, in any smaller disk we can

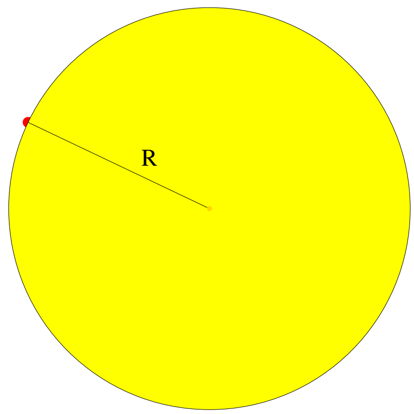


FIGURE 4. Disk of convergence of a Taylor series where the yellow region is a domain of analyticity and the red dot is a singularity.

apply Theorem 6.25 above. If the radius of convergence were larger than r, f would be analytic in a larger domain since convergent power series are, as we have seen, analytic.

Example. Consider the function $\frac{1}{1+z^2}$. Its Taylor series at $z = 0$ is

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty}(-1)^{k+1}z^{2k}, \quad \text{convergent for } |z| < 1$$

and on the boundary of the disk of convergence there are singularities of $\frac{1}{1+z^2}$, namely $z = \pm i$.

7. More properties of analytic functions

Assume f is analytic in $D(z_0, \varepsilon)$ and all derivatives of f are zero at z_0. Then f is zero in the whole of $D(z_0, \varepsilon)$ (check). More is true.

Proposition 7.28. *Assume f is analytic in a* domain *$\mathcal{D}$ and all derivatives at $z_0 \in \mathcal{D}$ of f are zero. Then f is identically zero in $\mathcal{D}$.*

Proof. For any $z \in \mathcal{D}$ there is a polygonal line P joining z_0 to z: segments $[z_{j-1}, z_j]$, $j = 1, \ldots, n$ (with $z_n = z$) and disks $\mathbb{D}(z_j, r_j) \subset \mathcal{D}$. (see Proposition 42.18).

............mine

Elementary geometry arguments show that we can find some $\rho > 0$ so that any disk centered at a point on P and of radius ρ is contained in $\mathcal{D}$. We cover P by a finite number of disks of radius ρ, centered at the equally spaced points on P, so that the center of each disk is contained in the previous one (for example, their centers are $\rho/2$ distance apart). The first disk is centered at z_0. Then f is identically zero on the first disk. This means that f and all its derivatives are zero at the center of the second disk, hence f is identically zero on the second disk as well. The argument is continued up to the last disk, showing that $f(z) = 0$.

...........yours

Since $\partial\mathcal{D} = (\mathbb{C} \setminus \mathcal{D}) \cap \overline{\mathcal{D}}$, then $\partial\mathcal{D}$ is compact, $\mathrm{dist}(P, \partial\mathcal{D}) := a > 0$ and P is contained in $D = \cup_{z\in P} D(z, a) \subset \mathcal{D}$. We can choose a finite subset $\{z_i = z_1, ..., z_n = z_f\} \in P$ (where z_i are considered as ordered successively on P) such that $P \subset \cup_{z_1,...,z_n} D(z_i, a) \subset \mathcal{D}$. The set $\{z : f(z) = 0\}$ is closed since f is continuous.

Either either f is identically zero on $\mathcal{D}$, or there is a smallest j so that f is not . Now, $f \equiv 0$ in $D(z_{j-1}, a)$ and since D is a covering of P, $D(z_{j-1}, a) \cap D(z_j, a) = S \neq \emptyset$. By elementary geometry, we can find a finite set of disks of radius $\varepsilon < a$, the first one contained in S and the center of every one of them contained in the previous disk, the last one centered at z_j. By local Taylor expansions of f in these disks we get a contradiction (how?).

Exercise 7.29. *** Permanence of relations. Use Proposition 7.28 to show that* $\sin^2 z + \cos^2 z = 1$ *in* $\mathbb{C}$. *Relations between analytic functions that hold in* $\mathbb{R}$ *extend in* $\mathbb{C}$. *Formulate and prove a theorem to this effect.*

See also §??.

Theorem 7.30 (Morera)**.** *Let f be continuous in a simply connected domain $\mathcal{D}$ and such that $\oint_\gamma f ds = 0$ for any simple piecewise differentiable closed curve γ contained in $\mathcal{D}$. Then f is analytic in $\mathcal{D}$. The same is true if we restrict the set of curves γ to triangles.*

Proof. Let $z_0 \in \mathcal{D}$ and let $F(z) = \int_{z_0}^{z} f(s)ds$. Here the integral is along any path from z_0 to z which is contained in $\mathcal{D}$; note that the value of the integral does not depend on the choice of the path, by the assumption of the theorem.

Then F is continuously differentiable in $\mathcal{D}$ and $F' = f$ (this is a straightforward calculation: check). By Theorem 6.23 f is analytic. ☐

Exercise 7.31. *Find a similar argument in the case when the set of curves γ is restricted to triangles.*

We have now three equivalent views of analytic functions: as differentiable functions of z, as sums of power series, and as continuous functions with zero loop integrals. All these characterizations are quite valuable.

Theorem 7.32 (Weierstrass's theorem)**.** *Assume that f_n are analytic in the domain Ω and converge uniformly on any compact set in Ω to f. Then f is analytic in Ω. Furthermore, $f_n' \to f'$ uniformly on any compact set in Ω.*

Note 7.33. *Clearly this implies that for any $k \in \mathbb{N}$ $f_n^{(k)} \to f^{(k)}$ uniformly on any compact set in Ω.*

Proof. Let T be a triangle contained in the compact $K \subset \Omega$. Then, by analyticity,

$$\int_T f_n(z)dz = 0 \tag{7.2}$$

Uniform convergence that implies f is continuous and

$$\int_T f(z)dz = 0 \tag{7.3}$$

Using Morera's theorem, we see that f is analytic. The properties of the derivatives are immediate, by Cauchy's formula. ▮

Corollary 7.34. *Assume γ is a piecewise differentiable curve and that $f(\cdot, t)$ is analytic in a neighborhood of γ for any t. If γ is of infinite length, assume additionally that the radius of analyticity has a nonzero lower bound ε along the curve and $\int_\gamma |f(z+a,t)|d|t|$ exists for any a with $|a| < \varepsilon$ to apply Fubini's theorem.*

Proof. Use Cauchy's formula for the derivative and justify the interchange of orders of integration. ▮

Corollary 7.35 (Liouville's theorem)**.** *A function which is* **entire** *(meaning analytic in all of $\mathbb{C}$) and bounded in $\mathbb{C}$ is constant.*

Proof. Let M be the maximum of $|f|$ in $\mathbb{C}$. We have, by Theorem 6.25

$$f'(z) = \frac{1}{2\pi i}\oint_{C(0;\rho)} \frac{f(s)}{(s-z)^2}ds \tag{7.4}$$

and thus

$$|f'(z)| \leqslant \frac{1}{2\pi}M\frac{1}{\rho^2}2\pi\rho = M/\rho \tag{7.5}$$

Since this is true for any ρ, no matter how large, it follows that $f'(z) \equiv 0$. Then f is a constant. □

Exercise 7.36. ** Show that an entire function other than a polynomial must grow faster than any power of $|z|$ along some path as $z \to \infty$.*

8. The fundamental theorem of algebra

One classical application of Liouville's Theorem is the Fundamental Theorem of Algebra: *a polynomial $P_n(z)$ of degree n has exactly n roots in $\mathbb{C}$, counting multiplicity.*

For its proof, note that it is enough to show that any nonconstant polynomial has at least one root $\mathbb{C}$; then the argument can be completed by induction on the degree of the polynomial.

Exercise 8.37. *Let P be a nonconstant polynomial. Then there exists an R s.t. $1/P$ is analytic in the domain $\{z : |z| > R\}$ and $1/P(z) \to 0$ as $|z| \to \infty$.*

Now it is clear that any polynomial nonconstant $P_n(z)$ must have a root, since otherwise $1/P_n(z)$ would be entire and bounded (check this, using for instance Exercise 8.37).

9. Harmonic functions

A real-valued, C^2 function $u(x, y)$ which satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{9.2}$$

in some domain U is called *harmonic* in U.

Theorem 9.38. *Let $\mathcal{D}$ be a simply connected domain in $\mathbb{C}$. A function u is harmonic in $\mathcal{D}$ if and only if u is the real part of an analytic function: $u = Re(f)$ with f analytic in $\mathcal{D}$; f is unique up to an arbitrary imaginary constant.*

Note 9.39. Of course, in the theorem above, "real part" can be replaced by "imaginary part".

Proof. If $u = \mathrm{Re}(f)$ then $u \in C^\infty$ (check this, for instance by using the Taylor series of f). Then (9.2) follows immediately from the CR equations. In the opposite direction, consider the field $\mathbf{E} = (-u_y, u_x)$. We check immediately that this is a potential field and thus $\mathbf{E} = \nabla v$ for some v (unique up to an arbitrary constant). But then, by the CR theorem, $u + iv$ is analytic in $\mathcal{D}$. □

In other words, for any harmonic function u, there exists a function v, harmonic on the same domain, and unique up to an additive constant, so that $u + iv$ is analytic. The function v is called *harmonic conjugate of u*.

Lemma 9.40. *Let $f = u + iv$ be analytic near a point z_0 and assume $f'(z_0) \neq 0$. Then the curves $u(x,y) = u(x_0,y_0)$ and $v(x,y) = v(x_0,y_0)$ exist near z_0, they are smooth and orthogonal to each-other.*

The fact that $f'(z_0) \neq 0$ implies, by C-R that $\nabla u, \nabla v$ are nonzero at x_0, y_0. The rest follows from the implicit function theorem in $\mathbb{R}^2$, and from $\nabla u \cdot \nabla v = 0$, a consequence of C-R.

9.1. **Digression: Potential and Hamiltonian flows.** Consider an autonomous system of ODEs in a domain $\mathcal{D}$: $\dot{x} = E_1(x,p); \dot{p} = E_2(x,p)$ is a Hamiltonian system if there is an $H \in C^1(\mathcal{D})$ s.t. $E_1 = \frac{\partial H}{\partial p}$ and $E_2 = -\frac{\partial H}{\partial x}$. It is a potential system if there is a $V \in C^1(\mathcal{D})$ s.t. $E_1 = \frac{\partial V}{\partial x}$ and $E_2 = \frac{\partial V}{\partial p}$. We see that a system is *both* potential and Hamiltonian if there exist two functions H and V s.t. $\frac{\partial H}{\partial p} = \frac{\partial V}{\partial x}$ and $\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial p}$. If H and V are smooth enough, then a system is both potential and Hamiltonian **iff** H and V are harmonic functions, the real and imaginary partof an analytic function. In Hamiltonian systems, H is a conserved quantity that is $\frac{d}{dt}H(x(t),y(t)) = \frac{\partial}{\partial H}\dot{x} + \frac{\partial}{\partial H}\dot{p} = 0$ for any solution, as can be easily checked. In gradient systems, $\langle \dot{x}, \dot{p}\rangle$ (when nonzero) clearly gives the direction of steepest ascent of V at the point $\langle x, p\rangle$.

10. The maximum modulus principle

An analytic function in a domain $\mathcal{D}$ can attain its maximum absolute value only on the boundary of $\mathcal{D}$:

Theorem 10.41. *Assume f is analytic and nonconstant in the domain $\mathcal{D}$. Then $|f|$ has no maximum point in $\mathcal{D}$, unless f is a constant.*

Usually the proofs use Cauchy's formula. Look up these other proofs, because they extend to harmonic functions in more than two dimensions.

We will give a proof based on Taylor series.

Proof. Assume that $z_0 \in \mathcal{D}$ is a point of maximum. (Recall that $\mathcal{D}$ is by definition open). The result is easy if $\max_{\mathcal{D}} |f| := M = 0$ (check). Replacing f by f/M and z by $z - z_0$ without loss of generality, we can assume that $M = 1$ and $z_0 = 0$. If f is not 1 everywhere, then there exists $k > 0$ so that the Taylor coefficient c_k of f at 0 is nonzero, and in some $D(0,\rho)$ we have

$$f(z) = 1 + c_k z^k + c_{k+1}z^{k+1} + \cdots = 1 + c_k z^k E(z);\ E(z) := 1 + d_k z + d_{k+1}z^2 + \cdots$$

Let z_1 be s.t. $c_k z_1^k \in \mathbb{R}^+$ and small enough s.t. $\operatorname{Re}(E(z_1)) > 0$ (check that this is possible) we get $|f(z_1)| > 1$, a contradiction. $\square$

Exercise 10.42. *Show that if* $|f|$ *has a* minimum *in* $\mathcal{D}$*, then this minimum is zero.*

Let f be analytic in a neighborhood of z_0 and $k \in \mathbb{N}$ be the least positive index for which $c_k \neq 0$. A direction $d \in \mathbb{C}$ is a *steepest ascent* direction at z_0 of f if $c_k d^k \in \mathbb{R}^+$. We saw that these exist at any point in the domain of analyticity of f. As its name suggests, this is the local direction of the fastest increase of the modulus of f. A curve γ that follows at each point a steepest ascent direction (is tangent to $d = d(z)$ for all $z \in \gamma$ is a *steepest ascent curve.*

Exercise 10.43. ** Find the maximum and minimum values of* $|\sin z|$ *inside the closed unit disk.*

Harmonic functions in a domain $\mathcal{D}$ also attain their maximum *as well as the minimum* value on $\partial\mathcal{D}$.

Theorem 10.44. *Assume* u *is harmonic and non-constant in* $\mathcal{D}$*. Then* u *has no minimum or maximum in* $\mathcal{D}$*.*

Proof. Let $u = \mathrm{Re}(f)$ and define $g = e^f$. We saw that g is analytic in $\mathcal{D}$. By the properties of the exponential that we have shown already, we have $e^f = e^u e^{iv}$; $|e^f| = e^u$ and then u has a maximum if and only if $|g|$ has a maximum. But this cannot happen strictly inside $\mathcal{D}$. For the minimum, note that $\min(u) = -\max(-u)$ □

10.1. **Application.** The soap film picked up by a thin closed wire has the minimum possible area compatible with the constraint that it is bordered by the wire, since the potential energy is proportional to the surface area. It then follows easily that the shape function u satisfies Laplace's equation. This is shown in many books (e.g. in Fisher's book on complex analysis ref??). It follows from Theorem 10.44 that this minimal surface is flat if the wire is flat. This is probably not a surprise. We will however be able to solve Laplace's equation with any boundary constraint, and this will provide us with a lot of insight on these minimal surfaces.

10.2. **Principal value integrals.** Suppose that f is analytic in a domain containing the simple closed piecewise differentiable curve C. By Cauchy's theorem we have

$$\frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}ds = \begin{cases} f(z) \text{ if } z \text{ is inside } C \\ \quad 0 \text{ otherwise} \end{cases} \tag{10.2}$$

What if z lies *on* C? Then of course the integral is not defined as it stands. A number of reasonable definitions can be given though, and they agree as far as they apply. In one such definition a symmetric segment of the curve centered at z of length ε is cut and then ε is taken to zero, giving the "Cauchy principal part integral" denoted $P\oint$ (and in many other ways). Another definition is to take the half sum of the integral on a curve circumventing z from the outside and of the integral on a curve circumventing z from the inside.

Definition 10.45. *If f is integrable on any $[a, c_n]$ and on any $[d_n, b]$ if $a < c_n < d_n < b$ and $\lim c_n = \lim d_n = L$, then*

$$P \int_a^b f(s)ds = \lim \varepsilon \downarrow 0 \int_a^{L-\varepsilon} f(s)ds + \int_{L+\varepsilon}^b f(s)ds \tag{10.3}$$

if the limit exists.

Exercise 10.46. Show that if C is a smooth closed curve and f is analytic in a neighborhood of C, then

$$\frac{1}{2\pi i} P \oint_C \frac{f(s)}{s-z} ds = \frac{1}{2} f(z) \tag{10.4}$$

and it coincides with the symmetric cutoff value defined above. The same equality holds up to a sign, if C is a compact piece of a smooth curve, approached from one side– the sign depends on the side. One of many ways to prove this is on p. 94.

Exercise 10.47. Show that if f is integrable on (a, b) then

$$P \int_a^b f(s)ds = \int_a^b f(s)ds \tag{10.5}$$

11. Linear fractional transformations: a first look

Exercise 11.48. *** Let $a \in (0, 1)$, $\theta \in \mathbb{R}$. Show that*

$$z \mapsto T(z) := e^{i\theta} \frac{a+z}{1+az} \tag{11.2}$$

is a one-to-one transformation of the closed unit disk onto itself.

12. Poisson's formula

Proposition 12.49. *Assume u is harmonic in the open unit disk and continuous in the closed unit disk. Then*

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})dt \tag{12.2}$$

Proof. If v is the harmonic conjugate of u then $f := u+iv$ is analytic in the open unit disk, and we have by Cauchy's formula for any $\rho < 1$,

$$u(0) + iv(0) = f(0) = \frac{1}{2\pi i}\oint_{C(0;\rho)} \frac{f(s)}{s}ds = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it})dt \tag{12.3}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{it})dt + i\frac{1}{2\pi}\int_0^{2\pi} v(\rho e^{it})dt$$

We get (12.2) by taking the real part of (12.3) and passing to the limit $\rho \to 1$. □

Exercise 12.50. ** (i) Let u as in Proposition 12.49 and T as in Exercise 11.48. Show that*

$$U(z) = u(T(z)) \tag{12.4}$$

is harmonic in the open unit disk and continuous in the closed unit disk.

(ii) Show that, if $z_0 = ae^{i\theta}$ we have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\frac{a+e^{is}}{1+ae^{is}}\right) ds \tag{12.5}$$

Proposition 12.51 (Poisson's formula)**.** *Let u be as in Proposition 12.49 and $z_0 = ae^{i\theta}$ with $a < 1$. We have*

$$u(ae^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-a^2}{1-2a\cos(t-\theta)+a^2}u(e^{it})dt \tag{12.6}$$

Conversely, if $u(e^{it})$ is continuous, then $u(ae^{i\theta})$ defined by (12.6) *is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.*

The proof of (1) is left any easy exercise:

Exercise 12.52. *Prove* (12.6) *by making the change of variable*

$$e^{i\theta}\frac{a+e^{is}}{1+ae^{is}} = e^{it} \tag{12.7}$$

in (12.5).

The proof of (ii) is not so immediate, and we will return to it in §30.1.

12.1. **The Dirichlet problem for the Laplacian in $\mathbb{D}$.** Formula (12.6) gives the solution of Laplace's equation in two dimensions with Dirichlet boundary conditions, namely with u specified on the boundary , when the domain is $\mathbb{D}$. A simple change of variables adapts this formula to any disk. More generally, we will see that the formula can be adjusted to accommodate for the general case of the domain lying in the interior of any simple, closed, piecewise differentiable curve. This is a consequence of the *Riemann mapping theorem*.

12.2. **The Neumann problem for the Laplacian in $\mathbb{D}$.** This is another important problem associated with Laplace's equation, finding its solution v on a specified domain, with given normal derivative v_n its boundary. Let for example the domain be the unit disk $\mathbb{D}$, with its boundary S^1, the unit circle, where we assume that v_n is continuous up to the boundary. Since v is harmonic in the unit disk, it has a harmonic conjugate $u \in C^2(\mathbb{D})$. If $x^2 + y^2 = 1$ then the normal derivative of v equals the tangential derivative of u:

$$v_n = \langle x, y\rangle \cdot \langle v_x, v_y\rangle = \langle x, y\rangle \cdot \langle -u_y, y_x\rangle = \langle y, -x\rangle \cdot \langle u_x, u_y\rangle = u_t$$

We now note that there is a necessary condition on v for the Neumann problem to have a solution: u being well-defined, it does not change after one loop and thus

$$\oint u_t ds = 0 = \oint v_n ds \tag{12.8}$$

Given that $v_n = u_t$, v_n on the boundary determines u_t and thus u on the boundary up to an additive constant, and using Poisson's formula (12.6) we get u in D, therefore v.

13. Isolated singularities, Laurent series

By definition f has an *isolated singularity* at z_0 if f is analytic in a disk $D(z_0, \rho) \setminus \{z_0\}$ for some $\rho > 0$. Note that we allow for the possibility that z_0 is a point of analyticity of f, or, to be precise, that there exists an extension of f analytic in $D(z_0, \rho)$. For example the functions $e^{1/z}$ and $1/\sin z$ have an isolated singularity at zero, whereas the singularity of $\ln z$ is not isolated (we will see that ln is not well defined in $\mathbb{D} \setminus \{0\}$).

Proposition 13.53 (Laurent series)**.** *A function f analytic in $\mathbb{D}_\rho(z_0) \setminus \mathbb{D}_{\rho'}(z_0)$ where $0 < \rho' < \rho$ has the convergent representation*

$$f(z) = \sum_{k\in\mathbb{Z}} a_k (z - z_0)^k, \qquad \text{for } z \in \mathbb{D}_\rho(z_0) \setminus \mathbb{D}_{\rho'}(z_0) \tag{13.2}$$

where

$$a_k = \frac{1}{2\pi i} \oint \frac{f(s)}{(s - z_0)^{k+1}} ds, \qquad k \in \mathbb{Z}$$

with the integral taken on any circle around z_0 of radius less that ρ.

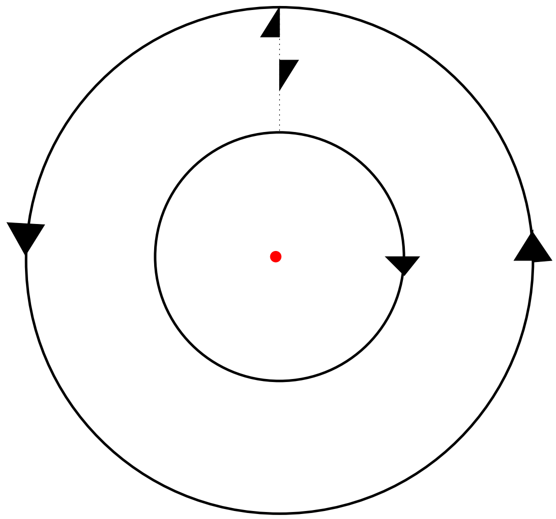


FIGURE 5. Circles of integration and cut in the proof of the Laurent series expansion.

Note. The number a_{-1} is called the *residue* of f at z_0: a_{-1} =Res$(f; z_0)$.
Proof. Replacing $z - z_0$ by z we see that there is no loss of generality in assuming $z_0 = 0$.

Consider the annulus between two circles C_o, C_i in $\mathbb{D}_\rho \setminus \mathbb{D}_{\rho'}$, as in Fig. 5 ($C_o$ is the outside circle, and C_i is the interior one). Make a cut in the annulus as shown. The remaining region is simply connected and Cauchy's formula applies there:

$$f(z) = \frac{1}{2\pi i} \oint_{C_o} \frac{f(s)}{s - z} ds - \frac{1}{2\pi i} \oint_{C_i} \frac{f(s)}{s - z} ds \tag{13.3}$$

where we used the fact that the boundary of the cut annulus is the closed path composed of C_o, ran counterclockwise from the cut point back to it, followed by the segment ℓ along the cut, then C_i going clockwise, and back to the starting point following $-\ell$. The rest of the proof is similar to Taylor theorem's proof and is left as an exercise below. $\square$

Exercise 13.54. *Complete the proof of formula (13.2) by expanding the integrands in (13.3) in powers of z/s and s/z respectively, and estimating the remainders as we did for obtaining formula (6.2).*

Note. Convince yourselves that (13.3) gives a decomposition of f into a part f_1 analytic in the disk enclosed by C_o and a function f_2 analytic in $1/z$ outside the disk enclosed but C_i. In [3] this decomposition is used for a nice proof of (13.2). $\square$

Definitions. An isolated singularity z_0 of f is a *pole of order M* if $a_k = 0$ for all $k < -M$, it is a *removable singularity* if it is "a pole of order 0" (in this case, f extends to a function $\tilde{f}$ analytic in the whole disk and given by the Taylor series of f at z_0) and an *essential singularity* otherwise. By slight abuse of notation we typically do not distinguish $\tilde{f}$ from f itself.

For example $e^{1/z}$ has an essential singularity at $z = 0$. Application of (13.2) yields

$$e^{1/z} = \sum_{k=0}^{\infty} z^{-k}/k! \tag{13.4}$$

Note. The part of the Laurent series containing the terms with negative k is called the principal part of the series.

Note. Laurent series are of important theoretical value. However, calculating effectively a function near the singularity from its Laurent series is another matter and it is usually not very practical to use Laurent series for this purpose. A Laurent series is *antiasymptotic*: its convergence gets slower as the singularity is approached.

14. *Laurent series and Fourier series

In this subsection we rely a bit more heavily on functional analysis. However, the results are not used in the rest of the book, and this part can be safely skipped. Let $f \in C^2(\mathbb{R})$ be $2\pi-$periodic and $c_k := \frac{1}{2\pi}\int_0^{2\pi} f(s)e^{-iks}ds$. Let $\tilde{f}(e^{is}) = f(s)$; we see that $\tilde{f}$ is well defined on the unit circle. By integration by parts we can check that for some constant C and all k we have $|c_k| \leqslant Ck^{-2}$. Then

$$\sum_{k=0}^{\infty} c_k z^k =: g(z)$$

converges absolutely and uniformly in $\overline{\mathbb{D}}$, and thus g is analytic in $\mathbb{D}$ and continuous up to the boundary. We see that

$$\oint_{\mathbb{D}} \frac{g(s) - \tilde{f}(s)}{s^{k+1}} = 0\,\forall k \geqslant 0 \text{ and } \oint_{\mathbb{D}} \frac{g(s)}{s^{k+1}} ds = 0\,\forall k < 0$$

Similarly, define

$$\sum_{k=-\infty}^{-1} c_k z^{-k} =: h(z)$$

If $H(z) = h(1/z)$ we have $H(\zeta) = \sum_{n=1}^{\infty} c_{-m}\zeta^m$ and H is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Now,

$$c_{-m} = \frac{1}{2\pi i}\oint_{\mathbb{D}} \frac{H(s)}{s^{m+1}} ds = -\frac{1}{2\pi i}\oint_{\mathbb{D}} \frac{h(1/t)}{t^{-m+1}} dt$$

and also

$$\oint_{\mathbb{D}} H(s)s^{n-1}ds = 0 \ \forall n \geqslant 1 \Leftrightarrow \oint_{\mathbb{D}} h(t)t^{-k-1}ds = 0 \ \forall k \geqslant 1$$

Let $U(z) = h(z) + g(z)$ defined for $z \in \partial\mathbb{D}$. We have

$$\oint (U(s) - \tilde{f}(s))s^k ds = 0 \ \forall k \in \mathbb{Z}$$

Let $F(x) = U(e^{ix})$. Then $F - f$ is orthogonal on e^{ikz} for all k and thus is zero everywhere. One way to see this is to use Parseval's theorem (convince yourself that this is not a circular proof). But since F and f are continuous, we have $f = F$.

15. Calculating the Taylor series of simple functions

One easy way to calculate Taylor series is to use §2.5.

Example. (1) The Taylor series of the function $z^{-1}\sin z$ is

$$\frac{\sin z}{z} = 1 - z^2/6 + z^4/120 + \cdots \tag{15.2}$$

(2) The Taylor series of the function $z/\sin z$ is

(15.3)
$$\frac{z}{\sin z} = \frac{1}{1 - (z^2/6 - z^4/120 + \cdots)} = 1 + (z^2/6 - z^4/120 + \cdots) +$$
$$(z^2/6 - z^4/120 + \cdots)^2 + \cdots = 1 + z^2/6 - z^4/120 + z^4/36 + \cdots$$
$$= 1 + z^2/6 + 7z^4/360 + \cdots$$

The first function defined is entire; the second one is not. What is the radius of convergence of the second series?

Exercise 15.55. * *Find the integral of* $1/\cos z$ *on a circle of radius* $1/2$ *centered at* $z_0 = \pi/2$.

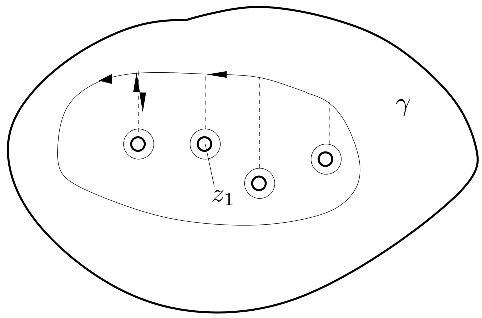


FIGURE 6. Multiply connected domains

16. Residues and integrals

Proposition 16.56. *Let $\mathcal{D}$ be a simply connected domain. Consider a function f which is analytic in the domain $\mathcal{D} \setminus \cup_{k=1}^{n} D_k$ where D_k are disjoint disks centered at $z_k \in D$, and consider a simple closed curve piecewise differentiable γ which encircles each D_k once, see Figure 11. We have*

$$\oint_\gamma f(s)ds = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f)_{z=z_k} \tag{16.2}$$

Exercise 16.57. *Prove Proposition 16.56 by deforming and cutting the curve of integration appropriately.*

Example. Calculate

$$\oint \frac{dz}{\sin^3 z}$$

on a circle of radius $1/2$ around the origin.
Solution. We have, in $D(0, 1/2)$,

(16.3)

$$\frac{1}{\sin^3 z} = \frac{1}{(z - z^3/6 + z^5/120 \cdots)^3} = \frac{1}{z^3}\frac{1}{(1 - z^2/6 + z^4/120 \cdots)^3} = \frac{1}{z^3}\left(1 + z^2/2 + 17z^4/120\right) + \cdots)$$

and thus the residue of $\sin^{-3}(z)$ at $z = 0$ is $1/2$ and the integral equals πi.

Exercise 16.58. *Show that if f has a pole of order m at $z = z_i$ then*

$$\mathrm{Res} f_{z=z_i} = \frac{[(z - z_i)^m f(z)]^{(m-1)}_{z=z_i}}{(m-1)!} \tag{16.4}$$

by applying Laurent's formula near $z = z_i$.

17. Integrals of trigonometric functions

Contour integration is very useful in calculating or estimating Fourier coefficients of periodic functions. Consider the integral

$$I = \int_0^{2\pi} \frac{\cos(nt)}{2+\cos t}dt = \mathrm{Re}\,(J);\;\; J := \int_0^{2\pi} \frac{e^{int}}{2+\cos t}dt \tag{17.2}$$

Let $z = e^{it}$. Then

$$J = -i \oint_C \frac{z^{n-1}}{2+(z+1/z)/2}dz = -2i \oint_C \frac{z^n}{z^2+4z+1}dz \tag{17.3}$$

where C is the unit circle. The roots of z^2+4z+1 are $-2 \pm \sqrt{3}$ and only one, $z_0 = -2+\sqrt{3}$ lies in the unit disk. Thus,

$$J = -2i \cdot 2\pi i \frac{z_0^n}{2z_0+4} \;\Rightarrow I = \frac{4\pi z_0^n}{2z_0+4} \tag{17.4}$$

18. Counting zeros and poles

Notations and definitions. (1) Assume f is analytic in a disk $\mathbb{D}_\rho(z_0)$ and $f(z_0) = 0$. Then, in $\mathbb{D}_\rho(z_0)$ we have

$$f(z) = \sum_{k=1}^{\infty} c_k (z-z_0)^k \tag{18.2}$$

If f is not identically zero then there exists some k_0 such that $c_{k_0} \neq 0$ (see Proposition 7.28). The smallest such k_0 is called the *order* (or *multiplicity*) of the zero z_0. For a meromorphic function g (see p. 13), the order of a pole at z_0 is the multiplicity of the root of $1/g$ at z_0.

Exercise 18.59. (The zeros of an analytic function are isolated) *Assume $f \not\equiv 0$ is analytic near z_0 and $f(z_0) = 0$. Use Taylor series to show that there is some disk around z_0 where $f(z) = 0 \Rightarrow z = z_0$.*

Assume f is meromorphic in $\mathcal{D}$; let γ be a piecewise differentiable simple closed curve contained in $\mathcal{D}$ together with its interior Γ. Note that by assumption the region of analyticity of f strictly exceeds Γ. For the purpose of the next proposition, the assumptions can be relaxed, allowing γ to be the boundary of the analyticity domain of f if we impose continuity conditions on f and f'. Check this.

Theorem 18.60 (counting zeros and poles)**.** *Let N be the total number of zeros of f in Γ counting multiplicities and let P be the number of*

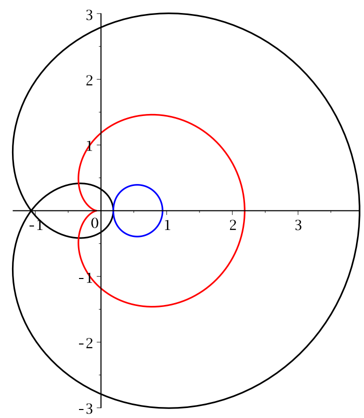


FIGURE 7. The image of the circles of radius 1/4 (innermost curve), of radius 3/4 (middle curve) and of radius 5/4 all centered at zero under the map $(z-1)(z-1/2)$.

poles, each pole being counted p times if it has order p. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(s)}{f(s)}ds = N - P \tag{18.3}$$

Proof. The function f'/f is also meromorphic. It has a pole of order 1 and residue n_i at a zero of order n_i of f and a pole of order 1 and residue $-p_i$ at a pole of order p_i of f (check!). The rest follows from (16.2).

Note 18.61 (The argument principle)**.** If we take, formally for now, $g = \ln f$, then $g' = f'/f$ and then (18.3) shows that the change in $\ln f$ as we traverse positively γ is $N-P$. Another formulation is that we take the image of a parametrization of γ under f. Then, $N-P$ counts the number of times the image turns around zero.

Exercise 18.62. *Let* $f(z) = \exp(1/z) - 1$*; clearly* 0 *is an essential singularity and Proposition 18.60 does not apply. Find however, as a function of* $\varepsilon > 0$*, how many times the curve* $\{f(\varepsilon e^{it}) : t \in [0, 2\pi)\}$ *turns around zero.*

18.1. **Hurwitz's theorem.** This theorem shows that a uniform limit of nonzero functions is nonzero. More precisely,

Theorem 18.63 (Hurwitz)**.** *If f_n are analytic and nonzero in a domain $\Omega \subset \mathbb{C}$ and f_n converge to $f \not\equiv 0$ uniformly on compact sets, then $f(z)$ has no zeros on Ω either.*

Proof. We do the analysis in the neighborhood of a point z_0; as usual we can take $z_0 = 0$. Since $f \not\equiv 0$, there is a $\mathbb{D}_\delta$ such that $f(z) \neq 0$ on $\partial\mathbb{D}_\delta \setminus \{0\}$ (apply Corollary 2.7). Since $f_n \to f$ uniformly (together with f_n' by Weierstrass's Theorem 7.32), f_n'/f_n converge uniformly as $n \to \infty$ to f'/f on the circle $\partial\mathbb{D}_{\delta/2}(0)$. The rest follows from Theorem 18.60. ∎

18.2. **Rouché's Theorem.**

Theorem 18.64 (Rouché)**.** *Assume f and h are analytic in the interior Γ of the piecewise differentiable simple closed curve γ, continuous in the closure $\overline{\Gamma}$ and that on γ we have $|h| < |f|$. Then the number of zeros of f and $f + h$ in Γ is the same (we can think of $f + h$ as a "small" perturbation of f).*

Proof. Note that all the assumptions hold in a small neighborhood of γ too. Since $0 \leqslant |h| < |f|$, f can have no zeros on γ. We have

$$f + h = f \cdot (1 + h/f) = fQ \Rightarrow \frac{f' + h'}{f + h} = \frac{f'}{f} + \frac{Q'}{Q} \tag{18.4}$$

Since we have $|h/f| < 1$ the series $q = \sum_{k=1}^{\infty} k^{-1}(-1)^{k+1}(h/f)^k$ is analytic function in a neighborhood of γ (see Corollary **??**), and we have $q' = Q'/Q$ (check!). But then, evidently, $\oint q' = 0$ and the proposition follows.

Exercise 18.65. ** Reformulate and prove the proposition when f and h, f' and h' are continuous up to the boundary γ but not necessarily beyond.*

19. Inverse function theorem

Theorem 19.66. *Assume f is analytic at z_0 and $f'(z_0) = a \neq 0$. Then there exists a disk $\mathbb{D}_\varepsilon(z_0)$ such that f is invertible from $\mathbb{D}_\varepsilon(z_0)$ to $\mathcal{D} = f(\mathbb{D}_\varepsilon(z_0))$ and the inverse is analytic.*

Without loss of generality, we may assume that $z_0 = 0$ and $f(z_0) = 0$. We have $f(z) = az + z^2 g(z)$ where $g(z) \to g(0)$ as $z \to 0$. We want to find a disk of injectivity for f. Take $M > |g(0)|$ and take $\varepsilon < 5^{-1}|a|M^{-1}$ small enough so that $|g| < M$ in $\mathbb{D}_{3\varepsilon}$. Let $z_1 \in \mathbb{D}_\varepsilon$ and $f(z_1) = w$. We show that $f(z) = w$ and $z \in \mathbb{D}_\varepsilon$ implies $z = z_1$ which means that f is one-to-one from the disk $\mathbb{D}_\varepsilon$ to $\mathcal{D} = f(\mathbb{D}_\varepsilon)$. We have $f(z) - w = az + z^2 g(z) - az_1 - z_1^2 g(z_1) = a(z - z_1) + z^2 g(z) - z_1^2 g(z_1)$. We apply Rouché's theorem in $f(\mathbb{D}_{3\varepsilon})$: since $|z_1| < \varepsilon$ and $|z| = 3\varepsilon$, we have $|a(z-z_1)| > 2|a|\varepsilon$. On the other hand, $|z^2 g(z) - z_1^2 g(z_1)| < 9\varepsilon^2 M + \varepsilon^2 M$. By direct calculation we see that Rouché's theorem applies, with $f = a(z - z_1)$ and $h = z^2 g(z) - z_1^2 g(z_1)$, if $10\varepsilon^2 M < 2|a|\varepsilon$ which holds by construction. But the equation $a(z - z_1) = 0$ has only one root in $\mathbb{D}_{3\varepsilon}$ and thus so does $f(z) - f(z_1) = 0$. (Note that z_1 is in the smaller disk $\mathbb{D}_{3\varepsilon}$.) Differentiability is shown by a direct verification, as in usual calculus –check!).

20. Analytic continuation

Assume that f is analytic in $\mathcal{D}$ and f_1 is analytic in $\mathcal{D}_1$, $\mathcal{D}_1 \supset \mathcal{D}$ and $f = f_1$ in $\mathcal{D}$. Then f_1 is an *analytic extension* of f. We also say that f_1 has been obtained from f by *analytic continuation*.

The point of view favored by Weierstrass was to regard analytic functions as properly defined chains of Taylor series, up to a natural equivalence (more about this later), each one of them being the analytic continuation of the adjacent ones. If f is analytic at z_0, then there exists a disk of radius ε centered at z_0 such that f is the sum of this series; we take ε_0 to be the largest ε with this property. If we take a point z_1 inside this disk, f is analytic at z_1 too, and thus near z_1 it is given by a series centered at z_1. The disk of convergence of this series is, as we know, at least equal to the distance $d(z_1, \partial\mathbb{D}_\varepsilon(z_0))$, but in general it could be larger. (Convince yourselves that this is the case with the function $1/(1+z)$ if we take a disk centered at $z = 0$ and then a disk centered at $z = 1/2$.) In the latter case, we have found a function f_1, piecewise given by the two Taylor series, which is analytic in the union $\mathbb{D}_\varepsilon(z_0) \cup \mathbb{D}_{\varepsilon_1}(z_1)$.

Uniqueness. If there is an analytic continuation in $\mathbb{D}_\varepsilon(z_0) \cup \mathbb{D}_{\varepsilon_1}(z_1)$, then it is unique (use Proposition 7.28 to show this).

In fact, we can continue this process and define chains $z_0, z_1, ...$ such that f is analytic in $\mathbb{D}_{\varepsilon_i}(z_i)$. These are elements of a "global analytic function". This "global analytic function" is not necessarily a function, since the chains may intersect each other while the value of the continuation of f in the overlap region can be different.

It is useful to experiment with this procedure on $\log(1 + z)$ which we have defined as an analytic function for $|z| < 1$. What happens if we take a chain around $z = -1$?

One can also find that there is a region in $\mathbb{C}$ where the function is well defined by this procedure, but no Taylor disk crosses the boundary. Then we have found a maximal region of analyticity, the boundary of which is called "natural boundary" or "singularity barrier". More generally a simple curve, closed or not, is a natural boundary if f is analytic on (possibly one-sided) neighborhood of the curve, no point on γ at which f has a limit (note: it can have no limit at any point) is a point of analyticity of f. The standard example of such a function is $f(z) = \sum_{k=0}^{\infty} z^{2^k}$: we have $f(z) \to \infty$ as $z \to 1$, and also as $z \to -1$ to i and $-i$ and more generally as $z \to e^{2\pi i N/2^M}$, $(N, M) \in \mathbb{N}^2$, which form a dense set on the unit circle. This precludes analyticity at *any* point on the circle (why is that?).

Exercise 20.67. *** Consider the rational numbers $r = p/q$ (we assume p and q are relatively prime) and associate to it $N_{pq} = 7^p 5^q$ (check that this is injective as a function from $\mathbb{Q}$ to $\mathbb{N}$). Take the function*

$$f(z) = \sum_{N_{pq}}^{\infty} \frac{2^{-N_{pq}}}{z - p/q} \tag{20.2}$$

Show that the series converges for $z \in \mathbb{C} \setminus \mathbb{R}$ and that $\mathbb{R}$ is a natural boundary for f.

How can this example can be modified to obtain an analytic function f in any domain bounded by a simple closed curve γ, and γ is a natural boundary of f?

21. The Schwarz reflection principle

Assume f is analytic in the domains $\mathcal{D}_1, \mathcal{D}_2$ which have a common piece of boundary, a piecewise differentiable curve γ. Assume further that f is continuous across γ. Then, by Morera's theorem, f is analytic in $\mathcal{D}_1 \cup \mathcal{D}_2$ (check this statement). This allows us to do analytic continuation, in some cases.

Theorem 21.68 (The Schwarz reflection principle)**.** *Assume f is analytic in a domain $\mathcal{D}$ in $\mathbb{H}^u$ (see p. 42.2) whose boundary contains an interval $I \subset \mathbb{R}$ and assume f is continuous on $\mathcal{D} \cup I$ and real valued on I. Then f has analytic continuation across I, in a domain $\mathcal{D} \cup \mathcal{D}^*$ where $\mathcal{D}^* = \{\overline{z} : z \in \mathcal{D}\}$.*

Note 21.69. see §30.5 for a generalization of this result.

Proof. Consider the function $F(z)$ equal to f in $\mathcal{D} \cup I$ and equal to $\overline{f(\overline{z})}$ in $\mathcal{D}^* \cup I$. This function is continuous in $\mathcal{D} \cup I \cup \mathcal{D}^*$ (explain this continuity). It is also analytic in $\mathcal{D}^*$ as it can be immediately seen using a local Taylor series argument. Now Morera's theorem applies: integrals along closed curves completely contained in $\mathcal{D}$ or $\mathcal{D}^*$ are evidently zero, whereas since a curve crossing I can be split into two integrals, with

I as the splitting, traversed twice, in opposite directions (where is the fact that f is real on I used?). Check the details.

Note. When we learn more about conformal mappings, we shall see that much more generally, a function admits a continuation across a curve γ if the curve is an analytic arc (we will define this precisely) and $f(\gamma)$ is an analytic arc as well.

Example 21.70. The square root function defined by $\sqrt{z} = \rho^{1/2}e^{i\phi/2}$ if $z = \rho e^{i\phi}, \phi \in [0, \pi)$ is analytic in the upper half-plane and continuous down to $[0, \infty)$ and real-valued there, and thus can be continued analytically in the lower half plane by Schwarz reflection. What is the continuation? Let $z = \rho e^{-i\theta} \in \mathbb{H}_l$ $(\theta \in (0, \pi))$. Then $\overline{z} = \rho e^{i\theta} \in \mathbb{H}^u$ where $\sqrt{\ }$ was defined: $\sqrt{\overline{z}} = \rho^{1/2}e^{i\theta/2}$. Then $\overline{\sqrt{\overline{z}}} = \rho^{1/2}e^{-i\theta/2}$. Note that although $\sqrt{\ }$ extends to the lower half plane as well, it is *not* continuous in $\mathbb{C}$: the limits as z approaches $\mathbb{R}^-$ from above and from below exist, but they are different. We can continue the square root further, however. Note that the lower half plane continuation has a limit which is purely imaginary on $\mathbb{R}^-$. Then $i\sqrt{z}$ is purely real on $\mathbb{R}^-$ and the Schwarz reflection principle applies. It is easy to see that the resulting analytic continuation to $\mathbb{H}^u$ will be different (by a sign) from the originally defined $\sqrt{\ }$. The square root function is a *branched function*; we will look more carefully at this in the next section.

22. Multi-valued functions

As we discussed, as a result of analytic continuation in the complex plane we may get a *global analytic function* which is not necessarily a function on $\mathbb{C}$ since the definition is path–dependent; the function is thus defined on a space of paths or curves, modulo homotopies.

As long as the domain of continuation is simply connected, we still get a function in the usual sense:

Exercise 22.71. **** Assume that f is analytic in $\mathbb{D}_\varepsilon(z_0)$ and that we have and two piecewise differentiable curves γ_1 and γ_2 joining z_0 to z which can be continuously deformed into each-other and furthermore analytic continuation exists along each intermediate curve:

That is, there is a smooth map $\gamma : [0, 1]^2 \mapsto \mathbb{C}$ such that $\gamma(s, 0) = z_0 \ \forall s \in [0, 1]$ and $\gamma(s, 1) = z_1 \forall s \in [0, 1]$ and furthermore f admits analytic continuation from z_0 to z_1 along $t \mapsto \gamma(s, t), t \in [0, 1]$ for any $s \in [0, 1]$.

Assume that $\mathcal{D} = \gamma((0, 1)^2)$ is simply connected. Show that there is an analytic function F in $\mathcal{D}$ which coincides with f in $\mathbb{D}_\varepsilon(z_0)$. As we know, this continuation is then unique. *(Rough sketch: consider the* *first curve which by compactness is covered by a finite number of disks of analytic continuation. Choose an intermediate curve close enough so that it is well covered by the same disks. From this point, it should be straightforward.)*

The simplest example is perhaps the logarithm. In real analysis $\ln x = \int_1^x s^{-1} ds$. Clearly the function z^{-1} is analytic in $\mathbb{C} \setminus \{0\}$ and we can define

$$\ln^{[C]} z = \int_{1;C}^{z} s^{-1} ds \tag{22.2}$$

where integration starts at 1 and is performed along the curve $\mathbb{C}$. This integral only depends on the homotopy class of the curve C in $\mathbb{C} \setminus \{0\}$.

Let C be a piecewise differentiable curve starting at 1 which does not contain 0. Drop for now the superscript: $\ln^{[C]} z = \ln z$ (this is customary, but we keep in mind that dependence on C persists). The function $e^{\ln z}$ is well defined along the curve C and analytic in a neighborhood of any point in C. We find

$$\left(\frac{e^{\ln z}}{z}\right)' = 0 \tag{22.3}$$

and thus

$$e^{\ln z} = z e^{\ln 1} = z \tag{22.4}$$

Can you prove (22.4) using permanence of relations?

The log thus defined is the inverse of the exponential. Therefore, if we write $z = \rho e^{i\phi}$ then by Exercise 2.9

(22.5)

$$\rho e^{i\phi} = e^{\ln \rho + i\phi} = e^{\ln z} \quad \Leftrightarrow \ln z = \ln \rho + i\phi + 2N\pi i \quad (\text{for some } N \in \mathbb{Z})$$

for any choice of the curve of integration.

Thus, we have proved

Proposition 22.72. *For any piecewise differentiable curves C_1 and C_2 we have there is an $N \in \mathbb{Z}$ so that $\ln^{[C_1]} z - \ln^{[C_2]} z = 2N\pi i$.*

This is the log "function" as a *global analytic function*. It has a *branch point* at $z = 0$ and it is *multivalued*. We sometimes say it is defined in $\mathbb{C}$ up to an additive integer multiple of $2\pi i$. If we choose a value of N, then we have chosen a "branch" of the log. But what does this choice mean?

Alternatively, as in Exercise 22, we can take a simply connected domain in $\mathbb{C}$ and define a *function* ln, relative to that domain.

Lemma 23.73. *Let $\mathcal{D}$ be a simply connected domain in $\mathbb{C}$ not containing zero. Then, for $a \in \mathcal{D}$, the function $\ln_{\mathcal{D}} z = \int_a^z s^{-1} ds$ is well defined and analytic in $\mathcal{D}$, it is injective and its inverse is $a^{-1}e^z$.*

Proof. The function $\ln_{\mathcal{D}} z := \int_a^z s^{-1} ds$ is well defined and analytic in $\mathcal{D}$ Evidently, by the argument leading to (22.4) we have $e^{\ln z} = z/a$ showing injectivity. ∎

Note 23.74. (i) The log in Lemma 23.73 is the *branch of the log* associated to $\mathcal{D}$. When no confusion is possible we simply drop the subscript $\mathcal{D}$.

(ii) Often $\mathbb{C} \setminus \overline{\mathbb{R}^-}$ is the domain needed, and this is the "natural branch" of the log.

The price that we pay if we need to restrict ln to a simply connected domain $\mathcal{D}$ not containing zero is that there is nothing special about $\partial\mathcal{D}$ except for 0: the log will have analytic continuation through $\partial\mathcal{D}$. No singularity exists, except for zero; in *the natural branch, defined in* $\mathbb{C} \setminus \overline{\mathbb{R}^-}$, $\mathbb{R}^-$ does not have singular points. The problem is that the analytic continuation of log through $\mathbb{R}^-$ is different, by $\pm 2\pi i$ from the definition of the log that we already had on the other side. This multivaluedness shows you that $\mathbb{C} \setminus \overline{\mathbb{R}^-}$ is a maximal domain *in* $\mathbb{C}$ of analyticity of the chosen branch.

Look at the figures below, and try to understand for which values of z we get the same value for the different branches of $\ln z$ defined here.

23.1. **Generalization: log of a function.** If g is a function defined in a region in $\mathbb{C}$ we can define $\ln g$ by

$$\ln g = \int_a^z \frac{g'(s)}{g(s)} ds \tag{23.2}$$

Now, depending on the properties of g, the homotopy classes will be in general more complicated.

For instance, if $g = g_1$ is a rational function, all the zeros and poles $S = \{z_i, p_j\}$ of g_1 are points where the integral, thus the log, is not defined. We are now dealing with homotopy classes in $\mathbb{C} \setminus S$.

It is convenient to define a branch of $\ln g_1$ by cutting the plane along rays originating at the points in S. Convince yourselves that this can

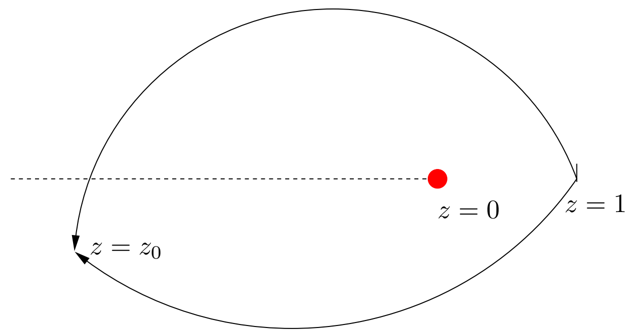


FIGURE 8. The natural branch of the log, with a cut along $\overline{\mathbb{R}^-}$; Im $\ln z \in (-\pi, \pi)$. $\mathbb{R}^-$ is not special, analytic continuation along the upper and lower paths exist, but differ by $2\pi i$.

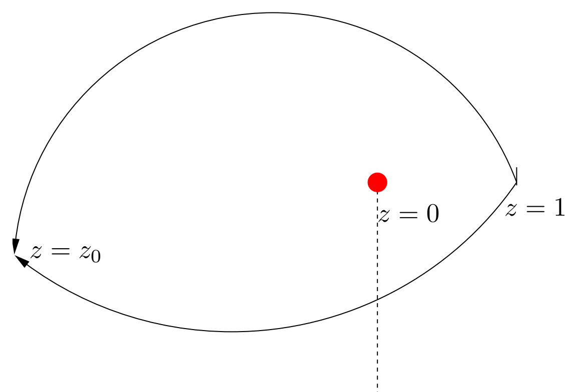


FIGURE 9. A branch of the log with a cut along $\overline{i\mathbb{R}^-}$; Im $\ln z \in (-\pi/2, 3\pi/2)$. Analytic continuation along the upper and lower paths still differ by $2\pi i$.

be done so that the remaining region $\mathcal{D}$ is simply connected. Then $\log g_1$ is well defined in $\mathcal{D}$ and analytic.

23.2. **General powers of z.** Once we have defined the log, it is natural to take

$$z^\alpha = e^{\alpha \ln z} \tag{23.3}$$

Since $\ln z$ is defined along a piecewise differentiable curve, modulo homotopies in $\mathbb{C} \setminus \{0\}$, so is z^α. For a general $\alpha \in \mathbb{C}$, the multivaluedness of z^α is inherited from the multivaluedness of the log: $e^{\alpha(\ln z + 2N\pi I)} = e^{\alpha \ln z} e^{2N\pi i\alpha}$. Note however that if $p \in \mathbb{Z}$ then the value

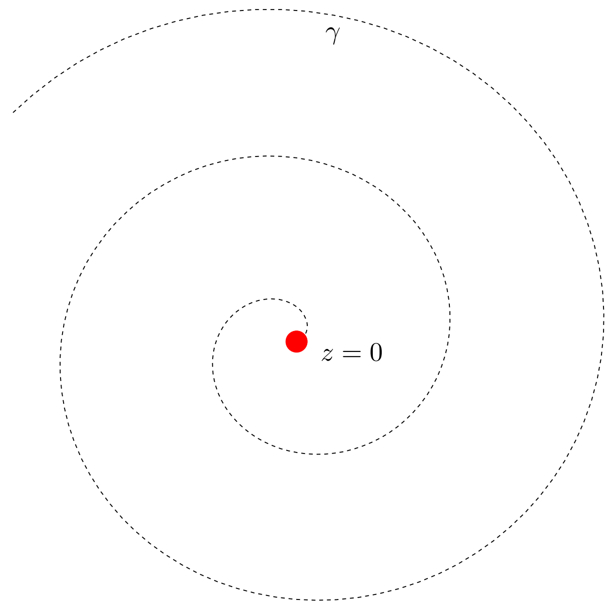


FIGURE 10. Yet another branch, in $\mathbb{C}$ cut along a spiral γ. Now $\operatorname{Im} \ln z$ is unbounded.

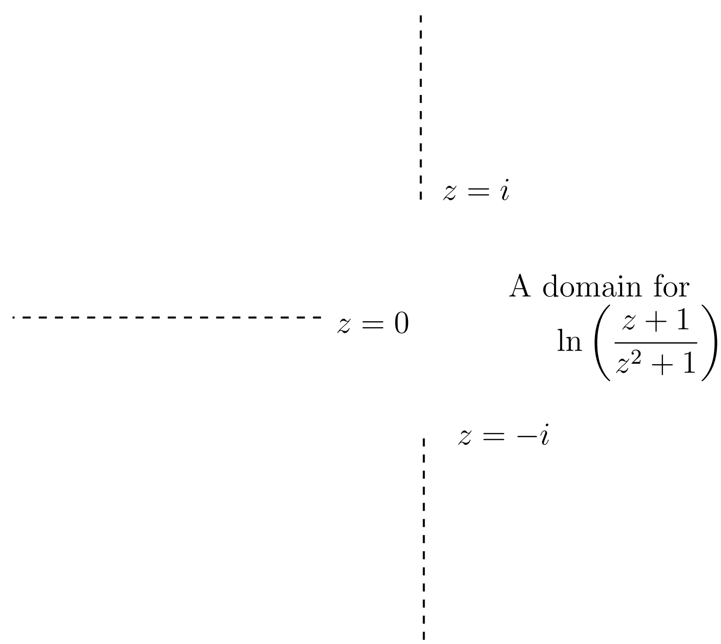


FIGURE 11. Cuts defining the log of a rational function.

does not depend on the homotopy class and the definition (23.3) defines a function in $\mathbb{C}\setminus\{0\}$, with a pole at zero if $p<0$ and a removable singularity if $p\geqslant 0$. Convince yourselves that this definition coincides with the usual power, defined algebraically.

Another special case is that when $\alpha = p/q$, p, q relatively prime integers. In this case $e^{2(p/q)\pi i}$ only takes q distinct values.

Note. Beware of possible pitfalls.

$$e^{\ln z_1+\ln z_2} = e^{\ln z_1}e^{\ln z_2} = z_1z_2 \tag{23.4}$$

However, this does not mean $\ln z_1+\ln z_2 = \ln z_1z_2$, but just that

$$\ln z_1+\ln z_2 = \ln z_1z_2 + 2N\pi i \tag{23.5}$$

For the same reason, $z^{\alpha_1}z^{\alpha_2}$ is not necessarily $z^{\alpha_1+\alpha_2}$. Note the fallacious calculation[2]

$$1=\sqrt{1}=\sqrt{(-1)(-1)}=\sqrt{-1}\sqrt{-1}=i\cdot i=-1 \quad (?!) \tag{23.6}$$

24. Evaluation of definite integrals

Contour integrals, and because of this, many definite integrals for which the endpoints are at infinity, or at special singular points of functions can be evaluated using the residue theorem. We have the following simple consequence of this theorem.

Proposition 24.75. *Let R be a rational function, continuous on $\mathbb{R}$ and such that $\exists C\in\mathbb{R}^+$ s.t. $|R(z)|\leqslant C|z|^{-2}$ in $\mathbb{C}$. (Convince yourself that this bound holds if the numerator has degree lower by two than the denominator.) Then*

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i\sum_{z_i\in\mathbb{H}^u} Res(R; z=z_i) \tag{24.2}$$

where z_i are poles of R.

Exercise 24.76. *The upper half plane is evidently not special; formulate and prove a similar result for the lower half plane $\mathbb{H}_l$. Note that if all residues are in the $\mathbb{H}^u$ (or all in the $\mathbb{H}_l$) the integral is zero.*

[2] This calculation was done as presented by an early version of a computer algebra program.

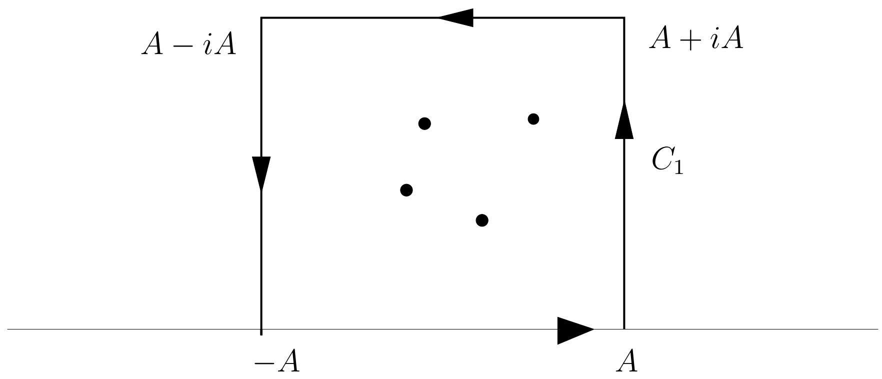


FIGURE 12.

Proof. Under the given assumptions, we take as a contour the square in the figure below and write

$$\begin{aligned}
(24.3)\quad \int_{-\infty}^{\infty} R(x)dx &= \lim_{A\to\infty}\int_{-A}^{A} R(x)dx \\
&= \lim_{A\to\infty}\oint_{[-A,A]\cup C_1} R(z)dz - \lim_{A\to\infty}\int_{C_1} R(z)dz
\end{aligned}$$

$$= 2\pi i \sum_{z_i\in\mathbb{H}^u} \mathrm{Res}(R; z = z_i) - \lim_{A\to\infty}\int_{C_1} R(z)dz = 2\pi i \sum_{z_i\in\mathbb{H}^u} \mathrm{Res}(R; z = z_i)$$

since

$$(24.4)\qquad \left|\int_{C_1} R(z)dz\right| \leqslant const A^{-2}(3A) = 3A^{-1} \to 0 \text{ as } A \to \infty$$

Note It is useful to interpret the method used above as starting with the integral along the real line and pushing this contour towards $+i\infty$. Every time a pole is crossed, a residue is collected. Since there are only finitely many poles, from a certain "height" on the contour can be pushed all the way to infinity, and that integral vanishes since the integrand vanishes at a sufficient rate.

Example. Find

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$$

Solution The singularities of R in the upper half plane are at $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$ with residues $1/[(1+x^4)']_{z=z_i}$. The result is $I = \pi/\sqrt{2}$.

25. Certain integrals with rational and trigonometric functions

We focus on integrals often occurring in integral transforms, of a type which can be reduced to

$$\int_{-\infty}^{\infty} e^{iax} Q(x) dx \tag{25.2}$$

$a > 0$, where Q has appropriate decay so that the integral makes sense. We would like to push the contour, as above, towards $+i\infty$ since the exponential goes to zero in the process. We need Q to satisfy decay and analyticity assumptions too, for this process to be possible. Jordan's lemma provides such a result suitable for applications.

Lemma 25.77 (Jordan)**.** *Assume $a > 0$ and that Q is analytic in the domain $\mathcal{D} = \{z : \operatorname{Im}(z) \geqslant 0, |z| > c\}$ and that γ in $\mathbb{H}^u$ (or $\mathbb{H}_l$) is the semicircle of radius $\rho > c$ centered at zero. Assume furthermore that $Q(z) \to 0$ as $|z| \to \infty$ in $\mathcal{D}$. Then,*

$$\int_\gamma e^{iaz} Q(z) dz \to 0 \quad \textit{as} \quad \rho \to \infty \tag{25.3}$$

Proof. Choose $\varepsilon > 0$ and let ρ_0 be such that $|Q(z)| < \varepsilon$ for all z with $|z| > \rho_0$. Then, for $\rho > \rho_0$ and γ as above we have

$$\begin{aligned} \left| \int_\gamma e^{iaz} Q(z) dz \right| &= \left| \int_0^\pi e^{ia\rho e^{i\phi}} Q(\rho e^{i\phi}) \rho i e^{i\phi} d\phi \right| \\ &\leqslant \varepsilon \int_0^\pi \rho e^{-\rho a \sin\phi} d\phi = 2\varepsilon \int_0^{\frac{\pi}{2}} \rho e^{-\rho a \sin\phi} d\phi \end{aligned} \tag{25.4}$$

To calculate the last integral we bound below $\sin\theta$ by $b\theta$ for some $b > 0$. By the symmetry $\sin t = \sin(\pi - t)$ the integral is twice the one on $[0, \pi/]$. By an elementary calculation we see that $t^{-1}\sin t$ is decreasing on $[0, \pi/2]$ and thus $\sin\theta \geqslant 2\theta/\pi$ for θ in $[0, \pi/2]$ and we get

$$\left| \int_\gamma e^{iaz} Q(z) dz \right| \leqslant \varepsilon \int_0^{\pi/2} 2\rho e^{-2\rho a\phi/\pi} d\phi \leqslant \frac{\varepsilon\pi}{a} \tag{25.5}$$

and the result follows. ∎

Proposition 25.78. *Assume $a > 0$ and Q is a rational function continuous on $\mathbb{R}$ and vanishing as $|z| \to \infty$ (that is, the degree of the denominator exceeds the degree of the numerator). Then*

$$\int_{-\infty}^{\infty} Q(x) e^{iax} dx = 2\pi i \sum_{z_i \in \mathbb{H}^u} \operatorname{Res}(Q(z)e^{iaz}; z = z_i) \tag{25.6}$$

The proof is left as an exercise: it is a simple combination of Jordan's lemma and of the arguments in Proposition 24.75. (Note also that $\int_1^\infty z^{-1}e^{iz}$ exists.)

Example Let $\tau > 0$ and find

$$I = \int_0^\infty \frac{\cos \tau x}{x^2+1} dx \tag{25.7}$$

Solution. The function is even; thus we have

$$2I = \int_{-\infty}^\infty \frac{\cos \tau x}{x^2+1} dx = \mathrm{Re} \int_{-\infty}^\infty \frac{e^{i\tau x}}{x^2+1} dx \tag{25.8}$$

which is of the form in Proposition 25.78 and thus a little algebra shows

$$I = \frac{\pi}{2} e^{-\tau}$$

Note that we have calculated the cos Fourier transform of an even function which is *real–analytic* (this means it is analytic in a neighborhood of the real line). The result is exponentially small as $\tau \to \infty$. This is not by accident: formulate and prove a result of this type for cos transforms rational functions with no poles on $\mathbb{R}$.

Example([10] p. 116) Assume $\mathrm{Re}\, z > 0$. Show that

$$I(z) = \int_0^\infty t^{-1}(e^{-t} - e^{-tz}) dt = \log z \tag{25.9}$$

Solution (for another solution look at the reference cited) Note that the integrand is continuous at zero and the integral is well defined. Furthermore, it depends analytically on z and Corollary 7.34 applies. We have

$$I'(z) = \int_0^\infty e^{-tz} dt = z^{-1} \quad \Leftrightarrow I(z) = \log z + C \tag{25.10}$$

Check that the constant C is zero.

Example: A common definite integral. Show that

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} \tag{25.11}$$

Solution This brings something new, since a naive attempt to write

$$\int_{-\infty}^\infty \frac{\sin t}{t} dt = \mathrm{Im} \int_{-\infty}^\infty \frac{e^{it}}{t} dt \quad (??) \tag{25.12}$$

cannot work as such, since the rhs is ill–defined. But we can still apply the ideas of the residue calculations in these lectures. Here is how.

(1) Use the box argument (see figure below) or be creative with Jordan's lemma to show that

$$\int_{-\infty}^{\infty}\frac{\sin t}{t}dt=\int_{-\infty+i}^{\infty+i}\frac{\sin t}{t}dt=\int_{-\infty}^{\infty}\frac{\sin(t+i)}{t+i}dt$$

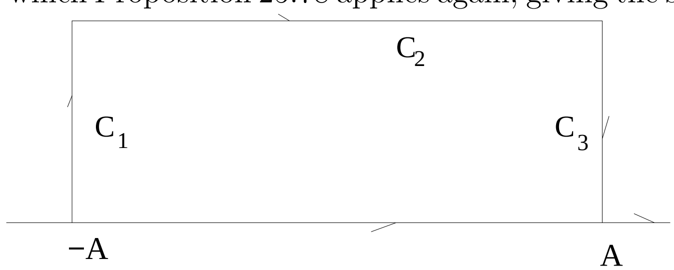
(2) Now we can write

$$\int_{-\infty}^{\infty}\frac{\sin(t+i)}{t+i}dt=\int_{-\infty}^{\infty}\frac{e^{i(t+i)}-e^{-i(t+i)}}{2i(t+i)}dt=\int_{-\infty}^{\infty}\frac{e^{i(t+i)}}{2i(t+i)}dt-\int_{-\infty}^{\infty}\frac{e^{-i(t+i)}}{2i(t+i)}dt$$

The first integral is zero, by Proposition 25.78. The last term equals

$$\overline{\int_{-\infty}^{\infty}\frac{-e^{i(t-i)}}{-2i(t-i)}dt}$$

to which Proposition 25.78 applies again, giving the stated result (check!)



Exercise 25.79. *** Find*

$$\int_0^{\infty}\frac{\sin^4 t}{t^4}dt$$

26. Integrals of branched functions

We now show that, for $\alpha\in(0,1)$ we have

$$\int_0^{\infty}\frac{t^{-\alpha}}{t+1}dt=\frac{\pi}{\sin\pi\alpha} \tag{26.2}$$

Note that the integrand has an integrable singularity at $t=0$ and decays like $t^{-\alpha-1}$ for large t, thus the integral is well defined. The integral is performed along $\mathbb{R}^+$ so we know what $t^{-\alpha}$ means. We extend $t^{-\alpha}$ to a global analytic function; it has a branch point at $t=0$ and no other singularities. Consider the region in the figure below. $t^{-\alpha}$ is analytic in $\mathbb{C}\setminus\mathbb{R}^+\setminus\{0\}$. Note first that the integral along any ray $\rho e^{it},\rho\in[0,\infty]$ equals the limit when $0<\varepsilon\to 0$ of the integral along $\rho e^{it},\rho\in[\varepsilon,\infty]$. Thus

$$\oint_{\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3}\frac{t^{-\alpha}}{t+1}dt=2\pi i\mathrm{Res}\left(\frac{t^{-\alpha}}{t+1};z=-1\right)=2\pi i e^{-\pi i\alpha} \tag{26.3}$$

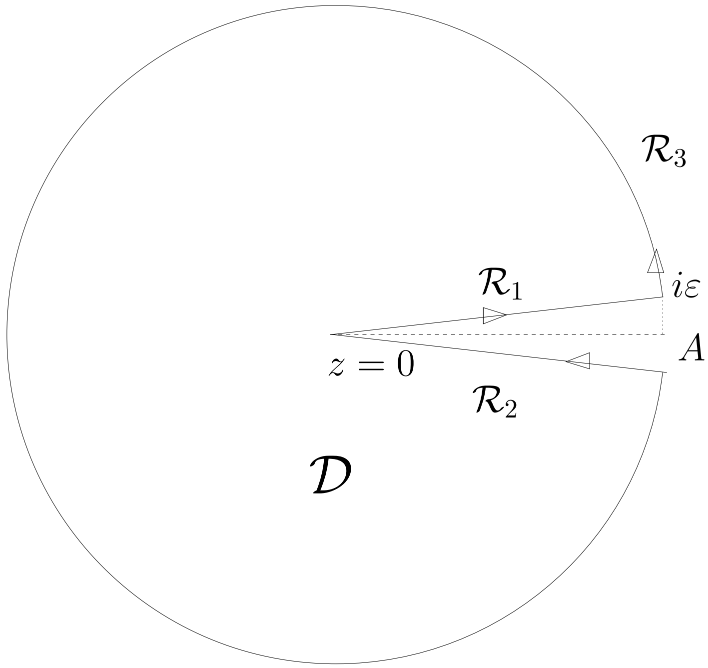


FIGURE 13.

In the limit $\varepsilon \to 0$ we get (check)

$$\int_0^A \frac{t^{-\alpha} - t^{-\alpha}e^{-2\pi i\alpha}}{t+1} dt + \int_{\mathcal{R}_3} \frac{t^{-\alpha}}{t+1} dt = 2\pi i e^{-\pi i\alpha}$$

In the limit $A \to \infty$, $\int_{\mathcal{R}_3}$ vanishes and $\int_0^A$ converges to $\int_0^\infty$. We get

$$\int_{\mathbb{R}^+} \frac{t^{-\alpha} - t^{-\alpha}e^{-2\pi i\alpha}}{t+1} dt = (1 - e^{-2\pi i\alpha}) \int_{\mathbb{R}^+} \frac{t^{-\alpha}}{t+1} dt = 2\pi i e^{-\pi i\alpha} \tag{26.4}$$

The rest is straightforward.

More generally, we have the following result.

Proposition 26.80. *Assume* $\operatorname{Re} a \in (0,1)$ *and* Q *is a rational function which is continuous on* $\mathbb{R}^+$ *and is such that* $x^a Q(x) \to 0$ *as* $x \to 0$ and *as* $x \to \infty$. *Then*

$$\int_0^\infty x^{a-1} Q(x) dx = -\frac{\pi e^{-\pi i a}}{\sin a\pi} \sum Res(z^{a-1}Q(z); z_i) \tag{26.5}$$

where z_i *are the poles of* Q.

Exercise 26.81. *** Prove Proposition 26.80.*

Exercise 26.82. *** Let $a \in (0,1)$. Calculate*

$$P\int_0^\infty \frac{x^{a-1}}{1-x}dx$$

where P denotes the Cauchy principal part, as defined before.

Exercise 26.83.

$$\int_0^\infty \frac{x^{-1/2}\ln x}{x+1}dx$$

(There is a simple way, using the previous results.)

27. Conformal Mapping

Laplace's equation in two dimensions

$$\Delta f = f_{xx} + f_{yy} = 0 \tag{27.2}$$

describes a number of problems in physics; it describes for instance the flow of an incompressible fluid, the space dependence of the electric potential in a region where the density of charges, ρ is zero and the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ fields are time-independent. For the latter problem, Maxwell's equations are $\nabla \cdot \mathbf{E} = \varepsilon_0^{-1}\rho = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0$. The second equation implies $\mathbf{E} = -\nabla V$, for some V (called potential) and the first equation gives $\Delta V = 0$. Since the electric field is produced by charges, the boundary conditions are expected physically to determine the solution. A typical problem would be to solve eq. (27.2) with $u = V$ in $\mathcal{D}$ with V given on $\partial\mathcal{D}$ (Dirichlet problem).

In the case of two-dimensional incompressible fluid flow, let $\langle v, u\rangle$ be the velocity field. Incompressibility translates into

$$\operatorname{div}\langle u, v\rangle = v_x + u_y = 0 \tag{27.3}$$

while the fact that the flow is irrotational implies

$$\nabla \times \mathbf{V} = 0 \Rightarrow u_x - v_y = 0 \tag{27.4}$$

(27.3) and (27.4) imply that $\langle v, u\rangle$ are harmonic conjugates. In any simply connected domain, (27.4) implies $\mathbf{V} = \nabla\varphi$ for some φ called *velocity potential*. We can check that $\Delta\varphi = 0$, thus φ is harmonic. Its harmonic conjugate ψ is called *the stream function*. In the physical applications above, the ODE system associated with $\mathbf{V}$ and $\mathbf{E}$ are *both* potential and gradient. In the case of fluid flow, the lines of constancy of ψ are parallel to the flow, see §9.1. If the fluid flows in some domain $\mathcal{D}$, a natural boundary condition is that the fluid does not flow through $\partial\mathcal{D}$, that is $\langle v, u\rangle \cdot \langle n_1, n_2\rangle = 0$ where $\langle n_1, n_2\rangle$ is the normal direction to the boundary; this is also known as a no-penetration condition.

Laplace's equation where the normal derivative is given on $\partial\mathcal{D}$ is called a Neumann problem.

We already know the general solution of the Dirichlet problem when $\mathcal{D}$ is a disk, (12.7). The solution of the Dirichlet problem exists and is unique in any connected domain $\mathcal{D}$ with smooth enough boundary and continuous data on the boundary.

27.1. **Uniqueness.** We can show uniqueness at this stage. For if we had two solutions u_1, u_2 then $u = u_1 - u_2$ would satisfy (27.2) with $u = 0$ on ∂D. But a harmonic function reaches both its maximum and minimum on the boundary. Thus $u \equiv 0$. A similar argument shows that in the Neumann problem, u is determined up to an arbitrary constant.

Example 27.84. The Faraday cage. (In two dimensions) explain why a region surrounded by a conductor does not feel the electrical influence of static outside charges.

Solution. The electric potential along a conductor, at equilibrium, is zero. For otherwise, there would be a potential difference between two points, thus an electric current $i = V/R$ where R is the resistivity. This would contradict equilibrium.

Thus we deal with (27.2) with $V = C$ on $\partial\mathcal{D}$. Since $V = C$ is a solution, it is *the* solution. But then $\mathbf{E} = -\nabla V = 0$ which we wanted to prove $\Box$.

27.2. **Existence.** We have already mentioned that the Dirichlet problem has a unique solution in any disk, if the boundary condition is continuous. What about other domains?

It is often the case in PDEs that a symmetry group exist and then it is very useful in solving the equation and/or determining its properties.

It turns out that (27.2) has a *huge* symmetry group: the equation is *conformally invariant*. This means the following.

Proposition 27.85. *If u solves (27.2) in $\mathcal{D}$ and $f = f_1 + if_2$ is analytic and such that $f : \mathcal{D}_1 \to \mathcal{D}$, then $u(f_1(s,t), f_2(s,t))$ is a solution of (27.2) in $\mathcal{D}_1 := f^{-1}(\mathcal{D})$.*

Proof. Let $D \subset \mathcal{D}$ be a disk. We know that u has a harmonic conjugate v determined up to an additive constant. Let $g = u + iv$. Then g is analytic in D. Let $D_1 = f^{-1}(D)$, which is an open set in $\mathcal{D}_1$ since in particular f is continuous. Then the composite function $g(f)$ is analytic in $\mathcal{D}_1$, and in particular $u(f_1(s,t), f_2(s,t))$ and $v(f_1(s,t), f_2(s,t))$ satisfy the CR equations in D_1. But then $u(f_1(s,t), f_2(s,t))$ is harmonic in D_1. Since this holds near any point in $\mathcal{D}_2$, the statement is proved.

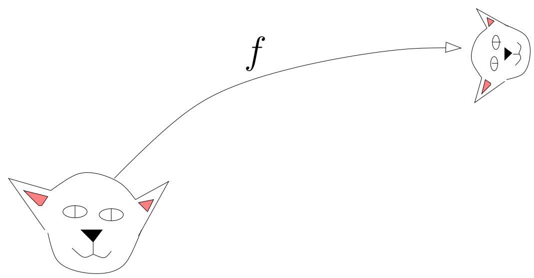


FIGURE 14. To be made rigorous in the sequel.

We will be mostly interested in analytic *homeomorphisms* which have many nice properties. Two regions that are analytically homeomorphic to each-other are called *conformally equivalent*.

The Riemann mapping theorem, which we will prove later, states that any simply connected domain other that $\mathbb{C}$ itself is conformally equivalent to the unit disk. The boundary of the region is then mapped onto the unit circle. The "orbit" of the disk under the group of conformal homeomorphisms group contains every simply connected region other that $\mathbb{C}$ itself.

The conformal group is large enough so that by its action we can solve Laplace's equation in any simply connected domain (the boundary has to be smooth enough for the boundary condition to make sense; $C^{1,\alpha}$ is sufficient.)

This is one of many motivations for a careful study of conformal maps.

27.3. **Heuristics.** Let f be analytic at z_0, $f'(z_0) = a \neq 0$ (w.l.o.g. $z_0 = 0$, $f(0) = 0$) and consider a tiny neighborhood $\mathcal{N}$ of zero. If z_β are points in $\mathcal{N}$ then

$$f(z_\beta) \approx a z_\beta \tag{27.5}$$

All these points get multiplied by the *same* number a. Multiplication by a complex number rescales it by $|a|$ and rotates it by $\arg a$. If we think of z_β as describing a figure, then $f(z_\beta)$ describes the same figure, rotated and rescaled. The shape (form) of the figure is thus preserved and the transformation is *conformal*.

Since a tiny square of side ε becomes a square of side $|a\varepsilon|$ areas are changed by a factor of $|a^2|$.

We make this rigorous in what follows.

27.4. **Preservation of angles.** Assume f is analytic in a disk D and that $f' \neq 0$. The angle between two smooth curves $\gamma(t)$ and $\Gamma(t)$ which cross at a point $z = \gamma(t_0) = \Gamma(t_1)$ (w.l.o.g. we can take $t_0 = t_1 = 0$)is by definition the angle between their tangent vectors, that is $\arg\gamma'(0) - \arg\Gamma'(0)$, assuming of course that these derivatives don't vanish.

The angle between the images of these curves is given by

(27.6)
$$\arg[f(\gamma)'(0)] - \arg[f(\Gamma)'(0)] = \arg[f'(\gamma(0))\gamma'(0)] - \arg[f'(\Gamma(0))\Gamma'(0)]$$
$$= \arg f'(\gamma(0)) + \arg\gamma'(0) - \Big(\arg f'(\Gamma(0)) + \arg\Gamma'(0)\Big) = \arg\gamma'(0) - \arg\Gamma'(0)$$

That is to say the image of two curves intersecting at an angle α is a pair of curves intersecting at the same angle α. Plreservation of angles means that a small enough domain is transformed into a similar one, only rotated and rescaled.

27.5. **Rescaling of arc length.** The arc length along a curve $\gamma(t)$ is given by

$$L(\gamma) = \int_a^b |\gamma'(t)|dt =: \int_\gamma d|z| \tag{27.7}$$

If f is analytic, then
(27.8)
$$L(f(\gamma)) = \int_a^b |f(\gamma)'(t)|dt = \int_a^b |f'(\gamma(t))|\,|\gamma'(t)|dt = \int_\gamma |f'(z)|d|z|$$

and thus the arc length is locally stretched by $|f'(z)|$.

27.6. **Transformation of areas.** The area of a set A is

$$\iint\limits_A dxdy \tag{27.9}$$

while after the transformation $(x, y) \mapsto (u(x, y), v(x, y))$ the area becomes

$$\iint\limits_{f^{-1}(A)} |J|dudv \tag{27.10}$$

where the Jacobian J is, using the CR equations, $|f'|^2$ (check!).

Note 27.86. It is interesting to remark that it is enough that (u, v) is a smooth transformation that preserves angles for $u+iv$ to be analytic. It is also enough that it rescales any figure by the same amount for it to be analytic or anti-analytic ($\overline{f}$ is analytic), see [3], p 74. This gives a very nice characterization of analytic functions: they are those which are "locally Euclidian".

Note 27.87. Observe that we did not require f to be globally one-to-one. The simple fact that f is analytic with nonzero derivative makes it conformal. We need to impose bijectivity for two regions to be conformally *equivalent*. On the other hand, if f is an analytic homeomorphism between $\mathcal{D}_1$ and $\mathcal{D}_2$ then f is conformal (that is, $f' \neq 0$ in $\mathcal{D}$). This follows from the following proposition.

Proposition 27.88. *Assume that $f : \mathcal{D}_1 \mapsto \mathcal{D}_2$ is analytic, that for some $z_0 \in \mathcal{D}_1$ we have $f^{(j)}(z_0) = 0$ if $j = 1, ..., m-1$ and $f^{(m)}(z_0) = a \neq 0$. W.l.o.g. assume $z_0 = 0$ and $f(z_0) = 0$. Then, in some disk $D(\varepsilon, 0)$, f is m-to-one, that is for any $0 \neq w \in f(D(\varepsilon, z_0))$ the set $f^{-1}(w)$ consists in precisely m different points.*

Proof. In a neighborhood of 0 we have, with $a \neq 0$

$$f(z) = z^m(a + b_1 z + b_2 z^2 + \cdots) = z^m g(z)$$

and $g(z)$ is analytic, $g(0) = a$ and thus $g \neq 0$ in some disk $D = D(0, \varepsilon_1)$. Then in $D(0, \varepsilon_1)$ $\ln g$ is well defined (by $\int g'/g$) and analytic and so is therefore $h = \exp(m^{-1} \ln g) = g^{1/m}$. For a given ζ, our equation becomes $(zh(z))^m = \zeta$, equivalent to m equations, $zh(z) = y_m := |\zeta|^{\frac{1}{m}} e^{im^{-1}\phi + \frac{2\pi ik}{m}}$, $k = 0, 1, ..., m-1$, where $\phi = \arg\zeta$. The function zh is analytic at zero and $(zh)'(0) = h(0) \neq 0$. By the inverse function theorem, the equation $zh = y_m$ has exactly one solution if $|y| = |\zeta|$ is small enough. $\square$.

27.7. **The open mapping theorem.**

Theorem 27.89. *Let $\mathcal{D}$ be a domain and $f : \mathcal{D} \to \mathbb{C}$ be a nonconstant analytic function. Then f is open, that is, the images of open sets are open.*

Proof. We want to check that a small neighborhood of any $x_0 \in \mathcal{D}$ is mapped into an open set. The composition of open maps is clearly open. By composition with translations (clearly open) we can set $x_0 = 0$, $f(x_0) = 0$. If $f'(0) \neq 0$ f is bijective between a disk $\mathbb{D}_\varepsilon$ into $f(\mathbb{D}_\varepsilon)$ and (since the inverse is continuous) f is open. If $f' \neq 0$ then, by Proposition 27.88 it can be written as $z^{m-1}(zg(z))$ with $g(0) \neq 0$ for

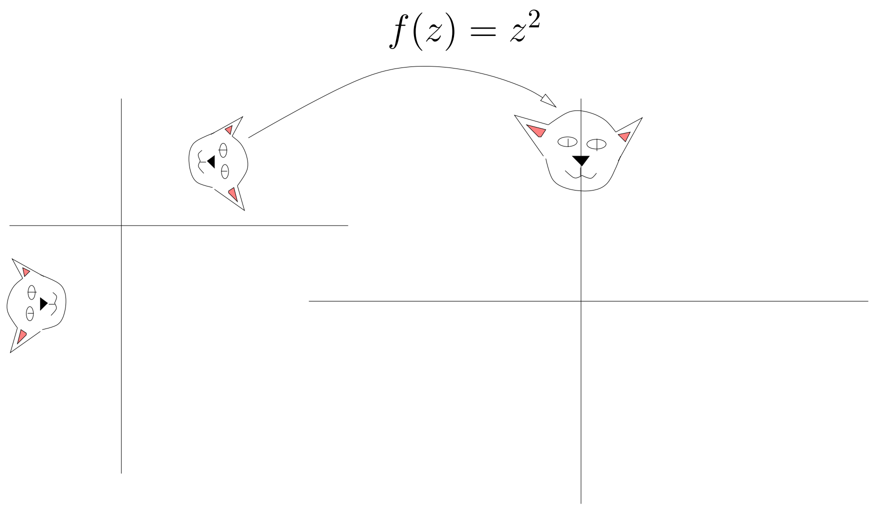


FIGURE 15. To be made rigorous in the sequel.

some $m \geqslant 1$ (check). Since $[zg(z)]'(0) = g(0) \neq 0$, $zg(z)$ is invertible and the result is immediate. ∎

28. Linear fractional transformations (Möbius transformations)

We assume some familiarity with these transformations, and we review their properties.

A linear fractional transformation (LFT) is a map of the form

$$S(z) = \frac{az+b}{cz+d} \tag{28.1}$$

where $ad - bc \neq 0$. If $c = 0$ we have a linear function. If $c \neq 0$ we write

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c^2(z+d/c)} \tag{28.2}$$

and we see that S is meromorphic, with only one pole at $z = -d/c$. It is also clear from (28.2) that $S(z_1) = S(z_2)$ iff $z_1 = z_2$ and in particular $S'(z) \neq 0$. They are one-to-one transformations on the Riemann sphere too (that is, the point at infinity included). LFTs are conformal.

Proposition 28.1. *LFTs form a group with respect to function composition.*

Proof: Exercise.

Exercise 28.2. *Show that $z \mapsto 1/z$ maps a line or a circle into a line or a circle. Hint: Show first that the equation*

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$ and $|\beta|^2 > \alpha\gamma$ is the most general equation of a line or a circle. Then apply the transformation to the equation.

As a result we have an important property of LFTs

Proposition 28.3. *An LFT maps a line or a circle into a line or a circle.*

Proof. A general LFT is the composition of Euclidian transformations and inversion:

$$z \mapsto w_1 = z + d/c \mapsto w_2 = c^2 w_1 \mapsto w_3 = \frac{1}{w_2} \mapsto w_4 = -(ad - bc)w_3 \mapsto w_5 = w_4 + \frac{a}{c} \tag{28.3}$$

It is clear that the statement holds for all linear transformations. We only need to show that this is also the case for inversion, $z \mapsto 1/z$. This follows from Exercise 28.2 □

28.1. **Finding specific LFTs.** As we know from elementary geometry a line is determined by two of its points and a circle is determined by three. We now show that for any two circles/lines there is a LFT mapping one into the other, and in fact they can be determined explicitly. Let z_1, z_2, z_3 be three points in $\mathbb{C}$. Then the transformation

$$S = \frac{z_1 - z_3}{z_1 - z_2}\,\frac{z - z_2}{z - z_3} \tag{28.4}$$

maps z_1, z_2, z_3 into $1, 0, \infty$ in this order; this is easy to check. If one of z_1, z_2, z_3 is ∞, we pass the transformation to the limit. The transformations in these limit cases are is

$$\frac{z - z_2}{z - z_3}, \quad \frac{z_1 - z_3}{z - z_3}, \quad \frac{z - z_2}{z_1 - z_2} \tag{28.5}$$

respectively.

Exercise 28.4. *Check that a LFT that takes $(1, 0, \infty)$ into itself is the identity.*

To find a transformation that maps z_1, z_2, z_3 into $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3$ in this order, clearly we apply $\tilde{\tilde{S}} := \tilde{S}^{-1}S$. By Exercise 28.4 this transformation is unique.

28.1.1. *Cross ratio.* If $z_i, i = 1...4$ are four distinct points and $w_i = S(z_i)$ then (check!)

$$\frac{w_1 - w_2}{w_1 - w_3}\frac{w_3 - w_4}{w_2 - w_4} = \frac{z_1 - z_2}{z_1 - z_3}\frac{z_3 - z_4}{z_2 - z_4}$$

This is often a handy way to determine the image of a fourth point when the transformation is calculated using three points.

Exercise 28.5. Associate to a LFT the coefficients matrix

$$\hat{M} := \frac{az + b}{cz + d} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{28.6}$$

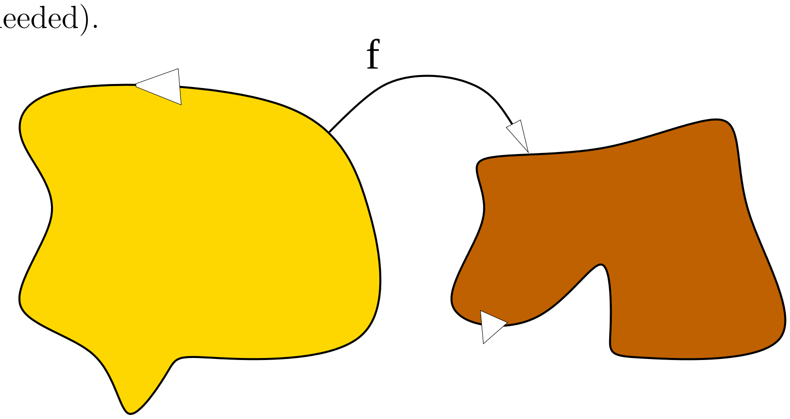
If T_1 and T_2 are LFTs, then show that

$$\hat{M}(T_1 \circ T_2) = \hat{M}(T_1)\hat{M}(T_2) \tag{28.7}$$

where the product on the right side of (28.7) is the usual matrix product.

28.2. **Mappings of regions.** We know that LFTs are *conformal and one-to-one* and transform circles/lines onto circles/lines. What about their interior? We look at this problem more generally.

By definition a curve is traversed in anticlockwise direction if the parameterization is such that the interior is to the left of the curve as the parameter increases (brush up the notions of orientation etc. if needed).



Proposition 28.6. *Assume that $f : \mathcal{D} \mapsto \mathcal{D}_1$ is analytic and γ is a simple piecewise differentiable* closed *curve contained in $\mathcal{D}$ together with its interior.*

If f is one-to-one from γ to $f(\gamma)$, then f maps one-to-one conformally Int*(γ) onto $f($*Int*$(\gamma))$ and preserves the orientation of the curve.*

Proof. From the assumptions it follows that $f(\gamma)$ is a simple curve. Let $w_0 \in$Int$(f(\gamma))$. Cauchy's formula implies

$$\frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w-w_0} = 1 \tag{28.8}$$

On the other hand by assumption f is one-to-one on γ and we can change variables $w = f(z)$, $z \in \gamma$, and we get

$$1 = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)dz}{f(z)-w_0} = \frac{1}{2\pi i}\int_\gamma \frac{(f(z)-w_0)'dz}{f(z)-w_0} \tag{28.9}$$

and by Proposition 18.60 (and since f is analytic) this shows that $f(z)-w_0$ has exactly one zero in Int(γ), or there is exactly one z_0 such that $f(z_0) = w_0$. Then f is conformal, one-to-one onto between Int(γ) and $f(\text{Int}(\gamma))$. This also shows that f preserves orientation, otherwise the integral would be -1.

Exercise 28.7. * (i) Find a LFTs that maps the unit disk onto the upper half plane.

(ii) Find a LFTs that maps the disk $(x-1)^2 + (y-2)^2 = 4$ onto the unit circle and the center is mapped to $i/2$.

(iii) Find the most general linear fractional transformation that maps the unit disk onto itself.

28.3. As usual, we let $\mathbb{D}$ be the open unit disk.

Theorem 28.8 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic and such that $f(0) = 0$. Then*

(i)

$$|f(z)| \leqslant |z| \tag{28.10}$$

for all $z \in D$.

(ii) If there is some $z_0 \in \mathbb{D}$ such that for $z = z_0$ we have equality in (28.10) then $f(z) = e^{i\phi}z$ for some $\phi \in \mathbb{R}$.

(iii) $|f'(0)| \leqslant 1$ and if equality holds then again $f(z) = e^{i\phi}z$ for some $\phi \in \mathbb{R}$.

Proof. (i) Since $f(0) = 0$, the function $f(z)/z$ extends analytically in $\mathbb{D}$. By the maximum modulus principle,

$$\left|\frac{f(z)}{z}\right| \leqslant \lim_{r\uparrow 1}\max_{|z|=r}\left|\frac{f(z)}{z}\right| = 1$$

(ii) If z_0 is such that equality in (28.10) holds, then z_0 is a point of maximum of $|f(z)/z|$, which cannot happen unless $f(z)/z = C = f(z_0)/z_0$.

(iii) The inequality follows immediately from (28.10). Assume $f'(0) = e^{i\phi}$, $\phi \in \mathbb{R}$. If $f(z) \not\equiv e^{i\phi} z$, then we can write

$$f(z)/z = e^{i\phi}(1 + z^m e^{i\psi} h(z))$$

where h is analytic and $h(0) \in \mathbb{R}^+$. If we then take $z = \varepsilon \exp(-i\psi/m)$ with ε small enough we contradict (i). $\Box$

Corollary 28.9. *If h is an* automorphism *of the unit disk and $h(0) = 0$ then $h(z) = e^{i\phi} z$ for some $\phi \in \mathbb{R}$.*

Proof. We must have, by Theorem 28.8 $|h(z)| \leqslant |z|$. But the inverse function h^{-1} is also an automorphism of the unit disk and $h^{-1}(0) = 0$. Thus $|h^{-1}(z)| \leqslant |z|$ for all z, in particular $|z| = |h^{-1}(h(z))| \leqslant |h(z)|$ or $|z| \leqslant |h(z)|$. Thus $|h(z)| = |z|$ for all z and the result follows from Theorem 28.8 (ii).$\Box$

28.4. **Automorphisms of the unit disk.** We have seen that

$$S(z) = e^{i\phi} \frac{z - \alpha}{1 - \overline{\alpha} z} \quad \text{with } \phi \in \mathbb{R} \text{ and } |\alpha| < 1 \tag{28.11}$$

maps the unit disk one-to-one onto itself.

The converse is also true:

Theorem 28.10. *Any automorphism f of $\mathbb{D}$ into itself is of the form (28.11) with $\alpha = f^{-1}(0)$.*

Proof. The function $h = S \circ f^{-1}$ is an automorphism of the unit disk and $h(0) = S(\alpha) = 0$. But then Corollary 28.9 applies and the result follows. $\Box$.

Exercise 28.11. Show that the automorphisms of the upper half plane are of the form $\frac{az+b}{cz+d}$ with $ad - bc > 0$ (see also Exercise 28.5). The automorphism is unique, the identity, if $\varphi(0) = 0$ and $\varphi(1) = 1$.

28.5. **Miscellaneous transformations.** We illustrate below a number of useful transformations; the references, esp. [1], [8] for more examples. A good number of interesting domains can be mapped to the unit disk using combinations of these transformations. Note that by Proposition 28.6 it suffices to examine carefully the way the boundaries are mapped to understand the action of a map on a whole domain.

28.5.1. *The Joukovski transformation.* This is an interesting map which straightens the region in the upper half plane above the unit circle (of course, by slight modifications, you can choose other radii or centers along $\mathbb{R}^+$) to the upper half plane. It is given by

$$z \mapsto z + \frac{1}{z} \tag{28.12}$$

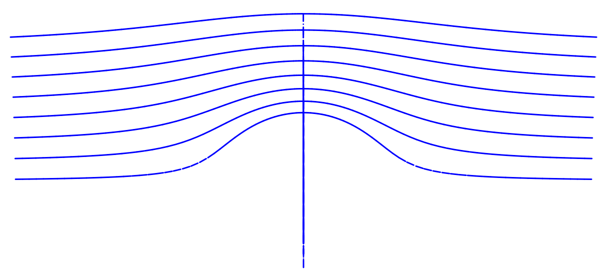
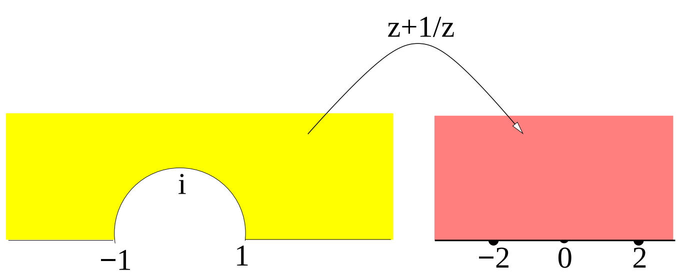
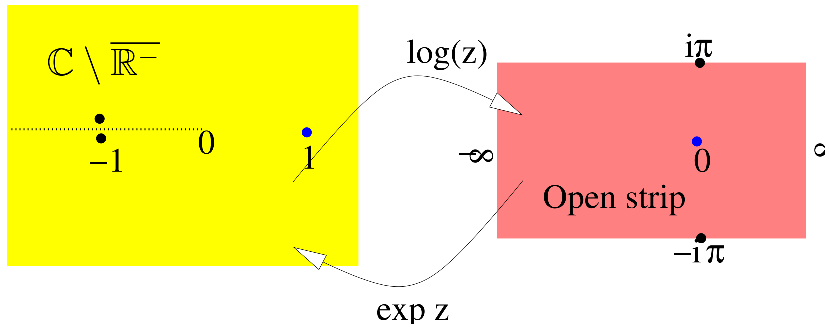


FIGURE 16. Velocity field lines in Joukovski's domain

Exercise 28.12. *Explain the effect of this map on the region depicted.*



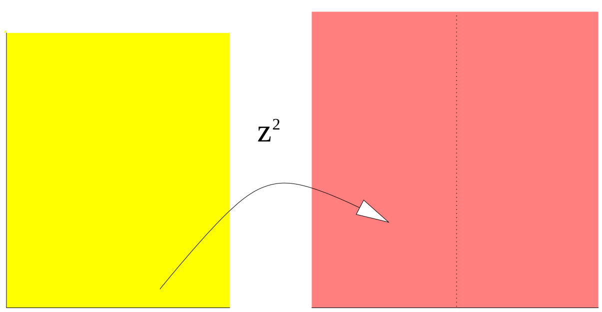
As a nice application, we can find the flow lines of a river passing above a cylindrical obstacle. Indeed, the free flow in the upper half plane is horizontal, with constant velocity, say one. More precisely, $\langle 1, 0\rangle$, coming from the velocity potential $\varphi(x, y) = y$ is a harmonic function and the associated velocity field is $\langle 1, 0\rangle$. It satisfies the boundary condition $\partial\varphi/\partial x = 0$, the tangential derivative is zero, which means that the vertical velocity is zero, the non-penetration condition. The associated analytic function, up to an irrelevant constant, is just $f(z) = z$. If $J = J_1 + iJ_2$ denotes the Joukovski transformation, then $J_2(x_1, y_1)$ is harmonic and satisfies the no-penetration condition since the composition $I \circ J$ is conformal. The fluid flow lines are then given by $J_2(x_1, y_1) = const.$ plotted with Maple in Fig 28.5.1.



Exercise 28.13. *

Find an explicit formula for flow lines in the previous example.

Other common mappings are depicted in the following figures. The transformation of the half plane into the disk is also called the Cayley transform. It plays an important role in the theory of unbounded operators. It is natural seek a Möbius transformation to map $\mathbb{R}$ onto $\partial\mathbb{D}$. Now a point is on the circle iff it is of the form $x/\overline{x}$ for some nonzero x, and the transformation should then be of the form $(az-i)/(az+i)$ with a real (which we can choose, up to rescaling, to be 1) [3].



Exercise 28.14. ** *(i) Check the transformation in the figures above.*

(ii) Draw a similar picture for the mapping $\sin z$ *from the upper half strip bordered by the half-lines* $x = \pm\pi/2, y > 0$.

(iii) Find a conformal homeomorphism of the quarter disk $|z| < 1, \arg(z) \in (0, \pi/2)$ *onto the upper half plane.*

(iv) Find a conformal homeomorphism of the half disk $|z| < 1, \arg(z) \in (0, \pi)$ *onto the half strip* $x < 0, y \in (0, \pi)$.

(iv) Find a conformal homeomorphism of the right half plane ($\mathbb{H}^+$) *with a cut along* $[0, 1]$ *in* $\mathbb{H}^+$.

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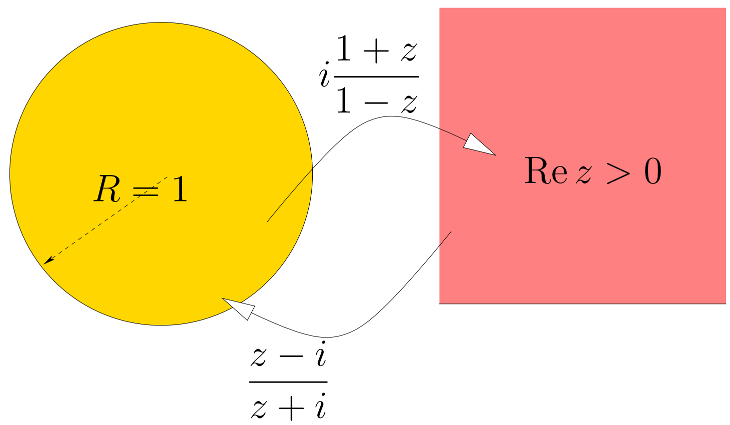
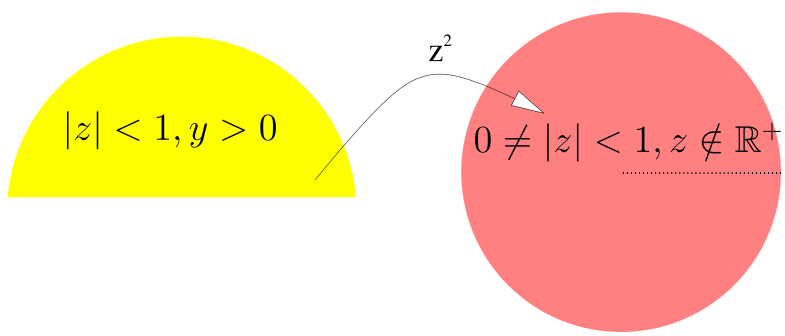
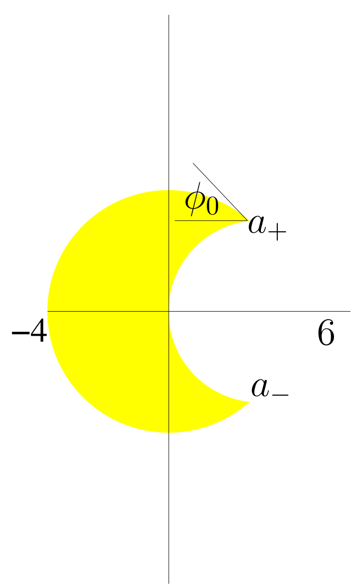
[3] This "qualitative" construction is based on a suggestion of I. Glogic.







Example 1 It is useful to remark that we can find linear fractional transformations which map a region between two circles into a half plane (or disk) using very simple transformations. Let us map the "moon crescent" M below into a half plane.

The equations of the circles are $x^2 + y^2 = 16$ and $(x-3)^2 + y^2 = 9$. Solving these equations for the intersection points $a_\pm$ we get $a_\pm = 8/3 \pm 4i\sqrt{5}/3$. The angle between the circles equals the angle between $a_+ - 3$ and a_+ that is $\arg[(a_+ - 3)/a_+] = \arctan(\sqrt{5}/2)$. The idea is that if we map by an LFT one of the intersection points to infinity, the arccircles become lines for which question is easier.

If we map $a_- \mapsto 0$, $0 \mapsto 1$ and $a_+ \mapsto \infty$ by a LFT, concretely

$$\frac{a_+}{a_-}\frac{z - a_-}{z - a_+} \tag{28.13}$$

then both arccircles become *rays* (since they end at ∞). The small arc becomes $\mathbb{R}^+$ and the larger one a ray of angle ϕ_0 (by conformality at a_-: check these statements).

To transform this sector of opening π_0 to the upper half plane we simply use a ramified transformation $z \mapsto z^{\pi/\phi_0}$.

Example 2 Solve $\Delta u = 0$ in the region $|z| < 1, \arg(z) \in (0, \pi/2)$ such that on the boundary we have: $u = 1$ on the arc and $u = 0$ otherwise.

Solution. Strategy: We find conformal homeomorphism of this region into the strip $\{z = x + iy : y \in (0,1)\}$ such that the arc goes into $y = 1$ and the segments into $y = 0$. The solution of the problem in this region is clear: $u = \operatorname{Im} z$. Then we map back this function through the transformations made.

How to find the transformation? We are dealing with circles, strips, etc so it is hopeful we can get the job done by composing elementary transformations. There is no unique way to achieve that, but the end result must be the same.

(1) The transformation $z \mapsto z^2$ opens up the quarter disk into a half disk. On the boundary we still have: $u = 1$ on the arc and $u = 0$ otherwise.

(2) We can now open the half disk into a quarter plane, by sending the point $z = 1$ to infinity, as in Example 1, by a linear fractional transformation. We need to place a pole at $z = 1$ and a zero at $z = -1$. Thus the second transformation is $z \mapsto i\frac{1+z}{1-z}$. The segment starting at -1 ending at 1 is transformed in a line too, and the line is clearly $\mathbb{R}^+$ since the application is real and positive on $[0, 1)$ and 1 is a pole. What about the half circle? It must become a ray since the image starts at $z = 0$ and ends at infinity. Which line? The image of $z = i$ is $w = i$. Now we deal with the first quadrant with boundary condition $u = 1$ on $i\mathbb{R}^+$ and $u = 0$ on $\mathbb{R}^+$.

(3) We open up the quadrant onto the upper half plane by $z \mapsto z^2$.

(4) We now use a rescaled log to complete the transformation. The composite transformation is

$$\frac{2}{\pi}\ln\left(\frac{1+z^2}{1-z^2}\right)$$

Exercise** The temperature distribution also satisfies Laplace's equation. (1) Map M onto a strip as in Example 2. What is the distribution of temperature in the domain M if the temperature on the larger arc is 1 and 0 on the smaller one? What shape do the lines of constant temperature have?

(2) What is the distribution of temperature in the domain and with the boundary conditions described in example 2? Draw an approximate picture of the lines of

29. The Riemann Mapping Theorem

Using elementary transformations we can conformally map a quite limited family of domains onto $\mathbb{D}$; with the maps we used in the previous section, the boundary is always very simple. The Schwarz-Christoffel formulas in §30,generally nonelementary provide conformal maps between any polygons and the upper half plane (which, as we saw can be mapped onto the unit disk). In principle, however, any simply connected domain other than $\mathbb{C}$ can be mapped onto the $\mathbb{D}$:

Theorem 29.1 (Riemann Mapping theorem)**.** *Given any simply connected domain $\mathcal{D}$* other than $\mathbb{C}$ *there is an analytic homeomorphism between $\mathcal{D}$ and $\mathbb{D}$.*

This map is unique if for some $z_0 \in \mathcal{D}$ it is normalized by the conditions $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}^+$.

The Riemann mapping theorem was stated by Riemann in 1851 for domains with piecewise regular boundary, and he provided a proof based on solving a Dirichlet problem for the Laplacian. The method used, the *Dirichlet's principle* was not quite right. Nonetheless, Riemann's approach can be made to work, and the idea is worth discussing. Roughly, with $z_0 \in \mathcal{D}$ one looks for a map in the form

$$\psi(z) = (z - z_0)\exp(f(z));\ \ f = u + iv$$

with f analytic in $\mathcal{D}$, or equivalently u harmonic in $\mathcal{D}$. The boundary condition is $u(z) = -\ln|z - z_0|$ on $\partial\mathcal{D}$. The Poisson equation was solved through the variational reformulation of the Dirichlet problem using *Dirichlet's principle*. However, while the Dirichlet functional is bounded below and has an infimum, it may may not reach this infimum. Weierstrass found that indeed this *can* happen for the Dirichlet functional. The first rigorous proof in full generality was given by Caratheodory in 1912.

*

The proof of this major theorem involves concepts and results that are very important and useful of their own. We will study these in detail.

29.1. **Equicontinuity.** We look at functions $f : M \mapsto M'$ where M, M' are metric spaces. We recall that if the metrics are d and d', a function is uniformly continuous if

$$\forall\delta\ \exists\varepsilon\ \Big(\forall(z,z_0)\in M^2,\ d(z,z_0)<\varepsilon\Rightarrow d'(f(z),f(z_0))<\delta\Big) \tag{29.1}$$

We can assume that the metric d' is a **bounded function**, for we can always replace it by $d'' = d'/(1+d')$ (check that d'' is a metric, topologically equivalent to d' [4]) and convergence with respect to d'' is the same as convergence with respect to d'.

Definition. An equicontinuous family $\mathcal{F}$ is a collection of continuous functions with the same continuity parameters at every point:

$$\forall x\exists\delta(x)s.t.\forall\varepsilon>0,\forall y\&\forall f, d(x,y)<\delta\Rightarrow d'(f(x),f(y)<\varepsilon \tag{29.2}$$

Definition 29.2. *Let M, M' be a complete metric spaces. Then a collection of continuous functions $\mathcal{F}$ from M to M' is a* **normal family** *if it is pre-compact, i.e., if any sequence $\{f_n\}_{n\in\mathbb{N}}\subset\mathcal{F}$ contains a subsequence that converges uniformly on compact subsets of M.*

Exhaustion by compact sets. We note that if the metric space Ω is $\mathbb{C}$, $\mathbb{R}^n$ or a subset M of these, we write

$$M=\bigcup_{n\in\mathbb{N}}K_n,\quad K_n=\{x\in M: d(x,0)+1/d(x,\partial\Omega)\leqslant n\} \tag{29.3}$$

where d is the usual *Euclidian distance*. Note that the sets K_n are closed and bounded, therefore compact; their union covers M.

Metrizability of the topology of uniform convergence on compact sets On each K_n in (29.3) we define the distance between two functions f and g in a manner analogous to the L^∞ distance:

$$\delta_n(f,g)=\sup_{x\in K_n} d''(f(x),g(x)) \tag{29.4}$$

and we create a distance on the whole of M which takes advantage of the compact exhaustion:

$$\rho(f,g)=\sum_{n=1}^{\infty}2^{-n}\delta_n(f,g) \tag{29.5}$$

(recall that $d''\leqslant 1$).

---

[4]In separable spaces, i.e. ones which contain a countable dense set, equivalence holds iff convergence in one metric is equivalent to convergence in the other.

Exercise 29.3. *Check that ρ is a metric on $C(M)$– the continuous functions on M. Check that convergence with respect to ρ is equivalent with uniform convergence on compact sets. Check that $\mathcal{F}$ is a complete metric space if M' is a complete metric space.*

Theorem 29.4. *A family $\mathcal{F}$ is normal iff its closure $\overline{\mathcal{F}}$ with respect to ρ is compact.*

Proof. This follows from the fact that a space is compact iff any sequence has a convergent subsequence. □

29.1.1. *The Ascoli-Arzelà Theorem.*

Theorem 29.5 (Ascoli-Arzelà)**.** *A family $\mathcal{F}$ of continuous functions in the region $\Omega \subset \mathbb{C}$ with values in a metric space M' is* **normal** *in Ω iff the following conditions are both satisfied:*
(i) $\mathcal{F}$ is equicontinuous at any $x \in \Omega$.
(ii) $\forall z \in \Omega\ \exists K_1$ compact in M' such that $\forall f \in \mathcal{F}, f(z) \in K_1$.

This is a standard theorem. We leave the proof for the Appendix, §42.3.

Proposition 29.6. *Let now $M' \subset \mathbb{R}^n$ and $\mathcal{F}$ be a normal family from Ω to M'. Let $K \subset \Omega$ be compact. Then the bound on $f(z)$ can be made z independent in K:*

$$\sup_{z\in K, f\in\mathcal{F}} |f(z)| = m < \infty \tag{29.6}$$

Proof. Since $\mathcal{F}$ is a normal family, for any point a we can find $\delta(a)$ such that

$$\forall b, \forall f,\ |a-b| < \delta \Rightarrow |f(a) - f(b)| < 1 \tag{29.7}$$

Extract a finite covering of K from the balls above, let a_j be the centers of the balls and δ_0 be the smallest δ in the finite cover. We denote $m_j = \sup\{|f(a_j)| : f \in \mathcal{F}\}$ and $m = 1 + \max_j\{m_j\}$. Then, for any $x \in K$ there is an a_j such that $|x - a_j| < \delta_0$. Therefore, by the choice of m and a_j (29.7) we have, for any $f \in \mathcal{F}$,

$$|f(x)| \leqslant |f(x) - f(a_j)| + |f(a_j)| = 1 + m_j \leqslant m \quad \Box \tag{29.8}$$

Theorem 29.7 (Montel)**.** *Consider a domain $\mathcal{D} \subset \mathbb{C}$ and assume $\mathcal{F}$ is a family of analytic functions $\mathcal{D}$ such that for every compact $K \subset \mathcal{D}$ we have* $\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} = m(K) < \infty$. *Then the family is normal.*

Proof. If the derivatives of the functions in $\mathcal{F}$ are equibounded in K, then equicontinuity follows easily (check).

Let K be a compact set in $\mathcal{D} \subset \mathbb{C}$ and let $0 < r = d(K, \partial\mathcal{D}$. Let $K' = \{z : d(z, K) \leqslant r/2$. K is clearly compact and contained in $\mathcal{D}$. For $f \in \mathcal{F}$ and $z \in K$ we have

$$|f'(z)| \leqslant \frac{1}{2\pi}\left|\oint_{\partial\mathbb{D}_{r/2}(z)} \frac{f(s)}{(s-z)^2} ds\right| \leqslant \frac{m(K')}{r} \tag{29.9}$$

proving the result. ▮

29.2. **The Riemann Mapping Theorem.**

Definition 29.8. (i) $\mathcal{D}$ and $\mathcal{D}'$ will be called **conformally equivalent** if there is a map which is analytic together with its inverse (biholomorphism) from $\mathcal{D}$ onto $\mathcal{D}'$. (ii) An analytic function is univalent (or *schlicht*) if $g(z_1) = g(z_2) \Rightarrow z_1 = z_2$.

Theorem 29.9 (Riemann mapping theorem)**.** *Given any nonempty simply connected domain $\Omega \subset \mathbb{C}$ other than $\mathbb{C}$ itself, a point $z_0 \in \Omega$ and the normalization conditions $\varphi(z_0) = 0, \varphi'(z_0) \in \mathbb{R}^+$ there exists a unique biholomorphism $\varphi(z)$ between Ω and $\mathbb{D}$.*

Note 29.10. The fact that $\mathbb{C}$ must be an exception follows from the fact that an entire bounded function is constant.

Corollary 29.11. *$\mathbb{D}$ above can be replaced by other domains such as $\mathbb{H}^u$. Indeed we can use the composition $\varphi \circ T_1 \circ T_2$ where $T_1(z) = \frac{1-z}{1+z}$ which transforms $\mathbb{H}^u$ into $\mathbb{D}$, and T_2 is any automorphism of $\mathbb{H}^u$, $T_2(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ (cf. Exercise 28.11).*

Proof of the corollary. This is straightforward and we leave it as an exercise. ▮

Proof of the Riemann Mapping Theorem
Uniqueness. This part is easier. If φ_1 and φ_2 are two functions with the stated properties, then $S := \varphi_1(\varphi_2)^{-1}$ is a biholomorphism of the unit disk and $S(0) = 0$. By Schwarz's lemma $S(z) = ze^{i\theta}$ for some θ. $S'(0) > 0$ implies $\theta = 0$.
Existence.

Note 29.12. Clearly, in the proof we can replace the arbitrary domain $\mathcal{D}$ with any set conformally equivalent to it. Therefore, we first simplify the domain by elementary transformations: it turns out that it is enough to prove the result for $\Omega \subset \mathbb{D}$ with $z_0 = 0$.

Note 29.13. By assumption there is a point in $\mathbb{C} \setminus \Omega$, which, by translation if needed, we can assume to be 0. $\Omega \neq \emptyset \Rightarrow a \in \Omega, a \neq 0$. The transformation $z \to z/a$ maps it to 1.

Lemma 29.14. *(i) There is a biholomorphic branch of the log ($\log z = \int_1^z s^{-1}ds$) between Ω and $\log(\Omega)$ (with* exp *as its inverse, of course).*

(ii) $\mathbb{C} \setminus \log(\Omega)$ contains an open disk.

(iii) There is a bounded set Ω' conformally equivalent to Ω.

(iv) There is a biholomorphism g s.t. $g(\Omega) \subset \mathbb{D}$ with $g(z_0) = 0, g'(0) > 0$.

Proof. (i) See Lemma 23.73 and Note 29.13.

(ii) Let $L = \log(\Omega)$. We claim that $L \cap (2\pi i + L) = \emptyset$.[5] Indeed, if $z \in L$, then $z = \ln x$ for some x and if $z + 2\pi i \in L$ too, then $e^{z+2\pi i} = x$ in contradiction with Lemma 23.73. $2\pi i + L$ is a nonempty open set. [6]

(iii) By translation if needed, we can assume that the missing disk is $\mathbb{D}_\varepsilon(0)$ for some ε. Now the function $z \mapsto z^{-1}$ biholomorphically maps Ω onto a set $\Omega' \subset \mathbb{D}$. After the transformations, z_0 is mapped to some $a \in \mathbb{D}$.

(iv) An automorphism A of the disk maps any $\emptyset \neq \Omega' \subset \mathbb{D}$ onto $\Omega'' \subset \mathbb{D}$ and z_0 to zero. $A'(a)$ can be changed by multiplication by $e^{i\theta}$ to make the derivative of the composition of all the maps above positive at z_0. ∎

We can thus assume wlog that $\Omega \subset \mathbb{D}$ and $z_0 = 0$.

Heuristics of the rest of the proof. Since $0 < |x| < 1 \Rightarrow 1 > \sqrt{|x|} > |x|$. If $\Omega \neq \mathbb{D}$, we should be able to expand Ω by taking appropriate square roots. By Schwarz's lemma we expect that we "ultimately" reach the sought-for transformation when $\varphi(0)$ has been maximized.

Definition 29.15. *Let $\mathcal{F}$ be the set of biholomorphic maps between Ω and a subset of $\mathbb{D}$, which vanish at zero and have positive derivative there.*

$\mathcal{F}$ is nonempty since the identity is in $\mathcal{F}$.

Proposition 29.16. *(i) $M := \sup\{f'(0)|f \in \mathcal{F}\}$ is attained by an F in $\mathcal{F}$.*

(ii) If $F(\Omega) \neq \mathbb{D}$, then there is an $F_1 \in \mathcal{F}$ with $F_1'(0) > M$ (a contradiction which finishes the proof of the theorem).

Proof. (i) For any $f \in \mathcal{F}$, by assumption, $|f(z)| < 1\ \forall z \in \Omega$. By Montel's Theorem 29.6 $\mathcal{F}$ is a normal family, thus if $f_n \in \mathcal{F}$ and $f_n'(0) = m_n \to M$ then $\{f_n\}_{n\in\mathbb{N}}$ has a convergent subsequence to a function f which, by Hurwitz's theorem (18.63) is a biholomorphism. By Weierstrass's theorem (7.32) $f'(0) = M$.

---

[5]If S is a set, $x + S := \{x + z : z \in S\}$.

[6]Argument streamlined by Irfan Glogic.

(ii) Let $a \in \mathbb{D} \setminus F(\Omega)$. First we use an automorphism of $\mathbb{D}$ to map Ω to a set in $\mathbb{D} \setminus \{0\}$:

$$f_1(z) = \frac{F(z) - a}{1 - \overline{a}F(z)} \tag{29.10}$$

Since $f_1(\Omega) \subset \mathbb{D} \setminus \{0\}$, there is a branch of the square root in $f_1(\Omega)$. Now we can define

$$f_2(z) = \sqrt{f_1(z)}, \ \forall z \in \Omega \tag{29.11}$$

Note that f_2 is a biholomorphism and that $f_2(\Omega) \subset \mathbb{D}$. We see that $f_2(0) = \sqrt{-a} := b \neq 0$. We now move b to zero and change the phase of the derivative to zero through another automorphism of the disk:

$$F_1(z) = \frac{f_2(z) - b}{1 - \overline{b} f_2(z)} \frac{|f_2'(b)|}{f_2'(b)}, \ \forall z \in \Omega \tag{29.12}$$

A straightforward calculation shows that

$$F_1'(0) = \frac{1 + |a|}{2\sqrt{|a|}} F'(0) > F'(0) = M \tag{29.13}$$

∎

Exercise 29.17. If we start with a simply connected domain $\Omega \subsetneq \mathbb{D}$, take a point in $\mathbb{D} \setminus \Omega$, apply the transformations in (ii), and then repeat this process indefinitely, does the process necessarily converge to φ? (Hint: think of what happens if a is on the boundary of Ω.)

30. Boundary behavior.

30.1. **Another look at Poisson's formula.** Check that Poisson's formula (12.6) can be written also, with $z = ae^{i\theta}$, as

$$u(z) = \operatorname{Re} f(z); \ f(z) := \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{s + z}{s - z} u(s) \frac{ds}{s} \tag{30.1}$$

Proposition 30.1. *(i) The* operator L *defined by* $Lu =: L(u) = \operatorname{Re} f(z)$ *with* f *as in* (30.1) *is linear (for any constants* a, b *and piece-wise continuous functions* u *and* v, $L(au + bv) = aL(u) + bL(v)$ *and positive* ($u > 0 \Rightarrow Lu > 0$.

(ii) For any continuous u, $L_t u$ *is analytic in* $\mathbb{D}$, *and with* $u = C =$ *const*. $(L_t C)(z) = (LC)(z) = 1 \forall |z| < 1$.

(iii) $\max_{\partial\mathbb{D}} |u| \leqslant M$ *then* $\sup_{\mathbb{D}} |Lu| \leqslant M$ *(the operator has norm one).*

Proof. (i) This follows directly from (12.6)).

(ii) By linearity, it is enough to check this when $C = 1$. We decompose the integrand as:

$$\frac{s+z}{s(s-z)} = \frac{2}{s-z} - \frac{1}{s} \tag{30.2}$$

Now, by residues, if $u = 1$, $f = 1$ and thus $u = 1$ on $\overline{\mathbb{D}}$.

(iii) Indeed, $M - u$ is nonnegative and so is $M + u$ thus $-M = L(-M) \leqslant Lu \leqslant LM = M$. ▮

Let also L_t be defined by $Lu = \operatorname{Re} f(z)$ with f as in (30.1).

Lemma 30.2. *For any continuous u, $L_t u$ is analytic in $\mathbb{D}$, and with $u = 1$ $L_t u = Lu(z) = 1 \forall |z| < 1$.*

Note also that

Theorem 30.3. *Assume $g(s)$ is continuous on $\partial\mathbb{D}$. Then Lg is harmonic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$ and, if $z_0 \in \partial\mathbb{D}$, then $(Lg)(z_n) \to u(z_0)$ as $z_n \to z_0$ if $|z_n| < z_0 \, \forall n$.*

We can assume wlog that $z_0 = 1$ and that $g(1) = 0$. Indeed, the latter can be arranged by taking $\hat{g} = g - g(1)$ and remembering that $L(g(1)) = g(1)L(1) = g(1)$. Choose a small arc $C_\varepsilon \subset \partial\mathbb{D}$ centered at 1 where $|g| < \varepsilon$. Define $g_1 = g$ on C_ε and zero on $C_2 = \partial\mathbb{D} \setminus C_\varepsilon$ and $g_2 = g - g_1$. As $z \to 1$, $L_t g_2$ is zero: indeed $L_t g_2$ is analytic at 1 and thus $\lim L_t g_2(z_n)$ exists as $z_n \to 1$. But

$$\operatorname{Re} \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}} = 0 \text{ if } \theta \neq \phi \text{ (both in } [0, 2\pi)) \tag{30.3}$$

Thus $\lim_{z_n \to 1} (L_t g_2)(z_n) = 0$. Now, since $|g_1| < \varepsilon$ on $\partial\mathbb{D}$, by Proposition 30.1 $|L_t g_1)(z_n)| < \varepsilon$ finishing the proof.

30.2. **Behavior at the boundary of biholomorphisms: a general but relatively weak result.** We derive an easy but useful result [3]: if $\mathcal{D}$ is a simply connected domain and φ maps it conformally onto $\mathbb{D}$, then $\varphi(z)$ approaches $\partial\mathbb{D}$ as z approaches $\partial\mathcal{D}$, in a sense defined below (which does not imply that $\varphi(z)$ converges).

Let $\mathcal{D}$ be a domain. Informally, a sequence or an arc approaches the boundary if eventually recedes away from any point in the region. Convince yourselves that the precise definition below corresponds to this intuitive description if $\mathcal{D}$ to be D, $\mathbb{H}^u$ or a cut disk.

Definition 30.4. *A sequence $z_n \to \partial\mathcal{D}$ as $n \to \infty$ if for any compact set $K \subset \mathcal{D}$ there exists n_0 such that for all $n > n_0$ we have $z_n \notin K$. Similarly, for an arc $\gamma : [0,1] \to \mathcal{D}$, $\gamma(t) \to \partial\mathcal{D}$ if $\forall K \, \exists t_0 \in (0,1)$ s.t. $\gamma(t) \notin K$ if $t > t_0$.*

Theorem 30.5. *If $\varphi : \mathcal{D} \to \mathcal{D}'$ is a domain biholomorphism and $z \to \partial\mathcal{D}$, then $\varphi(z) \to \partial\mathcal{D}'$.*

Proof. We prove the statement for sequences; the one for arcs is almost identical. Since φ is biholomorphic, any compact covering of $\mathcal{D}$ generates a compact covering of $\mathcal{D}'$ and vice-versa. Let $z_n \to \partial\mathcal{D}$ and let $K' \subset \mathcal{D}'$ be any compact set. By definition,

$$z_n \to \partial\mathcal{D} \;\Rightarrow \exists n_0 \; s.t. \forall n > n_0 \; z_n \notin K$$

Since φ is one to one, $\varphi(z_n) \notin K'$ either.

Corollary 30.6. *If $\varphi : \mathcal{D} \to \mathbb{D}$ is a biholomorphism, then $|\varphi(z)| \to 1$ as $z \to \partial\mathcal{D}$.*

30.3. **A reflection principle for harmonic functions.** Let $O^u \subset \mathbb{H}^u$ be such that $\partial O^u \supset I := [a,b] \in \mathbb{R}$. We denote by O_l the reflection of O^u across I.

Theorem 30.7. *Assume v is harmonic in O^u continuous on $O^u \cup I$ and $v = 0$ on I. Then v extends to a harmonic function on $O^u \cup I \cup O_l$.*

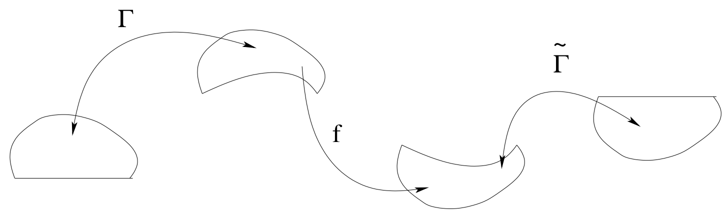
Note 30.8. *Theorem 21.68 is an easy consequence of Theorem 30.7.*

Proof. [6] As in the Schwarz reflection principle, the extension of v is defined through

$$v(\overline{z}) := -v(\overline{z}) \; \forall z \in O_l \tag{30.4}$$

The property of a function being harmonic is a local one, so it is enough to check that this is for some family of disks covering O. For disks in O^u this is so by assumption while for disks $\subset O_l$ it follows directly from (30.4). Consider now a disk $\mathbb{D}_\varepsilon$ containing part of I as its diameter. Laplace's equation $\Delta v = 0$ with the continuous boundary solution $v(z), z \in \partial\mathbb{D}_\varepsilon \cap \mathbb{H}^u$ and $-v(\overline{z}), z \in \partial\mathbb{D}_\varepsilon \cap \mathbb{H}_l$ has a unique solution V in $\mathbb{D}_\varepsilon$, see §27.1 and Proposition **??**. The boundary condition is odd w.r.t. y, and, by uniqueness (since $-V(\overline{z})$ is also a solution), V is odd too, thus zero on I. It is then a harmonic function in $\mathbb{D}_\varepsilon$ which must coincide with $v(z)$ in $\mathbb{D}_\varepsilon \cap \mathbb{H}^u$ since they satisfy the same boundary condition (on $\partial\mathbb{D}_\varepsilon \cap \mathbb{H}_l \cup \partial\mathbb{D}_\varepsilon \cap I$) and similarly, it coincides with $-v(\overline{z})$ in the reflected domain. ∎

Exercise 30.9. *Show that if $f = u + iv$ is analytic in a domain O^u as in the theorem and v is zero on I, then u is continuous at I.*



Note 30.10. The following interesting construction is given in [3], p.27. As usual, if f is analytic, then $\overline{f}$ is analytic too, where $\overline{f}(z) = \overline{f(\overline{z})}$. (Note that Check that

$$2u(x, y) = f(x + iy) + \overline{f}(x - iy) \tag{30.5}$$

for all x, y for which the rhs makes sense. (Note that $\overline{f}(x - iy) = \overline{f}\overline{z}$ is *not* analytic, but this does not matter in the following). Assuming also that the expression $u(z/2, z/2i)$ makes sense, we have (check!):

(30.6)

$$2u(z/2, z/2i) = f(z) + \overline{f}(0) = f(z) + C \Rightarrow f(z) = 2u(z/2, z/2i) - C$$
$$(\Rightarrow v(z) = \mathrm{Im}\,(2u(z/2, z/2i) - C))$$

In particular, no integration is needed to get f or v from u. This certainly works by the principle of permanence of relations for rational functions, or other simple functions. Can you make this work in general?

30.4. **Real-analytic functions.**

Definition 30.11. *A function f is real-analytic on $I := [a, b]$ if for any $t_0 \in I$ we have $\gamma(t) = \sum_k c_k(t - t_0)^k$, where the series converges.*

Since I is compact, by the finite covering property we see that f extends to an analytic function in a neighborhood of I, which provides an **equivalent definition** of real-analyticity.

Definition 30.12. *A proper analytic arc is the image of some I under a real analytic function $\gamma : I \to \mathbb{C}$ with the property $|\gamma'| > 0$ on I.*

An equivalent definition of a proper analytic arc is that γ extends to an isomorphism between a neighborhood of I and its image.

30.5. **An extension of the Schwarz reflection principle.** *Informally* this states that if f is analytic in a domain $\mathcal{D}$ which contains an analytic arc γ , if f is continuous up to γ and if the image of γ is an analytic arc, then f extends analytically beyond γ.

Definitions Let γ be a proper analytic arc bounding a domain $\mathcal{D}$. Then $\gamma : I \to \mathbb{C}$, $I = [a, b]$ is a one-sided boundary of $\mathcal{D}$ if for any

$z_0 \in \gamma(I)$ there is a disk $\mathbb{D}_\varepsilon(z_0)$ s.t. either $\gamma^{-1}(\mathbb{D}_\varepsilon(z_0)) \cap \mathcal{D} \subset \mathbb{H}^u$ or $\gamma^{-1}(\mathbb{D}_\varepsilon(z_0)) \cap \mathcal{D} \subset \mathbb{H}_l$. [7] In this case, we also say that $\mathcal{D}$ lies *on one side* of γ. As usual, by analytic continuation of a function f across a curve, we mean that there is an analytic function $\hat{f}$ in a neighborhood of the curve which coincides with f wherever they are both defined. Note that the analytic continuation is unique. The theorem below is the "conformally mapped" Schwarz reflection principle.

Theorem 30.13 (Analytic reflection across arcs)**.** *Let $\mathcal{D}$ be a domain and assume that $\mathcal{D}$ lies on one side of the proper analytic arc γ. Let f be analytic in $\mathcal{D}$ and continuous on $\mathcal{D} \cup \gamma$, assume that $f(\gamma) \subset \tilde{\gamma}$ where $\tilde{\gamma}$ is a proper analytic arc, and finally that $f(\mathcal{D})$ lies on one side of $\tilde{\gamma}$. Then f extends analytically across γ.*

Proof. By assumption γ and $\tilde{\gamma}$ are images of the closed intervals I and I' under isomorphisms, $\Gamma, \tilde{\Gamma}$ respectively. Now $\tilde{\Gamma}(f(\gamma)) = \tilde{I}$; Furthermore $\psi = \tilde{\Gamma} \circ f \circ \Gamma$ is defined on an open subset $\mathcal{O}$ of, say, $\mathbb{H}^u$ and is continuous up to I, and ψ is real valued on I. Then, by the Schwarz reflection principle, ψ extends analytically to the reflection across I of $\mathcal{O}$. The desired analytic continuation is $\Gamma^{-1} \circ \psi \circ \tilde{\Gamma}^{-1}$. ∎

Theorem 30.14. *Let $\mathcal{D} \subsetneq \mathbb{C}, \mathcal{D} \neq \emptyset$ be a simply connected domain and φ a biholomorphism between $\mathcal{D}$ and $\mathbb{D}$. If $\partial\mathcal{D}$ contains a proper analytic arc γ, lies on one side of γ, then*

(i) φ extends analytically across γ

(ii) if $z_n \to z_0 \in \gamma$, then $\varphi(z_n) \to \varphi(z_0) \in \gamma$.

(iii) Furthermore, this extension is one-to-one on s, thus in a neighborhood of $\mathcal{D} \cup s$.

Note 30.15. Applying a linear fractional transformation (nondegenerate ones are one-to-one we see that a similar statement holds if $\mathbb{D}$ is replaced by a half-plane.

In particular, if the boundary of $\mathcal{D}$ is piecewise analytic then φ extends analytically to a domain $\mathcal{D}' \supset \overline{\mathcal{D}}$ and in particular it is continuous in $\overline{\mathcal{D}}$. (It is also biholomorphic in some domain $\mathcal{D}'' \supset \overline{\mathcal{D}}$.)

Proof. (i) Follows from Theorem 30.13.

(ii) follows immediately from (i).

(iii) Assume we had $f'(z_0) = 0$ for some $z_0 \in \gamma$; Then $f(z)$ is a piece of an analytic arc near $f(z_0)$. $f(z_0)$ divides γ in two subarcs whose relative angle at z_0 is π. Then in a neighborhood of z_0 f, by Proposition 27.88 is n-to-one for some $n \geqslant 2$, and the two subarcs would

[7] Recall that γ extends to an isomorphism.

be mapped by $(f')^{-1}$ into $2n$ subarcs of $\partial\mathbb{D}$ and the angle between two successive ones being π/n (check!). But this violates the fact that φ is one-to-one in the interior and maps $\partial\mathcal{D}$ to $\partial\mathbb{D}$ as proved in (i). We know that $|f| = 1$ on $\partial\mathcal{D}$ and thus $\ln f$ is defined in a neighborhood of any point $z_0 \in \gamma$. If $|f'| > 0$ on γ, this is true, by continuity, in a neighborhood of γ. ∎

30.6. **Behavior at the boundary, a stronger result.** We recall that a Jordan curve in $\mathbb{C}$ is a *continuous map* γ defined (say) on $[0,1]$ with values in $\mathbb{C}$ which is injective, that is $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$ or $t_1 = 0$ and $t_2 = 1$ where in the latter case it is a closed Jordan curve. We also recall that a closed Jordan curve divides $\mathbb{C}$ into exactly two regions, one bounded and one unbounded. The bounded region is called the interior of the curve. A Jordan domain is the interior of a Jordan curve.

A point $z_0 \in \partial\mathbb{D}$ is called accessible if there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ and a continuous function $\gamma : [0,1] \to \mathcal{D}$ which passes through all z_n ($\forall n \exists t_n \in [0,1]$ s.t. $\gamma(t_n) = z_n$. If $\mathcal{D}$ is the interior of any Jordan curve, then any point on $\partial\mathcal{D}$ is accessible (why is this true?).

Theorem 30.16 (Boundary behavior)**.** *Let $\mathcal{D}$ be a bounded domain. (i) If z_0 is an accessible point of $\partial\mathcal{D}$ then the biholomorphism φ with the unit disk has a limit (call it $\varphi(z_0)$) as $z \to z_0$, $z \in \mathcal{D}$ and $\varphi(z_0) \in \partial\mathbb{D}$.*

(ii) If z_1 and z_2 are accessible points of $\partial\mathcal{D}$ then $\varphi(z_1) \neq \varphi(z_2)$.

In particular, if $\mathcal{D}$ is the interior of a simple, closed Jordan curve, then the map φ extends to a continuous $1-1$ function in $\overline{\mathcal{D}}$.

For space limitations we do not prove this interesting result; a proof, essentially based on the first proofs by Lindelöf and Koebe, is found in [6]. We shall not use conformal maps of this generality. In §31 we will find that for polygons, φ can be expressed by quadratures.

30.6.1. *A negative result.* The following is a standard example of an inaccessible point [8] shows that, in full generality, continuity of the conformal mapping cannot be expected. Such is the case of the *inaccessible point* below. Take $\mathcal{D}$ to be an open horizontal rectangle with a vertex at zero from which vertical line segments of length, say, $1/2$ have been removed, see Fig. 17 (check that 0 is inaccessible!). The image of 0 on $\partial\mathbb{D}$ (which wlog can be taken as $z = 1$) must be a point of discontinuity of the conformal map φ into the unit disk, for $\varphi^{-1}(1) = 0$ and in any neighborhood of 1 there are infinitely many points where φ^{-1} is $1/2$.

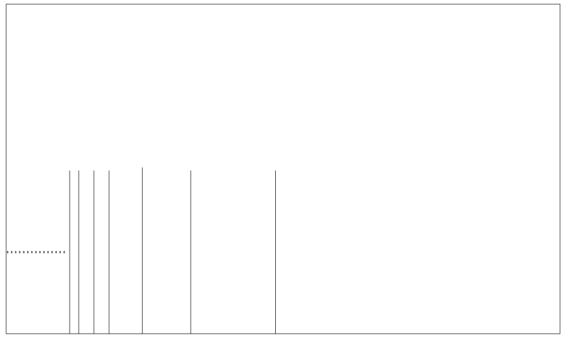


FIGURE 17. The origin is inaccessible

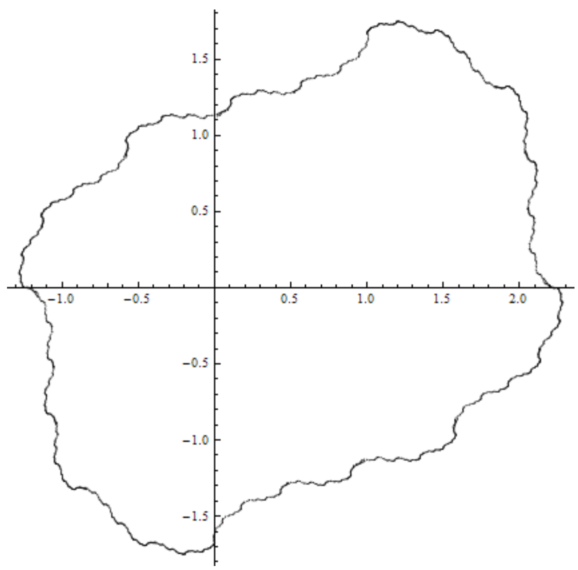


FIGURE 18. The conformal map of the unit disk through F

Illustration of nonanalytic behavior at all points the conformal image of the unit disk.. Figure 18 shows the conformal image of the unit disk under the map F defined by the functional equation $F(\lambda z) = \lambda F(z)(1 - F(z)), \forall z \in \mathbb{D},\ F'(0) = 1$ for $\lambda = 0.5i$. The unit disk is a natural boundary of F. The interior J of the curve corresponds to the points in $z \in \mathbb{C}$ for which the solution of the one step recurrence $x_{n+1} = \lambda x_n(1-x_n);\ x_0 = z$ converges to zero. J is the *Julia set* for the iteration of the quadratic map $z \mapsto \lambda z(1 - z)$. See [5].

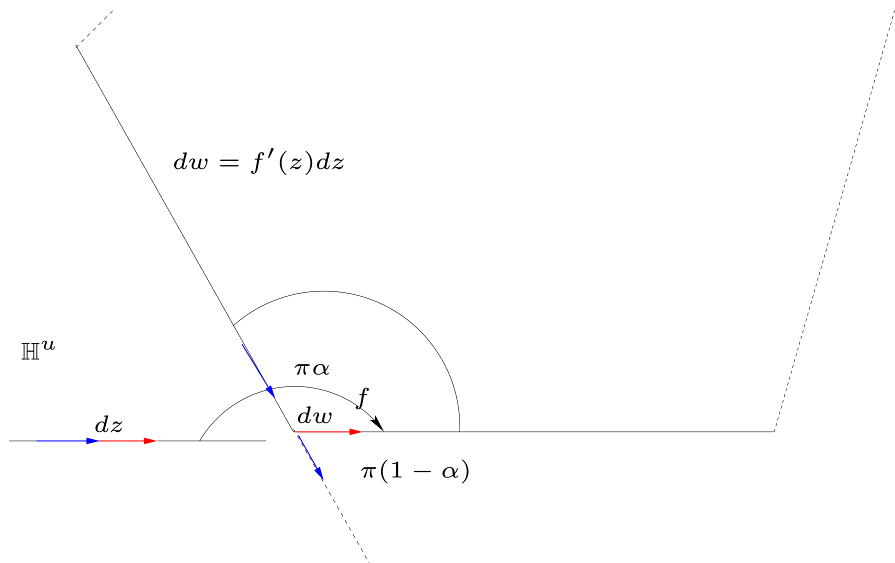


FIGURE 19. Schwarz-Christoffel transformations: two adjacent sides of the polygon.

31. CONFORMAL MAPPINGS OF POLYGONS AND THE SCHWARZ-CHRISTOFFEL FORMULAS

For polygonal regions, the conformal map to the unit circle (or to $\mathbb{H}^u$ obviously) can be done by quadratures. The transformation is still usually nonelementary, but the integral representation gives us enough control to describe the transformation quite well.

31.1. **Heuristics.** If f is biholomorphic at z_0, the angle between the tangent of a curve γ through z_0 and the tangent to its image through f is $\arg f'(z_0)$; we write this in differential form, $dw = f'(z)dz$. We want to map $\mathbb{H}^u$ to the interior of a polygon. We then choose the positive orientation when traversing $\partial\mathbb{H}^u$ (which leaves the domain to its left): this means traversing the boundary from $\mathbb{R}^-$ to $\mathbb{R}^+$. For later convenience we denote by $\pi\alpha_i$ the interior angles of the polygon.

Wlog we can place a vertex at zero, and rotate the polygon so that one side is in $\mathbb{R}^+$. The red arrow indicates the positive orientation of the polygon. Suppose that we want to map 0 to 0 and the segment in blue on $\partial\mathbb{H}^u$ to the blue segment on the polygon, see Fig19. We see that, to the left of $z = 0$, dw is rotated by $-\pi(1-\alpha)$ with respect to dz, while to the right of $z = 0$ (red arrow) dz and dw are parallel. A transformation that behaves like this on the boundary is $f'(z) = z^{\alpha-1}$. We see that indeed the argument of f' ($\operatorname{Im} \ln f'$ which exists locally since $f' \neq 0$) does not change except at the singularity, $z = 0$:

$$(31.1) \qquad (\ln f')' = \frac{\alpha - 1}{z} \in \mathbb{R} \text{ (since } z \in \mathbb{R}) \Rightarrow d\arg f' = 0 \text{ for } z \neq 0$$

Proposition 31.1. *Any transformation f of a one-sided neighborhood $\mathcal{N}$ in $\mathbb{H}^u$ of a segment $I = [-a, b]$; $a, b > 0$ which maps 0 to the vertex $w = 0$ of the polygon and is continuous up to the boundary has the property $f(z) = z^{\alpha-1}h(z)$ where h is holomorphic in $\mathcal{N} \cup I \cup \overline{\mathcal{N}}$. In particular, $f'(z) = z^{\alpha-1}\sum_{k=0}^{\infty} h_k z^k$ where $h(0) \neq 0$ and the sum is convergent.*

Definition 31.2. *We will call functions which are (locally) of the form $(z - z_0)^a H(z)$ with H holomorphic,* **ramified-analytic**.

Proof. Let $H = \int z^{-(\alpha-1)} f'$. Note that $dH = z^{-(\alpha-1)} f' dz$ maps the blue vector on the left side of the polygon $\mathcal{N}$ into a vector parallel to the red one, is continuous down to I. The image of I through $H = \int z^{-(\alpha-1)} f'$ will, by construction, be an interval in $\mathbb{R}$. Schwarz's reflection principle and the continuity of H ensure that H has analytic continuation in $\mathcal{N} \cup I \cup \overline{\mathcal{N}}$. Thus H' is analytic too. Since the transformation h is one-to one ($H(z)$ is strictly monotonic in $z \in I$, implying $H' = h \neq 0$. ∎

More generally, a transformation of the form

$$(31.2) \qquad f' = \prod_{i=1}^{n-1} (z - a_i)^{\alpha_i - 1}$$

satisfies

$$(31.3) \qquad (\ln f')' = \sum_{i=1}^{n-1} \frac{\alpha_i - 1}{z - a_i} \in \mathbb{R} \ \Rightarrow d\arg f' = 0 \text{ for } z \neq a_i$$

and $\arg f'$ changes by $\pi(1 - \alpha_i)$ upon traversing a_i. The points $A_i := f(a_i)$ on the polygon are the only ones where $\arg f'$ changes, and it changes by $+\pi(1 - a_i)$ (check the signs!). If the polygon P is a closed curve, then infinity must be mapped into one of the vertices of P, and, since f is conformal (to be proven later) it cannot be one of the $\{A_j, j = 1..n-1\}$ so it must be A_n. Recall that the sum of exterior angles of a closed polygon is 2π, and thus $B = \sum_1^{n-1} \beta_i = 1 + \alpha_n$, $\beta_i := a_i - 1$. Note also that $1 - a_i/z$ is analytic at infinity: as $z \to \infty$ we have

(31.4)

$$f' = z^B \prod_{i=1}^{n-1} (1 - a_i/z)^{\beta_i} = \zeta^{-B} g(\zeta); \; g(\zeta) = \prod_{i=1}^{n-1} (1 - a_i\zeta)^{\beta_i}, \; \zeta = 1/z$$

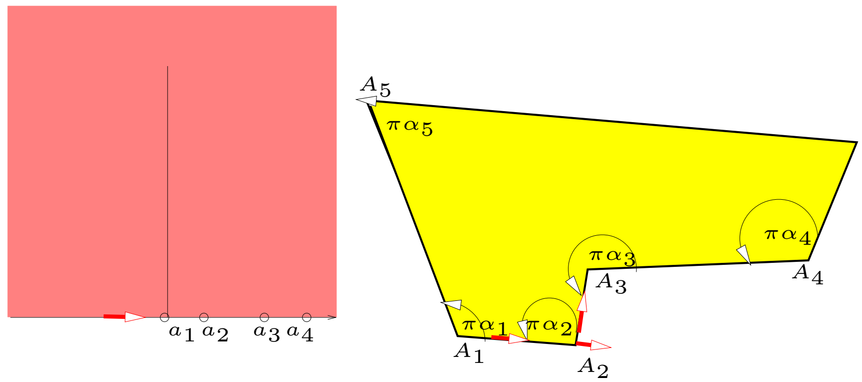


FIGURE 20. Schwarz-Christoffel transformations: a closed polygon.

If $|\zeta| < \varepsilon < \max\{1/|a_i|, i = 1, ..., n-1\}$, g is analytic and $\operatorname{Re} g(\zeta) > 0$, implying that $\ln g$ is well defined. If z traverses $\partial\mathbb{D}_R$ from $+R$ to $-R$ where $R = 1/\varepsilon$ in a positive direction the change in $\arg f'$ comes solely from z^B and it equals πB, the same change in argument that $(z-A)^{1-\alpha_n}$ would produce. This "closes the polygon" with the last vertex $A_n = \lim_{z\to\infty} f(z)$. See figure. In the case of a closed polygon we see that f is bounded on $\mathbb{R}$. In the case of an open polygonal line, the arguments are similar.

Theorem 31.3. *(i) Let P be any polygon (closed or open) with n vertices $A_1, ..., A_n$ and interior angles $\pi\alpha_1, ..., \pi\alpha_n$. There is a choice of $a_1, ..., a_{n-1} \in \mathbb{R}$ and $C, C' \in \mathbb{C}$ such that the function*

$$\Phi = z \mapsto C \int_0^z \prod_{k=1}^{n-1} (s - a_k)^{\alpha_k - 1} ds + C' \tag{31.5}$$

maps $\mathbb{H}^u$ into P (and $\Phi(a_i) = A_i$).

(ii) The function mapping $\mathbb{D}$ conformally into P is given by

$$\Phi = z \mapsto C \int_0^z \prod_{k=1}^{n} (b_k - s)^{\alpha_k - 1} ds + C' \tag{31.6}$$

(iii) Moreover, all transformations between $\mathbb{H}^u$ or $\mathbb{D}$ and polygons are of this form.

Remark 31.4. (i) It is important to note what freedom we have in such transformations. Suppose we want to map a triangle Δ. All triangles with same angles are similar, and a mapping between two similar triangles reduces to scaling, rotation and translation. Thus we need to understand one triangle with given angles α_i. We take say

$a_1 = 0$ and $a_2 = 1$, use the α's, and see what triangle Δ_1 is obtained; note that it must be *similar* to Δ. Then, we can choose C and C' so that we remap Δ_1 to Δ. Thus we are able to map any triangle to $\mathbb{H}^u$, prescribing the position of the images of the vertices at will.

(ii) For $n > 3$, we can still place three points at will but the position of the fourth one etc cannot be chosen arbitrarily. We see that we have the freedom of $n-1$ *real* constants, the $a_i's$, and of two complex ones C and C', a total of $n+3$ real constants, whereas an arbitrary closed polygon has $2n$ real constants as degrees of freedom (the position of its vertices); $2n > n+3$ if $n > 3$. The a_k for $k \geqslant 4$ are called accessory parameters, and they are determined by the polygon and the values of $a_1, ..., a_3$; for all but very simple cases, the accessory parameters are calculated using special functions or (in general, if $n > 4$) numerically. Symmetries help, if there are any.

Given the disparity in the number of available parameters versus the degrees of freedom of the problem, it is not a priori clear that the Schwarz-Chrsistoffel transformation should always work.

First proof of the Schwarz-Christoffel formula. We analyze closed polygons, open ones being similar. We can first arrange that one vertex is at zero and a second one at 1. Indeed, we can transform P by translation and multiplications by a constant (changing C, C' in the Theorem) into one geometrically similar to it, $\tilde{P}$ that has these properties. By composition to the right with $az+b$ we can arrange that 0 and 1 in $\mathbb{H}^u$ are mapped into 0 and 1 in $\tilde{P}$. Note that by Proposition 31.1 the function $F(z) = f'(z)\prod_{i=1}^{n-1}(z-a_i)^{1-\alpha_i}$ is analytic in $\overline{\mathbb{H}^u}$, does not change its phase except perhaps at a_i. But at every a_i, F is real and positive. Thus F is real on the real line and extends to an entire function. For the behavior at ∞ note that the transformation $\zeta = -1/z$, $f(z) = G(\zeta)$. $z \mapsto -1/z$ is an automorphism of $\mathbb{H}^u$ which maps ∞ to 0. Traversing $\mathbb{R}^+$ in z from 0 to ∞, returning to $-\infty$ on a "large" circle and moving in $(-\infty, 0)$ to the right corresponds to ζ traversing $(-\infty, \infty)$ from negative to positive values. To straighten the angle at a_n ($= 0$ in ζ) we need, as in Proposition 31.1 to multiply $G'(\zeta)$ by $\zeta^{1-\alpha_n}$: $\zeta^{1-\alpha_n}\frac{dG}{d\zeta} = H(1/z)$ where H is analytic in particular bounded. Since $\frac{d}{d\zeta}G = \zeta^{-2}f'(-1/\zeta)$ it follows that $z^{1+\alpha_n}f'$ is analytic at infinity.

Now, $1-\alpha_1+...+1-\alpha_n = 2$ implying $1-\alpha_1+...+1-\alpha_{n-1} = 1+\alpha_n$ and thus

$$(31.7)\quad (z-a_1)^{1-\alpha_1}\cdots(z-\alpha_{n-1})^{1-\alpha_{n-1}}f' \sim const.z^{1+\alpha_n}f' \text{ as } z \to \infty$$

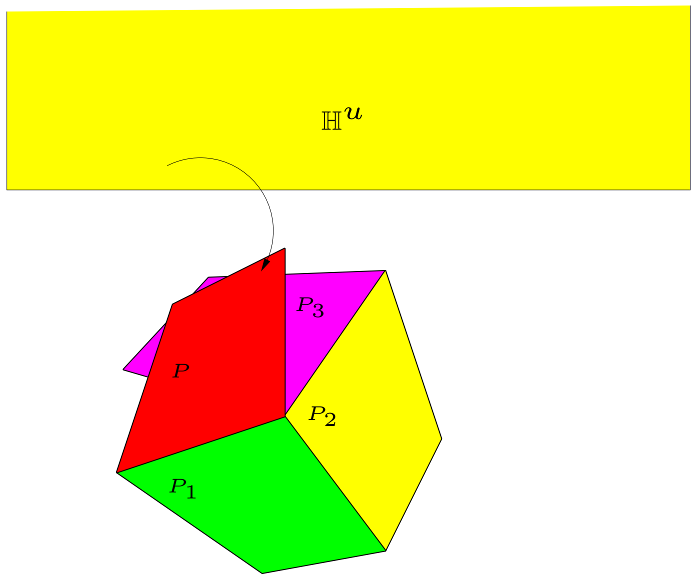


FIGURE 21. Successive reflections.

is entire and bounded, thus constant and hence

$$(z - a_1)^{1-\alpha_1} \cdots (z - \alpha_{n-1})^{1-\alpha_{n-1}} f' = C \tag{31.8}$$

Corollary 31.5 (Analytic structure of f)**.** *The function f is ramified-analytic (cf. Definition 31.2). In a neighborhood of a_j we have*

$$f = C_{1j} x^{\alpha_j} H_j + C_{2j} \tag{31.9}$$

where H_j is holomorphic.

Proof. This follows by straightforward integration of (31.8). ∎

∎

"Geometric" proof of the Schwarz-Christoffel formula. This proof is largely based on [8]. A slightly different argument is used in [3]. We prove the statement for a closed polygon, the one for open ones being very similar. Examine again Example 21.70, since we shall use the reflection principle in a somewhat similar way.

Let f be the conformal map between $\mathbb{H}^u$ and P. Since f transforms an interval I of $\mathbb{R}^+$ into the line segment ℓ bounding P and P_1, it has analytic continuation across I, and in fact the image of $\mathbb{H}^u$ is P and that of $\mathbb{H}^u$ is P_1. This continuation can be reflected back, and we obtain P_2. By simple geometry, the two yellow polygons P and P_2 are Euclidian transformations of each-other, of the form $P_2 = aP + c$ for some constants a and c with $|a| = 1$. Let f_2 be the analytic continuation of f to P_2: we must have $f_2 = af + c$, and in general $f_{2n} = a_n f + b_n$ with $|a_n| = 1$.

Note 31.6. It does *not* follow that f has analytic continuation through the whole of $\mathbb{R}$ into $\mathbb{H}_l$ (the image of $\mathbb{R}$ is generally a nontrivial polygon which is not an analytic arc). The analytic continuation can generally only be performed across segments of the form $I_i = (a_i, a_{i+1})$ where $f(a_i) = A_i, f(a_{i+1}) = A_{i+1}$, and we can reflect sets *not* containing $\mathbb{R}$, such as $\mathbb{H}^u$. Indeed, the image of the analytic arc I_i is an analytic arc, namely the side A_iA_{i+1} of the polygon. But if we took instead $(a_i - \varepsilon, a_{i+1}$, its image is a broken line, not an analytic arc. This was to be expected given Proposition 31.1. We are dealing with a branched-analytic function. A second reflection, around a different interval will generally produce a function f_2 on $\mathbb{H}^u$ *different from* f.

The continuation of f to P_1, f_1, and the continuation of f to P_{2n+1} satisfy $f_{2n+1} = A_n f_1 + B_n$ where $|A_n| = 1$. Any successive reflections of P about sides results in a polygon similar to P or P_f where P_f is the flip of P about one side (any side will have the same effect). Any continuation $\tilde{f}$ by successive reflections about sides of the polygon is either a flip relates to f or the continuation of f across one of the sides of P by $\tilde{f} = Af_1 + B$ or $A'f + B'$, where $|A| = |A'| = 1$. The Riemann mapping theorem implies that $f' \neq 0$ in $\overline{P} \setminus V$ where V are the vertices, where f might be singular.

P and P_1 differ by a reflection. Then, the range $f(P_1)$ is the image of a reflected region from $\mathbb{H}^u$ in $\mathbb{H}^l$. Note that $g(z) = f''(z)/f'(z) = f_1''(z)/f_1'(z)$ is analytic at any z except for $a_1, ..., a_n$. By Proposition 31.1, g has a pole of order one with residue $\alpha_i - 1$ at the a_i. Since a_i are the only singularities of g, g is single valued and extends to $\mathbb{C}$. Thus,

$$G(z) = g(z) - \sum_{j=0}^{n-1} \frac{a_k - 1}{z - a_k} \tag{31.10}$$

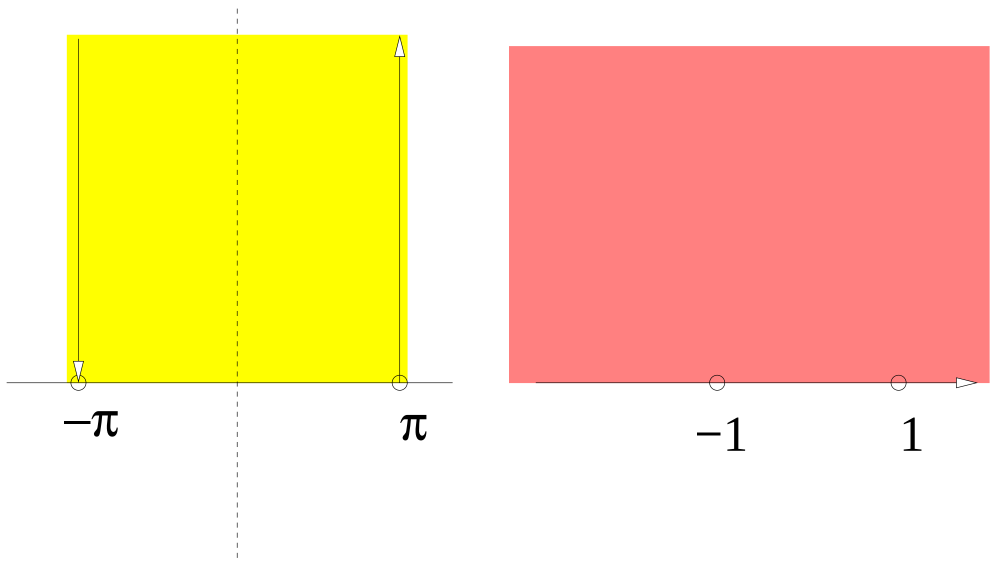
is entire. If infinity corresponds to a vertex, then a similar argument to the one used in Proposition 31.1 gives $f''/f' \sim z^{-1}$ for large z. If instead f is analytic at infinity then $f(z) = \sum_{k=0}^{\infty} c_k z^{-k}$ whence $g(z) \sim const./z$ for large z. Thus

$$\frac{f''(z)}{f'(z)} = \sum_{j=0}^{n-1} \frac{a_k - 1}{z - a_k} \tag{31.11}$$

Integrating 31.11 the conclusion follows. ∎

Note 31.7. The second order linear ODE obtained after multiplying (31.11) by f' is a *Fuchsian equation*: it has only *regular singularities* in $C^* = C \cup \infty$ [4].

31.2. **Another look at the sine function.** *Problem.* Map the strip indicated into $\mathbb{H}^u$ preserving the points marked with circles and the positive orientation.



Solution The α's at $-\pi$ and π are both $1/2$. We apply formula (31.5) with $a_1 = -1$, $a_2 = 1$ and the integrand is then $(s^2-1)^{-1/2}$. Eq. (31.5) therefore gives, for two arbitrary constants,

$$\Phi = C \arcsin z + C' \tag{31.12}$$

and therefore our map $f = \Phi^{-1}$ has the general form

$$\Phi^{-1}(w) = \sin(cw + c') \tag{31.13}$$

We have now to choose c and c' to match the prescribed points. We must have $\sin(-\pi c + c') = -1$ and $\sin(c\pi + c') = -1$; the choice $c' = 0$ and $c = 1/2$ matches these conditions. We get

$$f(w) = \sin(w/2) \tag{31.14}$$

32. Mapping of a rectangle: Elliptic functions

We map $\mathbb{H}^u$ in a rectangle. All the α's in (31.5) are $1/2$, as in §31.2. We choose three a_k as simple as possible, $0, 1,$ and $\rho > 1$, and study the resulting rectangle. The freedom allows us to place three vertices wherever we want; we choose $C = 1$ and $C' = 0$. The integrand is $s^{-1/2}(s-1)^{-1/2}(s-\rho)^{-1/2}$. If $z \in \mathbb{H}^u$ then so are s, $s-1$ and $s-\rho$. We take a branch of the square root with cuts in $\mathbb{H}_l$, say $-i\mathbb{R}^+$, $1-i\mathbb{R}^+$ and $\rho - i\mathbb{R}^+$. The Schwarz-Christoffel transformation is

$$\Phi(z) = \int_0^z \frac{ds}{\sqrt{s}\sqrt{s-1}\sqrt{s-\rho}} \tag{32.1}$$

a nonelementary *elliptic integral*. To find the range of Φ, we evolve s on $\mathbb{R}^+ + i\varepsilon$ and pass to the limit $\varepsilon \to 0$. For instance, on $(0,1)$ the arguments of $s, s-1$ and $s-\rho$ are $0, \pi, \pi$ resp., and thus $\arg(\Phi) = 0 - \pi/2 - \pi/2$. On $I_2 = (1,\rho)$ $\arg\Phi = 0 + 0 - \pi/2$, on $I_3 = (\rho,\infty)$ $\arg\Phi = 0$ and finally on $I_4 = (-\infty, 0)$ $\arg\Phi = -3\pi/2$. Therefore, starting with $z=0$ and evolving towards $+\infty$ and then from $-\infty$ to 0, $\Phi(z)$ traverses positively the boundary of the rectangle with vertices

$$(0,0), -(K_1,0), -(K_1, K_1+iK_2), -(0, iK_2); \tag{32.2}$$

$$\text{where } K_1 = \int_0^1 \frac{dx}{\sqrt{x(1-x)(\rho-x)}}, K_2 = \int_1^\rho \frac{dx}{\sqrt{x(x-1)(\rho-x)}}$$

and both integrands are positive on the given interval. In particular, we must have

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)(\rho-x)}} = \int_\rho^\infty \frac{dx}{\sqrt{x(x-1)(x-\rho)}} \tag{32.3}$$

$$\int_{-\infty}^1 \frac{dx}{\sqrt{-x(1-x)(\rho-x)}} = \int_1^\rho \frac{dx}{\sqrt{x(x-1)(\rho-x)}} \tag{32.4}$$

where the integral is well defined since for large s the integrad is bounded by $const. s^{-3/2}$. The fact that the polygon closes is immediate:

$$\int_{-\infty}^\infty \frac{dt}{\sqrt{t(1-t)(\rho-t)}} = 0 \tag{32.5}$$

(The branch cuts are in $\mathbb{H}_l$ and contour of integration can be pushed up to $i\infty$.)

Exercise 32.1. *Prove one of these identities without using the Schwarz-Christoffel theory. This can be done by changes of variables or using* (32.5).

32.1. **Differential equation.** We write

$$\frac{d\Phi}{dz} = \frac{1}{\sqrt{z(z-1)(z-\rho)}} \quad \text{or} \quad \left(\frac{dz}{d\Phi}\right)^2 = z(z-1)(z-\rho) \tag{32.6}$$

differentiating with respect to ρ and dividing by $dz/d\Phi$ we get

$$z'' = \frac{3}{2}z^2 - (\rho+1)z + \frac{\rho}{2} \tag{32.7}$$

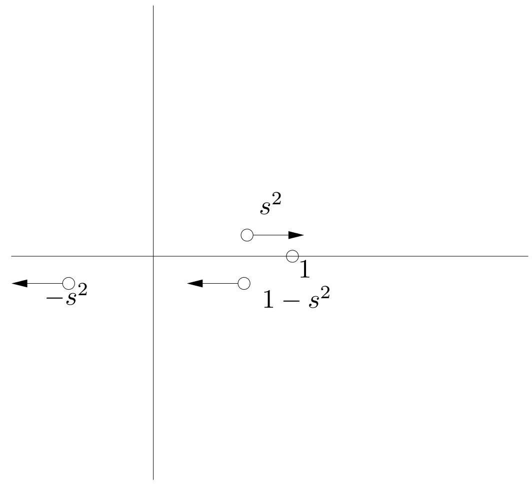


FIGURE 22. The evolution of s^2, $-s^2$ and $1-s^2$.

a second order autonomous equation. A linear change of variables of the form $z(\Phi) = ay(b\Phi)+c$ brings it to the canonical form of the elliptic equation:

$$y'' = 6y^2 \tag{32.8}$$

with general solution $y(x) = \wp(x+C_1; 0; C_2)$, where $\wp$ is the Weierstrass elliptic function. A great deal of information can be extracted from this equation alone, and we will return to it later.

32.2. **The symmetric version of the elliptic integral.** The double symmetry of the rectangle suggests a symmetric choice of a_i. The following integral is called *incomplete elliptic integral of the first kind*

$$F(\phi, k) = \int_0^{\sin\phi} \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{32.9}$$

Here $k \in (0,1)$. Since we are interested in the analytic behavior of this function, we will work instead with

$$f(z, k) = \int_0^{z} \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{32.10}$$

This transformation is similar to but not quite in the form of (31.5). The square roots are combined in pairs and the signs are different. As before, we want to understand the image $f(\mathbb{H}^u)$. To clarify the branches needed, we note that the integrand is bounded by $const./s^2$ at infinity, and the integral along $\mathbb{R}^+$ equals the integral along $\mathbb{R}^+ + i\varepsilon$, $\varepsilon > 0$; we take the latter and pass to the limit $\varepsilon \to 0$ after we have clarified the phase of the integrand. To indicate the direction of the

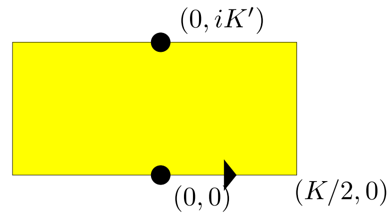


FIGURE 23. The fundamental rectangle.

limit, we write that the final integral is on $\mathbb{R} + i0^+$. In view of the symmetry $f(z,k) = f(-z,k)$ we only need to understand the map in the first quadrant.

The branched function here is the square root. We make branch cuts in such a way as to allow z (and thus s) to evolve in the first quadrant. Note that, as s evolves on $[0,a) + i0^+$, $g_1 := (1-s^2)$ *and* $g_2 := (1-k^2s^2)$ evolve on $[0,a) - i0^+$. Thus the cuts *of the square root* have to be *upward*, that is in $\mathbb{H}^u$.

On $[0,1)$ we take the natural, positive, value of the square root. The first change of argument of g_1, g_2 is when $a = 1$, where the become $-\pi^+, 0^-$ resp. A second one occurs when $a = 1/k$ when the arguments become $-\pi^+, -\pi^+$ Consequently, the integrand has argument 0 on $(0,1)$, $e^{i\pi^-/2}$ on $(1,1/k)$ and $e^{i\pi^-}$ on $(1/k,\infty)$. As a result, the integral $f(z,k)$ evolves counter-clockwise on a contour starting along $\mathbb{R}^+$ and changing direction twice by $+\pi/2$. As mentioned we can complete the evolution of f by symmetry. Traversing $\mathbb{R}$ gives

$$\int_{-\infty}^{\infty} \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} = 0 \tag{32.11}$$

since, once more the contour can be pushed up towards $i\infty$, and the figure closes (how come it is possible to use *four* points instead of *three*?). $\Phi(\mathbb{H}^u)$ it is then a rectangle with vertices

$$(-K/2,0), (K/2,0), (K/2,iK') \text{ and } (-K/2,iK) \tag{32.12}$$

where

$$K = \int_{-1}^{1} \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{32.13}$$

$$K' = \int_{1}^{1/k} \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{32.14}$$

Observe that for $s > 1/k$ we can write

$$\sqrt{1-s^2}\sqrt{1-k^2s^2} = -ks^2\sqrt{1-s^{-2}}\sqrt{1-(ks)^{-2}} \tag{32.15}$$

(where the square roots on the right side are positive) and that the function $\sqrt{1-\zeta}\ \zeta = 1/s^2$ is *analytic* for $|\zeta| < 1$. Thus $f(z,k)$ is analytic at infinity and for $z > 1/k$, writing $\int_0^z = \int_0^\infty - \int_z^\infty$ we get

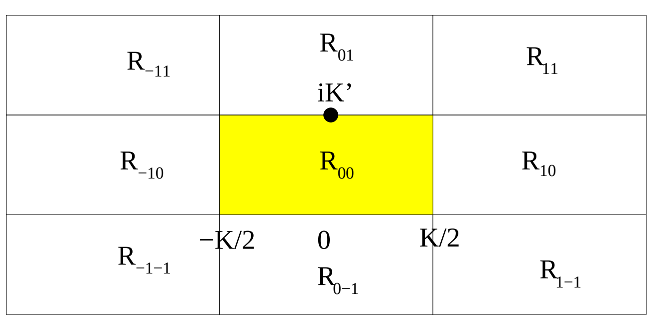
$$(32.16)\quad f(k,z) = iK' - \int_z^\infty -\frac{1}{ks^2} - \left(\frac{k^{-1}+k^{-3}}{2s^4} + \cdots\right) ds$$
$$= iK' + \frac{1}{kz} + \frac{k^{-1}+k^{-3}}{6z^3}\cdots$$

showing that f is analytic at infinity.

Exercise 32.2. *** Find changes of variables that connect* (32.11) *to* (32.1)

32.3. **Continuation to the whole of $\mathbb{C}$. Double periodicity.** It is interesting to see what happens to f by Schwarz reflection (of $\mathbb{H}^u$) which is more conveniently seen in the f plane. If the sequence of reflections is about $(-K/2, K/2), (K/2, K/2+iK'), (K/2, 3K/2), (K/2, K/2+iK')$ we return to the original domain. Since the values in each analytic reflection are symmetric with respect to the reflecting axis, it is easy to see by counting that the function continued through the four reflections above coincides with the original one. The continuation is consistent with the original definition. This is by no means guaranteed for a general function reflected back on itself, see Example (21.70). The *inverse function* $E = f^{-1}$ is, by the same argument, periodic with periods $2K$ horizontally and $2K'i$ vertically, that is, E is a doubly periodic function:

$$(32.17)\qquad \mathcal{E}(z) = \mathcal{E}(z+2K) = \mathcal{E}(z+2iK')$$



Nonconstant meromorphic functions with two periods in $\mathbb{C}$ are **elliptic functions**. The fundamental parallelogram associated to an elliptic function f is the one with sides T_1, T_2 where T_1, T_2 are the periods. Think whether the periods can be both real.

By Liouville's theorem elliptic functions must have at least one pole in the fundamental parallelogram. Since the integral $\oint f(s)ds$ along the boundary of the fundamental parallelogram vanishes by periodicity, the function must have at least two poles, counting multiplicity.

32.4. **Schwarz triangle functions and hypergeometric functions.** The upper half plane is homeomorphically mapped on a triangle of angles $\alpha_1\pi, \alpha_2\pi, \pi(1-\alpha_1-\alpha_2)$ by a Schwarz-Christoffel transformation which has no auxiliary parameters, as we discussed:

$$\Phi(z) = \int_0^z s^{\alpha_1-1}(s-1)^{\alpha_2-1}ds \tag{32.18}$$

(where we chose $C = 1, C' = 0$ for simplicity). In this case too we can apply the reflection-continuation procedure of §32.3. We now imagine the reflections having a common vertex. To insure a single valued function upon successive reflections about the sides, we must return to the starting triangle with no overlap or gap. Two successive reflections amount to a rotation with twice the angle at the common vertex. We must thus have $\alpha_i = 1/n_i,\ n_i \in \mathbb{N}$. In algebraic terms the reflections must generate a finite group, a special case of a *Coxeter group*. The constraints are thus:

$$\alpha_1 + \alpha_2 + \alpha_3 = 1; \quad 1/\alpha_i \in \mathbb{N} \tag{32.19}$$

There are only finitely many solutions $(\alpha_1, \alpha_2, \alpha_3)$ of this equation (why?). These are: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (equilateral triangle) $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ (half of an equilateral triangle) and $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ (isosceles right triangle). Then the reflected images cover the whole plane and the mapping functions are restrictions of meromorphic functions. These are special cases of the *Schwarz triangle functions.*

Each triangle function corresponds to an elliptic function. We will return to this topic later.

Differential equation. We can derive an equation with polynomial coefficients for Φ as follows. In $\Phi' = z^{\alpha_1-1}(z-1)^{\alpha_2-1}$ we write $\Phi = z^{\alpha_1}\Phi_1$ and get, after dividing by z^{α_1-1},

(32.20)

$$z\Phi_1' + \alpha_1\Phi_1 - (1-z)^{\alpha_2-1} = 0 \Rightarrow \frac{z}{(1-z)^{\alpha_2-1}}\Phi_1' + \frac{\alpha_1}{(1-z)^{\alpha_2-1}}\Phi_1 - 1 = 0$$

which we differentiate one more time to eliminate the constant 1 and we get

$$z(1-z)\Phi_1'' + [\alpha_1 + 1 - (2-\alpha_2-\alpha_1)z+]\Phi_1' + \alpha_1(\alpha_2-1)\Phi_1 = 0 \tag{32.21}$$

The differential equation for the hypergeometric function ${}_2F_1(a,b;c;z)$ is the *Riemann equation*

$$z(1-z)h'' + [c-(a+b+1)z]\,h' - abh = 0 \tag{32.22}$$

From the way we went from (32.20) to (32.21) (or by direct verification) we see that one solution is $z^{-\alpha_1}$. Comparing with (32.22) the second one is

$$ {}_2F_1(\alpha_1, 1-\alpha_2; \alpha_1+1; z) \Rightarrow \Phi = z^{\alpha_1} \cdot {}_2F_1(\alpha_1, 1-\alpha_2, \alpha_1+1, z) \tag{32.23}$$

That is, in this case, the Schwarz-Christoffel transformation is a ratio of two independent solutions of the special hypergeometric equation (32.21).

In general, the map from $\mathbb{H}^u$ into a *curvilinear triangle*, one whose sides are arccircles is given by the ratio of two independent solutions of (32.22), where the angles α, β, γ are related to a, b, c by (cf. [8])

$$a = \tfrac{1}{2}(1+\beta-\alpha-\gamma),\ b = \tfrac{1}{2}(1-\alpha-\beta-\gamma),\ c = 1-\alpha \tag{32.24}$$

Note 32.3. To see qualitatively why that is, before we work out the mathematical details, since the sought-for function f transforms segments into arccircles, Möbius transforms of f map real segments into real segments. Möbius transforms are conformal wherever defined, so they preserve angles, and thus f should be ramified analytic at three points, say $\{0,1,\infty\}$ where it must be ramified-analytic.

Secondly, note that any function which is real-analytic (with real values) on some interval I and s.t. $f' \neq 0$ on I conformally maps a neighborhood of I into a neighborhood of $f(I)$ (where, of course, $f(I)$ is an interval in $\mathbb{R}$).

Finally, this approach would apply to any curvilinear polygon, and the final ODE that we construct will still be second order linear, but the solutions are generally quite complicated.

To find an f as in Note 32.3, a natural candidate would be a ratio of two solutions of a real-valued linear second order ODE with analytic coefficients, whose solutions have only ramified singularities, at in $\{0,1,\infty\}$. Such an ODE will have two linearly independent solutions, f_1 and f_2 [8]. Define $S = \{a_1 f_1 + a_2 f_2 | A_{1,2} \in \mathbb{C}\}$ (this of course is the space of all solutions). Then,

(1) $f_{1,2}$ are analytic in $\mathbb{C}^* \setminus \{0,1,\infty\}$ while at the points $0,1,\infty$ they are *ramified-analytic*;
(2) on each interval $(-\infty,0)$, $(0,1)$ and $(1,\infty)$ there are two positive linearly independent functions in S;

[8]Meaning that $A, B \in \mathbb{C}$ and $Af_1 + Bf_2 = 0 \Rightarrow A = B = 0$.

(3) S is invariant under Schwarz reflections [9]. This is because any analytic continuation of a solution is a solution, by permanence of relations.
(4) $(f_1/f_2)' \neq 0 \, \forall x \in \mathbb{R}$. [10]

Then some ratio F_1/F_2 of two linearly independent $F_{1,2} \in S$ maps the upper half plane into a curvilinear triangle, that is, one whose sides are line segments or arccircles (a line segment is a limiting case of an arccircle, so we will call arccircle too). Indeed, take F_1 and F_2 in S linearly independent, choose one of the intervals in (2) above, and let $\tilde{F}_1$ and $\tilde{F}_2$ the functions which are real valued on the interval I. We choose $\tilde{F}_1$ and $\tilde{F}_2$ s.t. $\tilde{F}_1/\tilde{F}_2$ is bounded at zero. By assumption, $F_{1,2} = A_{1,2}\tilde{F}_1 + B_{1,2}\tilde{F}_2$ and

$$\frac{F_1}{F_2} = \frac{1 + a\tilde{F}_1/\tilde{F}_2}{b + c\tilde{F}_1/\tilde{F}_2} \tag{32.25}$$

Now, $\tilde{F}_1/\tilde{F}_2$ is real-valued and one-to-one, and thus the right side of (32.25) is a Möbius transformation of a line segment: an arccircle. The image of $\mathbb{R}$ through F_1/F_2 consists of three arccircles, a general curvilinear triangle, provided that the singularities are such that F_1/F_2 are continuous on $\mathbb{R}$.

We now show that (32.22) is such an equation, for a, b, c satisfying (32.24). The following integral representation due to Euler can be checked to solve (32.22):

$$ {}_2F_1(a, b; c; z) = K \int_0^1 H(z, x)dx, \tag{32.26}$$
$$\text{where } H(z, x) = x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} \text{ assuming } c > b > 0$$

The standard choice of K, immaterial here, is $K^{-1} = B(b, c-b)$ with B the Beta function [11]. For the integral to exist, we need

$$b - 1 > -1, c - b - 1 > -1, -a > -1 \tag{32.27}$$

which are satisfied if $\alpha + \beta + \gamma < 1$, $\alpha, \beta, \gamma \in \mathbb{R}^+$, which is the case for a hyperbolic triangle (with concave sides) as can be verified from

[9] Equivalently, the space generated by $f_{1,2}$, $S := \{C_1 f_1 + C_2 f_2 : C_1, C_2 \in \mathbb{C}\}$ is closed under continuations at the branch points (e.g., $f_1(ze^{2\pi i}) = C_1 f_1(z) + C_2(z)$).

[10] This in fact is implied by linear independence. Check this by first showing that two functions are linearly independent *iff* their Wronskian $W(f_1, f_2) := f_1' f_2 - f_2' f_1 \neq 0$.

[11] The formula is valid under the more general condition $\mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0$, but here we only need real a, b, c.

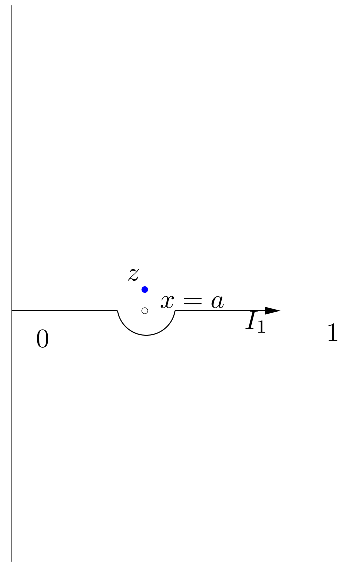


FIGURE 24. Hypergeometric contours

(32.24). Under this same condition, we have

$$(b-1)+c-b-1-a < -1 \Rightarrow H(x,z) \sim s^{-p},\ p > 1 \text{ as } x \to \infty \tag{32.28}$$

Singularities of $_2F_1$ For general a, b, c the behavior of h in (32.22) at the singular points of the ODE, $0, 1, \infty$ [12] follows general results about regular singular points of ODEs (Frobenius theory, cf. [4]); we will not assume this here however. We will find the behavior of the solutions in three different way, to illustrate various approaches.

Directly from the integral representation, the behavior can be calculated in the following way.

Clearly, (32.26) and Corollary 7.34 imply that $_2F_1$ is analytic except possibly on $\mathbb{R}^+$. If $z \to a \in \overline{\mathbb{R}^+} \setminus \notin \{1\}$ then we can use the analyticity of the integrand at a to first homotopically deform the contour as shown in Fig. 24. The new integral (of course, equal to the original one) is manifestly analytic in z near a. Thus the only possible singularity is at 1 (since the integrand is manifestly analytic for small z, cf. Corollary 7.34).

At $z = 0$ we get

$$\frac{{}_2F_1(a,b;c;z)}{K} = \int_0^1 x^{b-1}(1-x)^{c-b-1}dx > 0 \tag{32.29}$$

[12]Writing the equation in the form $h'' + Q(x)h + R(x) = 0$, $\{0,1\}$ are singular points of P, Q. Ditto after the change of variable $z = 1/t$, $y(z) = t^{-c}Y(t)$ at $t = 0$.

To find the type of behavior at $z = 1$ it is convenient to take $z = 1-\varepsilon$ and change the variable to $x = 1 - s$ in the integral:

$$(32.30)\quad K\int_0^1 h(z,(1-s))ds = K\int_0^1 (1-s)^{b-1}s^{c-b-1}(\varepsilon+s-\varepsilon s)^{-a}ds$$

By (32.28) We can then push the contour up toward $+i\infty$,
(32.31)

$$\int_0^1 h(z,(1-s))ds = \left(\int_0^{+i\infty} - \int_1^{+i\infty}\right)(1-s)^{b-1}s^{c-b-1}(\varepsilon+s-\varepsilon s)^{-a}ds$$

The second integral is analytic in ε by the same Corollary 7.34.

In the first one, we change variable to $s = \varepsilon u$, to get

$$(32.32)\quad -\varepsilon^{c-b-1-a+1}\int_0^{+i\infty}(1-\varepsilon u)^{b-1}u^{c-b-1}(1+u-\varepsilon u)^{-a}ds$$
$$= -\varepsilon^{c-b-a}\int_0^{-\infty}(1-\varepsilon u)^{b-1}u^{c-b-1}(1+u-\varepsilon u)^{-a}ds$$

where the contour change is justified by (32.28).

Exercise 32.4. *What is the phase of the last integral in* (32.32)*?*

The integral in (32.32) is analytic in ε and thus

(32.33) ${}_2F_1(a,b;c;z) = A_1 + A_2(1-z)^{\gamma}$ where $A_{1,2}$ are analytic at 1 and $A_2(1) = \int_1^{+\infty} u^{c-b-1}(1+u)^{-a}ds > 0$. A second solution of (32.22) is obtained, cf. [8], by noticing that the substitution $h(z) = g(1-z)$ leads to the equation (32.22) with c replaced by $C = a+b+1-c$. The integral representation (32.26) still holds with c replaced by $C = 1-\gamma$, and converges under the same conditions on α,β,γ as (32.26). The behavior of the integrand at infinity is still of the form in (32.28). Thus the same analysis applies, to show that there is a second solution which is analytic at $z = 1$ and has a singularity at zero, with $c-b-a$ replaced by $1-c = \alpha$. This also suggests making the substitution $h_1 = z^{1-c}h = z^{\alpha}h$ directly into (32.22); we get an equation of the form (32.22), with A, B, C replaced by

$$(32.34)\qquad A = a+1-c, B = b+1-c, C = c-2$$

which has an integral representation of the form (32.26) valid under the same conditions on α,β,γ. Thus, near $z = 0$ we have two linearly independent solutions, A_1 analytic and A_2 of the form $z^{\alpha}A(z)$ with A analytic and $A(0) \neq 0$. These are clearly linearly independent since the second solution is not analytic at zero. In the same way, there are two solutions, one analytic and positive at $z = 1$ and another one of the form $z^{\gamma}A(z)$, A analytic and positive for $z < 1$.

From the ODE. Here we are assuming knowledge of basic properties of linear ODEs: a linear combination (with constant coefficients) of solutions is a solution (this can be checked directly) and the fact that the space of solutions of a second order linear equation is two-dimensional (i.e., there are exactly two free constants, or, in other words, there are two linearly independent solutions which form a basis in the space of all solutions).

With F_{01} given by (32.26), we notice as before that F is analytic near zero and $F(0) > 0$. We look for a second solution in the form $F_{02} = F_{01}g$. The equation for g is

(32.35)
$$\frac{g''}{g'} = -2\frac{F'}{F} + q(x);\ \left(q(x) := \frac{(a+b+c)x-c}{x(1-x)}\right) = -2\frac{F'}{F} - \frac{c}{x} + A(x)$$
$$\Rightarrow g' = -\frac{c}{x}A_1(x) \Rightarrow F_{01}(x) = x^{-c+1}A_2(x) = x^{\alpha}A_2(x);$$

with A, A_1, A_2 analytic (check the conclusions above). Since F is real-valued for real $z \in (0,1)$, g is also real-valued, and it is an independent solution (it has a manifestly different behavior at zero). At $x = 1$ we make the substitution $z = 1 - y$ and we get

(32.36)
$$y(1-y)h'' + [C - (a+b+1)y]\,h' - abh = 0,\ C = a+b+c-1 = 1-\gamma$$

In the same way as above we get two independent real valued solutions for y real, $F_{10}(y)$ analytic and a second one F_{11} of the form $y^{\gamma}A_1$ with A_1 analytic. They are in general different from the solutions $F_{00}(y)$ and $F_{00}(y)$.

Finally the substitution $h(z) = z^a H(1/z)\ z = 1/Z$ in (32.22) results in an equation of the same type as (32.22). The ratio of two solutions behaves like $z^{-\beta}$ for large z. Now, any transformation of the type F_1/F_2 where F_1 and F_2 are real maps the interval $(0,1)$ into an interval. A different choice f_1/f_2 is a Möbius transformation of F_1/F_2, and thus it maps the interval $(0,1)$ into an arccircle. Keeping one combination F_1/F_2 chosen so that $F_1/F_2 \to 0$ as $z \to \infty$ will then map $\mathbb{R}$ into a curvilinear triangle, of angles α, β, γ.

Note 32.5. In the simplest nondegenerate case, Frobenius theory allows to determine the behavior of solutions of linear meromorphic ODEs, Lf, with regular singularities, say at z_0 by a very simple method: take $L(z-z_0)^r$ and keep only the lowest power of $(z-z_0)$. This gives a quadratic equation for r. If the solutions $r_{1,2}$ do not differ by an integer, then $Lf = 0$ has two linearly independent solutions in the form $(z-z_0)^r_{1,2}A_{1,2}(z)$, A analytic at z_0.

Note 32.6. The equation satisfied by f_1/f_2, a ratio of solutions of the hypergeometric equation is

$$\{w,z\} = \frac{1-\alpha^2}{2z^2} + \frac{1-\gamma^2}{2(z-1)^2} + \frac{\alpha^2+\gamma^2-\beta^2-1}{2z(z-1)} = 0 \tag{32.37}$$

where $\{w,z\}$ is the important *Schwarzian derivative* [8]:

$$\{w,z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 \quad \left(w' = \frac{dw}{dz}\right) \tag{32.38}$$

The Schwarzian derivative is *invariant under any Möbius transformations of w* as expected from our discussion and can be checked by straightforward calculation.

33. Riemann-Hilbert problems: an introduction

An impressive number of problems can be reduced to so-called Riemann Hilbert problems, and for many of them the only known way of solution is via the associated Riemann-Hilbert problem.

Problems which can be solved with RH techniques include

(1) integrable models such as the transcendental Painlevé equations e.g. $y'' = 6y^2 + x$ (P_I) and many others;

(2) relatedly, inverse scattering problems: find, from the scattering data the potential $q(x)$ in the time–independent Schrödinger equation

$$\psi_{xx} + (k^2 + q(x))\psi = 0 \tag{33.1}$$

(3) questions in orthogonal polynomials, random matrices, combinatorial probability;

(4) the nonlinear initial value problem for the KdV (Korteweg–deVries) equation

$$u_t + u_{xxx} + uu_x = 0;\;\; u(x,0) = u_0(x),\; u \to 0 \text{ as } |x| \to \infty \;\; (x \in \mathbb{R}, t \in \mathbb{R}^+) \tag{33.2}$$

(5) integral equations of the type

$$f(t) + \int_0^\infty \alpha(t-t')f(t')dt' = \beta(t) \tag{33.3}$$

(under suitable integrability conditions)

(6) finding the inverse Radon transform, a transform which is measured in computerized tomography,

33.1. A simple Riemann problem.

Perhaps the simplest RH problem is: given a simple smooth contour C and $f(t)$ a suitably regular function on C, find an analytic function whose jump across C is f:

$$\Phi^+(t) - \Phi^-(t) = f(t) \tag{33.4}$$

33.2. **Generalization: $\overline{\partial}$ (DBAR) problems.** A particular case of a RH problem is to find an analytic function with a given jump across the real line:

$$\Phi^+(x) - \Phi^-(x) = \phi(x) \tag{33.5}$$

with $\Phi^\pm$ analytic in the $\mathbb{H}^u$ ($\mathbb{H}_l$) respectively.

If we let Φ be defined by Φ^+ in the $\mathbb{H}^u$ and by Φ^- in the $\mathbb{H}_l$, then we have

$$\frac{\partial \Phi}{\partial y} = \frac{1}{2} f(x)\delta(y) \tag{33.6}$$

A general $\overline{\partial}$ problem would be, given g, to solve

$$\frac{\partial \Phi}{\partial \overline{z}} = g(x, y) \tag{33.7}$$

in some region $\mathcal{D} \subset \mathbb{C}$.

34. Cauchy type integrals

We recall that a function is Hölder continuous of order λ on a smooth curve C if

$$\exists\ \Lambda > 0 \text{ s.t. } \forall x, y \in C,\ |f(x) - f(y)| = \Lambda |x - y|^\lambda \tag{34.1}$$

The condition implies continuity if $\lambda > 0$ and it is nontrivial if $\lambda \leqslant 1$ (if $\lambda > 1$ then $df/ds = 0$).

Let C for now be a compact curve and ϕ be Hölder continuous on C. Then the function

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(s)}{s - z} ds \tag{34.2}$$

is manifestly analytic for $z \notin C$ (you can check this by Morera's theorem using Fubini).

34.1. **Asymptotic behavior for large z.**

Exercise 34.1. *Show that $\Phi(z)$ is* analytic at infinity *in z and that*

$$\Phi(z) = -\frac{\int_C \phi(s) ds}{2\pi i} \frac{1}{z} \left(1 + g(1/z)\right) \quad \text{as} \quad z \to \infty \tag{34.3}$$

as where $g(1/z) \to 0$ as $z \to \infty$.

Let C be a simple smooth contour and let t be an interior point of C. By this we mean that $C = \{\gamma(s) : s \in [0, 1]\}$ and $t = \gamma(s_1)$ with $s_1 \in (0, 1)$. We can then draw a small circle centered at t which intersects L in two points (a_1 and a_2). One arc of circle together with the curve segment between a_1 and a_2 form a closed Jordan curve, and so does the other arc circle and the curve segment between a_1 and a_2. A sequence approaches C from the left side if it eventually belongs to the closed Jordan curve whose interior is to the left of C as the curve is traversed positively (a similar definition applies to right limits).

34.2. **Regularity and singularities.** Let us first take a simple example, in which ϕ is analytic in a neighborhood $\mathcal{N}$ of $(0,1)$ (note that this does not exclude multi-valued functions singular at zero and real analytic at any $x \in (0,1)$ s.a. $\ln z$), and assume that 0 is an integrable singularity (that is, $\phi \in L^1((0,1))$). Then, the same argument we used to analyze hypergeometric functions shows that $\Phi(z)$ is analytic in $\mathbb{H}^u \cup \mathbb{H}_l \cup \mathcal{N}$. The only possible singularities are at 0 and 1. If z approaches a point a in (the interior of) $(0,1)$, say from $\mathbb{H}^u$, then we can, once more as in the analysis of hypergeometric functions locally deform the contour *before* z "touches a" and obtain that the integral equals $\phi(a) + A(z)$ with a analytic near a. What about $A(a)$? We can use Exercise 10.46 to see that $A(a) = \pm\frac{1}{2}\phi(a) + P\frac{1}{2\pi i}\int_0^1 \phi(s)(s-t)^{-1}ds$.

The same, interestingly, holds more generally:

Theorem 34.2 (Plemelj's formulas)**.** *Assume ϕ is Hölder continuous on the simple smooth curve $\mathcal{C}$, let t be an interior point of $\mathcal{C}$ and z_n approach $t \in \operatorname{Int}\mathcal{C}$ from the left (right). If $\mathcal{C}$ is not bounded, assume also that $\phi \in L^1(\mathcal{C})$. Then, with the $\pm$ sign being $+$ for left limit and $-$ for right limit,*

$$\lim_{n\to\infty} \Phi(z_n) = \Phi^{\pm}(t) \tag{34.4}$$

where

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i}P\int \frac{\phi(s)}{s-t}ds \tag{34.5}$$

and

$$P\int \frac{\phi(s)}{s-t}ds \tag{34.6}$$

(cf. Definition 10.45) exists.

Note 34.3. *(i) The property is local. It is easy to see that it is enough to prove it for the compact pieces of the curve $\mathcal{C}$.*

(ii) A similar statement can be obviously made when t is approached along a curve, since all limits along subsequences coincide.

It is clearly enough to show the formula as the contour is approached from the left. It is also easy, and left as an exercise to extend the proof from the case when $\mathcal{C}$ is a piece of $\mathbb{R}$, say $[-1,1]$ to a more general bounded curve (open or not): parametrize the curve and do similar estimates).

We reduce to the case when ϕ is a constant in the following way. Let $z_n = t_n + iy_n = t + \delta_n + iy_n$ where $\delta_n \to 0$ and $y_n \downarrow 0$. We write

$$\frac{\phi(s)}{s - t_n - iy_n} = \frac{\phi(s) - \phi(t_n)}{s - t_n - iy_n} + \frac{\phi(t_n)}{s - t_n - iy_n} \tag{34.7}$$

where the first function on the right side is expected to be more regular in the limit $z_n \to t$. Consider thus the auxilliary function

$$\Psi_n := \frac{\phi(s) - \phi(t_n)}{s - t_n - iy_n} \tag{34.8}$$

We first obtain a result about Ψ_n:

Proposition 34.4. *(i) Let $z_n = t + \varepsilon_n + iy_n$, $\varepsilon_n \to 0$ and $y_n \downarrow 0$. We have, as $z_n \to t$ in this way,*

$$\lim_{z_n \to t} \int_{-1}^{1} \Psi_n(s)ds = \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds \tag{34.9}$$

Proof. We write I for the characteristic function of the interval $[-1,1]$, *i.e.* $I(x) = 1$ if $x \in [-1,1]$ and zero otherwise. Denote

$$G_n(u) = \frac{\phi(u + t_n) - \phi(u)}{u + iy_n}; \ G(u) = \frac{\phi(u + t) - \phi(u)}{u} \tag{34.10}$$

For n large enough s.t. $1 + |t_n| < 2$ we change variables to $s = t_n + u$, and

(34.11)
$$\int_{-1}^{1} \Psi_n(s)ds = \int_{-a}^{a} \Psi_n(s)I(s)ds = \int_{-a}^{a} \frac{\phi(u + t_n) - \phi(t_n)}{u + iy_n} I(t_n + u)du$$
$$= \int_{-a}^{a} G_n(u)I(t_n + u)du$$

Now we note that for all u we have

$$\begin{aligned} &|\phi(u + t_n) - \phi(t_n)| \leqslant \Lambda |u|^{\lambda}, \ |u + iy_n| \geqslant |u| \\ &\qquad \Rightarrow |G_n(u)I(t_n + u)| \leqslant \Lambda |u|^{\lambda - 1} \in L^1([-a,a]) \end{aligned} \tag{34.12}$$

Note also that for any $u \neq 0$ we have

$$G_n(u)I(u + t_n) \to G(u)I(u + t) \text{ as } n \to \infty \tag{34.13}$$

The conditions of the dominated convergence theorem are met and thus

(34.14)
$$\int_{-a}^{a} G_n(u)I(t_n+u)du \to \int_{-a}^{a} G(u)I(t+u)du = \int_{-a}^{a} G(t-s)I(s)ds$$
$$= \int_{-1}^{1} \frac{\phi(s)-\phi(t)}{s-t}ds$$

∎

To deal with the term

$$\frac{\phi(t_n)}{s-t_n-iy_n}$$

as we will see, we only need to find out what happens when $\phi = 1$. The limit to evaluate is then

$$\lim_{z_n\to z} \int_{-1}^{1} \frac{1}{t-t_n-iy_n}dt \tag{34.15}$$

Proposition 34.5.

$$\lim_{z_n\to t} \int_{-1}^{1} \frac{1}{s-z_n}ds = \pi i + P\int_{-1}^{1} \frac{1}{s-t}ds \tag{34.16}$$

and the last principal part integral exists and equals

$$P\int_{-1}^{1} \frac{1}{s-t}ds = \ln\left(\frac{1-t}{1+t}\right), \ \ (t\in(-1,1)) \tag{34.17}$$

where the branch of the log is the natural one: positive for positive arguments.

Proof. We have, by analyticity and homotopic deformation for z to the left of the curve,

$$\int_{-1}^{1} \frac{1}{s-z}ds = \int_{\mathcal{C}_\varepsilon} \frac{1}{s-z}ds \tag{34.18}$$

where $\mathcal{C}_\varepsilon$ is the contour depicted in Fig. 25, where a line segment of length 2ε centered at zero is replaced by an semicircle of radius ε in $\mathbb{H}_l$. The integral around the half circle $\mathcal{C}_{l;\varepsilon}$ is easily calculated in the limit $z_n \to t$, by parametrization,

$$\lim_{z_n\to t} \int_{\mathcal{C}_{l;\varepsilon\varepsilon}} \frac{1}{s-z_n}ds = \lim_{z_n\to t} \int_{-\pi}^{0} i\frac{\varepsilon e^{i\phi}}{\varepsilon e^{i\phi}-\delta_n-iy_n}d\phi = \pi i \tag{34.19}$$

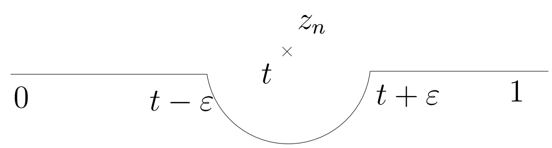


FIGURE 25. Contour $\mathcal{C}_\varepsilon$ in Prop. 34.4.

by dominated convergence which also shows that

$$P\int_{-1}^{1}\frac{1}{s-t}ds = \lim_{\varepsilon\to 0}\left[\int_{\mathcal{C}_\varepsilon}\frac{1}{s-t}ds - \int_{\mathcal{C}_{l;\varepsilon}}\frac{1}{s-t}ds\right] \tag{34.20}$$

exists, since the first term on the right side of (34.20) does not depend on ε whereas for the second limit we have already shown that the limit is πi.

The existence of the limit can be shown by explicit integration as well, giving (34.17). ∎

Corollary 34.6. *Under the conditions of Theorem 34.2,*

$$P\int_{-1}^{1}\frac{\phi(s)}{s-t}ds = \int_{-1}^{1}\frac{\phi(s)-\phi(t)}{s-t}ds - \phi(t)P\int_{-1}^{1}\frac{1}{s-t}ds \tag{34.21}$$

exists.

Proof. This is immediate from Exercise 10.47 and the results we obtained so far. ∎

34.3. **Proof of Plemelj's formulas.**

Proof. We have

$$
\begin{aligned}
(34.22)\quad & \lim_{z_n\to t}\int_{-1}^{1}\frac{\phi(s)}{s-z_n}ds \\
&\overset{34.4}{=}\lim_{z_n\to t}\int_{-1}^{1}\frac{\phi(s)-\phi(t_n)}{s-z_n}ds+\lim_{z_n\to t}\int_{\mathcal{C}_\varepsilon}\frac{\phi(t_n)}{s-z_n}ds \\
&\overset{34.5}{=}\int_{-1}^{1}\frac{\phi(s)-\phi(t)}{s-t}ds+\phi(t)\lim_{z_n\to t}\int_{\mathcal{C}_\varepsilon}\frac{1}{s-z_n}ds \\
&\overset{34.5}{=}\int_{-1}^{1}\frac{\phi(s)-\phi(t)}{s-t}ds+\pi i\phi(t)+\phi(t)P\int_{-1}^{1}\frac{1}{s-t}ds \\
&\overset{34.6}{=}P\int_{-1}^{1}\frac{\phi(s)}{s-t}ds-\phi(t)P\int_{-1}^{1}\frac{1}{s-t}ds+\pi i\phi(t)+\phi(t)P\int_{-1}^{1}\frac{1}{s-t}ds \\
&=P\int_{-1}^{1}\frac{\phi(s)}{s-t}ds+\pi i\phi(t)
\end{aligned}
$$

where we indicated the propositions and corollary used in the calculation. ▮

Note 34.7. The function defined by the Cauchy type integral (34.2) is called **sectionally analytic**. With the convention about the sides of the curve mentioned before, functions that are boundary values of Cauchy type integrals are sometimes denoted $\oplus$ and $\ominus$ functions. respectively.

A straightforward calculation shows that the following result holds.

Theorem 34.8 (Existence)**.** *Under the conditions of Theorem 34.2, the function in* (34.2) *solves the Riemann-Hilbert problem in §33.1.*

34.4. **Examples.**

34.4.1. *A very simple example.* As usual S_1 is the unit circle. Find a function Φ analytic in $\mathbb{C}\setminus S_1$ such that along S_1 we have

$$\Phi^+(t)-\Phi^-(t)=1 \tag{34.23}$$

Note that for this problem the set of analyticity is not a domain but a union of two disjoint domains. There is no reason to think of Φ as one analytic function, unless the two pieces were analytic continuations of each other across S_1, which they cannot be.

For the problem in this section, Plemelj's formula reads

$$\Phi(z)=\frac{1}{2\pi i}\int_{S_1}\frac{1}{s-z}ds \tag{34.24}$$

Clearly, if z is inside the unit disk, which, according to our convention, is to the left of S_1 oriented positively, we have

$$\Phi_{\text{in}}(z) = \frac{1}{2\pi i}\int_{S_1} \frac{1}{s-z}ds = 1 \tag{34.25}$$

Likewise, if z is outside the unit disk we have

$$\Phi_{\text{out}}(z) = \frac{1}{2\pi i}\int_{S_1} \frac{1}{s-z}ds = 0 \tag{34.26}$$

Both Φ_{in} and Φ_{out} are analytic, but not analytic continuations of eachother, so in this case our sectionally analytic function is really a pair of distinct analytic functions. We leave the question of uniqueness to the next subsection when the contour is open and which leads to a more interesting discussion.

34.4.2. *Another simple example.* Find a function Φ analytic in $\mathbb{C} \setminus [-1,1]$ such that along $[-1,1]$ we have

$$\Phi^+(t) - \Phi^-(t) = 1 \tag{34.27}$$

34.4.3. *A* <u>*(but not the)*</u> *solution.* According to Plemelj's formulas this function is given by

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{1}{s-z}ds \tag{34.28}$$

(It is clear Φ is well defined and analytic in $\mathbb{C} \setminus [-1,1]$, which is now a region.)

34.4.4. *Formula for the solution in §34.4.3.* Can we say more about this function? For $z \in (1,\infty)$, for example, we can calculate the integral explicitly and it gives

$$\Phi(z) = -\frac{1}{2\pi i}\ln\left(\frac{z-1}{z+1}\right) \tag{34.29}$$

with the usual branch of the log, which makes Φ negative for $z > 1$. Since $(z-1)(z+1)^{-1} = 1 - 2z^{-1}(1+z^{-1})^{-1}$, Φ admits a convergent series representation in powers of $1/z$ for $|z| > 1$ and thus it is analytic in $\{z : |z| > 1$. It coincides by construction with our Φ on $(1,\infty)$. They are therefore identical to each other on $\mathbb{C} \setminus [-1,1]$.

34.4.5. *A verification.* As an exercise of working with branched functions, let's check that indeed the function in (34.29) solves our problem. Let

$$w = \frac{z-1}{z+1}$$

To see what argument we get for $z \in (-1, 1)$ one way is the following. We remember that $z \in \mathbb{H}^u$ and we chose the branch with $\operatorname{Im} \ln((z-1)/(z+1) \to 0$ as $z \downarrow (1, \infty)$. Since everything is continuous for $|z| > 1$ we can simply say that $\operatorname{Im} \ln((z-1)/(z+1)) = 0$ if $z > 1$. We then take $z = \varepsilon e^{i\phi}$ with $\phi \in [0, \pi]$ and this clearly corresponds to analytic continuation in $\mathbb{H}^u$ from $z > 1$ to $z < 1$. Thus,

$$\ln\left(\frac{z-1}{z+1}\right) \to \ln\left(\frac{1-t}{1+t}\right) + i\pi \tag{34.30}$$

and similarly, as z approaches $[-1, 1]$ from below we get

$$\ln\left(\frac{z-1}{z+1}\right) \to \ln\left(\frac{1-t}{1+t}\right) - i\pi \tag{34.31}$$

and indeed $\Phi^+ - \Phi^- = 1$ along $[-1, 1]$. Note that only the points $\{-1, 1\}$ are special, not the whole segment. In any event, we can define Φ as one analytic function on $\mathbb{C} \setminus [-1, 1]$ or, more globally, on the universal covering of $\mathbb{C} \setminus \{-1, 1\}$.

34.4.6. *Calculating principal value integrals.* Plemelj's formulas help us calculate principal values integrals as well, sometimes in a simpler way. Let $\mathcal{C}$ be a simple smooth closed curve and assume that $f(z)$ is analytic in $\operatorname{Int}(\mathcal{C})$ and Hölder continuous in the closure of $\operatorname{Int}(\mathcal{C})$.

Exercise 34.9. Show that

$$\frac{1}{2\pi i} P \int_{\mathcal{C}} \frac{f(s)}{s-t} ds = \frac{1}{2} f(t) \tag{34.32}$$

a "limiting case" of a Cauchy formula.

34.4.7. *Uniqueness issues.* Is the solution of our problem unique? Certainly not. We can add to Φ any analytic function with isolated singularities at ± 1. Can we achieve uniqueness in such a problem? Yes, if we rule out this freedom by providing conditions at infinity and near the endpoints of the curve.

Theorem 34.10 (Uniqueness)**.** *Consider the problem in §33.1 with the following further conditions:*

(1) $\Phi(z) \to 0$ *as* $|z| \to \infty$;

(2) $(1 \mp z)\Phi(z) \to 0$ *as* $z \to \pm 1$

FIGURE 26. Contour used for applying Morera's theorem.

Then the solution is unique, namely (34.2).

Proof. The Φ in (34.2) satisfies this condition, as it is easy to verify. Assume Φ_1 is another solution with the same properties. Then $f = \Phi - \Phi_1$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and continuous on $(-1, 1)$, entailing continuity in $\mathbb{C} \setminus \{-1, 1\}$. To apply Morera's theorem, show as an exercise that the contour integral of $\Phi - \Phi_1$ on a circle of radius ε tends to zero as $\varepsilon \to 0$, thus is zero on any closed contour in $\mathbb{C}$ (see Fig. 26). Since $\lim_{z\to\infty}(\Phi(z) - \Phi_1(z)) = 0$, then $\Phi - \Phi_1 = 0$. ∎

Exercise 34.11. Show that any simple smooth curve in $\mathbb{C}$ is the natural boundary of many analytic functions, even ones that are in $C^\infty(\mathbb{R}^2)$ (that is, as functions of *two real* variables, (x, y), and in the sense of lateral limits of derivatives). What other curves, or, more generally, boundaries of domains can be natural boundaries?

35. Extensions

35.1. **Scalar homogeneous RH problems.** This is a problem of the type

$$\Phi^+ = g\Phi^- \quad \text{on} \quad \mathcal{C} \tag{35.1}$$

where $\mathcal{C}$ is a smooth simple closed contour, g nonzero on $\mathcal{C}$ and satisfying a Hölder condition on $\mathcal{C}$. We are looking for solutions of *finite order* and assume that the *index of g w.r.t. $\mathcal{C}$* is k. We now explain these last two notions.

35.1.1. *Index of a function with respect to a curve.* We first need to define the index of a function ϕ with respect to a closed curve.

Assumption 35.1. *$\mathcal{C}$ is a smooth closed curve, ϕ is Hölder continuous of exponent α and constant A along $\mathcal{C}$ and* $\min_{\mathcal{C}} |\phi| = a > 0$.

If ϕ is in fact differentiable, the definition of the index is simply

$$\mathrm{ind}_{\mathcal{C}}\phi := \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\phi'(s)}{\phi(s)} ds \tag{35.2}$$

(If ϕ is meromorphic inside of $\mathcal{C}$, then clearly $\mathrm{ind}_{\mathcal{C}}\phi = N - P$, the number of zeros minus the number of poles inside $\mathcal{C}$.) If ϕ is not differentiable, we can still define the index by noting that in the differentiable case $\phi'/\phi = (\ln\phi)'$ and then

$$\mathrm{ind}_{\mathcal{C}}\phi := \frac{1}{2\pi i}[\log\phi]_{\mathcal{C}} \tag{35.3}$$

the total variation of the argument of ϕ when $\mathcal{C}$ is traversed once. Because of the nonvanishing of ϕ, a branch of the log can be consistently chosen and followed along $\mathcal{C}$. Let $\Gamma = \max_{t\in[-1,1]}|\gamma'(t)|$. We choose ε such that $A\Gamma\varepsilon^\alpha < a/2$ and then we have $\phi(\gamma(t+\varepsilon)) = \phi(\gamma(t))+\delta$ where $|\delta| < a/2$. If we partition $[0,1]$ in intervals of size ε and choose a branch of $\log(\phi(\gamma(0)))$ we can calculate inductively the log in any interval of size ε by taking $0 < \varepsilon' < \varepsilon$ and writing $\phi(\gamma(k\varepsilon+\varepsilon'))) = \phi(\gamma(k\varepsilon)) + \delta'$, noting that $|\delta'| < |\phi(\gamma(k\varepsilon))|$ and thus

$$\log\Big(\phi(\gamma(k\varepsilon+\varepsilon'))\Big) = \log\Big(\phi(\gamma(k\varepsilon))\Big) + \log\Big(1+\delta'/\phi(\gamma(k\varepsilon))\Big) \tag{35.4}$$

can be calculated by Taylor expanding the last log. Since the log and ϕ are well defined, and the condition $\exp(\log\phi(z)) = \phi(z)$ is preserved in the process, the index of ϕ must be an integer.

35.1.2. *Degree of a function at infinity.* By definition Φ has degree k at infinity if for some $C \neq 0$ we have

$$\Phi(z) = Cz^k + O(z^{k-1}) \quad \text{as} \quad z \to \infty \tag{35.5}$$

The function Φ has finite degree at infinity if $\Phi = o(z^m)$ for some m.

35.1.3. *Solution to the homogeneous RH problem.* First we note that if Φ is a solution and P is a polynomial of order m, then by homogeneity ΦP is also a solution.

Let us assume $\mathcal{C}$ is a simple smooth closed curve. Without loss of generality we assume $0 \in\mathrm{Int}(\mathcal{C})$. We can rewrite the problem as

$$\Phi^+(t) = (t^{-k}g(t))(t^k\Phi^-) \quad \text{on} \quad \mathcal{C} \tag{35.6}$$

or

$$\ln\Phi^+(t) = \ln(t^{-k}g(t)) + \ln(t^k\Phi^-) \quad \text{on} \quad \mathcal{C} \tag{35.7}$$

or finally, with obvious notation,

$$\Gamma^+(t) = f(t) + \Gamma^-(t) \quad \text{on} \quad \mathcal{C} \tag{35.8}$$

The reason we form the combination $t^{-k}g(t)$, where we choose k to be the index of ϕ w.r.t $\mathcal{C}$, is to ensure Hölder continuity of f. Otherwise, since $\arg\phi$ changes by $2k\pi$ upon traversing $\mathcal{C}$ f would have a jump discontinuity. But we already know a solution to (35.8), given by Plemelj's formulas

$$\Gamma(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(s)}{s-z}ds \tag{35.9}$$

We recall that $\Gamma^- = O(z^{-1})$ for large z, and thus $\log(z^k\Phi^-) = O(z^{-1})$ too. This means that $z^k\Phi^- \to 1$ as $z \to \infty$, or, which is the same, $\Phi^- = z^{-k} + o(z^{-k})$ for large z. We finally get the solution of degree m at infinity,

$$\Phi(z) = X(z)P_{m+k}(z) \tag{35.10}$$

where

$$X = \begin{cases} e^{\Gamma(z)}, & z \text{ inside } \mathcal{C} \\ z^{-k}e^{\Gamma(z)}, & z \text{ outside } \mathcal{C} \end{cases} \tag{35.11}$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\log(s^{-k}g(s))}{s-z}ds \tag{35.12}$$

The polynomial P is appended to ensure, if possible, the desired behavior at infinity. We will not, for reasons of space, discuss uniqueness issues here.

35.1.4. *Ingomogeneous RH problems.* These are equations of the form

$$\Phi^+ = g\Phi^- + f \tag{35.13}$$

again under suitable assumptions on g and f. These can be brought to Plemelj's formulas in the following way. We first solve the homogeneous problem

$$\Psi^+ = g\Psi^- \tag{35.14}$$

and look for a solution of (35.13) in the form $\Phi = U\Psi$. We get
(35.15)

$$U^+\Psi^+ = gU^-\Psi^- + f \Rightarrow U^+g\Psi^- = U^-g\Psi^- + f \Rightarrow U^+ - U^- = \frac{f}{g\Psi^-}$$

which is of the form we already solved.

Exercise 35.2. ** Check that f/Ψ^+ is Hölder continuous.*

35.2. **Applications.**

35.2.1. *Ingomogeneous singular integral equations.* These are equations of the form

$$a(t)\phi(t) + b(t)P\int_{\mathcal{C}} \frac{\phi(s)}{s-z}ds = c(t) \tag{35.16}$$

with a, b, c Hölder continuous and the further condition $i\pi a(t) \pm b(t) \neq 0$. We attempt to write, guided by Plemelj's formulas

$$\phi(t) = \Phi^+(t) - \Phi^-(t) \tag{35.17}$$

and

$$\Phi(z) = \frac{1}{2\pi i}\int \frac{\phi(s)}{s-z}ds \tag{35.18}$$

and then

$$P\int_{\mathcal{C}} \frac{\phi(s)}{s-t}ds = i\pi\Big[\Phi^+(t) + \Phi^-(t)\Big] \tag{35.19}$$

where, of course ϕ is still unknown. The equation becomes

$$a(t)\Big[\Phi^+(t) - \Phi^-(t)\Big] + b(t)i\pi\Big[\Phi^+(t) + \Phi^-(t)\Big] = c(t) \tag{35.20}$$

or

$$\Phi^+(t)(a(t) + b(t)i\pi) + \Phi^-(t)(b(t)i\pi - a(t)) = c(t) \tag{35.21}$$

or, finally,

$$\Phi^+(t) = \frac{a(t) - b(t)i\pi}{a(t) + b(t)i\pi}\Phi^-(t) + \frac{c(t)}{a(t) + b(t)i\pi} \tag{35.22}$$

which is of the form (35.13) which we addressed already. Care must be taken that the chosen solution Φ is such that $\Phi^+ + \Phi^-$ has the behavior (34.3) at infinity. Then the substitution is a posteriory justified. We did not discuss whether there are other solutions of the integral equation. A complete discussion of this and related equations can be found in [7].

The number of applications of RH techniques is impressive. Many interesting examples are given in [1]. The inversion formula for the Fourier transform can be easily obtained after reformulating the question as an RH problem. The inverse Radon transform can be solved similarly. The (more complicated) matrix RH problems allow for many more problems to be solved: integrable ODEs and PDEs, inverse scattering and so on.

We choose one of the applications in [1], the solution of the Dirichlet problem for the Laplacian in the upper half plane, with condition f on the boundary, $\mathbb{R}$. The problem can be reformulated as a Riemann-Hilbert problem of the form (35.22), see [1], but in this case, the solution is obtained easily from Plemelj's formulas.

For this purpose we look for an analytic function in $\mathbb{H}^u$ generated by u. As we know, this is

$$\Phi^+ = u + iv \tag{35.23}$$

where v is the harmonic conjugate of u, unique up to a constant. Now note that the function

$$\Phi(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(s)}{s-z}ds \tag{35.24}$$

is analytic in the upper half plane and, if f is Hölder continuous, then by Plemelj's formulas we have

$$\lim_{z\downarrow t\in\mathbb{R}} \Phi(z) = f(t) + \frac{1}{\pi i}P\int_{\mathbb{R}} \frac{f(s)}{s-t}ds \tag{35.25}$$

In particular $u = \operatorname{Re}\Phi$ is harmonic and has the limit $f(x,y)$ as $(x,y) \to (t,0)$. Now, simply writing $z = x + iy$ and taking the real part we get the solution in the Poisson kernel form,

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)d\tau}{(\tau-x)^2+y^2} \tag{35.26}$$

The condition that f is Hölder can be relaxed to mere continuity as follows. To better adapt to the limit $y \to 0$ we change variable to $t = x + \beta y$ and obtain

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x+\beta y)}{\beta^2+1}d\beta \tag{35.27}$$

which, by dominated convergence, has $f(x)$ as the limit when $y \to 0$.

36. Entire and Meromorphic functions

Analytic and meromorphic functions share with polynomials and rational functions a number of very useful properties, such as decomposition by partial fractions and factorization. These notions have to be carefully analyzed though, since questions of convergence arise.

36.1. **A historical context.** Finding the *exact* value of the sum

$$S := \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{36.1}$$

known as the "Basel problem" had been open for almost a century, in spite of efforts by great mathematicians of the time (the Bernoulli brothers, Leibniz, Goldbach, Stirling, Moivre and others) before Euler solved it in 1735 whe he was 28; because the problem stumped so many brilliant minds, this attracted a lot of attention. Euler's solution, well before there was any systematic theory of complex functions, proceeds as follows. Take the function

$$f(x) = \frac{\sin x}{x} \tag{36.2}$$

This has a Taylor series which converges for all $x \in \mathbb{C}$. *If* f were a polynomial with roots at $z = a_i$, then we would be able to write

$$f(x) = A \prod (x - a_i) \tag{36.3}$$

Assuming the same were true for an "infinite order polynomial" and noting that the roots of $\sin(x)/x$ are at $n\pi$, $n \in \mathbb{Z} \setminus \{0\}$ we get

$$\frac{\sin x}{x} = \prod_{n \geqslant 1} \left(1 - \frac{x}{n\pi}\right)\left(1 + \frac{x}{n\pi}\right) = \prod_{n \geqslant 1} \left(1 - \frac{x^2}{(n\pi)^2}\right) \tag{36.4}$$

Expanding out and collecting the coefficient of x^2 we get

$$\frac{\sin x}{x} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \cdots\right) = -\frac{1}{\pi^2} S \tag{36.5}$$

On the other hand from the Maclaurin expansion of $f(x)$ this coefficient must also equal $-\frac{1}{3!}$. Thus

$$S = \frac{\pi^2}{6} \tag{36.6}$$

In fact, Euler went further and calculated $\sum \frac{1}{n^{2k}}$ for $k \geqslant 1$, in principle for all even k. This was fine by the standards of the day, though it led to some criticisms that prompted more rigorous proofs later by Euler. Weierstrass was apparently inspired by this solution when he developed the theory of decomposition of entire functions as products. For a rigorous proof of (36.5) see §37.2

36.2. **Partial fraction decompositions.** First let $R = P_0/Q = P_1 + P/Q$ be a rational function. where P_i and Q are polynomials and $\deg(P) < \deg(Q)$. We aim at a partial fraction decomposition of R; if $\deg(Q) = 0$ there is nothing further to do. Otherwise let $z_1, ..., z_n$, $n \geqslant 1$, be the zeros of Q, where we don't count the multiplicities, and let n_j be the multiplicities of these roots. Let's look at the singular part of the Laurent expansion of P/Q at z_j:

$$\frac{P}{Q} = \sum_{k=1}^{n_j} \frac{c_{jk}}{(z - z_j)^k} + \text{analytic at } z_j \tag{36.7}$$

We claim that

$$\frac{P}{Q} = \sum_{j=1}^{n} \sum_{k=1}^{n_j} \frac{c_{jk}}{(z - z_j)^k} \tag{36.8}$$

Indeed,

$$E(z) := \frac{P}{Q} - \sum_{j=1}^{n} \sum_{k=1}^{n_j} \frac{c_{jk}}{(z - z_j)^k} \tag{36.9}$$

is an entire function. By assumption, $P/Q \to 0$ as $z \to \infty$ and the rhs of (36.8) also, clearly, goes to zero as $z \to \infty$. Thus $E(z) \to 0$ as $z \to \infty$ and therefore $E \equiv 0$. Let's try a less trivial example. The function

$$\frac{\pi^2}{\sin^2 \pi z} \tag{36.10}$$

has zeros for $z_i = N$, $N \in \mathbb{Z}$, and the singular part of the Laurent series at $z = N$ is, as it can be quickly checked

$$\frac{1}{(z - N)^2} \tag{36.11}$$

We claim that in fact

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{k \in \mathbb{Z}} \frac{1}{(z - k)^2} \tag{36.12}$$

and in fact the proof is similar to that for rational functions. We first note that the series on the rhs of (36.12) converges in $\mathbb{C} \setminus \mathbb{Z}$, uniformly on compact sets; it thus defines an analytic f function in $\mathbb{C} \setminus \mathbb{Z}$. Let $E(z)$ be the difference between the lhs and rhs of (36.12).

(i) If $z = x + iy$, $x \in [0, 1]$ and $|y| > 1$ large, the terms of the series are bounded by

$$\frac{1}{1 + (k - x)^2}$$

as can be seen by rearranging the terms in the denominators, and in particular, by dominated convergence, for fixed x, $f \to 0$ as $y \to \infty$.

(ii) With $z = x + iy$ we have

$$|\sin(x + iy)^2| = \frac{1}{2}\Big(\cosh(2y) - \cos(2x) \Big) \geqslant \frac{1}{2} (\cosh(2y) - 1)$$

(iii) Clearly $E(z)$ is periodic with period 1 and it is, by construction and the form of f, an entire function.

(iv) By analyticity, $E(z)$ is bounded in the rectangle $\{(x, y) : |x| \leqslant 1, |y| \leqslant 1\}$ and by (i) and (ii) it is also bounded in the strip $\{(x, y) : |x| \leqslant 1, |y| > 1$ and since it is periodic, it is bounded in $\mathbb{C}$. Thus E is a constant. It can only be zero, since by (i) and (ii) $E \to 0$ as x is fixed and $y \to \infty$.

36.3. **The Mittag-Leffler theorem.** How generally is it possible to decompose meromorphic functions by partial fractions? Completely general, as we'll see in a moment, provided we are careful with the issues of convergence. We can't naively just write, in the same spirit,

$$\frac{\pi}{\sin \pi z} \stackrel{??}{=} \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{z-n} \tag{36.13}$$

because clearly the series in (36.13) *diverges for all* z. But provided we add and subtract terms so as to ensure convergence, the partial fraction decomposition is general. The theorem below shows that for any sequence of one-sided Laurent series centered at the points $b_n \in \mathbb{C}$ with no accumulation point, there is a meromorphic function having exactly that singular behavior and analytic elsewhere. Conversely, any meromorphic function can be decomposed as a sum of its negative powers-part of its Laurent series at the poles and an entire function. More precisely,

Note 36.1. *The condition that* $\{b_n\}_{n\in\mathbb{N}}$ *has no accumulation point in* $\mathbb{C}$ *automatically implies that* $|b_n| \to \infty$ *as* $n \to \infty$.

Theorem 36.2 (Mittag-Leffler)**.** (i) Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers with no accumulation point (cf. Note 36.1) in $\mathbb{C}$ and let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials without constant term. Then there are meromorphic functions f in $\mathbb{C}$ such that the only poles are at $z = b_n$ and let the singular part of f at $z = b_n$ is $P_n((z-b_n)^{-1})$.

(ii) Conversely let f be meromorphic in $\mathbb{C}$ with poles only at $z = b_n$ and with the singular part of the Laurent expansion at b_n $P_n((z-b_n)^{-1})$. Then there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ and an entire function g such that

$$f = \sum_{n\in\mathbb{N}} \left[P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right] + g(z) := S(z) + g(z) \tag{36.14}$$

where the series converges uniformly on compact sets in $\mathbb{C} \setminus \{b_n\}_{n\in\mathbb{N}}$.

Proof. We start by proving (ii). We can assume without loss of generality that $b_n \neq 0$ for if say $b_1 = 0$ then we can prove the theorem for $\tilde{f} = f - P_1(1/z)$.

The possible of divergence of the infinite series $\sum_{n\in\mathbb{N}} P_n((z-b_n)^{-1})$ is due to the behavior for large n of the terms of the series.

To ensure convergence, it is then natural to subtract from each P_n, $n \in \mathbb{N}$, a sufficient number of terms of the convergent series in $1/b_n$

(thought of as a variable, for the moment; note that we have analyticity at infinity in b_n). It is easy to see that the terms in the expansion in $1/b_n$ are the same as the Taylor series terms *in z for small z*.

For each b_n we take a disk of radius $R = |b_n/2|$ and denote by p_n the Taylor polynomial of order m_n at zero of $P_n((z-b_n)^{-1})$. We take $|z| < R/2$ and use integration on the circle of radius R to obtain from (6.7)

$$\left|P_n((z-b_n)^{-1}) - p_n(z)\right| \leqslant C_n 2^{-m_n-1} \tag{36.15}$$

where C_n only depends on the maximum of $|P_n|$ on the circle of radius R. Thus, we can choose m_n so that

$$\left|P_n((z-b_n)^{-1}) - p_n(z)\right| \leqslant 2^{-n}; \quad \forall z \text{ s.t. } |z| < |b_n/4| \tag{36.16}$$

Now we look at the series

$$f_1 = \sum_{n=1}^{\infty} \Big[P_n((z-b_n)^{-1}) - p_n(z)\Big] \tag{36.17}$$

and fix an R and analyze the series for $z \in \mathbb{D}_R$. We split the sum (36.16) into two parts:

(36.18)
$$f_1 = \sum_{n:|b_n|\leqslant 4R} \Big[P_n((z-b_n)^{-1}) - p_n(z)\Big] + \sum_{n:|b_n|>4R} \Big[P_n((z-b_n)^{-1}) - p_n(z)\Big]$$

The first sum is finite, while for the second one we have $|z| < |b_n/4|$ and the estimate (36.16) applies. Thus (36.17) is convergent away from the poles uniformly on compact sets. Clearly, $f - f_1$ is entire.

(i) We note that, by the same arguments, the function

$$h(z) = \sum_{n\in\mathbb{N}} P_n((z-b_n)^{-1}) - p_n(z) \tag{36.19}$$

constructed in (ii) is analytic in $\mathbb{C} \setminus \{b_n\}_{n\in\mathbb{N}}$ and has the required singular Laurent part. ∎

36.4. **Further examples.** Lets' look at $f(z) = \pi\cot(\pi z)$. This function has simple poles with residue 1 in $\mathbb{Z}$ and is analytic in $\mathbb{C} - \mathbb{Z}$. If to each term the series of Laurent polynomials,

$$\sum \frac{1}{z-n}$$

we add $1/n$ we get a convergent expansion, and thus

(36.20)
$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n(z-n)} + E(z)$$

with E entire. We could estimate the behavior at infinity as we did for $\pi^2/\sin^2(\pi z)$, but we an also find E by relating the two expansions: Note that

$$\pi(\cot \pi z)' = -\frac{\pi^2}{\sin^2 \pi z} \tag{36.21}$$

On the other hand, the series in (36.20) converges uniformly in $\mathbb{C} \setminus \mathbb{Z}$ and, by Weierstrass's theorem can be differentiated termwise.

We get, using (36.12),

$$S'(z) = -\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(z-n)^2} = \pi(\cot \pi z)' + \frac{1}{z^2} \tag{36.22}$$

and thus

$$\pi \cot \pi z = C + \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n}\right) \tag{36.23}$$

Combining pairwise the term with n with the term with $-n$ we get

$$\pi \cot \pi z = C + \frac{1}{z} + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{2z}{z^2 - n^2} \tag{36.24}$$

since the left side is odd, we must have $C = 0$ and thus

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{2z}{z^2 - n^2} \tag{36.25}$$

We can now use this identity to calculate easily some familiar sums. Note that the lhs of (36.25) has the Laurent expansion at $z = 0$

$$\pi \cot \pi z = \frac{1}{z} - \frac{\pi^2 z}{3} - \frac{\pi^4 z^3}{45} - \frac{2\pi^6 z^5}{945} - \cdots \tag{36.26}$$

Since the series on the rhs of (36.25) converges uniformly near $z = 0$, by Weierstrass's theorem it converges together with all derivatives. On the other hand we have

$$\frac{2z}{z^2 - n^2} = -2\left(\frac{z}{n^2} + \frac{z^3}{n^4} + \frac{z^5}{n^6} + \cdots\right) \tag{36.27}$$

and we get immediately,

$$\sum_{n\geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n\geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n\geqslant 1} \frac{1}{n^6} = \frac{\pi^6}{945} \cdots \tag{36.28}$$

Exercise 36.3. ** The definition of the Bernoulli numbers B_k is*

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1} \tag{36.29}$$

Show that

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=2^{2k-1}\frac{B_k}{(2k)!}\pi^{2k} \tag{36.30}$$

Also based on (36.25), or using the same approach as for (36.26), it is not difficult to show that

$$\frac{\pi}{\sin \pi z}=\lim_{m\to\infty}\sum_{n=-m}^{m}\frac{(-1)^n}{z-n}=\frac{1}{z}+\sum_{m=1}^{\infty}(-1)^m\frac{2z}{z^2-m^2} \tag{36.31}$$

giving a precise meaning to (36.13).

37. Infinite products

An infinite product is the limit

$$\prod_{n=1}^{\infty}p_n:=\lim_{k\to\infty}\prod_{n=1}^{k}p_n=\lim_{k\to\infty}\Pi_k \tag{37.1}$$

We adopt here the convention of existence of a nontrivial limit used in [3]. Evidently, if one of the factors is zero, the infinite product would be zero regardless of the behavior of the other terms. On the other hand, we will be able to express analytic functions as infinite products, and we should allow them to vanish. Then (37.1) is said to converge *iff only finitely many terms p_n are zero, and the rest of the product has a finite nonzero limit*. Omitting the zero factors and writing $p_n = P_n/P_{n-1}$, $P_0 = 1$ we see that the limit of Π_k is the same as the limit of the P_k, and thus $p_n \to 1$ is a necessary condition of convergence of the infinite product. We should then better write the products as

$$\prod_{n=1}^{\infty}(1+a_n) \tag{37.2}$$

and then a necessary condition of convergence is $a_n \to 0$.

Theorem 37.1. *The infinite product (37.2) converges iff*

$$\sum_{n=1}^{\infty}\ln(1+a_n) \tag{37.3}$$

converges. We use the principal branch of the log, extended by continuity when $\arg(z)\uparrow\pi$ *and omit, as before, the terms with* $a_n = -1$.

Proof. If the sum (37.3) converges, then P_n converges, since the exponential of a finite sum is a finite product.

In the opposite direction, a word of caution. We know that in the complex domain, $\ln ab$ is not always $\ln a + \ln b$. The limit of the sum will not, in general, be the log of the infinite product. So the reasonning is not that obvious.

Let now n_0 be large enough so that for all $n > n_0$, $|a_n| < \varepsilon$ and $|P_n - 1| < 1/4$. We first eliminate from the sum (and from the product) the first n_0 terms, since they do not contribute to convergence. Then, by crude estimates, we see that $\arg(1+a_n)$, $\arg(P_n)$ and $\arg((1+a_n)P_n)$ are all $< \pi/2$, and thus for all $n > n_0$ we have $\ln(P_{n+1}) = \ln(P_n)+\ln(1+a_n)$. The rest is immediate. ∎

Absolute convergence is easier to control in terms of series. An infinite product is absolutely convergent, by definition, iff

$$\sum_{n=1}^{\infty} |\ln(1 + a_n)| \tag{37.4}$$

is convergent.

Theorem 37.2. *The sum (37.4) is absolutely convergent iff $\sum a_k$ is absolutely convergent.*

Proof. Assume $\sum a_k$ converges absolutely. Then in particular $a_n \to 0$. Also, if $\sum_{n=1}^{\infty} \ln(1 + a_n)$ converges absolutely then $\ln(1 + a_n) \to 0$ and $a_n \to 0$. But (eliminating all the irrelevant zero terms which are zero) we have, as $n \to \infty$ $\lim_{n\to\infty} |a_n|^{-1} \ln(1 + |a_n|) = 1$, and the rest follows from the limit ratio theorem. ∎

Note 37.3. *Conditional (not absolute) convergence of $\sum a_n$ and of $\prod(1 + a_n)$ are unrelated notions. (Consider, e.g., the product $\prod(1 - (-1)^n n^{-1/2})$. Is the associated series $\sum(-1)^n n^{-1/2}$ convergent? Is the product convergent?)*

37.1. **Uniform convergence of products.**

Exercise 37.4. *** Assume that $p_n(z)$ are analytic in the region Ω and $f(z) = \prod_{n\geqslant 1} p_n(z)$ converges absolutely and uniformly on every compact set in the region Ω. Show that f is analytic in Ω. Show that*

$$f'(z) = \sum_{k=1}^{\infty} \prod_{n=1}^{\infty} \frac{p_k'}{p_k} p_n \tag{37.5}$$

where the sum is also uniformly convergent. Hint: Use Weierstrass's theorem.

37.2. **Example: the** sin **function.** We note that the zeros of $\sin \pi z$ are at the integers and we would like to write sin in terms of the products of these roots. We can start with (36.25) and note that

$$\ln(C\pi \sin \pi z)' = \pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{N}\setminus\{0\}} \frac{2z}{z^2 - n^2} \tag{37.6}$$
$$= (\ln z)' + \sum_{n \in \mathbb{N}\setminus\{0\}} \ln(z^2 - n^2)'$$

so we end up with the *formal* identity

$$C\pi \sin \pi z \overset{?}{=} z \prod_{n>0} C_n(z^2 - n^2) \tag{37.7}$$

(it is formal because, in principle, we are not allowed to combine the logs the way we did) where the constants C_n need to be chosen so that the product is convergent. Except for ensuring convergence the C_n are immaterial, since we already have one on the left side a free constant. A good choice is $C_n = -n^{-2}$ which gives us the tentative identity

$$C\pi \sin \pi z \overset{?}{=} z \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{37.8}$$

The constant C can only be $1/\pi^2$ if we look at the behavior near $z = 0$. Thus,

$$\frac{\sin \pi z}{\pi} \overset{?}{=} z \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{37.9}$$

This equality of course needs to be proved, but this is not difficult; the proof can be done in the same way as we proved equalities stemming from partial fractions decompositions.

First we note that the product on the rhs of (37.9) is absolutely and uniformly convergent on any compact z set; this can be easily checked. It thus defines an entire function $g(z)$. Motivated by the way we obtained this possible identity, let us look at the expression $f'/f - g'/g$ where $\pi f(z) = \sin \pi z$. We get, using Exercise 37.4,

$$f'/f - g'/g = \pi \cot \pi z - \frac{1}{z} + \sum_{n \in \mathbb{N}\setminus\{0\}} \frac{2z}{z^2 - n^2} = 0 \tag{37.10}$$

This means that

$$\frac{f'g - fg'}{fg} = 0 \tag{37.11}$$

in $\mathbb{C} \setminus \mathbb{Z}$ or, equivalently,

$$\frac{f'g - fg'}{g^2} = 0 = \left(\frac{f}{g}\right)' \tag{37.12}$$

or $f/g = const$; we already calculated the constant based on the behavior at zero, it is one. Thus indeed,

$$\frac{\sin \pi z}{\pi z} = \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{37.13}$$

proving (36.5).

How general is this decomposition possible? Again, if we are careful about convergence issues it is perfectly general. This is what we are going to study in the next subsection.

37.3. **Canonical products.** The simplest possible case is that in which we have a function with no zeros.

Theorem 37.5. *Assume f is entire and $f \neq 0$ in $\mathbb{C}$ Then f is of the form*

$$f = e^g \tag{37.14}$$

where g is also entire.

Proof. Since f'/f is entire and $\mathbb{C}$ is simply connected, $h(z) = \int_0^z f'(s)/f(s)ds$ is well defined and also entire. Now we note that $(fe^{-g})' = 0$ in $\mathbb{C}$ and thus $f = \exp(h + C)$ proving the result. Another proof is by using the monodromy theorem and the fact that $\log f$ has no singularities in $\mathbb{C}$. ∎

Assume now that f has finitely many zeros, a zero of order $m \geqslant 0$ at the origin, and the nonzero ones, possibly repeated are $a_1, ...a_n$.

Then

$$f = z^m \prod_{k=1}^{n} \left(1 - \frac{z}{a_n}\right) e^{g(z)}$$

where g is entire.

This is clear, since if we divide f by the prefactor of e^g we get an entire function with no zeros.

We cannot expect, in general, such a simple formula to hold if there are infinitely many zeros. Again we have to take care of convergence problems. This is done in a manner similar to that used in the Mittag-Leffler construction.

Theorem 37.6 (Weierstrass). *(i) If $(a_n)_{n\in S\subset\mathbb{N}}$ is a sequence with no accumulation point, then there exists an entire function with zeros at a_n and no other zeros.*

(ii) Assume f is an entire function with zeros at a_n. Then there exist integers m, m_n and an entire function $g(z)$ such that

$$f(z) = e^{g(z)} z^m \prod \left[\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \right] \tag{37.15}$$

Proof. This is a consequence of Mittag-Leffler. Indeed, note that if f is entire, then f'/f is meromorphic, with *simple poles* at every zero b_n of f, and the residue is the order of the zero. Then in the representation (36.14), $P_n((z-b_n)^{-1}) = m_n(z-b_n)^{-1})$ where $m_n \in \mathbb{N}$, giving

$$\frac{f'}{f} = \sum_{n\in\mathbb{N}} \left[\frac{m_n}{z-b_n} - p_n(z) \right] + g(z) := S(z) + g(z) \tag{37.16}$$

The fact that (37.15) holds is a simple exercise in integrating (37.16). Note that we can always take here (but not in Theorem 36.2) $m_n = n$ if we order the roots according to size, since the multiplicity of a_n cannot exceed n. ▮

Corollary 37.7. *Any meromorphic function is a ratio of entire functions.*

Proof. Let F be meromorphic with poles at b_n of order m_n. Let G be any entire function with zeros at b_n of order m_n. Then FG has only removable singularities. ▮

37.4. **Counting zeros of analytic functions. Jensen's formula.** The rate of growth of an analytic function is closely related to the density of zeros. We have a quite effective counting theorem, due to Jensen.

Theorem 37.8 (Jensen). *Assume $f \not\equiv 0$ is analytic in the closed disk $\overline{\mathbb{D}_r}$ and $f(z) = cz^m g(z)$ with $m \geqslant 0$ and $g(0) = 1$. Let a_i be the nonzero roots of f in $\mathbb{D}_r$, repeated according to their multiplicity. Then*

$$\ln|c| = -m\ln r - \sum_{i=1}^{n} \ln\left(\frac{r}{|a_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})| d\theta \tag{37.17}$$

Proof. The proof essentially boils down to the case where $f(0) \neq 0$ and it has no zeros inside the disk of radius r. In this simple case, a consistent branch of $\ln f$ can be defined inside $\mathbb{D}_r$ (see also the second proof of Theorem 37.5), and $\mathrm{Re}\ln f = \ln|f|$ is harmonic in $\mathbb{D}_r$. For

$r' < r$ we have be the mean value theorem for harmonic functions we have,

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r'e^{i\theta})|d\theta \tag{37.18}$$

Since f is analytic in the closed disk and $\ln|x|$ is in $L^1(\mathbb{R})$, it is easy to see by dominated convergence (check) that (37.18) holds in the limit $r = r'$ too, even if there are zeros on the circle of radius r:

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \tag{37.19}$$

Assume now f has zeros, with the convention in the statement of the theorem. We then build a function which has no zeros inside $\mathbb{D}_r$ and has the same absolute value for $|z| = r$. Such a function is

$$h(z) = \frac{r^m}{z^m} f(z) \prod_{i=1}^{n} \frac{r^2 - \overline{a_i} z}{r(z - a_i)} \tag{37.20}$$

Then

$$\ln|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|h(re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \tag{37.21}$$

The formula now follows by expanding out $\ln|h(0)|$. ∎

Corollary 37.9. *Assume f is analytic in the closed disk of radius R and $f(0) \neq 0$. Let $\nu(r)$ denote the number of zeros of f in the disk of radius $r \leqslant R$. Then*

$$\int_0^R \frac{\nu(x)}{x} dx \leqslant \ln \max_{|z|=R} |f(z)| - \ln|f(0)| \tag{37.22}$$

Of course, $\nu(x)$ is an increasing discontinuous function of x.

Proof. Note that

$$\ln(R/|a_i|) = \int_{|a_i|}^R \frac{dx}{x} = \int_0^R \chi_{[|a_i|,R]}(x)\frac{dx}{x}$$

Thus

$$\sum_{i=1}^n \ln\left(\frac{r}{|a_i|}\right) = \int_0^R \sum_{i=1}^n \chi_{[|a_i|,R]}(x)\frac{dx}{x} = \int_0^R \frac{\nu(x)}{x} dx$$

The rest follows immediately from (37.17). ∎

37.5. **Entire functions of finite order.** Let f be an entire function. We denote by $\|f\|_R$ the maximum value of $|f(z)|$ for $|z| \leqslant R$, or which is the same, $\|f\|_R$ is the maximum value of $|f(z)|$ for $|z| = R$. A function is of **order** $\leqslant \rho$ if for any $\varepsilon > 0$ there is some $c > 0$ such that for all R large enough we have

$$\|f\|_R \leqslant e^{cR^{\rho+\varepsilon}} \tag{37.23}$$

or equivalently

$$\ln \|f\|_R = O(R^{\rho+\varepsilon}) \tag{37.24}$$

Note 37.10. *We can always check the condition for $R \in \mathbb{N}$ large enough since $(N+1)^\rho = O(N^\rho)$.*

The function f has **strict order** $\leqslant \rho$ if there is some $c > 0$ such that for all R large enough we have

$$\|f\|_R \leqslant e^{cR^\rho} \tag{37.25}$$

A function has **order equal** ρ if ρ is the inf of ρ' s.t. (37.23) holds. A function has **strict order equal** ρ if ρ is the inf of ρ' s.t. (37.25) holds.

Example 37.11. *Assume $f(z)$ is entire, and for large $|z|$ there are positive constants C, c and ρ such that $|f(z)| \leqslant Ce^{c|z|^\rho}$. $R > 0$ there is a $c_2 > 0$ such that we have*

$$\nu(R) \leqslant c_2 R^\rho \tag{37.26}$$

Indeed, the zeros at zero do not change the shape of the identity, and we can thus assume $f(0) \neq$. Then,

$$c|R|^\rho \geqslant \int_0^R \frac{\nu(x)}{x} dx \geqslant \int_{R/2}^R \frac{\nu(x)}{x} dx \geqslant \frac{\nu(R/2)}{R} \frac{R}{2} \tag{37.27}$$

and the rest is immediate. The constants in the inequality can be optimized by choosing R/τ, $\tau > 1$ instead of $R/2$ and finding the best τ.

Theorem 37.12. *Let f be entire of strict order $\leqslant \rho$ and let $\{z_n\}$ be its nonzero zeros, repeated according to their multiplicity and ordered increasingly by their absolute value. Then for any $\varepsilon > 0$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\rho+\varepsilon}} \tag{37.28}$$

is convergent.

Proof. We can obviously discard the roots with $|z_i| \leqslant 1$ which are in finite number. Without loss of generality we assume there are none. We have, with $N \in \mathbb{N}$ and estimating the sum by annuli,

$$\sum_{|z_n|\leqslant N} \frac{1}{|z|^{\rho+\varepsilon}} \leqslant \sum_{k=1}^{N} \frac{\nu(k+1)-\nu(k)}{k^{\rho+\varepsilon}} \tag{37.29}$$

we can now use the method of Abel summation by parts. We write

(37.30)
$$\frac{\nu(k+1)-\nu(k)}{k^{\rho+\varepsilon}} = \nu(k+1)\left(\frac{1}{k^{\rho+\varepsilon}} - \frac{1}{(k+1)^{\rho+\varepsilon}}\right) + \left(\frac{\nu(k+1)}{(k+1)^{\rho+\varepsilon}} - \frac{\nu(k)}{k^{\rho+\varepsilon}}\right)$$

and note that by summation, the terms in the last parenthesis cancel out to

$$\frac{\nu(N+1)}{(N+1)^{\rho+\varepsilon}} - \nu(1)$$

Note that by usual calculus we have for some $\gamma = \gamma(k)$

$$\frac{\nu(k+1)}{k^{\rho+\varepsilon}} - \frac{\nu(k+1)}{(k+1)^{\rho+\varepsilon}} = \frac{(\rho+\varepsilon)\nu(k+1)}{(k+\gamma)^{\rho+\varepsilon+1}} \leqslant \frac{Ck^{\rho}(\rho+\varepsilon)}{k^{\rho+\varepsilon+1}} \tag{37.31}$$

and the sum converges. ∎

37.6. **Estimating analytic functions by their real part.**

Theorem 37.13 (Borel-Carathéodory)**.** *Let $f = u + iv$ be analytic in a closed disk of radius R. Let $A_R = \max_{|z|=R} u(z)$. Then for $r < R$ we have*

$$\max_{|z|\leqslant r} |f(z)| \leqslant \frac{2rA_R}{R-r} + \frac{R+r}{R-r}|f(0)| \tag{37.32}$$

Note that if f is entire, say, as $z \to \infty$ we have $|f| \to \infty$ then, since $|u| \leqslant |f|$, the theorem above shows that $\max|f|/\max u$ in a disk of radius R is $\leqslant 2$ as $R \to \infty$.

Proof. Assume first that $f(0) = 0$. Then $u(0) = 0$ and by the mean value theorem $A_R \geqslant 0$. If $A_R = 0$ then by the same argument $u \equiv 0$ on $\partial\mathbb{D}_R$ and by Poisson's formula $u \equiv 0$ in $\mathbb{D}_R$. Then $v \equiv \mathit{const} = 0$ since $f(0) = 0$, thus $f \equiv 0$ and the formula holds trivially.

We now take $A_R > 0$. Since the maximum of a harmonic function is reached on the boundary, we have $2A_R - u \geqslant u$ in $\mathbb{D}_R$ and the inequality is strict in the interior. Also note that if at some point $u < 0$, then again $2A_R - u = 2A_R + |u| \geqslant |u|$. The function

$$g(z) = \frac{1}{2A_R - f(z)} \frac{f(z)}{z} \tag{37.33}$$

is holomorphic in $\mathbb{D}_R$ and on the disk of radius R we have

$$|2A_R - f| = \sqrt{(2A_R - u)^2 + v^2} \geqslant \sqrt{u^2 + v^2} = |f| \tag{37.34}$$

and thus in $\mathbb{D}_R$ we have

$$|g(z)| = \left| \frac{1}{2A_R - f(z)} \frac{f(z)}{z} \right| \leqslant \frac{1}{R} \tag{37.35}$$

hence

$$\left| \frac{f(z)}{z} \right| \leqslant \frac{1}{R} |2A_R - f(z)| \leqslant \frac{1}{R}(2A_R + |f(z)|) \tag{37.36}$$

Solving for $|f(z)|$ we get

$$|f(z)| \leqslant \frac{2|z|A_R}{R - |z|} \tag{37.37}$$

as claimed. The general case is obtained by applying this inequality to $f(z) - f(0)$ (exercise). ▮

Corollary 37.14. *Assume $\rho \geqslant 0$, $f = u + iv$ is entire and as $|z| \to \infty$ we have*

$$|u(z)| \leqslant C|z|^\rho \tag{37.38}$$

Then f is a polynomial of degree at most ρ.

Proof. Let $R = 2|z|$. We have, from Theorem 37.13 for large $r = |z|$,

$$|f(z)| \leqslant \frac{2Crr^\rho}{r} + 3|f(0)| \leqslant C'r^\rho \tag{37.39}$$

The rest is standard. ▮

38. Hadamard's theorem

Let $\rho > 0$ and let k_ρ be the smallest integer *strictly* greater than ρ, $k_\rho = \lfloor \rho \rfloor + 1$. We consider again the truncates of the series of $-\ln(1-z)$, namely, with $k = k_\rho$,

$$P_k(z) = z + \frac{z^2}{2} + \cdots + \frac{z^{k-1}}{k-1} \tag{38.1}$$

Theorem 38.1 (Hadamard)**.** *Let f be entire of order ρ, let z_n be its nonzero zeros and let $k = k_\rho$. Then, with $m \geqslant 0$ the order of the zero of f at zero, there is a polynomial h of degree $\leqslant \rho$ such that*

$$f(z) = e^{h(z)} z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{P_k(z/z_n)} = e^{h(z)} E(z) \tag{38.2}$$

The proof of this important theorem requires a number of intermediate results, notably the *minimum modulus principle* proved in the following section, a very useful result in its own right.

Lemma 38.2. *Let ε be such that $\lambda := \rho + \varepsilon < k_\rho := k$. There is a $c > 0$ such that*

$$|(1-\zeta)\exp P_k(\zeta)| \leqslant \exp(c|\zeta|^\lambda) \tag{38.3}$$

Proof. For $|\zeta| \leqslant 1/2$ we have

$$\ln(1-\zeta) + P_k(\zeta) = \sum_{n=k}^{\infty} \frac{\zeta^n}{n} = \zeta^k C_k; \ |C_k| \leqslant \sum_{n=k}^{\infty} 2^{-n} \leqslant 2 \tag{38.4}$$

$$\Rightarrow (1-\zeta)e^{P_k(\zeta)} \leqslant e^{2|\zeta|^k} \leqslant e^{2|\zeta|^\lambda} \tag{38.5}$$

For $|\zeta| \in [1/2, 1]$ we have

$$\begin{aligned} |(1-\zeta)\exp P_k(\zeta)| &\leqslant \frac{1}{2} \exp\left[|\zeta|^k \left(\frac{1}{|\zeta|^{k-1}} + \cdots + \frac{1}{|\zeta|(k-1)}\right)\right] \\ &\leqslant \frac{1}{2}\exp(2^k|\zeta|^k) \leqslant \frac{1}{2}\exp(2^k|\zeta|^\lambda) \end{aligned} \tag{38.6}$$

For $|\zeta| > 1$ we have

(38.7)

$$|(1-\zeta)\exp P_k(\zeta)| \leqslant |(1-\zeta)| \exp\left[|\zeta|^{k-1}\left(\frac{1}{k-1} + \cdots + \frac{1}{|\zeta|^{k-2}}\right)\right]$$

$$\leqslant \exp\left(k|\zeta|^{k-1} + \ln|1+|\zeta||\right) \leqslant \exp\left(k|\zeta|^\lambda + \ln|1+|\zeta||\right) \leqslant \exp\left(C_2|\zeta|^\lambda\right)$$

for some C_2 independent of $\zeta, |\zeta| > 1$. This is because $t^{-\lambda}\ln(1+t)$ is continuous on $[1,\infty)$ and goes to zero at infinity (fill in the details). ∎

38.1. **Canonical products.** Take any sequence $\{z_n\}_n$ where the terms are ordered by absolute value, with the property that for some $\rho > 0$ and any $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\rho+\varepsilon}} < \infty \tag{38.8}$$

Definition 38.3. *The canonical product determined by the sequence* $\{z_n\}$*, denoted by* $E^{(k)}(z, \{z_n\})$ *or simply* $E(z)$ *is defined by*

$$E(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp[P_k(z/z_n)] \tag{38.9}$$

Theorem 38.4. $E(z)$ *is an entire function of order* $\leqslant \rho$.

Proof. Take again any ε be such that $\lambda := \rho + \varepsilon < k_\rho$. Then, by Lemma 38.2 we have

(38.10)
$$|E(z)| \leqslant \prod_{n=1}^{\infty} \exp(c|z/z_n|^\lambda) = \exp\left(c|z|^\lambda \sum_{k=1}^{\infty} |z_n|^{-\lambda}\right) \leqslant \exp\left(c_1|z|^\lambda\right)$$

proving, in the process, uniform convergence of the product. ∎

39. The minimum modulus principle; end of proof of Theorem 38.1

This important theorem tells us, roughly, that if a function does not grow too fast it cannot decrease too quickly either, aside from zeros. More precisely we have

Theorem 39.1 (Minimum modulus theorem)**.** *Let* f *be an entire function of order* $\leqslant \rho$. *As before, let* $\{z_n\}$ *be its zeros with* $|z_i| > 1$*, repeated according to their multiplicity and let* $\varepsilon > 0$. *At every root, take out a disk* $D(z_n, r_n)$ *with* $r_n = |z_n|^{-\rho-\varepsilon}$ *and consider the complement* U *in* $\mathbb{C}$ *of these disks. Then in* U*, for large* r *there is a constant* c *such that*

$$|f(z)| \geqslant \exp(-c|z|^{\rho+\varepsilon}) \quad \text{or} \quad \frac{1}{|f(z)|} = O(\exp(|z|^{\rho+\varepsilon})) \tag{39.1}$$

Proof. We start with the case when the entire function is a canonical product. We take $|z| = r$ and write

$$E(z) = \prod_{|z_n|<2r} E_k(z, z_n) \prod_{|z_n|\geqslant 2r} E_k(z, z_n) \tag{39.2}$$

and estimate the two terms separately. We note that in the second product, all ratios $\zeta =: \zeta_n = z/z_n$ have the property $|\zeta| \leqslant 1/2$. Taking one term of the product, we have to estimate *below*

$$E(\zeta) = (1 - \zeta)e^{P(\zeta)} \tag{39.3}$$

Since $|\zeta| \leqslant 1/2$, $\ln(1-\zeta)$ exists; we take the principal branch and write

$$(39.4)\quad \left|(1-\zeta)e^{P(\zeta)}\right| = \left|e^{\ln(1-\zeta)+P(\zeta)}\right| = \left|\exp\left(-\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right)\right|$$
$$= \exp\left(-\mathrm{Re}\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right) \geqslant \exp\left(-\left|\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right|\right) \geqslant \exp\left(-\sum_{n=k}^{\infty}\frac{|\zeta|^n}{n}\right)$$
$$\geqslant \exp\left(-|\zeta|^k\sum_{n=0}^{\infty}\frac{(1/2)^n}{n+k}\right) \geqslant e^{-2|\zeta|^k} \geqslant e^{-2|\zeta|^{\rho+\varepsilon}}$$

Thus for the second product in (39.2) we have

$$(39.5)\qquad \prod_{|z_n|\geqslant 2r} E_k(z,z_n) \geqslant \exp\left(-2|z|^{\rho+\varepsilon}\sum_{|z_n|\geqslant 2r}\frac{1}{|z_n|^{\rho+\varepsilon}}\right) \geqslant e^{-c|z|^{\rho+\varepsilon}}$$

since the infinite sum converges by Theorem 37.12.

We split the remaining region into $z_k \in [1,r]$ and $z_k \in (r,2r)$. Here the factors $(1 - z/z_k)$ have to be bounded from below, and it is here that we use the conditions on the removed disks. We have, on $[1,r]$,

(39.6)
$$\prod_{|z_n|\leqslant r} |1 - z/z_n| = \frac{|z-z_n|}{|z_n|} \geqslant \prod_{|z_n|\leqslant r} |z_n|^{-\rho-\varepsilon-1} \geqslant \prod_{|z_n|\leqslant r} r^{-\rho-\varepsilon-1} =$$
$$\left(r^{-\rho-\varepsilon-1}\right)^{\nu(r)} \gtrsim e^{-r^{\rho+\varepsilon}\ln r(\rho+\varepsilon+1)} \geqslant e^{-c_1 r^{\rho+\varepsilon'}}$$

for some C_1 and $\varepsilon' > \varepsilon$, since $(r^{\varepsilon-\varepsilon'}\ln r \to 0$ as $r\to\infty$.

On $(r,2r)$ we have

$$(39.7)\qquad |1 - z/z_n| = \frac{|z-z_n|}{|z_n|} \geqslant |z_n|^{-\rho-\varepsilon-1} \geqslant (2r)^{-\rho-\varepsilon-1}$$

and thus

(39.8)
$$\prod_{|z_n|<2r} |1 - z/z_n| \geqslant [(2r)^{-\rho-\varepsilon-1}]^{\nu(2r)} = e^{-\nu(2r)(\rho+\varepsilon+1)\ln(2r)} \geqslant e^{-c_6 r^{\rho+\varepsilon'}}$$

for any $\varepsilon' > \varepsilon$ if r is large enough.

We now examine the convergence improving factors, for $|z_n| < 2r$.

$$(39.9)\qquad \left|\sum_{|z_n|<2r} P_k(z/z_n)\right| \leqslant \left|\sum_{r<|z_n|<2r} P_k(z/z_n)\right| + \left|\sum_{|z_n|\leqslant r} P_k(z/z_n)\right|$$

For the first term on the right we note that when $|z/z_n| = |\zeta_n| =: |\zeta| < 1$ and we have

$$\left|\sum_{n=1}^{k-1} \frac{\zeta^n}{n}\right| \leqslant \sum_{n=1}^{k-1} \frac{1}{n} =: c_1 \tag{39.10}$$

and thus

$$\sum_{r<|z_n|<2r} |P_k(z/z_n)| \leqslant \nu(2r)c_1 \leqslant c_2 r^{\rho+\varepsilon} \tag{39.11}$$

For the second term on the right of (39.9) we note that $|z/z_n| \geqslant 1$ and thus, with $\zeta = z/z_n$ we have

$$\left|\sum_{n=1}^{k-1} \frac{\zeta^n}{n}\right| \leqslant |\zeta|^{k-1} \sum_{n=1}^{k-1} \frac{1}{n} =: c_1|\zeta|^{k-1} = c_1 r^{k-1}|z_n|^{-k+1} \tag{39.12}$$

We use Abel summation by parts (we are careful that r is not necessarily an integer)

$$\begin{aligned}\sum_{|z_n|\leqslant r} |z_n|^{-k+1} &\leqslant \sum_{m\leqslant r} \frac{\nu(m+1)-\nu(m)}{m^{k-1}} \\ &= \sum_{m\leqslant r} \nu(m+1)\left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}}\right) + \frac{\nu(r+1)}{r^{k-1}} - \nu(1) \\ &\leqslant \sum_{m\leqslant r} \nu(m+1)\left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}}\right) + \frac{\nu(r+1)}{r^{k-1}} \\ &\leqslant \sum_{m\leqslant r} \frac{kCm^{\rho+\varepsilon}}{m^k} + c_3 r^{\rho+\varepsilon-k+1} \\ &\leqslant C_1 \sum_{m\leqslant r} m^{\rho+\varepsilon-k} + c_3 r^{\rho+\varepsilon-k+1} \leqslant C_3 r^{\rho+\varepsilon-k+1}\end{aligned} \tag{39.13}$$

where we majorized the sum by an integral in the usual way. Multiplying by $c_1 r^{k-1}$ we get that the second term on the right of (39.9) is bounded by

$$C_4 r^{\rho+\varepsilon} \tag{39.14}$$

We now finish the proof of Theorem 38.1.

Proof. We take $\varepsilon > 0$ and $s = \rho+\varepsilon$. We order the roots nondecreasingly by $|z_n|$. For each root z_n we consider the annulus $A_n = \{z : |z| \in [|z_n| - 2|z_n|^{-s}, |z_n| + 2|z_n|^{-s}]$ (for the purpose of this argument, we can as well assume that all roots are on $\mathbb{R}^+$, since the angular position is irrelevant). Consider $J^c := \mathbb{R}^+ \setminus J$ where J is the union of all intersections of the A_n with $\mathbb{R}^+$. Since the Lebesgue measure of J does not exceed $4\sum_n |z_n|^{-s} < \infty$, there exist arbitrarily large numbers in the complement J^c. Take r be any number in J^c and consider the circle $\partial\mathbb{D}_r$. Consider the function $g = f/E$. g is clearly an entire function with no zeros. Then, by Theorem 37.5, $g = e^h$ with h entire. Since $\operatorname{Re} h \leqslant (C_f + C_E)r^{\rho+\varepsilon}$ for some $C_f + C_E$ independent of r in $\mathbb{D}_r$ for arbitrarily large r (check), we have by Corollary 37.14 that h is a polynomial of degree at most $\rho + \varepsilon$. Since ε is arbitrary, h is a polynomial of degree at most ρ. ∎

To finish the proof of the minimum modulus principle, we use Hadamard's theorem and the fact that e^{-h} satisfies the required bounds. (Exercise: fill in the details.) ∎

Example 39.2. Let us show that $f(z) = e^z - z$ has infinitely many roots in $\mathbb{C}$. Indeed, first note that $f(z)$ has order 1 since $|z| \leqslant e^{|z|}$ for all z. Suppose f had finitely many zeros. Then

$$e^z - z = P(z)e^{h(z)} \tag{39.15}$$

where $P(z)$ is a polynomial and $h(z)$ is a polynomial of degree one, and without loss of generality we can take $h(z) = cz$, $c = \alpha + i\beta$. As $z = t \to +\infty$ we have

$$P(t)e^{(c-1)t} = 1 - te^{-t} \to 1 \tag{39.16}$$

In particular $|P(t)|e^{(\alpha-1)t} \to 1$ which is only possible if $\alpha = 1$. But then $|P(t)| \to 1$ which is only possible if $P(t) = const = e^{i\phi}$. We are then left with

$$e^{i(\phi+t\beta)} = \cos(t\beta + \phi) + i\sin(t\beta + \phi) \to 1 \quad \text{as} \quad t \to +\infty \tag{39.17}$$

which clearly implies $\beta = 0$. Then $e^{i\phi} = 1$. We are left with the identity

$$e^z - z = e^z \ \forall z \tag{39.18}$$

which is obviously false.

Exercise 39.3. ** Let $P \not\equiv 0$ be a polynomial. Show that the equation $e^z = P(z)$ has infinitely many roots in $\mathbb{C}$.*

Exercise 39.4. *** (i) Rederive formula (37.13) using Hadamard's theorem.*

(ii) Write down a product formula of the function

$$f(z) = \sin z + 3\sin(3z) + 5\sin(5z) + 7\sin(7z)$$

The final formula should be explicit except for arcsins of roots of a cubic polynomial.

39.1. **Some applications.**

Corollary 39.5 (Borel)**.** *Assume that ρ is not an integer and f has order strictly ρ. Then f takes every value in $\mathbb{C}$ infinitely many times.*

Proof. It suffices to show that such a function has infinitely many zeros, since f and $f - z_0$ have the same strict order. Assume to get a contradiction f had finitely many zeros. Then $g(z) = f(z)\prod_{i=1}^{n}(z - z_i)^{-1}$ would be entire, with no zeros, and as it is easy to check, of order strictly ρ. Then $g = e^h$ with h a polynomial whose degree can only be an integer. ∎

Let $\exp^{(n)}$ be the exponential composed with itself n times.

Corollary 39.6 (A weak form of Picard's theorem)**.** *A nonconstant entire function which is bounded by $\exp^{(n)}(C|z|)$ for some n and large z takes every value with at most one exception.*

Proof. We prove this by induction on n. We first show that a nonconstant entire function of finite order takes every value with at most one exception. Assume a is an exceptional (*lacunary*) value. Then $f(z) - a$ is entire with no zeros, thus of the form e^h with h a polynomial, $f = e^h - a$. If the degree of h is zero, then f is a constant. Otherwise, we must show that $e^h - a$ takes all values with at most one exception ($-a$ of course), or, which is the same, e^h takes all values with at most one exception. The equation $e^h = b$, $b \neq 0$ is solved if $h - \ln b$ has roots, which is true by the fundamental theorem of algebra.

Assume now the property holds for $n \leqslant k-1$ and we wish to prove it for $n = k$. Let f be an entire function bounded by $\exp^{(n)}(C|z|)$ which avoids the value a. Then $f - a$ is entire with no zeros, $f - a = e^h$ with h entire. It is easy to show that h is bounded by $\exp^{(n-1)}(C|z|)$. Thus it avoids at most one value, by the induction hypothesis. The equation $e^h - a = b$, for $b \neq -a$ always has a solution. Indeed, if $\ln(b-a)$ is not an avoided value of h this is obvious. On the other hand, if $\ln(b-a)$ is avoided by h, then again by the induction hypothesis $\ln(b-a) + 2\pi i$ is not avoided. ∎

Exercise 39.7. ** Show that the equation

$$\cos(z) = z^4 + 5z^2 + 13 \tag{39.19}$$

has infinitely many roots in $\mathbb{C}$.

Exercise 39.8. ** *(Bonus)* Show that the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{39.20}$$

takes every complex value infinitely many times. (Hint: using L'Hospital show that $\mathrm{erf}(is)/(e^{s^2}/s) \to const.$ as $s \to +\infty$.)

We will return to the error function later and use asymptotic methods to locate, for large x, these special points.

40. The Phragmén-Lindelöf Theorem

Theorem 40.1 (Phragmén-Lindelöf)**.** *Let U be the open sector between two rays from the origin, forming an angle π/β, $\beta > \frac{1}{2}$. Assume f is analytic in U, and continuous on its closure, and for some $C_1, C_2, M > 0$ and $\alpha \in (0, \beta)$ it satisfies the estimates*

$$|f(z)| \leqslant C_1 e^{C_2|z|^\alpha}; \quad z \in U; \quad |f(z)| \leqslant M; \quad z \in \partial U \tag{40.1}$$

Then

$$|f(z)| \leqslant M; \quad z \in U \tag{40.2}$$

Proof. By a rotation we can make $U = \{z : 2|\arg z| < \pi/\beta\}$. Making a cut in the complement of U we can define an analytic branch of the log in U and, with it, an analytic branch of z^β. By taking $g = f(z^{1/\beta})$, we can assume without loss of generality that $\beta = 1$ and $\alpha \in (0, 1)$ and then $U = \{z : |\arg z| < \pi/2\}$. Let $\alpha' \in (\alpha, 1)$ and consider the analytic function

$$e^{-C_2 z^{\alpha'}} f(z) \tag{40.3}$$

Since $|e^{-C_2 z^{\alpha'}}| < 1$ in U (check) and $|e^{-C_2 z^{\alpha'} + C_2 z^\alpha}| \to 0$ as $|z| \to \infty$ on the half circle $|z| = R, \mathrm{Re}\, z \geqslant 0$ (check), the usual maximum modulus principle completes the proof. ∎

40.1. **An application to Laplace transforms.**

We will study Laplace and inverse Laplace transforms in more detail later. For now let $F \in L^1(\mathbb{R})$. Then it by Fubini and dominated convergence, the Laplace transform

$$\mathcal{L}F := \int_0^\infty e^{-px} F(p)dp \tag{40.4}$$

is well defined and continuous in x in the closed $\mathbb{H}^+$ and analytic in the open RHP (the open $\mathbb{H}^+$). (Obviously, we could allow $Fe^{-|\alpha|p} \in L^1$ and then $\mathcal{L}F$ would be defined for Re $x > |\alpha|$.) F is uniquely defined by its Laplace transform, as seen below.

Lemma 40.2 (Uniqueness)**.** *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of x with an accumulation point. Then $F = 0$ a.e.*

We will from now on write $F = 0$ on a set to mean $F = 0$ *a.e.* on that set.

Proof. By analyticity, $\mathcal{L}F = 0$ in the open RHP and by continuity, for $s \in \mathbb{R}$, $\mathcal{L}F(is) = 0 = \hat{\mathcal{F}}F$ where $\hat{\mathcal{F}}F$ is the Fourier transform of F (extended by zero for negative values of p). Since $F \in L^1$ and $0 = \hat{\mathcal{F}}F \in L^1$, by the known Fourier inversion formula [?], $F = 0$. ∎

More however can be said. We can draw interesting conclusions about F just from the rate of decay of $\mathcal{L}F$.

Proposition 40.3 (Lower bound on decay rates of Laplace transforms)**.** *Assume $F \in L^1(\mathbb{R}^+)$ and for some $\varepsilon > 0$ we have*

$$\mathcal{L}F(x) = O(e^{-\varepsilon x}) \quad as \quad x \to +\infty \tag{40.5}$$

Then $F = 0$ on $[0, \varepsilon]$.

Proof. We write

$$\int_0^\infty e^{-px}F(p)dp = \int_0^\varepsilon e^{-px}F(p)dp + \int_\varepsilon^\infty e^{-px}F(p)dp \tag{40.6}$$

we note that

$$\left|\int_\varepsilon^\infty e^{-px}F(p)dp\right| \leqslant e^{-\varepsilon x}\int_\varepsilon^\infty |F(p)|dp \leqslant e^{-p\varepsilon}\|F\|_1 = O(e^{-\varepsilon x}) \tag{40.7}$$

Therefore

$$g(x) = \int_0^\varepsilon e^{-px}F(p)dp = O(e^{-\varepsilon x}) \quad \text{as} \quad x \to +\infty \tag{40.8}$$

The function g is entire (prove this!) Let $h(x) = e^{\varepsilon x}g(x)$. Then by assumption h is entire and uniformly bounded for $x \in \mathbb{R}$ (since by assumption, for some x_0 and all $x > x_0$ we have $h \leqslant C$ and by continuity $\max|h| < \infty$ on $[0, x_0]$). The function is also manifestly bounded for $x \in i\mathbb{R}$ (by $\|F\|_1$). By Phragmén-Lindelöf (first applied in the first quadrant and then in the fourth quadrant, with $\beta = 2, \alpha = 1$) h is bounded in the closed RHP. Now, for $x = -s < 0$ we have

$$e^{-s\varepsilon}\int_0^\varepsilon e^{sp}F(p)dp \leqslant \int_0^\varepsilon |F(p)| \leqslant \|F\|_1 \tag{40.9}$$

Again by Phragmén-Lindelöf (again applied twice) h is bounded in the closed $\mathbb{H}_l$ thus bounded in C, and it is therefore a constant. But, by the Riemann-Lebesgue lemma, $h \to 0$ for $x = is$ when $s \to +\infty$. Thus $h \equiv 0$. But then, with χ_A the characteristic function of A,

$$\int_0^\varepsilon F(p)e^{-isp}dp = \hat{\mathcal{F}}(\chi_{[0,\varepsilon]}F) = 0 \tag{40.10}$$

for all $s \in \mathbb{R}$ entailing the conclusion. ∎

Corollary 40.4. *Assume $F \in L^1$ and $\mathcal{L}F = O(e^{-AX})$ as $x \to +\infty$ for all $A > 0$. Then $F = 0$.*

Proof. This is straightforward. ∎

As we see, uniqueness of the Laplace transform can be reduced to estimates. Also, no two different $L^1(\mathbb{R}^+)$ functions, real–analytic on $(0, \infty)$, can have Laplace transforms within exponentially small corrections of each–other. This will play an important role later on.

40.2. **A Laplace inversion formula.**

Theorem 40.5. *Assume $c \geqslant 0$, $f(z)$ is analytic in the closed half plane $H_c := \{z : \mathrm{Re}\ z \geqslant c\}$. Assume further that $\sup_{c' \geqslant c} |f(c' + it)| \leqslant g(t)$ with $g(t) \in L^1(\mathbb{R})$. Let*

$$F(p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} f(x) dx =: (\mathcal{L}^{-1}F)(p) \tag{40.11}$$

Then for any $x \in \{z : \mathrm{Re}\ z > c\}$ we have

$$\mathcal{L}F = \int_0^{\infty} e^{-px} F(p) dp = f(x) \tag{40.12}$$

Proof. Note that for any $x' = x_1' + iy_1' \in \{z : \mathrm{Re}\ z > c\}$

$$\int_0^{\infty} dp \int_{c-i\infty}^{c+i\infty} \left| e^{p(s-x')} f(s) \right| d|s| \ \leqslant\ \int_0^{\infty} dp e^{p(c-x_1')} \|g\|_1 \ \leqslant\ \frac{\|g\|_1}{x_1' - c} \tag{40.13}$$

and thus, by Fubini we can interchange the orders of integration:

$$\begin{aligned} U(x') &= \int_0^{\infty} e^{-px'} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} f(x) dx \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dx f(x) \int_0^{\infty} dp e^{-px'+px} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(x)}{x'-x} dx \end{aligned} \tag{40.14}$$

Since $g \in L^1$ there must exist subsequences $\tau_n, -\tau_n'$ tending to ∞ such that $|g(\tau_n)| \to 0$. Let $x' > \mathrm{Re}\, x = x_1$ and consider the box $B_n = \{z : \mathrm{Re}\, z \in [x_1, x'], \mathrm{Im}\, z \in [-\tau_n', \tau_n]\}$ with positive orientation. We have

$$\int_{B_n} \frac{f(s)}{x' - s} ds = -f(x') \tag{40.15}$$

while, by construction,

$$\lim_{n\to\infty}\int_{B_n}\frac{f(s)}{x'-s}ds=\int_{x'-i\infty}^{x'+i\infty}\frac{f(s)}{x'-s}ds-\int_{c-i\infty}^{c+i\infty}\frac{f(s)}{x'-s}dx \tag{40.16}$$

On the other hand, by dominated convergence, we have

$$\int_{x'-i\infty}^{x'+i\infty}\frac{f(s)}{x'-s}ds\to 0 \quad\text{as}\quad x'\to\infty \tag{40.17}$$

∎

40.3. **Abstract Stokes phenomena.** This theorem shows that if an analytic function decays rapidly along some direction, then it increases "correspondingly" rapidly along a complementary direction. The following is reminiscent of a theorem by Carlson [9].

Theorem 40.6. *Assume $f\not\equiv 0$ is analytic in the closed $\mathbb{H}^+$ and that for all $a>0$ we have $f(t)=O(e^{-at})$ for $t\in\mathbb{R}^+, t\to\infty$. Then, for all $b>0$ the function*

$$e^{-bz}f(z) \tag{40.18}$$

is unbounded in the closed $\mathbb{H}^+$.

Proof. Assume that for some $b>0$ we had $|e^{-bz}f(z)|<M$ in the closed RHP. Then, the function

$$\psi(z)=\frac{e^{-bz}f(z)}{(z+1)^2} \tag{40.19}$$

satisfies the assumptions of Theorem 40.5. But then $\psi(z)=\mathcal{L}\mathcal{L}^{-1}\psi(z)$ satisfies the assumptions of Corollary (40.4) and $\psi\equiv 0$. ∎

Let $\alpha>2$.

Corollary 40.7. *Assume $f\not\equiv 0$ is analytic in the closed sector $S=\{z: 2|\arg z|\leqslant\pi/\alpha\}$, $\alpha>\frac{1}{2}$ and that $f(t)\leqslant Ce^{-t^\beta}$ with $\beta>\alpha$ for $t\in\mathbb{R}^+$. Then for any $\beta'<\beta$ there exists a subsequence $z_n\in S$ such that*

$$\left|f(z_n)e^{-z_n^{\beta'}}\right|\to\infty \quad \textit{as}\quad n\to\infty \tag{40.20}$$

Proof. This follows from Theorem 40.6 by simple changes of variables. ∎

Exercise 40.8. ** Carry out the details of the preceding proof.*

41. Asymptotic series

We have seen in the Schwarz-Christoffel section that the behavior of analytic functions near a point of nonanalyticity can be given by a series in noninteger powers of the distance to the singularity. The behavior can be more complicated, containing exponentially small corrections, logarithmic terms and so on. The series themselves may have zero radius of convergence. It is not the purpose of this part of the course to classify these behaviors, but it can be done for a fairly large class of functions. Here we look how simple behaviors can be determined for relatively simple functions.

Example 41.1. *Consider the following integral related to the so–called error function*

$$F(z) = e^{z^{-2}} \int_0^z s^{-2} e^{-s^{-2}} ds$$

It is clear that the integral converges at the origin, if the origin is approached through real values (see also the change of variable below). **Definition of** $F(z)$. *We define the integral* to $z \in \mathbb{C}$ as being taken on a curve γ with $\gamma'(0) > 0$, and define $F(0) = 0$.

Check that this is a consistent definition and the resulting function is analytic except at $z = 0$ (this is essentially the contents of Exercise 41.3 below.

What about the behavior at $z = 0$? It depends on the direction in which 0 is approached! Let's look more carefully. Replace z by $1/x$, make a corresponding change of variable in the integral and you are led to

$$E(x) = e^{x^2} \int_x^\infty e^{-s^2} ds =: \frac{\sqrt{\pi}}{2} e^{x^2} \operatorname{erfc}(x) \tag{41.1}$$

Let us take x (and thus z) real and integrate by parts m times

(41.2)

$$E(x) = \frac{1}{2x} - \frac{e^{x^2}}{2} \int_x^\infty \frac{e^{-s^2}}{s^2} ds = \frac{1}{2x} - \frac{1}{4x^3} + \frac{3e^{x^2}}{4} \int_x^\infty \frac{e^{-s^2}}{s^4} ds = \dots$$
$$= \sum_{k=0}^{m-1} \frac{(-1)^k}{2\sqrt{\pi}} \frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} + \frac{(-1)^m e^{x^2} \Gamma(m+\frac{1}{2})}{\sqrt{\pi}} \int_x^\infty \frac{e^{-s^2}}{s^{2m}} ds$$

On the other hand, we have, by L'Hospital

$$\left(\int_x^\infty \frac{e^{-s^2}}{s^{2m}} ds \right) \left(\frac{e^{-x^2}}{x^{2m+1}} \right)^{-1} \to \frac{1}{2} \text{ as } x \to \infty \tag{41.3}$$

and the last term in (41.2) is $O(x^{-2m-1})$ as well. On the other hand it is also clear that the series in (41.2) is alternating and thus

$$\sum_{k=0}^{m-1}\frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} \leqslant E(x) \leqslant \sum_{k=0}^{m}\frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} \tag{41.4}$$

if m is even.

Remark 41.2. *Using (41.3) and Exercise 41.13 below we conclude that $F(z)$ has a Taylor series at zero,*

$$\tilde{F}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2\sqrt{\pi}}\Gamma(k+\frac{1}{2})z^{2m+1} \tag{41.5}$$

that $F(z)$ is C^∞ on $\mathbb{R}$ and analytic away from zero.

Exercise 41.3. *** Show that $z=0$ is an isolated singularity of $F(z)$. Using Remark 41.2, show that F is unbounded as 0 is approached along some directions in the complex plane.*

Notes (1) It is *not* the Laurent series of f at 0 that we calculated! Laurent series converge. Think carefully about this distinction and why the positive index coefficients do not coincide.

(2) The rate of convergence of the Laurent series is slower as 0 is approached, quickly becoming numerically useless. By contrast, the precision gotten from (41.4) near zero is such that for $z=0.1$ the error in calculating f is of order 10^{-45} ! However, of course (41.4) is divergent and it cannot be used to calculate *exactly* for any nontrivial value of z.

(3) We have illustrated here a simple method of evaluating the behavior of integrals, the method of integration by parts.

41.1. **More general asymptotic series.** Classical asymptotic analysis typically deals with the qualitative and quantitative description of the behavior of a function close to a point, usually a singular point of the function. This description is provided in the form of an *asymptotic expansion.* The expansion certainly depends on the point studied and, as we have noted, often on the direction along which the point is approached (in the case of several variables, it also depends on the relation between the variables as the point is approached). If the direction matters, it is often convenient to change variables to place the special point at infinity.

Asymptotic expansions are formal series[13] of simpler functions f_k,

[13]That is, there are no convergence requirements. More precisely, they are defined as sequences $\{f_k\}_{k\in\mathbb{N}\cup\{0\}}$, the operations being defined in the same way as if they represented convergent series; see also §41.2.

$$\tilde{f} = \sum_{k=0}^{\infty} f_k(t) \tag{41.6}$$

in which each successive term is much smaller than its predecessors (one variable is assumed for clarity). For instance if the limiting point is t_0 approached from above along the real line this requirement is written

$$f_{k+1}(t) = o(f_k(t)) \quad \text{or} \quad f_{k+1}(t) \ll f_k(t) \quad \text{as} \quad t \downarrow t_0 \tag{41.7}$$

denoting

$$\lim_{t \to t_0^+} f_{k+1}(t)/f_k(t) = 0 \tag{41.8}$$

We will often use the variable x when the limiting point is $+\infty$ and z when the limiting point is zero. Simple examples are the Taylor series, e.g.

$$\sin z + e^{-\frac{1}{z}} \sim z - \frac{z^3}{6} + \ldots \quad (z \to 0^+)$$

and the expansion in the Stirling formula

$$\ln \Gamma(x) \sim x \ln x - x - \frac{1}{2} \ln x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}, \ x \to +\infty$$

where B_k are the Bernoulli numbers.

(The asymptotic expansions in the examples above are the formal sums following the "$\sim$" sign, the meaning of which will be explained shortly.)

Examples of expansions that are *not* asymptotic expansions are

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (x \to +\infty)$$

which converges to $\exp(x)$, but it is not an asymptotic series for large x since it fails (41.7); another example is the series

$$\sum_{k=0}^{\infty} \frac{x^{-k}}{k!} + e^{-x} \quad (x \to +\infty) \tag{41.9}$$

(because of the exponential terms, this is not an ordered *simple series* satisfying (41.7)). Note however expansion (41.9), *does* satisfies all requirements in the *left* half plane, if we write e^{-x} in the first position.

We also note that in this particular case the first series is convergent, and if we replace (41.9) by

$$e^{1/x} + e^{-x} \tag{41.10}$$

then (41.10) *is* a valid asymptotic expansion, of a very simple kind, with two nonzero terms. Since convergence is relative to a topology, this elementary remark will play a crucial role when we will speak of Borel summation.

Functions asymptotic to a series, in the sense of Poincaré. The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits:

Definition 41.4. *A function f is asymptotic to the formal series $\tilde{f}$ as $t \to t_0^+$ if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) =: f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{41.11}$$

We note that condition (41.11) can then be also written as

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{41.12}$$

where $g(t) = O(h(t))$ means $\limsup_{t \to t_0^+} |g(t)/h(t)| < \infty$. Indeed, this follows from (41.11) and the fact that $f(t) - \sum_{k=0}^{N+1} \tilde{f}_k(t) = o(\tilde{f}_{N+1}(t))$.

41.2. **Asymptotic power series.** In many instances the functions f_k are exponentials, powers and logarithms. This is not simply a matter of choice or an accident, but reflects some important fact about the relation between asymptotic expansions and functions which will be clarified later.

A special role is played by power series which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \ \ z \to 0^+ \tag{41.13}$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$) the series can be centered at t_0 (or $+\infty$, respectively).

Remark. If a c_k is zero then Definition 41.4 fails trivially in which case (41.13) is not an asymptotic series. This motivates the following definition.

Definition 41.5 (Asymptotic power series)**.** *A function possesses an asymptotic* power *series if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \tag{41.14}$$

We use the boldface notation $\boldsymbol{\sim}$ for the stronger asymptoticity condition in (41.11) when confusion is possible.
Example Check that the Taylor series of an analytic function at zero is its asymptotic series there.

In the sense of (41.14), the asymptotic power series at zero of e^{-1/x^2} is the zero series. It is however surely not the case that e^{-1/x^2} behaves like zero as $x \to 0$ on $\mathbb{R}$. Rather, in this case, the asymptotic *behavior* of e^{-1/x^2} is e^{-1/x^2} itself (only exponentials and powers involved).

Asymptotic power series form an algebra; addition of asymptotic power series is defined in the usual way:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} c'_k z^k = \sum_{k=0}^{\infty} (Ac_k + Bc'_k) z^k$$

while multiplication is defined as in the convergent case

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} c'_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j c'_{k-j}\right) z^k$$

Remark 41.6. *If the series $\tilde{f}$ is convergent and f is its sum (note the ambiguity of the "sum" notation) $f = \sum_{k=0}^{\infty} c_k z^k$ then $f \sim \tilde{f}$.*

The proof of this remark follows directly from the definition of convergence.

Lemma 41.7. *(Uniqueness of the asymptotic series to a function) If $f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$ as $z \to 0$ then the $\tilde{f}_k$ are unique.*

Proof. Assume that we also have $f(z) \sim \tilde{F} = \sum_{k=0}^{\infty} \tilde{F}_k z^k$. We then have (cf. (41.11))

$$\tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = o(z^N)$$

which is impossible unless $g_N(z) = \tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = 0$, since g_N is a polynomial of degree N in z. ∎

Corollary 41.8. *The asymptotic series at the origin of an analytic function is its Taylor series at zero. More generally, if F has a Taylor series at 0 then that series is its asymptotic series as well.*

The proof of the following lemma is immediate:

Lemma 41.9. *(Algebraic properties of asymptoticity to a power series)* *If* $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ *and* $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ *then*
(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$
(ii) $fg \sim \tilde{f}\tilde{g}$

Sometimes it is convenient to check a formally weaker condition of asymptoticity:

Lemma 41.10. *Let* $\tilde{f} = \sum_{n=0}^{\infty} a_n z^n$. *If* f *is such that there exists a sequence* $p_n \to \infty$ *such that*

$$\Big(\forall n \exists p_n\Big) \ s.t. \ \ f(z) - \tilde{f}^{[p_n]}(z) = o(z^n) \quad as \quad z \to 0$$

then $f \sim \tilde{f}$.

Proof. We let m be arbitrary and choose $n > m$ such that $p_n > m$. We have

$$f(z) - \tilde{f}^{[m]} = (f(z) - \tilde{f}^{[p_n]}) + (\tilde{f}^{[p_n]} - \tilde{f}^{[m]}) = o(z^m) \ (z \to 0)$$

by assumption and since $\tilde{f}^{[p_n]} - \tilde{f}^{[m]}$ is a polynomial for which the smallest power is z^{m+1} (we are dealing with truncates of the same series). ∎

41.3. **Integration and differentiation of asymptotic power series.** While asymptotic power series can be safely integrated term by term as the next proposition shows, differentiation is more delicate. In suitable spaces of functions and expansions, we will see the asymmetry largely disappears if we are dealing with analytic functions in suitable regions.

Anyway, for the moment note that the function $e^{-1/z}\sin(e^{1/z^2})$ is asymptotic to the zero power series as $z \to 0^+$ although the derivative is unbounded and thus not asymptotic to the zero series.

Proposition 41.11. *Assume* f *is integrable near* $z = 0$ *and that*

$$f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} \tilde{f}_k z^k$$

Then

$$\int_0^z f(s)ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{k+1} z^{k+1}$$

Proof. This follows from the fact that $\int_0^z o(s^n)ds = o(z^{n+1})$ as can be seen by immediate estimates. ▮

Asymptotic power series of analytic function, if they are valid in wide enough regions can be differentiated.

Asymptotics in a strip. Assume $f(x)$ is analytic in the strip $S_a = \{x : |x| > R, |\mathrm{Im}\,(x)| < a\}$. Let $\alpha < a$ and and $S_\alpha = \{x : |x| > R, |\mathrm{Im}\,(x)| < \alpha\}$ and assume that

$$f(x) \sim \tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k} \quad (|x| \to \infty, x \in S_\alpha) \tag{41.15}$$

It is assumed that that the limits implied in (41.15) hold uniformly in the given strip.

Proposition 41.12. *If (41.15) holds, then, for $\alpha' < \alpha$ we have*

$$f'(x) \sim \tilde{f}'(x) := \sum_{k=0}^{\infty} -\frac{kc_k}{x^{k+1}} \quad (|x| \to \infty, x \in S_{\alpha'})$$

Proof. We have $f(x) = \tilde{f}^{[N]}(x) + g_N(x)$ where clearly g is analytic in S_a and $|g_N(x)| \leqslant Const.|x|^{-N-1}$ in S_α. But then, for $x \in S_{\alpha'}$ and $\delta = \frac{1}{2}(\alpha - \alpha')$ we get

$$\begin{aligned} |g_N'(x)| = \frac{1}{2\pi}\left|\oint_{|x-s|=\delta} \frac{g_N(s)ds}{(s-x)^2}\right| &\leqslant \frac{1}{\delta}\frac{Const.}{(|x| - |\delta|)^{N+1}} \\ &= O(x^{-N-1}) \quad (|x| \to \infty,\ x \in S_{\alpha'}) \end{aligned}$$

By Lemma 41.10, the proof follows. ▮

Exercise 41.13. ** *Show that if $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ and $f'(x) \to L$ as $x \downarrow 0$, then f is differentiable to the right at zero and this derivative equals L. Use this fact, Proposition 41.12 and induction to show that the Taylor series at the origin of $F(z)$ is indeed given by (41.5).*

41.4. **Watson's Lemma.** In many instances integral representations of functions are amenable to Laplace transforms

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-xp} F(p)dp \tag{41.16}$$

The behavior of $\mathcal{L}F$ for large x relates to the behavior for small p of F.

It is shown in the later parts of this book that solutions of generic analytic differential equations, under mild assumptions can be conveniently expressed in terms of Laplace transforms.

For the error function note that

$$\int_N^\infty e^{-s^2}ds = N\int_1^\infty e^{-N^2u^2}du = \frac{\sqrt{x}e^{-x}}{2}\int_0^\infty \frac{e^{-xp}}{\sqrt{p+1}}dp; \quad x = N^2$$

For the Gamma function, writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ in

$$n! = \int_0^\infty e^{-t}t^n dt = n^{n+1}\int_0^\infty e^{n(-s+\ln s)}ds \tag{41.17}$$

we can make the substitution $t - \ln t = p$ in each integral and obtain

$$n! = n^{n+1}e^{-n}\int_0^\infty e^{-np}G(p)dp$$

Watson's Lemma

This important tool states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 41.14. *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants β_i with $\mathrm{Re}\,(\beta_i) > 0$, $i = 1, 2$. Then*

$$\mathcal{L}F \sim \sum_{k=0}^\infty c_k\Gamma(k\beta_1 + \beta_2)x^{-k\beta_1-\beta_2}$$

along any ray ρ in the open right half plane H.

Proof. Induction, using the simpler version, Lemma 41.15, proved below. □

Lemma 41.15. *Let $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ and assume*

$$F(p) \sim p^\beta \quad \text{as } p \to 0^+$$

with $\mathrm{Re}\,(\beta) > -1$. Then

$$\int_0^\infty F(p)e^{-px}dp \sim \Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $U(p) = p^{-\beta}F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose C and a positive so that $|F(p)| < C|p^\beta|$ on $[0, a]$. Since

$$\left|\int_a^\infty F(p)e^{-px}\mathrm{d}p\right| \leqslant e^{-|x|a\cos\phi}\|F\|_1 \tag{41.18}$$

we have by dominated convergence, and after the change of variable $s = p|x|$,

$$\begin{aligned} x^{\beta+1}\int_0^\infty F(p)e^{-px}\mathrm{d}p &= e^{i\phi(\beta+1)}\int_0^\infty s^\beta U(s/|x|)\chi_{[0,a]}(s/|x|)e^{-se^{i\phi}}\mathrm{d}s \\ &\quad + O(|x|^{\beta+1}e^{-|x|a\cos\phi}) \to \Gamma(\beta+1) \quad (|x|\to\infty) \end{aligned} \tag{41.19}$$

41.5. **Example: the Gamma function.** We start from the representation

$$\begin{aligned} n! = \int_0^\infty t^n e^{-t}dt &= n^{n+1}\int_0^\infty e^{-n(s-\ln s)}ds \\ &= n^{n+1}\int_0^1 e^{-n(s-\ln s)}ds + n^{n+1}\int_1^\infty e^{-n(s-\ln s)}ds \end{aligned} \tag{41.20}$$

On $(0,1)$ and $(1,\infty)$ separately, the function $s-\ln(s)$ is monotonic and we may write, after inverting $s-\ln(s)=t$ on the two intervals to get $s_{1,2}=s_{1,2}(t)$,

$$n! = n^{n+1}\int_1^\infty e^{-nt}(s_2'(t)-s_1'(t))dt = n^{n+1}e^{-n}\int_0^\infty e^{-np}G(p)dp \tag{41.21}$$

where $G(p)=s_2'(1+p)-s_1'(1+p)$. In order to determine the asymptotic behavior of $n!$ we need to determine the small p behavior of the function $G'(p)$

Remark 41.16. *The function $G(p)$ is an analytic function in $\sqrt{p}$ and thus $G'(p)$ has a convergent Puiseux series*

$$\sum_{k=-1}^\infty c_k p^{k/2} = \sqrt{2}p^{-1/2} + \frac{\sqrt{2}}{6}p^{1/2} + \frac{\sqrt{2}}{216}p^{3/2} - \frac{139\sqrt{2}}{97200}p^{5/2} + \ldots$$

Thus, by Watson's Lemma, for large n we have

$$n! \sim \sqrt{2\pi n}n^n e^{-n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \ldots\right) \tag{41.22}$$

Proof. We write $s = 1+S$ and $t = 1+p$ and the equation $s-\ln(s) = t$ becomes $S-\ln(1+S) = p$. Note that $S-\ln(1+S) = S^2U(S)/2$ where $U(0) = 1$ and $U(S)$ is analytic for small S; with the natural branch of the square root, $\sqrt{U(S)} = H(S)$ is also analytic. We rewrite $S-\ln(1+S) = p$ as $SH(S) = \pm\sqrt{2}\sigma$ where $\sigma^2 = p$. Since $(SH(S))'_{|S=0} = 1$ the implicit function theorem ensures the existence of two functions $S_{1,2}(\sigma)$ (corresponding to the two choices of sign) which are analytic in σ. The concrete expansion may be gotten by implicit differentiation in $SH(S) = \pm\sqrt{2}\sigma$, for instance. ▮

42. Appendix

42.1. Appendix to Chapter 7.

42.2. Some facts about the topology of $\mathbb{C}$. From a topological point of view, $\mathbb{C}$ is identical to $\mathbb{R}^2$. A region of $\mathbb{C}$ is called *open* if it contains together with any point z_0 all sufficiently close points, that is, it also contains a nonempty disk centered at z_0; intuitively, an open set is a region without its boundary. For example an *open disk*

$$\mathbb{D}(z_0, r) = \{z \in \mathbb{C} |\, |z - z_0| < r\} \tag{42.23}$$

a *punctured disk*

$$\mathbb{D}_p(z_0, r) = \{z \in \mathbb{C} |\, 0 < |z - z_0| < r\}$$

the upper half plane $\mathbb{H}^u := \{z : \mathrm{Im}\,(z) > 0\}$ and $\mathbb{C}$ are open, as is, trivially, the empty set $\emptyset$, but a *closed disk*

$$\overline{\mathbb{D}}(z_0, r) = \{z \in \mathbb{C} |\, |z - z_0| \leqslant r\}$$

is not open. The exterior of a closed disk, $\{z \in \mathbb{C} |\, |z - z_0| > r\}$, is open. A finite intersection of open sets is open. Clearly a set in $\mathbb{C}$ is open iff it is a (finite or infinite) union of open disks.

More generally, a **topology on a space** X consists of a family $\mathcal{O}$ of sets defined as open, which should have the following properties: (1) $X, \emptyset \in \mathcal{O}$, (2) $O_1, O_2 \in \mathcal{O} \Rightarrow O_1 \cap O_2 \in \mathcal{O}$ and (3) any union, finite or infinite of open sets $O_\alpha \in \mathcal{O}$ is open: $\cup_\alpha O_\alpha \in \mathcal{O}$. Complements of open sets are called *closed* sets. The whole X is both open and closed; so is its complement, $\emptyset$. The family $\mathcal{O}$ in the case of $\mathbb{C}$ can be taken to be the collection of all unions of open disks (42.23), for all $z_0 \in \mathbb{C}, r \in [0, \infty]$.

The *boundary* of the set S in $\mathbb{C}$, denoted ∂S, consists of all points z in $\mathbb{C}$ for which there are sequences contained in S which converge to z, as well as sequences in the exterior of S convergent to z. For example, the boundary of a disk is the circle surrounding it:

$$\partial\mathbb{D}(z_0, r) = C(z_0, r) := \{z \in \mathbb{C} |\, |z - z_0| = r\}$$

Also $\partial\overline{\mathbb{D}}(z_0,r)=C(z_0,r)$, but $\partial\mathbb{D}_p(z_0,r)=C(z_0,r)\cup\{0\}$.

A point z in $\mathbb{C}$ is an *accumulation point of the set* S if there is a sequence of points in S converging to z.

Note that a set O is open iff it contains no points in ∂O. At the opposite end, if a set contains *all* of its boundary points then it is closed.

If a set is defined by (finitely many) inequalities involving continuous functions, then the set is open only if all the inequalities are strict ($<$, $>$, or $\neq$), and it is closed if all are $\leqslant,\geqslant$ or $=$; the boundary is obtained by replacing all inequalities by equalities.

If X is a topological space and $X_1\subset X$ the *induced topology* in X_1 is $\{X_1\cap O|O\in\mathcal{O}\}$.

42.2.1. *Connected sets.* An open set O is *connected* if it is not the union of two disjoint nonempty open sets. More generally, a subset $X_1\in X$ is connected if it is *not* the disjoint union of two nonempty sets that are open in the induced topology on X_1. Equivalently, there is no subset of X_1 which is both open and closed in the induced topology (other than X_1 and the empty set). For example any disk in $\mathbb{C}$ is connected, and so is a punctured disk. See also Proposition 42.18 below.

A *domain* in $\mathbb{C}$ is by definition an open connected set.

Exercise 42.17. *Is the annulus* $\{z\in\mathbb{C}|\ r\le|z|<R\}$ *open? closed? connected? What is its boundary?*

A curve in $\mathbb{R}^2$ is often given using a parametrization, as the image of a pair of *continuous* real functions: $\{(x(t),y(t)):t\in[a,b]\}$. The same curve can obviously be the image many different maps. If at least one of these is differentiable, then the curve is called differentiable; $\{x(t)+iy(t):t\in[a,b]\}$ is the corresponding curve in $\mathbb{C}$.

A set S with the property that any two points is S can be connected by a curve in S is called *path connected*; it can be shown that a path connected set is necessarily connected. But the converse is not true, for example $S=\{(x,\sin\frac{1}{x})\,|\,x>0\}\cup\{(0,0)\}$ is connected, but not path connected. But:

Proposition 42.18. *Domains* $\mathcal{D}\subset\mathbb{C}$ *are path connected. The path can be chosen to be a polygonal line.*

Proof. Indeed, let $z,w\in\mathcal{D}$ be two arbitrary points. Collect the points which are path connected to z:

$$\mathcal{D}_z=\left\{u\in\mathcal{D}\,\middle|\,\exists\gamma:[a,b]\to\mathcal{D}\text{ continuous, with }\gamma(a)=z,\,\gamma(b)=u\right\}$$

Then $\mathcal{D}_z$ is open since for any $u\in\mathcal{D}_z$, there is a disk $D_\varepsilon(z)$ included in $\mathcal{D}$ (since $\mathcal{D}$ is open), and then z can be path connected to any point in this disk (the path connecting z to u followed by the segment form u to any point in the disk), hence $\mathcal{D}_z$ contains a disk centered at u. By the same argument also $\mathcal{D}_w$ is open, and since $\mathcal{D}$ is connected then there must be a point $u \in \mathcal{D}_z \cap \mathcal{D}_w$. But then the path going from z to u followed by the path from u to w connects z to w.

The path connecting two points can be chosen to be a polygonal line: Indeed, let $\gamma : [a, b] \to \mathcal{D}$ continuous, so that $\gamma(a) = z$ and $\gamma(b) = w$. Since $\mathcal{D}$ is open, every point along the path is contained in a disk included in $\mathcal{D}$: for all $t \in [a, b]$ there is $\varepsilon_t > 0$ so that $\mathbb{D}(\gamma(t), \varepsilon_t) \subset \mathcal{D}$. Since the image of γ is compact, and is included in the union of all these disk, then it is included in a finite number of them: there are $t_1, \dots, t_n$ so that $\gamma([a, b]) \subset \cup_{k=1}^{n} \mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$ and now γ can be replaced by segments in each disk. To be more precise in this construction, let $t_0 = a, t_{n+1} = b$ and let $\varepsilon_0, \varepsilon_{n+1} > 0$ so that $\mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$ for $k = 0$ and $k = n + 1$. Then $\gamma([a, b]) \subset \cup_{k=0}^{n+1} \mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$. We can remove any disk of the covering that is completed include in another disk, and we number the t_k in increasing order. Then the segments $[\gamma(t_{k-1}), \gamma(t_k)]$ are included in $\mathbb{D}(\gamma(t_{k-1}), \varepsilon_{t_{k-1}}) \cup \mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$ and form a polygonal line joining z and w. $\Box$

A *rectifiable curve* is a continuous curve $t \mapsto \gamma(t)$ (defined for $t \in [a, b]$) with finite length, meaning that the sup of the length of polygonal lines joining points of γ is finite. In $\mathbb{C}$ this means:

$$\sup \left\{ \sum_{i=0}^{n} |\gamma(t_i) - \gamma(t_{i+1})| : \ 0 = t_0 < t_i < t_{i+1} < t_n = b \forall i \in 1..n-1, \forall n \in \mathbb{N} \right\} < \infty$$

A piecewise differentiable curve with integrable γ' is easily checked to be rectifiable, and the length, defined by the sup above, also equals

$$l(\gamma) = \int_a^b |\gamma'(t)| dt$$

DEFINE winding number

42.3. **Proof of the Ascoli-Arzelà theorem.** *Necessity* (i) Suppose $\mathcal{F}$ is not equicontinuous on some compact K. Then on K

(42.24)

$$\exists (\varepsilon > 0, \ \{z_n\}, \{z_n'\}, \{f_n\}) \ s.t. (|z_n - z_n'| \to 0 \ \& \ d(f_n(z_n), f_n(z_n') > \varepsilon)$$

Since K is compact and $\mathcal{F}$ is normal from any sequence we can extract a convergent subsequence, which w.l.o.g. we can assume to be $\{z_n\}, \{z_n'\}, \{f_n\}$ themselves. Let $z_n \to z$, $f_n \to f$ ($z_n' \to z$ too). The limit f is continuous, thus uniformly continuous. We have

$$\lim_{n\to\infty}\sup_{x\in K} d'(f(x), f_n(x)) = 0$$

thus for n large enough,

$$d'(f_n(z'_n), f(z'_n)) < \frac{\varepsilon}{4},\ d'(f(z'_n), f(z)) < \frac{\varepsilon}{4}, \qquad (42.25)$$
$$d'(f(z), f(z_n)) < \frac{\varepsilon}{4} \quad \text{and} \quad d'(f(z_n), f_n(z_n)) < \frac{\varepsilon}{4}$$

implying by the triangle inequality,

$$d'(f_n(z'_n), f_n(z_n)) < \varepsilon$$

a contradiction.

(ii) Fix z and take $K = \overline{\{f(z) : f \in \mathcal{F}\}}$. Take a sequence $\{w_n\} \subset K$ By the definition of K, if $w_n \in K$ $\exists f_n \in \mathcal{F}$ such that $d(f_n(z), w_n) < 1/n$. By the normality of $\mathcal{F}$, there exists a subsequence of functions, w.l.o.g. $\{f_n\}$ themselves, $f_n \to f$. But then $w_n \to f(z)$ $\square$.

Sufficiency. The sufficiency of the two conditions is shown by Cantor's famous diagonal argument. Let $\{f_n\} \subset \mathcal{F}$. We take a countable everywhere dense set $\mathcal{Q} = \{z_k\}$ of points in Ω, e.g., those with rational coordinates and we let $\mathcal{K}$ be any compact in Ω. Take $z_1 \in \mathcal{Q}$. By (ii), there is a convergent subsequence $\{f_{n_{j1}}(z_1)\}_{j\in\mathbb{N}}$. Take now $z_2 \in \mathcal{Q}$. From $\{f_{n_{j1}}(z_2)\}$we can extract a subsequence $\{f_{n_{j2}}(z_2)\}_{j\in\mathbb{N}}$ which converges as well. So $\{f_{n_{j2}}(z)\}_{j\in\mathbb{N}}$ converges both at z_1 and z_2. Inductively we find a subsequence $\{f_{n_{jm}}(z)\}_{j\in\mathbb{N}}$ such that it converges at the points $z_1, ..., z_m$. But then, the subsequence $\{g_j\} := \{f_{n_{jj}}\}$ converges at all points in $\mathcal{Q}$. We aim to show that g_j *converges uniformly* in any compact set $K \in \Omega$. By equicontinuity,

(42.26)

$$\forall\varepsilon > 0\, \exists\delta\, s.t.\ \forall(a, b, f) \in K^2 \times \mathcal{F}(|a - b| < \delta \Rightarrow d(f(a), f(b)) < \frac{\varepsilon}{3})$$

Consider a finite covering of K by balls of radius $\delta/2$. Since $\mathcal{Q}$ is everywhere dense, there is a z_k in each of these balls. They are finitely many, so that for $l, m > n_0$,

$$d(g_l(z_k), g_m(z_k)) < \frac{\varepsilon}{3} \qquad (42.27)$$

On the other hand, any $a \in \mathcal{K}$ is, by construction, at distance at most δ from some z_k and thus by (42.26) (for any $f \in \mathcal{F}$, in particular) for g_{ni}, g_{nj} we have

$$d(g_l(a), g_l(z_k)) < \frac{\varepsilon}{3} \qquad (42.28)$$

$$d(g_m(a), g_m(z_k)) < \frac{\varepsilon}{3} \tag{42.29}$$

We thus see by the triangle inequality that

$$d(g_l(a), g_m(a)) < \varepsilon \tag{42.30}$$

Thus $g_n(a)$ converges. Convergence is uniform since the pair ε, δ is independent of a. □

43. Dominated convergence theorem

We state this theorem only for the real line

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