Course notes

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Some notations

$\mathcal{L}$	Laplace transform, §**??**
$\mathcal{L}^{-1}$	inverse Laplace transform, §1.58
$\mathcal{B}$	Borel transform, §2.1
$\mathcal{LB}$	Borel/BE summation operator, §**??** and §2.2b
p	usually, Borel plane variable
$\tilde{f}$	formal expansion
$H(p)$	Borel transform of $h(x)$
$\sim$	asymptotic to, §1.1a
$\lesssim$	less than, up to an unimportant constant, §1.1a
$\mathbb{D}_r$	The disk of radius r centered at 0
∂A	The boundary of the set A
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ $\mathbb{N}^+, \mathbb{R}^+$	the nonnegative integers, integers, rationals, real numbers, complex numbers, positive integers, and positive real numbers, respectively
$\mathbb{H}$	open right half complex-plane.
$\mathbb{H}_\theta$	half complex-plane centered on $e^{i\theta}$.
$\overline{S}$	closure of the set S.
C_a	absolutely continuous functions, [74]
$f * g$	convolution of f and g, §**??**
$L^1_\nu, \|\cdot\|_\nu$, $\mathcal{A}_{K,\nu}$, etc.	various spaces and norms defined in §2.6 and §2.7

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Chapter 1

Introduction

1.1 Expansions and approximations

Classical asymptotic analysis is a set of mathematical results and methods to find the limiting behavior of functions, near a point, most often a singular point. It is particularly efficient in the context of differential or difference equations when the function has no simple representation that immediately conveys the desired limiting behavior.

Asymptotic analysis may involve several variables; however, in this book, we will be mostly concerned with limiting behavior in one scalar variable; in the context of differential or difference equations, this can be the independent variable or a parameter.

1.1a Notation

Let the special point of analysis be $t_0 \in \mathbb{C}$.

Some common notations are: $f = O(1)$ if f is bounded near t_0 and $f = o(1)$ if $f \to 0$ as $t \to t_0$. More generally $f = O(g)$ if $f/g = O(1)$ and similarly $f = o(g)$ if $f/g = o(1)$. This requires $g(t) \neq 0$ near t_0. Whenever similar divisions occur, this condition is tacitly assumed.

We also write $f \ll g$ if $f = o(g)$. It is understood that g cannot vanish close to t_0. The notation $|f| \lesssim |g|$ is used to represent $|f| \leqslant C|g|$ in the domain of interest, where C is a constant whose value is immaterial. Clearly $|f| \lesssim |g|$ in a small neighborhood of t_0 is the same as $f = O(g)$. We write $f = O_s(g)$; when both $f = O(g)$ and $g = O(f)$ near t_0.

The point t_0 may be approached only from one direction, along a curve in $\mathbb{C}$ or even along a given sequence of points tending to t_0 and when such further restrictions are needed, they will be specified. For instance if $t_0 = 0$, then $t = o(1)$ as $t \to 0$ and $e^{-1/t} = o(t^m)$ for any m as $t \downarrow 0$ ($t \in \mathbb{R}^+$ decreases towards 0), but the opposite holds, $t^m = o(e^{-1/t})$, as $t \uparrow 0$.

1.1b Asymptotic expansions

A sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ such that $f_n \ll f_m$ if $n > m$ is called an asymptotic scale at $t = t_0$. In terms of it we can write the leading order behavior of a function, $f = f_0 + o(f_0)$ and also successively higher order corrections: $f = f_0 + f_1 + \cdots + f_n + o(f_n)$ etc. This process can continue for finitely many n or for all $n \in \mathbb{N}$. In a compact form, we write an asymptotic expansion as a formal sum,

$$\sum_{k=0}^{\infty} f_k(t) =: \tilde{f}, \quad \text{or} \quad \sum_{k=0}^{N} f_k(t) =: \tilde{f}_N \tag{1.1}$$

where no convergence condition is imposed, and define asymptoticity by the following.

Definition 1.2 *A function f is asymptotic to the formal series $\tilde{f}$ as $t \to t_0$ (once more, the approach of t_0 may have to be restricted to a generally complex curve) if*

$$f(t) - \sum_{k=0}^{M} f_k(t) = o(f_M(t)) \quad (\forall M \in \mathbb{N} \text{ or } \forall M \leqslant M_0 \in \mathbb{N}) \tag{1.3}$$

We shall assume, without any serious loss of generality that f_k exist for all $k \in \mathbb{N}$.

Condition (1.3) can be written in a number of equivalent ways, useful in applications, as the following result shows.

Proposition 1.4 *If $\tilde{f} = \sum_{k=0}^{\infty} f_k(t)$ is an asymptotic series as $t \to t_0$ and f is a function asymptotic to it, then the following characterizations are equivalent to each other and to (1.3).*

(i)

$$f(t) - \sum_{k=0}^{N} f_k(t) = O(f_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.5}$$

(ii)

$$f(t) - \sum_{k=0}^{N} f_k(t) = f_{N+1}(t)(1 + o(1)) \quad (\forall N \in \mathbb{N}) \tag{1.6}$$

(iii) There is function $\nu : \mathbb{N} \mapsto \mathbb{N}$, such that $\nu(N) \geqslant N$ and

$$f(t) - \sum_{k=0}^{\nu(N)} f_k(t) = O(f_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.7}$$

Condition (iii) seems strictly weaker, but it is not. It allows us to use less accurate estimates of remainders, provided we can do so to all orders.

PROOF We only show that (iii) implies f is asymptotic to $\tilde{f}$ since the other statements immediately follow from the definition. We may assume $\nu(N) > N$, as otherwise there is nothing to prove. Let $N \in \mathbb{N}$. We have

$$f(t) - \sum_{k=0}^{N} f_k(t) \;=\; f(t) - \sum_{k=0}^{\nu(N)} f_k(t) + \sum_{j=N+1}^{\nu(N)} f_j(t) \;=\; O\left(f_{N+1}(t)\right) \quad (1.8)$$

since in the sum $\sum_{j=N+1}^{\nu(N)}$ in (1.8), each term is $O(f_{N+1})$ while the number of terms is fixed, and thus the sum remains $O\left(f_{N+1}\right)$ as $t \to t_0$. ∎

Of course, in practice the asymptotic scale is chosen to consist of simple functions, such as powers, logs and exponentials, the behavior of which is manifest. Taylor series are perhaps the simplest nontrivial asymptotic expansions. The following is a way of restating Taylor's theorem with remainder.

Proposition 1.9 *Assume f is C^∞ in an interval containing t_0. Then*

$$f(t) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k \text{ as } t \to t_0 \quad (1.10)$$

Clearly, the asymptotic series of a function f converges to f **iff** f is analytic at t_0. Otherwise, the series is not convergent, or it converges to a function other than f (see Example 1.16).

Note 1.11 As mentioned in Definition 1.2 none of the f_k is allowed to vanish. For instance, although the function $\varphi = x \mapsto e^{-1/x^2}$ for $x \neq 0$ and $\varphi(0) = 0$ is in $C^\infty(\mathbb{R})$ and we have $(\forall n)\,(\varphi^{(n)}(0) = 0)$, we cannot write $\varphi \sim 0$ as $x \to 0$. This is a natural restriction since all the derivatives vanish at zero for many other functions, for instance $\psi = x \mapsto \sin(1/x)e^{-1/\sqrt{|x|}}$ for $x \neq 0$ and $\psi(0) = 0$ has the same property; yet ψ has a quite different behavior from φ as $x \to 0$. We will also define *asymptotic power series*, a weaker notion in which sense φ and ψ above will be represented by the same series.

Example 1.12 (A divergent asymptotic series) A simple example of a divergent asymptotic expansion is obtained by calculating the Taylor series of the function

$$f(z) = -\frac{1}{z}e^{1/z}\mathrm{Ei}\left(-\frac{1}{z}\right) = \int_0^\infty \frac{e^{-t}}{1+zt}dt;\ z > 0 \quad (1.13)$$

where $\mathrm{Ei}(\tau) = \int_{-\infty}^{\tau} t^{-1}e^t dt$, $(\tau < 0)$ is the exponential integral. The exponential decay of the integrand and elementary analysis show that f is C^∞ at zero from the right (in the sense of right derivatives [73]) and the derivatives are

$$f^{(k)}(z) = k!\int_0^\infty \frac{(-t)^k e^{-t}}{(1+zt)^{k+1}}dt \;\Rightarrow\; f^{(k)}(0) = k!(-1)^k \int_0^\infty t^k e^{-t}dt = (-1)^k (k!)^2 \quad (1.14)$$

and thus

$$f(z) \sim \sum_{k=0}^{\infty} (-1)^k k! z^k, \ z \downarrow 0 \tag{1.15}$$

a series with zero radius of convergence, or in short a *divergent series*.

Example 1.16 (A convergent asymptotic series) Since all derivatives of $e^{-1/z}$ vanish as $z \downarrow 0$ we have

$$\frac{1}{1-z} + e^{-1/z} \sim \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \ z \to 0^+ \tag{1.17}$$

Convergence of an asymptotic series does not thus imply that the function equals the sum of the series. Note also that here, as it is often done in practice, we have used the *same* notation $\sum_{k=0}^{\infty} z^k$ to mean two different things: an asymptotic series simply displaying the asymptotic scale involved, which is a formal object, and its *sum*, an actual function.

Example 1.18 (A convergent but *anti*asymptotic series) The following Laurent series converges in $\mathbb{C} \setminus \{0\}$:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k! z^k} = e^{-1/z} \tag{1.19}$$

Eq. (1.19) is **not** an asymptotic expansion as $z \to 0$. In (1.19), the functions $f_k := (-1)^k z^{-k}/k!$ satisfy $f_k \ll f_{k+1}$ as $z \to 0$ the **opposite** of what is required from an asymptotic series. We have $|e^{-1/z} - \sum_{k=0}^{M} (-1)^k/(k! z^k)| \gtrsim |z^{-M-1}|$ as $z \to 0$ which means that keeping the same number of terms, the approximation deteriorates as $z \to 0$.

In general, for understanding the behavior of a function near a point, an antiasymptotic series, even if convergent, is not very useful. We can see that if we try to determine whether

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k! + \sin k) z^k} \tag{1.20}$$

tends zero or not, as $z \to 0$ (note that for large k the Laurent coefficients of f in (1.20) are close to those in (1.19)).

By contrast, although (1.15) is divergent, by the definition of an asymptotic series, in (1.13) we see that $f(z) \to 1$ as $z \downarrow 0$, and that $f(z) - 1 = -z(1 + o(1))$ and so on.

Stirling's formula for $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, which will be derived in §2.4d, is an example of a divergent asymptotic expansion, where the scales involve powers of $1/x$ and logs:

$$\ln(\Gamma(x)) \sim (x - 1/2)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{\infty} c_j x^{-2j+1}, \ x \to +\infty \tag{1.21}$$

where $2j(2j-1)c_j = B_{2j}$ and $\{B_{2j}\}_{j\geqslant 1} = \{1/6, -1/30, 1/42...\}$ are Bernoulli numbers, [1], eq. 6.140. This expansion is asymptotic as $x \to \infty$: successive terms get smaller and smaller. For $x = 6$, truncating (1.21) at $j = 3$ we get $\Gamma(6) \approx 120.0000002$ (while $\Gamma(6) = 5! = 120$). Here, $j = 3$ was chosen using summation to the least term, a technique we will later explain (see Note 2.302 in §2.11); rigorous error bounds can be obtained using a form of alternating series criterion, see §2.11. Stirling's expansion converges for no x, since $\Gamma(x)$ has poles at all $x \in -\mathbb{N}$ (why is this an obstruction to convergence?).

Remark 1.22 Asymptotic expansions cannot be added (or subtracted), in general. Indeed, we note that $1/(1-z)$ has the same expansion (1.17) as $-e^{-1/z} + 1/(1-z)$, as $z \downarrow 0$. Subtracting these would give $e^{-1/z} \sim 0$, which is not a valid asymptotic expansion, see Note 1.11. This is one reason for considering, for restricted expansions, a weaker asymptoticity condition; see §1.1c.

Remark 1.23 Sometimes we encounter oscillatory expansions such as $\sin x(1 + a_1x^{-1} + a_2x^{-2} + \cdots)$ $(*)$ for large x, which, while very useful, have to be understood differently. They are not asymptotic expansions, as we saw in Note 1.11. Furthermore, usually the approximation itself is expected to fail near zeros of sin. However, if small neighborhoods of the zeros of sin are excluded, the expansion remains valid in the sense defined. Also, usually there are ways to present the asymptotics in a way that avoids these exclusions,(see §**??**).

1.1c Asymptotic power series

A special role is played by series in *powers* of a small variable, such as

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \ \ z \to 0^+ \tag{1.24}$$

With the transformation $z = t - t_0$, a series in powers of $t - t_0$ can be transformed into (1.24). With transformation $z = 1/x$, (1.24) may be transformed into a $1/x$-series or *viceversa.*

Definition 1.25 (Asymptotic power series) *A function is asymptotic to a series as $z \to 0$,* in the sense of power series *if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \quad \textit{as } z \to 0, \tag{1.26}$$

where, as for general asymptotic expansions, it may be necessary to restrict the approach $z \to 0$ to a particular set of curves.

Remark 1.27 If f has an asymptotic expansion (in the sense of Definition 1.2) that happens to be a power series, it is asymptotic to it in the sense of power series as well.

However, the converse is not true, unless all c_k are nonzero, *i.e.* it is possible that $f \sim \tilde{f} \equiv \sum_{k=0}^{\infty} c_k z^k$ in the power series sense, without $\tilde{f}$ being the asymptotic expansion in the sense of Definition 1.2.

For now, whenever confusions are possible, we will use a different symbol, $\sim_p$, for asymptoticity in the sense of power series.

Remark 1.28 Noninteger asymptotic power series, e.g., series of the form

$$z^{\alpha} \sum_{k=0}^{\infty} c_k z^{k\beta}, \quad \mathrm{Re}\,(\beta) > 0 \tag{1.29}$$

as well as asymptoticity of a function to (1.29) can be defined by easily adapting Definition 1.25, and replacing $O(z^N)$ by $O(z^{N\beta+\alpha})$ which is the same as $O(z^{\mathrm{Re}\,\alpha + N\mathrm{Re}\,(\beta)})$. More generally, in (1.29), instead of z^{α}, we could have other simple functions such as exponentials or logs.

The asymptotic power series at zero in $\mathbb{R}$ of e^{-1/z^2} is the zero series, which is not its asymptotic expansion in the sense of Definition 1.2, see again Note 1.11. The advantage of asymptotic power series however is the fact that they form a commutative algebra, with restricted inversion (if the constant term of $\tilde{g}$ is nonzero, then $1/\tilde{g}$ is also a power series).

1.1d Operations with asymptotic power series

Addition and multiplication of asymptotic power series are defined as in the convergent case:

$$A \sum_{k=0}^{\infty} c_k z^k + B \sum_{k=0}^{\infty} d_k z^k = \sum_{k=0}^{\infty} (Ac_k + Bd_k) z^k$$

$$\left(\sum_{k=0}^{\infty} c_k z^k \right) \left(\sum_{k=0}^{\infty} d_k z^k \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} c_j d_{k-j} \right) z^k$$

Remark 1.30 *If the series $\tilde{f}$ is convergent and f is its sum, $f = \sum_{k=0}^{\infty} c_k z^k$, (note the ambiguity of the sum notation), then $f \sim_p \tilde{f}$.*

The proof follows directly from the definition of convergence.

The proof of the following lemma is immediate:

Lemma 1.31 (Algebraic properties of asymptoticity to a power series) *If $f \sim_p \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ and $g \sim_p \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$, then*

(i) $Af + Bg \sim_p A\tilde{f} + B\tilde{g}$

(ii) $fg \sim_p \tilde{f}\tilde{g}$

(iii) $f/g \sim_p \tilde{f}/\tilde{g}$ *if* $d_0 \neq 0$

Corollary 1.32 (Uniqueness of the asymptotic series to a function) *If $f(z) \sim_p \sum_{k=0}^{\infty} c_k z^k$ as $z \to 0$, then the c_k are unique.*

PROOF Indeed, if $f \sim_p \sum_{k=0}^{\infty} c_k z^k$ and $f \sim_p \sum_{k=0}^{\infty} d_k z^k$, then, by Lemma 1.31 we have $0 \sim_p \sum_{k=0}^{\infty}(c_k - d_k)z^k$ which implies, inductively, that $c_k = d_k$ for all k. ◻

Of course, the asymptotic behavior of many functions, such as e^{1/z^2} near $z = 0$, cannot be described by power series. Also, asymptotic power series cannot distinguish between functions differing by a quantity which is $o(z^m)$ for all $m > 0$ as $z \to 0$. Indeed, we have the following result (see also Example 1.16)

Proposition 1.33 *Assume f and g have nonzero asymptotic power series as $z \to 0$ and $f - g = h$ where $h = o(z^m)$ for all $m > 0$ as $z \to 0$. Then the asymptotic series of f and g coincide.*

PROOF This follows straightforwardly from Definition 1.26 and the assumption on h. ◻

1.1d.1 Integration and differentiation of asymptotic power series

Asymptotic relations can be integrated termwise as Proposition 1.34 below shows.

Proposition 1.34 *Assume that*

$$f(z) \underset{p}{\sim} \tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k \text{ as } z \to 0^+$$

Then

$$\int_0^z f(s)ds \underset{p}{\sim} \int_0^z \tilde{f}(s)ds := \sum_{k=0}^{\infty} \frac{c_k z^{k+1}}{k+1} \text{ as } z \to 0^+$$

The direction $z \to 0^+$ can be replaced by $ze^{i\varphi} \to 0^+$, φ fixed depending on the properties of f.

PROOF This follows from the fact that for $n > 0$, $\int_0^{|z|} C|s|^n d|s| \leqslant C|z|^{n+1}$. ◻

Differentiation is a different issue. Many simple examples show that asymptotic series cannot be unrestrictedly differentiated. For instance $e^{-1/z^2} \sin e^{1/z^4} \sim_p 0$ as $z \to 0$ on $\mathbb{R}$, but the derivative is unbounded and thus it is not asymptotic to the zero series at zero.

1.1d.2 Asymptotics in regions in $\mathbb{C}$

Asymptotic power series of analytic functions can be differentiated if they hold in a region which is not too rapidly shrinking as $z \to 0$. This is so, since the derivative is expressible as an integral by Cauchy's formula. Such a region is often a sector or strip in $\mathbb{C}$, but can be allowed to be thinner. In the folowing, we formulate this condition in the variable $x = 1/z$:

Proposition 1.35 *Let $M \geqslant 0$, $a > 0$. Denote*

$$S_a = \{x : \quad |x| \geqslant R, \quad |x|^M |\mathrm{Im}\,(x)| \leqslant a\}$$

Assume f is continuous in S_a and analytic in its interior, and

$$f(x) \underset{p}{\sim} \sum_{k=0}^{\infty} c_k x^{-k} \quad \text{as } x \to \infty \text{ in } S_a$$

Then, for all $a' \in (0, a)$ we have

$$f'(x) \underset{p}{\sim} \sum_{k=0}^{\infty} (-kc_k) x^{-k-1} \quad \text{as } x \to \infty \text{ in } S_{a'}$$

PROOF Here, Proposition 1.4 (iii) will come in handy. Let $\nu(N) = N+M$. By the asymptoticity assumptions, for any N there is some constant $C(N)$ such that $|f(x) - \sum_{k=0}^{\nu(N)} c_k x^{-k}| \leqslant C(N)|x|^{-\nu(N)-1}$ (*) in S_a.

Let $a' < a$, take x large enough, and let $\rho = \frac{1}{2}(a - a')|x|^{-M}$; then check that $\mathbb{D}_\rho = \{x' : |x - x'| \leqslant \rho\} \subset S_a$. We have, by Cauchy's formula and (*),

$$\left| f'(x) - \sum_{k=0}^{\nu(N)} (-kc_k) x^{-k-1} \right| = \left| \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_\rho} \left(f(s) - \sum_{k=0}^{\nu(N)} c_k s^{-k} \right) \frac{ds}{(s-x)^2} \right|$$
$$\leqslant \frac{C(N)}{(|x|-1)^{\nu(N)+1}} \frac{1}{2\pi} \oint_{\partial \mathbb{D}_\rho} \frac{d|s|}{|s-x|^2} \leqslant \frac{2C(N)}{|x|^{\nu(N)+1}\rho} \leqslant \frac{4C(N)}{a-a'} |x|^{-N-1} \tag{1.36}$$

and the result follows. ∎

Note 1.37 Usually, we can determine from the context whether $\sim$ or $\sim_p$ should be used. Often in the literature, it is left to the reader to decide which notion is in use. After we have explained the distinction, we will do the same, so as not to complicate notation.

1.2 Asymptotics of integrals

Often when differential equations have closed form solutions, these can be expressed in terms of elementary functions or integral transforms of elementary functions. These integral representations allow for asymptotic analysis and more generally, global analysis of solutions, and, for this reason they are very important. Most "named" special functions have integral representations. For $\sigma = \pm 1$, the equation

$$x^2 y'' + xy' + \sigma(x^2 - \sigma\nu^2)y = 0; \quad \sigma = \pm 1 \tag{1.38}$$

is the Bessel equation [1]; if $\sigma = 1$, the solution which is regular at the origin is $J_\nu(x)$ – the Bessel function of the first kind and a linearly independent one is $Y_\nu(x)$ – the Bessel function of the second kind. For $\sigma = -1$ (1.38) is the modified Bessel equation; the solution which is regular at the origin is $I_\nu(x)$ – the modified Bessel function of the first kind and a linearly independent one is $K_\nu(x)$ – the modified Bessel function of the second kind. The Airy equation

$$y'' - xy = 0 \tag{1.39}$$

has solutions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, the Airy functions. The hypergeometric equation

$$x(x-1)y'' + [(a+b+1)x - c]y' + aby = 0 \tag{1.40}$$

has linearly independent solutions ${}_2F_1(a,b;c;x)$ and $x^{1-c}\,{}_2F_1(a-c+1, b-c+1; 2-c; x)$ where ${}_2F_1$ is a hypergeometric function. All these functions have integral representations, in fact a good number of representations suitable for different asymptotic regimes. For instance, see [27] 10.9.17, [8] (Equation 6.6.30, page 298),

$$J_\nu(z) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} \exp(z\sinh t - \nu t)dt; \quad \mathrm{Re}\ z\ >\ 0 \tag{1.41}$$

and [27] 9.5.4, and [8] (p. 313, Problem 6.75, with the change of integration variable $t \to -t$).

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp\left(t^3/3 - zt\right)\, dt, \tag{1.42}$$

Finally, for $|z| < 1$ we have, [27] 15.1.2 and 15.6.1,

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \ \mathrm{Re}(c) > \mathrm{Re}(b) > 0 \tag{1.43}$$

1.2a The Laplace transform and its properties.

The Laplace transform of a function F, denoted by $\mathcal{L}F$, is defined by

$$f(x) = \int_0^\infty e^{-xp} F(p)dp, \quad \mathrm{Re}\,(x) > \nu \geqslant 0 \tag{1.44}$$

Here it is assumed that F is locally integrable in $[0,\infty)$ and does not grow faster than exponentially, for instance

$$\|F\|_{\infty,\nu} = \sup_{p\geqslant 0} |F(p)|e^{-\nu p} < \infty \text{ or } \|F\|_{L^1_\nu} = \int_0^\infty |F(p)|e^{-\nu p}dp < \infty \tag{1.45}$$

(see §2.13a) for some $\nu \in \mathbb{R}$. Both ensure the existence of $\mathcal{L}f$ if $\mathrm{Re}\, x > \nu$.

As will be seen in the sequel, solutions of linear or nonlinear differential equations, including (1.42) and (1.41) above, can often be written as Laplace transforms of simpler functions. It is then important to understand the asymptotic behavior of Laplace transforms. A general asymptotic result is the following:

Lemma 1.46 *Under the assumption in* (1.45), *we have*

$$\int_0^\infty e^{-xp} F(p)dp \to 0 \text{ as } \mathrm{Re}\,(x) \to \infty \tag{1.47}$$

PROOF This follows from the dominated convergence theorem, see §2.13a. Indeed, $\int_0^\infty |e^{-xp}F(p)|dp \leqslant \int_0^\infty |e^{-x_0 p}F(p)|dp < \infty$ for $\mathrm{Re}\,(x) \geqslant x_0 > \nu$, and $e^{-xp}F(p) \to 0$ as $\mathrm{Re}\,(x) \to \infty$ for all $p \in (0,\infty)$. ∎

Furthermore, convergence is exponentially fast iff F is identically zero on some interval $[0,\varepsilon)$, where $\varepsilon > 0$ is independent of x as shown in the following proposition. For the notation, see §2.13a.

Proposition 1.48 *Assume that F is exponentially bounded in the sense of* (1.45)*; let $x_1 = \mathrm{Re}\,(x)$. Then*

$$\int_0^\infty e^{-xp} F(p)dp = o(e^{-x_1\varepsilon}) \quad \text{as } x_1 \to \infty \textbf{ iff } F = 0 \quad \text{a.e.}^1 \text{ on } [0,\varepsilon] \text{ as } x_1 \to \infty \tag{1.49}$$

Also, $\int_0^\infty e^{-xp}F(p)dp = O(e^{-x_1\varepsilon}) \Leftrightarrow \int_0^\infty e^{-xp}F(p)dp = o(e^{-x_1\varepsilon})$, *implying* $F = 0$ *a.e. on* $[0,\varepsilon]$.

PROOF (i) Assume that $F = 0$ a.e. on $[0,\varepsilon)$. This implies that

$$\int_0^\infty e^{-xp}F(p)dp = \int_\varepsilon^\infty e^{-xp}F(p)dp = e^{-x\varepsilon}\int_0^\infty e^{-xp}F(p+\varepsilon)dp = e^{-x_1\varepsilon}o(1) \tag{1.50}$$

as $x_1 \to \infty$ by Lemma 1.46.

(ii) For the converse, assume that $\int_0^\infty e^{-xp}F(p)dp = O(e^{-x_1\varepsilon})$. We write

$$\int_0^\infty e^{-xp}F(p)dp = \int_0^\varepsilon e^{-xp}F(p)dp + \int_\varepsilon^\infty e^{-xp}F(p)dp. \tag{1.51}$$

The rightmost integral in (1.51) is shown to be $o(e^{-x_1\varepsilon})$ by using the change variable $p \to p + \varepsilon$ and using Lemma 1.46. Thus

$$g(x) := e^{x\varepsilon}\int_0^\varepsilon e^{-xp}F(p)dp = O(1) \text{ as } x_1 = \text{Re } x \to +\infty \tag{1.52}$$

It is easy to see that g is entire. Furthermore, it is bounded for $x \in \mathbb{R}^+$ by (1.52) and also manifestly bounded for $x \in i\mathbb{R}$, and $x \in \mathbb{R}^-$. Since g is of exponential order 1, using the Phragmén-Lindelöff theorem in all of the four quadrants (see [21] pp. 19 and 23 for more details) shows g is bounded. From Liouville's theorem, g is a constant. The Riemann-Lebesgue lemma implies that g goes to zero as $x \to \infty$ along the imaginary line. Thus $g = 0$, implying $\int_0^\varepsilon F(p)e^{-px}dp = 0$, $\forall x \in \mathbb{C}$ implying that the Fourier transform $\int_{-\infty}^\infty e^{-itp}\chi_{[0,\varepsilon]}(p)F(p)dp = 0$ $\forall t \in \mathbb{R}$ and thus, by inverse Fourier transform, $F(p) = 0$ *a.e.* on $(0, \varepsilon)$. Now, (i) implies that $\int_0^\infty F(p)e^{-px}dp = o\left(e^{-\varepsilon x_1}\right)$. ∎

Corollary 1.53 (Injectivity of the Laplace transform) *Under the condition* (1.45)*, if* $\mathcal{L}F = 0$ *for all* $x > 0$*, then* $F = 0$ *a.e. on* $\mathbb{R}^+$*.*

PROOF Since, in particular, $\mathcal{L}F = O(e^{-xa})$ for any $a > 0$, from Proposition 1.48, $F = 0$ a.e. on $\mathbb{R}^+$. ∎

First inversion formula

Let $\mathcal{H}$ denote the space of analytic functions in the right half complex plane.

Proposition 1.54 *(i)* $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ *and* $\|\mathcal{L}F\|_\infty \leqslant \|F\|_1$.

(ii) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{L}(L^1(\mathbb{R}^+)) \subset \mathcal{H}$ *is invertible, and the inverse is given by*

$$F = \hat{\mathcal{F}}^{-1}\{\mathcal{L}F(i\cdot)\} \tag{1.55}$$

on $\mathbb{R}^+$ *where* $\hat{\mathcal{F}}$ *is the Fourier transform (in distributions if* $\mathcal{L}F \notin L^1(i\mathbb{R})$*).*

In the following, $\mathbb{H}$ will denote the open right half plane.

PROOF (i) The fact that $\mathcal{L}F$ is analytic in $\mathbb{H}$ follows from the exponential decay of the integrand: by dominated convergence we can differentiate in x under the integral sign. The estimate follows simply from the fact that $|e^{-xp}| < 1$.

(ii) We note that $(\mathcal{L}F)(it)$ exists since $F \in L^1$, and it is, by definition the Fourier transform of F extended by $F(p) = 0$ for $p < 0$. The rest is just Fourier inversion, in the in a generalized sense –in distributions–if $\mathcal{L}F \notin L^1(i\mathbb{R})$. ∎

Second inversion formula

Proposition 1.56 *(i) Assume f is analytic in an open sector $\mathbb{H}_\delta := \{x : |\arg(x)| < \pi/2 + \delta\}$, $\delta \geqslant 0$ and is continuous on $\partial\mathbb{H}_\delta$, and that for some $K > 0$ and any $x \in \overline{\mathbb{H}_\delta}$ we have*

$$|f(x)| \leqslant K(|x|^2 + 1)^{-1} \tag{1.57}$$

Then $\mathcal{L}^{-1}f$ is well defined by

$$F = \mathcal{L}^{-1}f = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}t\, e^{pt} f(t) \tag{1.58}$$

and

$$\int_0^\infty \mathrm{d}p\, e^{-px} F(p) = \mathcal{L}\mathcal{L}^{-1}f = f(x) \tag{1.59}$$

We have $\|\mathcal{L}^{-1}\{f\}\|_\infty \leqslant K/2$ and $\mathcal{L}^{-1}\{f\} \to 0$ as $p \to \infty$.

(ii) If $\delta > 0$, then $F = \mathcal{L}^{-1}f$ is analytic in the sector $S_\delta = \{p \neq 0 : |\arg(p)| < \delta\}$. In addition, $\sup_{S_\delta} |F| \leqslant K/2$ and $F(p) \to 0$ as $p \to \infty$ along rays in S_δ.

(iii) If $\mathcal{L}^-$ is taken on a vertical line through $x = c > 0$, then $F(p)e^{-cp} \to 0$ as $p \to \infty$ along rays in S_δ.

Note 1.60 We have assumed (1.57) for simplicity; however, it can be replaced by any bound that would allow for applying Jordan's lemma to deform the contour as in the proof below.

Note 1.61 One can easily adapt these results for Laplace/inverse Laplace transforms along other directions in $\mathbb{C}$. Assume that $f(xe^{i\varphi})$ is analytic in an open sector of opening $\pi/2 + \delta$ centered along the real line, and that the bound (1.57) holds in the closure of this sector. Then, check that

$$F(p') = (\mathcal{L}_\varphi^{-1})(f)(p') = \frac{1}{2\pi i}\int_\Gamma e^{xp'} f(x) dx \tag{1.62}$$

where Γ is a line from $(c - i\infty)e^{i\varphi}$ to $(c + i\infty)e^{i\varphi}$, exists and satisfies (ii), (iii) above, if $p' = pe^{-i\varphi}$.

PROOF Clearly, F in (1.58) is well-defined since $f(is) \in L^1(\mathbb{R})$. (i) We have

$$2\pi i\mathcal{L}\left[\mathcal{L}^{-1}f\right](x) = \int_0^\infty \mathrm{d}p\, e^{-px}\int_{-\infty}^\infty i\mathrm{d}s\, e^{ips} f(is) \tag{1.63}$$

$$= \int_{-\infty}^\infty i\mathrm{d}s\, f(is)\int_0^\infty \mathrm{d}p\, e^{-px}e^{ips} = \int_{-i\infty}^{i\infty} f(z)(x-z)^{-1}\mathrm{d}z = 2\pi i f(x) \tag{1.64}$$

where we applied Fubini's theorem[2] and then pushed the contour of integration past x to infinity. The norm of $\mathcal{L}^{-1}$ is obtained by majorizing $|f(x)e^{px}|$ by $K(|x^2|+1)^{-1}$. The behavior $\left[\mathcal{L}^{-1}f\right](p) \to 0$ as $p \to +\infty$ follows by applying Riemann-Lebesgue Lemma to (1.58).

(ii) For any $\delta' < \delta$ we have, by (1.57),

$$\int_{-i\infty}^{i\infty} \mathrm{d}s\, e^{ps} f(s) = \left(\int_{-i\infty}^{0} + \int_{0}^{i\infty} \right) \mathrm{d}s\, e^{ps} f(s)$$
$$= \left(\int_{-i\infty e^{-i\delta'}}^{0} + \int_{0}^{i\infty e^{i\delta'}} \right) \mathrm{d}s\, e^{ps} f(s) \quad (1.65)$$

Take any $p \in S_\delta$. Choose $\delta' < \delta$ so that $p \in S'_\delta$. Analyticity of (1.65) in $p \in S'_\delta$ is manifest, given the analyticity and exponential decay of the integrand. For the estimates on $F(p)$, we note that (i) applies to $f(xe^{i\varphi})$ if $|\varphi| < \delta$.

(iii) This follows simply by changing the integration variable to $p' = p + c$.

Many cases can be reduced to (1.57) after transformations. For instance if $f_1 = \sum_{j=1}^{N} a_j(1+x)^{-k_j} + f(x)$, **where $k_j > 0$ and f satisfies the assumptions above, then (1.58) and (1.59) apply to f_1, since they do apply, by straightforward verification, to the finite sum.

□

Proposition 1.66 *Let F be analytic in the open sector $S_p = \{e^{i\varphi}\mathbb{R}^+ : \varphi \in (-\delta, \delta)\}$ and such that $|F(|p|e^{i\varphi})| \leqslant g(|p|) \in L^1[0,\infty)$. Then $f = \mathcal{L}F$ is analytic in the sector $S_x = \{x : |\arg(x)| < \pi/2 + \delta\}$ and $f(x) \to 0$ as $|x| \to \infty, \arg(x) = \theta \in (-\pi/2 - \delta, \pi/2 + \delta)$.*

PROOF Because of the analyticity of F and the decay conditions for large p, the path of Laplace integration can be rotated by any angle φ in $(-\delta, \delta)$ without changing[3] $(\mathcal{L}F)(x)$. The fact that$g \in L^1$ also implies that The decay of $(\mathcal{L}F)(x)$ in x follows from Lemma 1.46 with x replaced by $xe^{-i\varphi}$ and φ chosen $\arg\left(xe^{-i\varphi}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ □

Note F need not be analytic at $p = 0$ for Proposition 1.66 to apply.

[2]This theorem addresses the permutation of the order of integration; see [74]. Essentially, if $f \in L^1(A \times B)$, then $\int_{A\times B} f = \int_A \int_B f = \int_B \int_A f$.

[3]The fact that $g \in L^1$ implies that $\liminf_{R\to\infty} Rg(R) = 0$; thus there is a subsequence R_n s.t. $R_n g(R_n) \to 0$. By straightforward estimates, or by Jordan's lemma, we see that the integral of Fe^{-px} along an arc of a circle of radius R_n goes to zero with n.

1.2b Watson's Lemma

Heuristics. Consider the following asymptotic problem: what is the behavior of

$$\int_{-a}^{a} e^{xG(p)}h(p)dp, \quad x \to +\infty$$

where $a > 0$, G is real valued, smooth enough, and has a unique maximum point $p_m \in [-a, a]$. Intuitively, it is clear that, as $x \to \infty$ most the contribution to the integral will come from an increasingly narrow region around p_m, since for fixed $p \neq p_m$, as $x \to \infty$, $e^{xG(p)} \ll e^{xG(p_m)}$.

Watson's lemma is a very useful tool to transform this intuition into proofs, as well as to deal with the asymptotics of a large class of other integrals arising in applications, which often can be transformed so that Watson's Lemma is applicable.

Before stating the theorem let us look at the following example, the asymptotics of the incomplete Gamma function which we will need later.

Lemma 1.67 *Let* $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\varphi}$, $\rho > 0$, $\varphi \in (-\pi/2, \pi/2)$ *and assume*

$$F(p) \sim p^{\beta} \quad \textit{as } p \to 0^+$$

with $\mathrm{Re}\,(\beta) > -1$. *Then*

$$\int_0^{\infty} F(p)e^{-px}dp \sim \Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. By definition $F(p) \sim p^{\beta}$ means $p^{-\beta}F(p) \to 1$ as $p \to 0$. We have

$$\int_0^{\infty} e^{-xp}F(p)dp = \frac{1}{x}x^{\beta}\int_0^{\infty}\frac{F(t/x)}{(t/x)^{\beta}}e^{-t}t^{\beta}dt \sim \frac{\Gamma(\beta+1)}{x^{\beta+1}} \ (x \to \infty) \tag{1.68}$$

where we used dominated convergence.

Corollary 1.69 *If* $x^{\alpha}\int_0^{\infty} e^{-xp}F(p)dp$ *has an asymptotic power series in* $z = x^{-\beta}$ *for some* β *with* $\mathrm{Re}\,\beta > 0$ *as* $\mathrm{Re}\,x \to \infty$, *then for any fixed* $\varepsilon > 0$, $x^{\alpha}\int_0^{\varepsilon} e^{-xp}F(p)dp$ *has an asymptotic power series as well, and the two power series agree.*

PROOF This is an immediate consequence of Propositions 1.48 and 1.33.
□

Watson's lemma allows us to integrate power series term by term as stated below.

Lemma 1.70 (Watson's lemma) *(i) Assume that* $\|F\|_{L^1_\nu} < \infty$ *(cf.* (1.45)*) and*

$$F(p) = p^{\alpha-1}\sum_{k=0}^{m} c_k p^{k\beta} + o(p^{\alpha-1+m\beta}) \ \textit{as } p \to 0^+ \textit{ for all } m \leqslant m_0 \in \mathbb{N}\cup\infty \tag{1.71}$$

for some α *and* β*, with* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$*. Then as* $x \to \infty$ *along an arbitrary ray in* $\mathbb{H}$*, we have*

$$f(x) := (\mathcal{L}F)(x) = \int_0^\infty e^{-xp} F(p)dp = \sum_{k=0}^{m} c_k \Gamma(k\beta+\alpha) x^{-\alpha-k\beta} + o\left(x^{-\alpha-m\beta}\right). \tag{1.72}$$

for any $m \leqslant m_0$*.*

(ii) If $F \in L^1(0,a)$ *and* (1.71) *holds, then the asymptotic expansion* (1.72) *holds for* $f(x) = \int_0^a F(p)e^{-px}dp$ *(*a *independent of* x*).*

PROOF The proof is straightforward from Lemma 1.67, Corollary 1.69 applied to $(F(p) - p^{\alpha-1}\sum_{k=0}^{m-1} c_k p^{k\beta})/c_m$, and Laplace transforming explicitly the finite sum of powers. ∎

Note 1.73 (i) Intuitively, we see that, for a fixed F, the larger $\operatorname{Re} x$ is, the more damped is the contribution of any region that is not very close to zero. The behavior of a Laplace transform is gotten from the immediate neighborhood of zero.

(ii) We see that the power series of F at *zero* can be Laplace transformed term by term to obtain the asymptotic expansion of $f(x)$ *as* $x \to \infty$ *along a ray in* $\mathbb{H}^+$. From the proof we see that the same conclusion would hold if instead of $p^{k\beta}$ we had other asymptotic representations of F for small p, for instance in terms of $p^{k\beta}(\log p)^m$.

Note 1.74 Watson's lemma holds for $\int_0^{ae^{i\theta}} F(p)e^{-px}dp$ as $|x| \to \infty$ if the asymptotic behavior (1.71) is valid along a ray $\arg p = \theta$, where $F \in L^1(0, ae^{i\theta})$ $\arg(x)$ satisfies $\theta + \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The proof is manifest by changing variables $p \to pe^{i\theta}$, $x \to xe^{-i\theta}$ and applying Lemma 1.70.

Exercise 1.75 (A generalization of Watson's Lemma) Assume that for some $\varepsilon > 0$, we have $\sup_{|z|<\varepsilon} \|F(\cdot;z)\|_{L^1_\nu} = C < \infty$ and that

$$F(p;z) = p^{\alpha-1} \sum_{\substack{0\leqslant k\leqslant m \\ 0\leqslant l\leqslant n}} c_{k,l}\, p^{k\beta_1} z^{l\beta_2} + o(p^{\alpha-1+m\beta_1} z^{n\beta_2})$$
$$\text{as } (p,z) \to (0^+,0) \text{ for all } (m,n) \leqslant (m_0,n_0) \in (\mathbb{N}\cup\infty)^2 \tag{1.76}$$

where $\operatorname{Re}\alpha, \operatorname{Re}\beta_1$ and $\operatorname{Re}\beta_2$ are positive. Then, show that

$$\int_0^\infty e^{-xp} F\left(p, \frac{1}{x}\right) dp = \sum_{\substack{0\leqslant k\leqslant m \\ 0\leqslant l\leqslant n}} \frac{c_{kl}\Gamma(k\beta_1+\alpha)}{x^{\alpha+k\beta_1+l\beta_2}} + o\left(x^{-\alpha-m\beta_1-n\beta_2}\right). \tag{1.77}$$

1.2c Laplace's method

1.2c.1 Laplace asymptotics, minimum of the exponential at an endpoint

Corollary 1.78 *Assume that F is continuously differentiable on $[0, a)$ (as usual when we close the interval we mean right derivative) and $F' > 0$ and g is continuous. Then*

$$\int_0^a e^{-\nu F(x)} g(x)dx \sim e^{-\nu F(0)} \frac{g(0)}{\nu F'(0)} \quad \textit{as } \nu \to \infty \tag{1.79}$$

PROOF By choosing $\tilde{F} = F(x) - F(0)$ we reduce to the case $F(0) = 0$. Since $F' > 0$, F is invertible near zero and, with $h(x) = F^{-1}(x)$, we have

$$\int_0^a e^{-\nu F(x)} g(x)dx = \int_0^{F(a)} e^{-\nu p} g(h(p))h'(p)dp \tag{1.80}$$

By continuity $g(h(p))h'(p) = g(0)h'(0) + o(1)$ as $p \to 0^+$. Noting that $h'(0) = 1/F'(0)$, the rest follows from Watson's lemma. ∎

Exercise 1.81 Assuming $F(0) = 0$ and $g = 1$, we could use the fact that

$$\lim_{\nu\to\infty} \nu \int_0^a e^{-\nu F(x)} g(x)dx = \frac{1}{F'(0)} \quad \text{as } \nu \to \infty \tag{1.82}$$

as a definition of $F'(0)$, namely,

$$F'(0) := \frac{1}{\lim_{\nu\to\infty} \nu \int_0^a e^{-\nu F(x)}dx} \tag{1.83}$$

Clearly, when $F \in C^1[0, 1)$ and $F'(0) \neq 0$, the limit exists. Show that the limit exists and it is zero even when $F'(0) = 0$, provided $F \in C^1[0, 1)$ and $F' > 0$ on $(0, 1)$.

Challenge: Let $F' \in C(0, a)$. Does the existence of a nonzero limit in (1.83) imply that $F'(x)$ has a limit when $x \to 0^+$?

1.2c.2 Laplace asymptotics, minimum of the exponential at an inner point

Corollary 1.84 *Assume that $a > 0$, F is twice continuously differentiable on $(-a, a)$ $F'(0) = 0$ and $F''(x) > 0$ on $(-a, a)$, and that g is continuous*[4]*. Then,*

$$\int_0^a e^{-\nu F(x)} g(x)dx \sim e^{-\nu F(0)} g(0) \sqrt{\frac{\pi}{2\nu F''(0)}} \;\; \textit{as } \nu \to \infty \tag{1.85}$$

[4]The function g may be complex valued.

and

$$\int_{-a}^{a} e^{-\nu F(x)} g(x)dx \sim e^{-\nu F(0)} g(0)\sqrt{\frac{2\pi}{\nu F''(0)}} \text{ as } \nu \to \infty \tag{1.86}$$

PROOF As in Corollary 1.78 without loss of generality we may assume $F(0) = 0$. Define $h(x) = \text{signum}(x)\sqrt{F(x)}$ and denote $\frac{1}{2}F''(0) = \lambda^2$. Clearly h is continuously differentiable away from zero. For x close to zero, we have $F(x) = \lambda^2 x^2 + o(x^2)$ and thus $h(x) = \lambda x + o(x)$ for small x. It is then easy to show that h is continuously differentiable on $(-a, a)$ and $h' > 0$. We calculate only the integral from 0 to a and prove the first part of the corollary. Indeed, the integral from $-a$ to 0 is treated similarly and has an equal contribution to the final estimate in (1.86) . We make the change of variables $h(x) = \sqrt{u}$ and and note that by continuity $g(\sqrt{u})/h'\left(h^{-1}(\sqrt{u})\right) \sim g(0)/h'(0)$ to obtain

$$\int_0^a e^{-\nu h^2(x)} g(x)dx = \int_0^{F(a)} e^{-\nu u} \frac{g(h^{-1}(\sqrt{u}))}{h'(h^{-1}(\sqrt{u}))} \frac{1}{2\sqrt{u}} du \sim \frac{g(0)\sqrt{2\pi}}{2\sqrt{\nu F''(0)}} \tag{1.87}$$

by Watson's lemma and the fact that $\Gamma(1/2) = \sqrt{\pi}$. ∎

Note: Only the leading order asymptotic calculations are given in Corollaries 1.78 and 1.84. Watson's Lemma can be used to determine higher order corrections in the asymptotic expansion if F and g are smooth enough near 0.

Exercise 1.88 *Formulate and prove a generalization of Lemma 1.84 for the case when* $F'(0) = \cdots = F^{(2m-1)}(0) = 0$ *and* $F^{(2m)}(0) > 0$.

Example: Asymptotics of the Γ function The Gamma function is defined by

$$\Gamma(x+1) \equiv x! = \int_0^\infty e^{-\tau}\tau^x d\tau = \int_0^\infty e^{x\log\tau} e^{-\tau} d\tau \tag{1.89}$$

for $x > -1$. [5] $x\log\tau - \tau$ is maximal when $\tau = x$. This suggests rescaling $\tau = x(1+t)$. This leads to

$$\Gamma(x+1) = x^{x+1}e^{-x}\int_{-1}^{\infty} \exp\left[-x\left(t - \log(1+t)\right)\right] dt \tag{1.90}$$

with a maximum of the integrand at $t = 0$. In this form, Corollary 1.84 applies and we get Stirling's formula,

$$\Gamma(x+1) \sim \sqrt{2\pi}x^{x+1/2}e^{-x} \tag{1.91}$$

[5] This representation is valid for complex x as well in the domain $\text{Re}\, x > -1$.

To get more terms in the asymptotic series, we introduce, in the spirit of the proof of Corollary 1.84,

$$q = \sqrt{t - \log(1+t)} = t \left[\frac{t - \log(1+t)}{t^2} \right]^{1/2} \tag{1.92}$$

We extend the term in square brackets by 1 at the removable singularity $t = 0$. It is readily checked that $t \mapsto q(t)$ an analytic change of variable near $t = 0$, with $q'(0) = 1/\sqrt{2}$. Further, $t \to q(t)$ is monotonic and maps the the real axis interval $(-1, \infty)$ to $q \in (-\infty, \infty)$. We define the unique inverse function to be $t = T(q)$ and obtain

$$\Gamma(x+1) = x^{x+1} e^{-x} \int_{-\infty}^{\infty} e^{-xq^2} \quad T'(q) dq \tag{1.93}$$

We decompose the integral in (1.93) as $\int_{-\infty}^{0} + \int_{0}^{\infty}$. We introduce change of variable $q = -\sqrt{p}$ in the first integral and $q = \sqrt{p}$ in the second to obtain

$$\Gamma(x+1) = \frac{1}{2} x^{x+1} e^{-x} \int_0^{\infty} \frac{e^{-px}}{\sqrt{p}} \left(T'(-\sqrt{p}) + T'(\sqrt{p}) \right) dp \tag{1.94}$$

Using Taylor series $T(q) = \sum_{j=1}^{\infty} 2^{j/2} b_j q^j$,

$$\frac{1}{2\sqrt{p}} \left(T'(-\sqrt{p}) + T'(\sqrt{p}) \right) = \sum_{j=1, j=odd}^{\infty} 2j b_j \quad (2p)^{j/2-1}. \tag{1.95}$$

It follows from Watson's Lemma that

$$\Gamma(x+1) \sim x^{x+1} e^{-x} \sum_{j=1, j=odd}^{\infty} 2^{j/2} \Gamma(j/2) j b_j x^{-j/2} \tag{1.96}$$

The first few b_j are easily computed by substituting a truncation of $t = b_1 q + b_2 q^2 + b_3 q^3 + ..$ into (1.92) and equating like powers of q and solving resulting equations. This gives $b_1 = 1$, $b_3 = \frac{1}{36}$, $b_5 = \frac{1}{4320}$, the even b_j's being inconsequential in (1.96). Using $\Gamma(1/2) = \sqrt{\pi}$, the first few nonzero terms are

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + O(x^{-3}) \right) \tag{1.97}$$

The three term evaluation at $x = 6$ gives 720.0088692 versus the exact value of 720. If the general term in the asymptotic expansion (1.96) is desired, we can use Lagrange formula for inversion of a series:

$$\begin{aligned} b_j = \frac{1}{2\pi i} \oint \frac{T(q)}{q^{j+1}} dq &= \frac{1}{2\pi i} \oint t^2 (1+t)^{-1} \left[2t - 2\log(1+t) \right]^{-j/2-1} dt \\ &= \frac{2^{-j/2-1}}{2\pi i} \oint \frac{(e^u - 1)^2}{(e^u - 1 - u)^{j/2+1}} du, \end{aligned} \tag{1.98}$$

where the closed loop contour integrals are assumed to circle the origin in the respective variables in the positive sense.

1.3 Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann-Lebesgue lemma.

Proposition 1.99 *Assume $f \in L^1[a,b]$. Then $\int_a^b e^{ixt} f(t)dt \to 0$ as $x \to \pm\infty$. The same is true if $\int_{-\infty}^{\infty} e^{ixt} f(t)dt$ for $f \in L^1(\mathbb{R})$.*

PROOF It is enough to show the result on a set which is dense[6] in L^1. Since trigonometric polynomials are dense in the continuous functions on a compact set[7], say in $C[a,b]$ in the sup norm, and thus in $L^1[a,b]$, while continuous functions with compact support are dense in $L^1(\mathbb{R})$, it suffices to look at trigonometric polynomials, thus (by linearity), at e^{ikx} for fixed k; for the latter we just calculate explicitly the integral; we have

$$\int_a^b e^{ixs} e^{iks} ds = O(x^{-1}) \quad \text{for large } x. \qquad \square$$

For the last statement, we can use the density of compactly supported functions in $L^1(\mathbb{R})$, or a direct argument: for any $\varepsilon > 0$ we can choose $T(\varepsilon)$ large enough so that for all $x \in \mathbb{R}$,

$$\left| \int_{-\infty}^{\infty} e^{ixt} f(t)dt - \int_{-T(\varepsilon)}^{T(\varepsilon)} e^{ixt} f(t)dt \right| < \varepsilon/2$$

and, by the first part of the theorem, we can choose $X(\varepsilon)$ so that for all $|x| > X(\varepsilon)$, $x \in \mathbb{R}$, we have

$$\left| \int_{-T(\varepsilon)}^{T(\varepsilon)} e^{ixt} f(t)dt \right| < \varepsilon/2$$

$\square$

[6] A set of functions f_n which, collectively, are arbitrarily close to any function in L^1. Using such a set we can write

$$\int_a^b e^{ixt} f(t)dt = \int_a^b e^{ixt}(f(t) - f_n(t))dt + \int_a^b e^{ixt} f_n(t)dt$$

and the last two integrals can be made arbitrarily small.

[7] One can associate the density of trigonometric polynomials with approximation of functions by Fourier series.

No rate of decay of the integral in Proposition 1.99 follows without further knowledge about the regularity of f. With some regularity we have the characterization in Proposition 1.103 below. But first, we need a technical lemma.

Lemma 1.100 *We have the following estimate*

$$\left|\frac{e^x-1}{x}\right| \leqslant \max\{e^{\operatorname{Re} x}, 1\} \tag{1.101}$$

PROOF Note that

$$\left|\frac{e^x-1}{x}\right| = \left|\int_0^1 e^{xs}ds\right| \leqslant \int_0^1 e^{s\operatorname{Re} x}ds \tag{1.102}$$

and the last integral is $\leqslant 1$ if $\operatorname{Re}(x) \leqslant 0$ and $< e^{\operatorname{Re} x}$ if $\operatorname{Re}(x) > 0$. ∎

Proposition 1.103 *For $\eta \in (0,1)$ let the $C^\eta[a,b]$ be the Hölder continuous functions of order η on $[a,b]$, i.e., the functions with the property that there is some constant $c > 0$ such that for all $x, x' \in [a,b]$ we have $|f(x) - f(x')| \leqslant c|x-x'|^\eta$.*

(i) We have

$$f \in C^\eta[a,b] \Rightarrow \left|\int_a^b f(s)e^{ixs}ds\right| \leqslant \frac{(b-a)}{2}c\pi^\eta x^{-\eta} + O(x^{-1}) \quad as \quad x \to \infty \tag{1.104}$$

(ii) If $f \in L^1(\mathbb{R})$ and $|x|^\eta f(x) \in L^1(\mathbb{R})$ with $\eta \in (0,1]$, then its Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f(s)e^{-ixs}ds$ is in $C^\eta(\mathbb{R})$.

(iii) Let $f \in L^1(\mathbb{R})$. If $x^n f \in L^1(\mathbb{R})$ with $n \in \mathbb{N}$ then $\hat{f}$ is in $C^n(\mathbb{R})$. If $f \in C^{n-1}(\mathbb{R})$ and $\forall j \leqslant n, f^{(j)} \in L^1(\mathbb{R})$, then $\hat{f}(x) = o(x^{-n})$ as $x \to \infty$.

(iv) If for $A > 0$, $e^{A|x|}f \in L^1(\mathbb{R})$ then $\hat{f}$ extends analytically in a strip of width A centered on $\mathbb{R}$. If $|f(ix+t)| \leqslant g(t)$ with $g \in L^1(\mathbb{R})$ for any $x \in [-A, A]$ and all $t \in \mathbb{R}$ then, for some $C > 0$, $|\hat{f}| \leqslant Ce^{-Ax}$.

PROOF (i) By rescaling, we can choose $[a,b] = [0,1]$. We have as $x \to \infty$ ($\lfloor \cdot \rfloor$ denotes the integer part)

$$\left|\int_0^1 f(s)e^{ixs}ds\right| =$$
$$\left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds + \int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1})$$
$$= \left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} \big(f(s)-f(s+\pi/x)\big)e^{ixs}ds\right| + O(x^{-1})$$
$$\leqslant \sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor} c\left(\frac{\pi}{x}\right)^{\eta}\frac{\pi}{x} \leqslant \frac{1}{2}c\pi^{\eta}x^{-\eta} + O(x^{-1}) \quad (1.105)$$

Exercise 1.106 Show that if $f \in L^1[a,b]$ then $\int_a^b |f(x)-f(x+\varepsilon)| \to 0$ as $\varepsilon \to 0$ (where we extend f by zero where undefined). (The Lebesgue differentiation theorem is one way.) Then, prove the Riemann-Lebesgue Lemma by adapting the argument above.

(ii) We see that

$$\left|\frac{\hat{f}(s)-\hat{f}(s')}{(s-s')^{\eta}}\right| = \left|\int_{-\infty}^{\infty}\frac{e^{-ixs}-e^{-ixs'}}{x^{\eta}(s-s')^{\eta}}x^{\eta}f(x)dx\right| \leqslant \int_{-\infty}^{\infty}\left|\frac{e^{-ixs}-e^{-ixs'}}{(xs-xs')^{\eta}}\right|\left|x^{\eta}f(x)\right|dx \quad (1.107)$$

is bounded. Indeed, for $|\varphi_1-\varphi_2|<1$ Lemma 1.100 implies

$$|\exp(i\varphi_1)-\exp(i\varphi_2)| \leqslant |\varphi_1-\varphi_2| \leqslant |\varphi_1-\varphi_2|^{\eta} \quad (1.108)$$

while for $|\varphi_1-\varphi_2| \geqslant 1$ we see that

$$|\exp(i\varphi_1)-\exp(i\varphi_2)| \leqslant 2 \leqslant 2|\varphi_1-\varphi_2|^{\eta} \quad (1.109)$$

(iii) Let

$$[D_h\hat{f}](x) := \frac{\hat{f}(x+h)-\hat{f}(x)}{h} = \int_{\mathbb{R}} -isf(s)e^{-ixs}\left(\frac{e^{-ihs}-1}{-ihs}\right)ds$$

and, by Lemma 1.100 we have

$$\left|\frac{e^{-ihs}-1}{-ihs}\right| \leqslant 1 \quad (1.110)$$

Since $s \mapsto -isf(s) \in L^1$, differentiability follows by dominated convergence, and we have

$$\hat{f}'(x) = -i\int_{\mathbb{R}} sf(s)e^{-ixs}ds \quad (1.111)$$

Again, since $s \mapsto -isf(s) \in L^1$, the first inequalities in (1.108) and (1.109), and dominated convergence can be applied to the right side of (1.111) to show continuity of the derivative. Higher order differentiation is shown inductively in the same way.

The second part is also shown by induction. We show this for $n = 1$. If $f \in L^1$ then there exist sequences $P_j \to \infty$ and $N_j \to -\infty$ as $j \to \infty$ such that $f(P_j)$ and $f(N_j) \to 0$ as $j \to \infty$. On the other hand

$$\int_{-\infty}^{\infty} f(s)e^{-ixs}ds = \lim_{j\to\infty} \int_{N_j}^{P_j} f(s)e^{-ixs}ds = \lim_{j\to\infty} f(s)\frac{e^{-ixs}}{-ix}\Big|_{N_j}^{P_j} \\ + \lim_{j\to\infty} \frac{1}{ix}\int_{N_j}^{P_j} f'(s)e^{-ixs}ds = \frac{1}{ix}\int_{-\infty}^{\infty} f'(s)e^{-ixs}ds \quad (1.112)$$

and, by the Riemann-Lebesgue lemma, the last integral goes to zero as $x \to \infty$.
(iv) Take any $x \in S_A := \{x \in \mathbb{C} : |\mathrm{Im}\, x| < A\}$. Choose $A' < A$ so that $x \in S_{A'}$. Choose $h \in \mathbb{C}$ so that $|h| \leqslant \frac{A-A'}{2}$. Then

$$D_h\hat{f}(x) := \frac{\hat{f}(x+h) - \hat{f}(x)}{h} = \int_{\mathbb{R}} f(s)e^{-ixs}\left(\frac{e^{-ihs}-1}{h}\right)ds$$

and by Lemma 1.100 and elementary estimates, $\left|e^{-ixs}s\left(\frac{e^{-ihs}-1}{-ihs}\right)\right| \leqslant Ce^{A|s|}$ and by the dominating convergence theorem $\hat{f}'(x) = \lim_{h\to 0}\left[D_h f\right](x) = \int_{\mathbb{R}} -isf(s)e^{-ixs}ds$ implying $\hat{f}$ is analytic in a strip of width A. The last part is proved similarly. First we choose a set of points P_j, N_j as above *but now for* g and estimate away the contribution of the integral outside $[N_j, P_j]$. Then we deform the contour of the integral on $[N_j, P_j]$ into a vertical segment from N_j to $N_j - iA$, the horizontal line $z = -iA + t$, $t \in [N_j, P_j]$ followed by the segment from $P_j - iA$ to P_j and take $j \to \infty$. We leave the details as an exercise. ∎

Note 1.113 In Laplace type integrals Watson's lemma implies that it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral, whereas in Fourier-like integrals, the considerably weaker decay (1.104) is optimal as seen in the exercise below.

Proposition 1.114 *Assume* $f \in C^{n-1}[a,b]$ *and* $f^{(n)} \in L^1([a,b])$. *Then we have*

$$\int_a^b e^{ixt}f(t)dt = e^{ixa}\sum_{k=1}^{n} c_k x^{-k} + e^{ixb}\sum_{k=1}^{n} d_k x^{-k} + o(x^{-n}) \\ = e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b + o(x^{-n}), \quad (1.115)$$

where $c_k = -f^{(k-1)}(a)/i^k$ *and* $d_k = f^{(k-1)}(b)/i^k$.

PROOF This follows by integration by parts and the Riemann-Lebesgue lemma since

$$\int_a^b e^{ixt} f(t)dt = e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b + \frac{(-1)^n}{(ix)^n}\int_a^b f^{(n)}(t)e^{ixt}dt \quad (1.116)$$

∎

Corollary 1.117 *(1) Assume $f \in C^\infty[0, 2\pi]$ is periodic with period 2π. Then $\int_0^{2\pi} f(t)e^{int}dt = o(n^{-m})$ for any $m > 0$ as $n \to +\infty, n \in \mathbb{Z}$.*

(2) Assume $f \in C_0^\infty[a, b]$ vanishes at the endpoints together with all derivatives; then $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ for any $m > 0$ as $x \to \pm\infty$.

Exercise 1.118 Show that if f is analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}_1}$, and for all a, b we have $|\int_a^b e^{ikx} f(e^{ix})dx| > C(a, b)x^{-\eta}$ for some $\eta \in (0, 1)$ and $C(a, b) > 0$, then $\partial\mathbb{D}_1$ is a natural boundary for f.

Exercise 1.119 Show that if f is analytic in a neighborhood of $[a, b]$ but not entire, then both series multiplying the exponentials in (1.115), i.e.

$$\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n} + \cdots,\ x \in \{a, b\}$$

have empty domain of convergence.

Exercise 1.120 In Corollary 1.117 (2) show that $\limsup_{x\to\infty} e^{\varepsilon|x|}|\hat{f}(x)| = \infty$ for any $\varepsilon > 0$ unless $f = 0$.

Exercise 1.121 For smooth f, the interior of the interval does not contribute because of cancellations: rework the argument in the proof of Proposition 1.103 under smoothness assumptions. If we write $f(s + \pi/x) = f(s) + f'(s)(\pi/x) + \frac{1}{2}f''(c)(\pi/x)^2$ cancellation is manifest.

Exercise 1.122 (*) (a) Consider the function f given by the *lacunary trigonometric series* $f(z) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta}e^{ikz}$, $\eta \in (0, 1)$. Show that $f \in C^\eta[0, 2\pi]$. We want to estimate $f(\varphi_1) - f(\varphi_2)$ in terms of $|\varphi_1 - \varphi_2|^\eta$, when $\varphi_1 - \varphi_2$ is small. We can take $\varphi_1 - \varphi_2 = 2^{-p}b$ with $|b| < 1$. Use the first inequality in (1.108) to estimate the terms in with $n < p$ and the simple bound $2/k^\eta$ for $n \geqslant p$. Then it is seen that $\int_0^{2\pi} e^{-ijs} f(s)ds = 2\pi j^{-\eta}$ (if $j = 2^m$ and zero otherwise) and the decay of the Fourier transform is exactly given by (1.104).

(b) Use Proposition 1.114 and the result in Exercise 1.122 to show that the function $f(t) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta}t^k$, analytic in the open unit disk, has no analytic continuation across the unit circle, that is, the unit circle is a *natural boundary.* For this, note that the lower bounds on the Fourier coefficients' decay holds if the function is restricted to any interval.

1.4 Steepest descent method

We seek to determine the asymptotic behavior of $I(\nu)$ as $\nu \to +\infty$, where

$$I(\nu) = \int_{\mathcal{C}} g(z)e^{\nu f(z)}dz \tag{1.123}$$

for f and g that are analytic in some some region of the complex plane[8], and $\mathcal{C}$ is some simple curve that may be finite or infinite. Further, we may assume f is not a constant, as otherwise the asymptotics is trivial. The problem is to determine the asymptotics of I as $\nu \to +\infty$. More generally, if $\nu \to \infty$ along some complex ray $\arg \nu = \varphi$, we can replace ν by $|\nu|$ and f by $e^{i\varphi}f$ to obtain asymptotics along complex rays.

The idea of the steepest descent method is to use the analyticity of the integrand in (1.123) in z to deform $\mathcal{C}$ homotopically into one or more paths, each of which characterized by $\operatorname{Im} f = C$, a constant, and to which Laplace's method applies.

Typically, $\mathcal{C}$ is homotopic to a finite number of finite or infinite piecewise smooth curves of constant imaginary part, each with finitely many non-differentiability points. As we will see in a moment, non-differentiable points of the steepest descent decomposition correspond to singularities of f and zeros[9]. We write

$$f(z) = u(x, y) + iv(x, y) \tag{1.124}$$

and note that $f' = 0$ implies that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$, and, since u and v are harmonic, such points are *saddle points*.

We define **special points** to be singularities of f, endpoints, saddle points and the point at infinity. If $f' \neq 0$, the path of constant imaginary part ($v = const$) is a smooth curve (since $\nabla v \neq 0$). Let $t \mapsto \gamma(t) = \alpha(t) + i\beta(t)$ be a parameterization one of these smooth pieces. We have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\alpha'(t) + \frac{\partial u}{\partial x}\beta'(t) \tag{1.125}$$

and also, since v is constant,

$$0 = v' = \frac{\partial v}{\partial x}\alpha'(t) + \frac{\partial v}{\partial y}\beta'(t) \tag{1.126}$$

At a point where, say $u_x := \partial u/\partial x \neq 0$ and $\alpha' \neq 0$ we solve for α' from (1.126), and use the Cauchy-Riemann equations to obtain

$$d\gamma = \frac{\alpha'}{u_x}\langle u_x, u_y \rangle \, dt \tag{1.127}$$

[8]The region of analyticity will be dictated by the need to deform C into one or more steepest descent paths and will depend on the specifics of the problem.

[9] It is understood that a zero of f is a point where f is analytic and $f' = 0$.

where $u_y = \partial u/\partial y$ and thus $d\gamma$ is tangent at every point to the steepest variation direction of u. If $\alpha'/u_x > 0$, it is a direction of steepest ascent of u, and of steepest descent otherwise. Between every two *special points* as defined above, we choose to traverse the curve in the steepest descent direction, reversing the sign of the integral if needed; hence the name "steepest descent" for the method. Note that the saddle points are of finite order since $(\forall n)(f^{(n)}(z_0) = 0)$ implies $f \equiv 0$.

For simplicity we assume for now that the homotopic deformation of $\mathcal{C}$ does not cross singularities of f. Between each two special points, the integral becomes

$$e^{i\nu C} \int_0^1 e^{\nu u(\alpha(t),\beta(t))} g(\gamma(t))\gamma'(t)dt \tag{1.128}$$

where C is the constant value of $v\ \langle \gamma_x', \gamma_y' \rangle = \gamma_x' + i\gamma_y'$ and similarly for g.

The integral (1.128) is one in which the exponent is monotonic and thus one-to-one. and all conditions of Laplace's method applies. In particular, we can take as a new variable $u(\alpha(t), \beta(t))$ and reduce the question to a Laplace transform of the type $\int_0^a e^{-u\nu} G(u)du$ for $a \in (0, \infty]$ to which Watson's lemma applies. Generally, multiple steepest descent paths, each with a different value of C, are involved in homotopic deformation of $\int_{\mathcal{C}}$; these paths may also join up at *sinks* where $\operatorname{Re} f \to -\infty$ such as ∞ or other singularities of f. Multiple descent paths will definitely be needed when $\operatorname{Im} f$ is different at the end points of $\mathcal{C}$, as in the example in §1.4a. In such cases, the calculation of $I(\nu)$ generally requires adding up the contributions from each steepest descent path $\int_{\mathcal{C}_s}$ in the manner outlined in the last paragraph. Therefore, the only new element in the steepest descent method is to determine steepest curves which are homotopically equivalent to the original path $\mathcal{C}$. It should be further noted that without homotopic deformation into descent paths, (1.123) will typically be an oscillatory integral; asymptotics obtained through the stationary phase method often leads to substantially weaker results, see note 1.113. The stationary phase method, however, does not require analyticity of f and g.

Note 1.129 Also, it is *important to note* that Watson's lemma applies in a half plane, and the resulting asymptotic expansion depends only on the behavior of the integrand near zero. If the curve of steepest descent starting at some point z_0 is clumsy, it can be replaced with a segment of line in the same direction, or even in the same open half-plane centered on the direction of steepest descent at z_0

1.4a Simple illustrative example

Consider

$$I(\nu) = \int_0^1 \frac{e^{i\nu z^2}}{z+1} dz \text{ for } \nu \to +\infty \tag{1.130}$$

This first example is taken to be as simple as possible, to the point of being a bit oversimplified. In particular, the stationary phase method (most often suboptimal in $\mathbb{C}$) would apply with the same result, and in the deformation of contour process we do not cross singularities of the integrand, nor do saddle points interfere with the deformation. Indeed, the steepest descent line at the saddle $z = 0$ is vertical, and, since each point on the curve is moved along a steepest descent path $z = 0$ simply moves up too. However, for $\arg\nu \neq 0$ (more precisely, when $\operatorname{Im}\nu = 0$) this situation changes. In the notation of (1.123), $f(z) = iz^2$, $g(z) = \frac{1}{z+1}$. Steepest descent paths emanating at $z = 0$ are determined by

$$\operatorname{Im} f = \operatorname{Im} f(0) = 0 \quad \text{implying } \operatorname{Re} z^2 = 0, \quad i.e.\ z = re^{\pm i\pi/4} \quad \text{for} \quad r \in (-\infty, \infty) \tag{1.131}$$

However, since $\operatorname{Re} f \to -\infty$, along the ray $z = \{e^{i\pi/4} : r \in [0, \infty)\}$ as $r \to \infty$, it follows that $\infty e^{i\pi/4}$ is a sink that is connected to $z = 0$ along the steepest descent path $z = re^{i\pi/4}$. The steepest descent path from the other end point $z = 1$ in the integral (1.130) is found by setting

$$\operatorname{Im} f = \operatorname{Im} f(1) = 1 \ \text{ implying } \ \operatorname{Re} z^2 = 1, \ i.e.\ x^2 - y^2 = 1 \tag{1.132}$$

A simple way to determine the local descent direction at a point z_0 is to analyze the differential $df = f'(z_0)dz$ and determine the direction of dz for which $df \in \mathbb{R}^-$ (note that $df = du$ since $dv = 0$). In our example $df = 2izdz = 2idz$ and $df < 0$ if $dx = 0, dy > 0$. Since only one branch of the hyperbola passes through (1, 0) and it asymptotes to $y = x$, *i.e.* approaches the sink $\infty e^{i\pi/4}$, by simple estimates a homotopic deformation of the $\int_0^1$ may be made to coincide with descent paths $z = re^{i\pi/4}$, $0 \leqslant r < \infty$ followed by integration along steepest descent path C that connects $\infty e^{i\pi/4}$ to 1 along the hyperbola[10] $x^2 - y^2 = 1$. Therefore,

$$I(\nu) = \int_0^{\infty e^{i\pi/4}} \frac{e^{i\nu z^2}}{1+z} dz + \int_C \frac{e^{i\nu z^2}}{1+z} dz \equiv I_1(\nu) + I_2(\nu) \tag{1.133}$$

For $I_1(\nu)$, using $z = re^{i\pi/4}$ for $0 < r < \infty$, we obtain after change of variable and application of Watson's Lemma

$$\begin{aligned} I_1(\nu) = e^{i\pi/4} \int_0^\infty \frac{e^{-\nu r^2}}{1 + re^{i\pi/4}} dr &= e^{i\pi/4} \int_0^\infty \frac{e^{-\nu p} dp}{2p^{1/2}[1 + p^{1/2} e^{i\pi/4}]} \\ &\sim \frac{1}{2} e^{i\pi/4} \sum_{j=0}^{\infty} (-1)^j \Gamma\left(\frac{j+1}{2}\right) e^{ij\pi/4} \nu^{-(j+1)/2} \end{aligned} \tag{1.134}$$

[10]We do not have the option of going along $re^{-i\pi/4}$ $, 0 < r < \infty$ since $\operatorname{Re} f \to +\infty$ and so contribution at $\infty e^{-i\pi/4}$ cannot be ignored as it can be for a *sink*.

For $I_2(\nu)$, we know that $-p := f(z) - f(1) = iz^2 - i$ is real valued and monotonically decreasing on the parabolic path C from $z = 1$ to $z = \infty e^{i\pi/4}$, since $f' \neq 0$ on this path. Therefore, solving for z, inversion leads to

$$z = Z(p) = (1 + ip)^{1/2}, \tag{1.135}$$

where we can readily check that for this branch of square-root, as $p \to +\infty$, $z \to \infty e^{i\pi/4}$ as required. Therefore,

$$I_2(\nu) = -e^{i\nu} \int_0^\infty \frac{e^{-p\nu}}{1 + Z(p)} Z'(p) dp. \tag{1.136}$$

Taylor expansion gives

$$\frac{Z'(p)}{1 + Z(p)} = \frac{i}{2}(1 + ip)^{-1/2} \left[1 + (1 + ip)^{1/2}\right]^{-1} = \sum_{j=0}^{\infty} a_j p^j, \tag{1.137}$$

where the first few coefficents are: $a_0 = \frac{i}{4}$, $a_1 = \frac{3}{16}$, $a_2 = -\frac{5i}{32}$, $a_3 = -\frac{35}{256}$. Applying Watson's Lemma to (1.136), it follows

$$I_2(\nu) \sim -e^{i\nu} \sum_{j=0}^{\infty} a_j \nu^{-j-1} \Gamma(j + 1), \tag{1.138}$$

The full asymptotic expansion of $I(\nu) = I_1(\nu) + I_2(\nu)$ is then obvious from (1.134) and (1.138).

Note 1.139 (1) The Taylor expansion in (1.137) can be written explicitly, and in a simple way, in terms of the binomial series by multiplying the numerator and the denominator by $\left[1 - (1 + ip)^{1/2}\right]$ and expanding it out.

(2)If we replace the integrand $\frac{e^{i\nu z^2}}{z+1}$ in (1.130), by $\frac{e^{i\nu z^2}}{z-z_0}$, where z_0 is in the upper-half plane region between $e^{i\pi/4}\mathbb{R}^+$ and steepest descent contour C connecting $\infty e^{i\pi/4}$ to 1, for *e.g.* $z_0 = \frac{1+i}{2}$, then the singulariy at $z = z_0$ interferes with the homotopic deformation into steepest descent paths. Nonetheless, since this singularity is a pole, after collecting residue at $z = z_0$, we can use the same descent paths as in Example 1.4a. Since $\operatorname{Im} z_0^2 > 0$, the residue contribution will be exponentially small in ν relative to (1.138) and (1.134). If this z_0 were a branch point instead, in addition to the steepest descent paths, the homotopically deformed path will include a contour that wraps around z_0. Nonetheless, as in the case of the pole, the contribution of the branch point is exponentially small in ν.

Note 1.140 The end result of this procedure, after changes of variables, is indeed a sum of Laplace transforms on $[0, a)$, $a \in [0, \infty]$ to which Watson's lemma applies.

1.4b Finding the steepest variation lines

As mentioned, the main challenge in evaluation of asymptotic behavior of

$$\int_{\mathcal{C}} g(z)\exp(\nu f(z))dz \tag{1.141}$$

is the determination of steepest descent paths decomposition of $\mathcal{C}$. We now discuss how steepest descent paths may be found when $f(z)$ is not as simple as the one in Example 1.4a.
In a nutshell, each point z_0 on the initial curve $\mathcal{C}$ is moved along the steepest descent curve γ passing through it, more precisely we look at the function $\gamma(z_0, t)$ as $t \to \infty$. To simplify the discussion, we will assume that both f and g are entire, and if parts of $\mathcal{C}$ extend to infinity, the integral along those parts converges. If the functions are not entire, then the contours can be deformed inside the domain of analyticity, and beyond that only in special cases, for instance when the singularities of g are poles or simple branch points. If an integral extends to infinity and the integral would not converge, then we truncate the contour at some large enough z_0 (see Note 1.151) at the price of introducing exponentially small relative errors in the estimates.

When v is very simple, as in 1.4a, one can just plot the curves $v(z) = C$. If not, we can use tools from elementary ODE analysis to find the steepest descent lines.

If along a curve $\gamma(t) = (x(t), y(t))$ we have $v(z) = C$, then

$$\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} = 0 \tag{1.142}$$

which happens along the solution curves of the system

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\mathrm{Re}\,(f'(z)) \\ \frac{dy}{dt} &= \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \mathrm{Im}\,(f'(z)) \end{aligned} \tag{1.143}$$

where we used $v = \mathrm{Im}\, f$ to write the system in terms of f'. We also chose the sign so that the flow of ODEs system is antiparallel to ∇u, thus along curves of steepest *descent*.

Note 1.144 The system (1.143) is autonomous, and the task is to draw the phase portrait. As a side remark, (1.143) is the characteristic system for the PDE

$$-\mathrm{Re}\, f' \frac{\partial v}{\partial x} + \mathrm{Im}\, f' \frac{\partial v}{\partial y} = 0 \tag{1.145}$$

The direction field is antiparallel with ∇u, that is, it points toward steepest descent directions of u and of $e^{\nu u}$. To draw the phase portrait more easily we note that:

1. Using the Cauchy-Riemann equations we see that Eq. (1.143) is at the same time a Hamiltonian system as well as a gradient one.

2. There are no closed trajectories since f, thus v, are not identically constant. Indeed, $v = \operatorname{Im} f$ is harmonic, and a harmonic function in a domain attains its maximum and minimum value on the boundary; since we are dealing with a level set of v, call it γ, if γ is closed then $\max v = \min v$ in the $\operatorname{int}(\gamma)$ implying that v is constant in an open set, thus constant everywhere, implying f is a constant.

3. As discussed, all critical points of the field ($x' = y' = 0$) are *saddle points*, the points of interest for our analysis. Indeed, v cannot have, by the maximum modulus principle already used in 2, any interior maxima or minima. (If f is not entire, then of course singularities of f are also singularities of the field.)

4. At a critical point z_0 we have

$$f'(z_0) = 0 \tag{1.146}$$

by (1.143) and (1.146), the local behavior of u near z_0 is

$$u(z) - u(z_0) = \frac{1}{k!}\operatorname{Re}\left(f^{(k)}(z_0)(z - z_0)^k\right)(1 + o(1)) \tag{1.147}$$

where k, generically $k = 2$, is the smallest such that $f^{(k)}(z_0) \neq 0$. Eq. (1.147) provides a simple way to plot the directions of steepest descent of u at z_0. These are the directions

$$f^{(k)}(z_0)(z - z_0)^k \in \mathbb{R}^- \tag{1.148}$$

5. Trajectories can only intersect at critical points of the field.

6. The properties above, together with the behavior of f at infinity completely determine the topology of the direction field.

7. To find the steepest descent line decomposition of a contour $\mathcal{C}$ we let every point $z_0 = x_0 + iy_0 \in \mathcal{C}$ flow along the steepest descent path passing through z_0. We write $(x_0, y_0) \mapsto (x(t; x_0), y(t; y_0))$ and we denote the set of such points by $\mathcal{C}(t)$. The connected components of the limiting set:

$$\{z : \lim_{t\to\infty} d(z, \mathcal{C}(t)) = 0\}$$

represent the sought-for decomposition.

8. By construction, on each $\mathcal{C}_i$, u is strictly monotonic and v, a constant, thus $f - f(z_i)$, with z_i an endpoint of $\mathcal{C}_i$, is one-to-one, and the change of variable defined by $\zeta = z \mapsto -(f(z) - f(z_i))$ where $\operatorname{Re} f$ attains a maximum, brings the integrals to a Watson's lemma form, see Note 1.153.

9. The asymptotic expansions are collected from the endpoints of the steepest descent lines from which u increases, since $e^{-\nu u}$ decreases rapidly starting from such a point.

We illustrate this on a simple example: we start with the integral

$$\int_{\infty e^{5i\pi/4}}^{\infty e^{i\pi/4}} e^{\nu(z^4/4-z)}dz \tag{1.149}$$

where $\nu \to +\infty$.

Because of the rapid decay in z, the integral converges. We want to find a curve homotopic to $\mathcal{C}$ that consists of paths of steepest descent of e^{-u}. In this example, (1.143) becomes

$$\begin{aligned}\frac{dx}{dt} &= 1 - x^3 + 3xy^2 \\ \frac{dy}{dt} &= 3x^2y - y^3\end{aligned} \tag{1.150}$$

The equilibria of (1.150) are, by (1.146) the solutions of $1-z^3 = 0$ ($z = x+iy$). These are $z_k = e^{2k\pi i/3}$, $k = 0, ..., 2$ and near a critical point the directions of descent are obtained from (1.147), $3z_k^2(z - z_k)^2 \in \mathbb{R}^-$.

For large $t = |t|e^{i\varphi}$, we have $f = -|t|^4 e^{4i\varphi}(1+o(1))$, and thus asymptotically there are, four curves of steepest descent, $\cos(4\varphi) = -1 + o(1)$ and four of steepest ascent, $\cos(4\varphi) = 1 + o(1)$. All needed qualitative features of the phase portrait, sketched in Fig. 1.1, follow from this information and the fact that trajectories do not intersect except at critical points. In the phase portrait, the arrows point towards steepest descent. We illustrate the detailed arguments that leads one to Fig. 1.1 by showing how we can argue where each of the two steepest descent and ascent lines emanating at the saddle $z_2 = e^{i4\pi/3}$ must end up. First, note that each of the descent paths must end up at sinks $\infty e^{-i\pi/4}$ or $\infty e^{-3i\pi/4}$ since the paths cannot cross the real axis since $y = 0$ is an invariant set of the dynamical system (1.150). Each of the two ascent paths at z_2 must end up at $-\infty$ or $-i\infty$, since they cannot cross the real axis or approach $+\infty$ without crossing the lower-half plane descent path emanating at the sadde $z_0 = 1$. Further, noting that the two ascent or the two descent paths cannot approach the same *sink* or *source* at ∞ without crossing each other, we are qualitatively led to Fig. 1.1.

Note 1.151 Note that if a path of integration starts at ∞ in some direction and ends at ∞ in some other direction, then for large t on the curve the arrows should point towards infinity to ensure convergence of the integral. This is indeed the case for (1.149). The steepest descent line decomposition for (1.149) consists of the curve L_1 joining $\infty e^{i\pi/4}$ to $\infty e^{-i\pi/4}$ passing through the saddle $z_0 = 1$ together with the curve L_2 connecting $\infty e^{-i\pi/4}$ to $\infty e^{i5\pi/4}$ passing through the saddle $z_2 = e^{4i\pi/3}$, as shown in Fig 1.1.

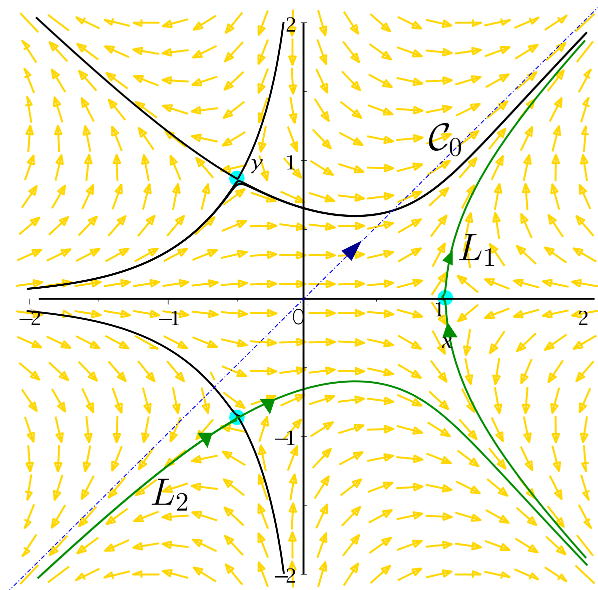

FIGURE 1.1: Phase portrait of (1.150). The dotted line $\mathcal{C}_0$ is the initial contour of integration, and the two curves L_1 and L_2 through the saddle points 1 and $e^{4\pi i/3}$ are its steepest descent decomposition. The light vectors point in steepest descent directions and the black curves are some trajectories of the system (1.150). See also Fig. 1.2.

Note 1.152 If the example above were modified to $\int_{\infty e^{5i\pi/4}}^{\infty e^{i\pi/4}} g(z)e^{\nu(z-z^4/4)}dz$, where $g(z)$ grows too fast along $\infty e^{-i\pi/4}$ to allow meaningful homotopic deformation as shown in Fig 1.1, for *e.g.* $g(z) = \exp\left[e^{-i\pi/6}z^6\right]$, then g participates in shaping the steepest descent lines for large z, and the saddle points for large z are calculated using $\nu f + \log g$ instead[11] of νf. This is similar to a steepest descent problem in which singularities are present, such as the one outlined in the next section. If only leading order asymptotics is needed, one can simply the paths L_1 and L_2 at some large enough z_{L_1}, z_{L_2} independent of ν. With such a choice, it is easily seen that the straight line path connecting the two points is exponentially small relative to the saddle point contributions.

Note 1.153 (Connection with Watson's Lemma) For a general entire f, the set of saddle points through which the steepest variation curve passes cannot have accumulation points, because of the assumed analyticity of f. Then along any steepest descent line, the equation $u(x(t), y(t)) = T$ has a unique solution, and $T(u)$ is smooth except at the saddle points where it has algebraic singularities. Furthermore, by construction, $\exp(iv(x(t), y(t)) =$

[11] Sometimes, it is possible to rewrite $\nu f + \log g$ in the form $\nu\tilde{f}$ by rescaling z.

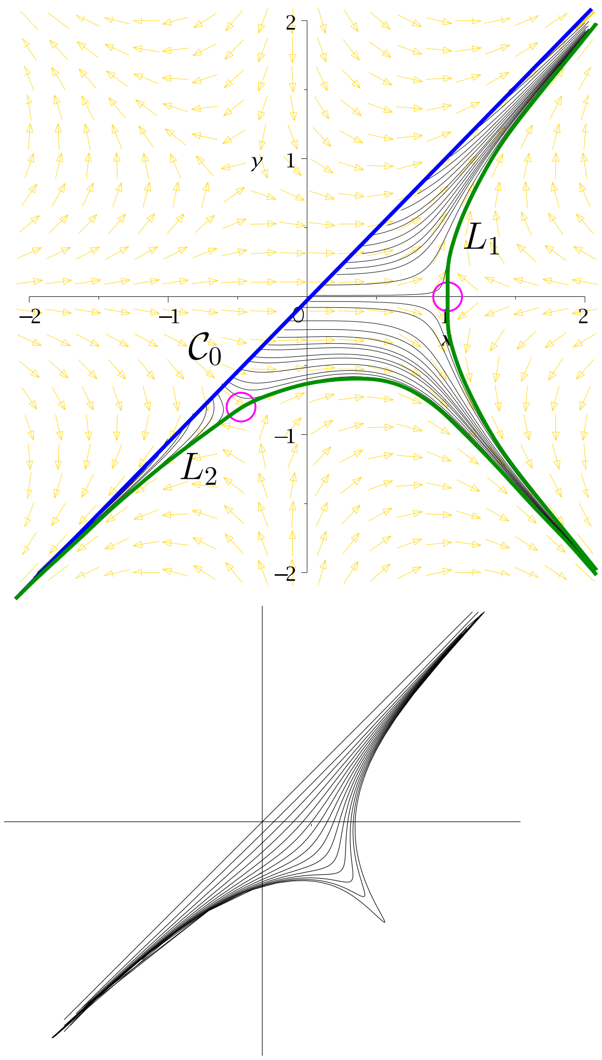

FIGURE 1.2: The original integration path in (1.149) is $\mathcal{C}_0$. The light arrows represent the vector field of steepest descent. Trajectories of the points of $\mathcal{C}_0$ are shown in black. The four curves emanating from the saddle points in quadrants I and IV are the limiting curves, which are lines of steepest descent. Note that the partition of $\mathcal{C}_0$ occurs along the *unstable* manifolds at the saddles. The lower picture shows the flow of the curve $y = x$ at various time intervals, as each point on the curve is moved along the steepest descent line through that point.

const along such a curve. The change of variables $f(z) = f(z_0) + t$ brings the problem to the Laplace form to which Watson's lemma applies.

Exercise 1.154 Complete the analysis of the example (1.149). Transform the final contour integrals into Laplace transforms (the integrand might be an implicitly defined function). Find the asymptotic behavior of (1.149) as $\nu \to \infty$, keeping only the two leading consecutive terms in the series expansions.

1.4b.1 A singular example

Consider the problem of finding the asymptotic behavior of the Taylor coefficients c_k for large k in the expansion

$$e^{\frac{1}{1-z}} = \sum_{k=0}^{\infty} c_k z^k, \quad |z| < 1 \tag{1.155}$$

We have

$$c_{k-1} = \frac{1}{2\pi i} \oint_{|s|=r<1} \frac{e^{\frac{1}{1-s}}}{s^k} ds = \frac{1}{2\pi i} \oint_{|s|=r<1} e^{\frac{1}{1-s} - k \ln s} ds \tag{1.156}$$

The rightmost integral is of the general form (1.141). What distinguishes this case from the case we considered throughout this section is that $g(z) = e^{\frac{1}{1-z}}$ has an essential singularity at $z = 1$.

The steepest descent lines of $f = -k \ln s$ are simply rays towards ∞, but *it is not possible to deform the $|s| = r$ path along these lines of steepest descent*, since the singularity at $z = 1$ is not integrable. The function g contributes nontrivially to the geometry of the curves of interest. We instead plot the steepest descent lines of $h(s;k) = \frac{1}{1-s} - k \ln s$ for fixed k and let $k \to \infty$; we see that $h(s;k)$ has two saddle points, at $s = 1 \pm k^{-1/2}(1 + o(1))$.

Both saddles are on $\mathbb{R}^+$, where $(1-s)^{-1} - k \ln s$ is real; two arcs connect the saddle points –above $\mathbb{R}^+$ and below it– as the imaginary part of h is zero at both saddle points, see Fig. 1.3; each arc is a *heteroclinic connection* [12] (how do you prove this?).

An initial circle of radius $r < 1$ moved by the steepest descent flow becomes, as $t \to \infty$ simply the union of the two arcs connecting the saddle points, see Fig. 1.3. To arrive at this conclusion we also used the fact that the integrals along $\mathbb{R}^+$ to the right of the saddle point $s = 1 + k^{-1/2}(1 + o(1))$ are traversed in opposite directions and cancel each-other; also the integrand decays rapidly on a circle of radius R: the contribution of the latter circle vanishes in the limit $R \to \infty$.

The behavior of c_k for large k stems from the behavior of h on a scale of order $k^{-1/2}$ near $s = 1$. The change of variables $s = 1 + u/\nu$, $\nu = k^{1/2}$ results

[12] A curve connecting two critical points of the field.

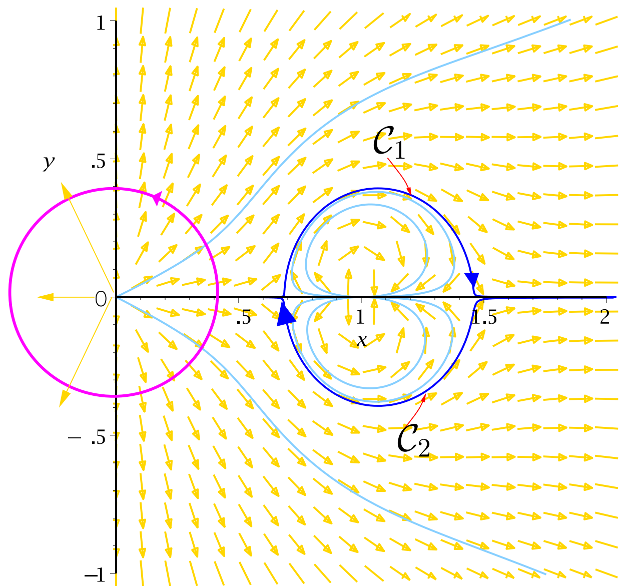

FIGURE 1.3: Steepest descent paths of $\frac{1}{1-s} - k \ln s$ for large k and the saddle points at $s = 1 \pm k^{-1/2}(1+o(1))$. The original contour of integration is the circle centered at 0. Its deformation along steepest descent paths consists of the union of the heteroclinic connections $\mathcal{C}_1$ and its reflection along $\mathbb{R}^+$, $\mathcal{C}_2$ ($k = 7$ in this picture; as $k \to \infty$, $\mathcal{C}_{1,2}$ shrink, and if rescaled, their shape approaches a half-circle).
.

in

$$c_{k-1} = \nu^{-1} \frac{1}{2\pi i} \int_{\mathcal{C}_1 \cup \mathcal{C}_2} \exp\left[-\nu(u+u^{-1}) - \nu^2[\ln(1+u/\nu) - u/\nu]\right] du \tag{1.157}$$

where we added and subtracted $-\nu u$ in preparation for expanding the log for large ν. We note that the function

$$z^{-2}[\ln(1+zu) - zu] = -\tfrac{1}{2}u^2 + \tfrac{1}{3}zu^3 + \cdots \tag{1.158}$$

is analytic at $z = 0$ and we can expand convergently in $z = 1/k$, as $k \to \infty$

$$\exp\left[-\nu^2[\ln(1+u/\nu) - u/\nu]\right] = e^{u^2/2}\left[1 + \frac{u^3}{3\nu} - \frac{u^4}{4\nu^2} + \frac{u^6}{18\nu^2} + \cdots\right] \tag{1.159}$$

We get

$$c_{k-1} = -\frac{1}{2\pi i\nu} \int_{\mathbb{T}} e^{-\nu(u+1/u)+u^2/2}\left[1 + \tfrac{1}{\nu}F_1(\frac{1}{\nu}, u)\right] du \tag{1.160}$$

where $\mathbb{T}$ is the unit circle, traversed anticlockwise and $F_1(z,u)$ is analytic in $(z,u) \in \mathbb{D}_{\frac{1}{2}} \times T$ where $\mathbb{D}_{\frac{1}{2}}$ is the disk of radius $1/2$ centered at zero and T is a neighborhood of the circle $\mathbb{T}$. The steepest descent lines are slightly changed into two half circles, as we expanded out a small term. Now the substitution $u+1/u = -2+v$ brings the integral to a form to which Exercise 1.75 applies. Indeed, to the leading order,

$$c_{k-1} \sim -\frac{1}{2\pi i\nu}\oint_{|u|=1} \exp\left[-\nu(u+1/u)+u^2/2\right] du$$
$$= -\frac{1}{2\pi\nu}\left\{\int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}\right\} \exp\left[-2\nu\cos\theta\right]\exp\left[i\theta + e^{2i\theta}/2\right] d\theta \quad (1.161)$$

The second integral gives exponentially large contribution relative to the first since $-2\nu\cos\theta$ is maximal at $\nu = \pi$. Using Laplace's method on this second integral gives, to leading order,

$$c_{k-1} = \frac{e^{2\sqrt{k}}}{2\sqrt{\pi e}k^{3/4}}(1+o(1)) \quad (1.162)$$

It is to be noted that the contribution from the saddle $u = +1$, corresponding to $\theta = 0$, is exponentially small in k relative to the contribution from $u = -1$ ($\theta = \pi$).

Higher order corrections are obtained more simply as follows. We note that $f(z) = \exp(1/(1-z))$ satisfies the ODE

$$(1-z)^2 f'(z) - f(z) = 0 \quad (1.163)$$

The general analytic theory of ODEs implies that there is a on-parameter family of solutions analytic at zero of the form $f(z) = C\sum_{k=0}^{\infty} c_k z^k$. Inserting the power series into (1.163) and collecting the like powers of z, we obtain recurrence relation for c_k

$$c_k = (2-1/k)c_{k-1} - (1-2/k)c_{k-2}, \quad k \geqslant 2 \quad (1.164)$$

with $c_0 = 1$ since we set $C = f(0) = e$. It follows that $c_1 = \frac{1}{2}c_0 = \frac{1}{2}$. As we will see in the sequel, the asymptotic behavior of c_k to all orders in $1/k$ for large k can be obtained by WKB from this relation up to a multiplicative constant, determined by comparing the WKB expansion to (1.164).

1.5 Regular versus singular perturbations

1.5a A simple model

Consider first two elementary problems: finding the roots of the polynomials $P_1(x;\varepsilon) = x^5 - x - \varepsilon$ and $P_2(x;\varepsilon) = \varepsilon x^5 - x - \varepsilon$ for small ε.

We see that $P_1(x;0)$ has five roots, $\rho = 0, \pm 1, \pm i$. We choose one of them, say $\rho = 1$ and look for roots of $P_1(x;\varepsilon)$ in the form $\rho(\varepsilon) = 1 + \sum_{k\geqslant 1} c_k\varepsilon^k$. Substituting in the equation $P_1 = 0$ we get $(4c_1 - 1)\varepsilon + (4c_2 + 10c_1^2)\varepsilon^2 + (4c_3 + 20c_1c_2 + 10c_1^3)\varepsilon^3 = 0$, and solving for the coefficients $c_1, \dots, c_3, \dots$ we get

$$c_1 = \frac{1}{4},\ c_2 = -\frac{5}{32},\ c_3 = \frac{5}{32}, \dots \tag{1.165}$$

The series of $\rho(\varepsilon)$ is actually convergent. It would not be very convenient to prove this directly from the recurrence relation, though this is possible. Instead, the (analytic) implicit function theorem applies at all 5 roots of $P_1(x,0)$; for instance, at $x = 1$, $P'(1,0) = 4$ and analytic solutions extend analytically since at $x = 1$ or at any other root of $P_1(x,0) = 0$ satisfies $5x^4 - 1 \neq 0$. One can also apply the contractive mapping principle by substituting $\rho = 1 + \delta$ into the equation, placing the largest term containing δ on the left side, and showing that the equation for δ is contractive for small δ, in a space of functions analytic in ε at $\varepsilon = 0$. We leave the details as an exercise. In this case the implicit function theorem is simple. For general nonlinear differential or difference systems, the difficulty lies elsewhere, and the contractive mapping theorem may be more convenient.

This is a typical behavior in regularly perturbed problems: the roots of the leading order equation $P_1(x;0)$ give the leading behavior of the actual roots of $P_1(x,\varepsilon)$ as $\varepsilon \to 0$.

By contrast, $P_2(x;0)$ has only one root, $x = 0$. Four solutions of the quintic polynomial $P_2(x,\varepsilon)$ are lost by setting $\varepsilon = 0$ in the equation; this is an example of singular perturtbation since $P_2(x;0)$ does not capture all the behavior of the five roots of $P_2(x,\varepsilon)$ as $\varepsilon \to 0$. In this simple problem, we can transform it into a regular perturbation one in $\varepsilon^{5/4}$ by using scaled variable $y = \varepsilon^{-1/4}x$. However, in anticipation of more complicated problems, it is useful to employ a more intuitive argument. As $\varepsilon \to 0$ some polynomial terms become relatively small compared to others. Evidently, one cannot have a single nonzero term $\gg$ all others in the limit. The method of dominant balance is a way to determine which terms (at least two, per the above) contribute to the leading order (or *dominant*) balance. We can take this example and illustrate the method, though the example is a bit too easy. Balancing $-x$ with ε is consistent since the result $x \sim \varepsilon$ is compatible with ignoring εx^5. However, this can only lead to the approximate determination of one root of the quintic polynomial. Clearly, other balances need to be investigated. Balancing εx^5 with ε leads to an inconsistency, since the ignored term $-x$ would turn out to be much larger. We must then have $\varepsilon x^5 \sim x$ or equivalently $\varepsilon x^4 \sim 1$. To obtain the higher order corrections, we substitute the scaling obtained from the leading order balance: $x = \varepsilon^{-1/4}y$ and we get

$$y^5 = y + \eta; \quad (\eta = \varepsilon^{5/4}) \tag{1.166}$$

Now the limiting ($\eta \to 0$) equation, $y^5 = y$, has five roots as expected of a quintic polynomial. Note that the approximation $x \sim \varepsilon$ is recovered as

$y \sim \eta$ and there is no need for a separate argument for that case. In fact, the equation (1.166) is $P_1(y;\eta) = 0$, now the implicit function theorem applies at each root. The roots are thus analytic in η, implying they have convergent expansion in powers of $\varepsilon^{5/4}$. For instance, the convergent expansion in η near $y = 1$ which can be obtained by substitution or by iteration, $\delta = \frac{1}{4}\eta - \frac{5}{32}\eta^2 + \frac{5}{32}\eta^3 + \ldots$.

1.6 Regular and singular perturbation equations in differential equations

By contrast, we will find that in singular perturbation of differential equations, where a small parameter typically multiples the highest derivative, the asymptotic expansions are generally divergent.

An equation can be regularly perturbed in some regimes and singularly perturbed in some others.

1.6a Formal and actual solutions

Consider the differential equation

$$\frac{df}{dz} = f + f^2 + zf^3; \quad f(0) = 1 \tag{1.167}$$

which we analyze in a neighborhood of $z = 0$. The general analytic theory of ODEs ensures existence, uniqueness and analyticity of the solution in a neighborhood of $z = 0$. We can calculate the power series solution in a number of ways, for instance by substituting $f(z) = \sum_{k=0}^{\infty} c_k z^k$ into (1.167) and identifying the coefficients c_k. We get

$$f(z) = 1 + 2z + \frac{7z^2}{2} + \frac{41z^3}{6} + \frac{57z^4}{4} + \cdots \tag{1.168}$$

If we write the equation in integral form

$$f(z) = 1 + \int_0^z \left[f(s) + f^2(s) + sf^3(s)\right] ds$$

and iterate,

$$f_{n+1}(z) = 1 + \int_0^z (f_n(s) + f_n^2(s) + sf_n^3(s))ds; \quad f_0(z) \equiv 1 \tag{1.169}$$

we can check that, for small z the sequence $\{f_k\}_k$ is uniformly Cauchy, and thus convergent. This can be seen using the fact that if a function h is bounded and integrable, then

$$\left|\int_0^z h(s)ds\right| \leqslant |z| \max_{|s|<|z|} |h(s)| \tag{1.170}$$

The recurrence (1.169) can be used to generate the power series at zero, by inductively replacing f_n by its Maclaurin series truncated to $O(z^n)$ and integrating the resulting series term by term. We will not go over the details here, as we will develop more general tools shortly.

1.6b Perturbed Hamiltonian systems

An interesting example is the pendulum of slowly variable length. A model equation is

$$\ddot{q} + \frac{g}{l_0 + \varepsilon t} q = 0 \tag{1.171}$$

where q is the generalized position, g is the gravitational acceleration and l_0 is the initial length. A proper treatment of this problem will have to wait until we study adiabatic invariants.

By changing units and ε we can assume without loss of generality $l_0 = g = 1$ (and denote the rescaled time still by t):

$$\ddot{y} + \frac{1}{1 + \varepsilon t} y = 0 \tag{1.172}$$

The limiting equation $\ddot{y} + y = 0$ has a two dimensional family of solutions, $y = A \sin t + B \cos t$. Assuming that $y(0) = 0$ and $\dot{y}(0) = 1$ we choose $q_0(t) = \sin t$. We look for solutions y in the form of power series in powers of ε,

$$y(t) = \sin t + \sum_{k=1}^{\infty} \varepsilon^k y_k(t) \tag{1.173}$$

Solving order by order in ε and using the initial condition $y(0) = 0$ and $\dot{y}(0) = 1$, translating to $y_k(0) = 0$, $y_k'(0) = 0$ for $k \geqslant 1$, we get

$$q(t) = \sin t + \left(\frac{1}{4} t \sin t - \frac{1}{4} t^2 \cos t\right) \varepsilon \\ + \left[\left(\frac{3}{32} - \frac{3}{32} t^2 - \frac{1}{32} t^4\right) \sin t - \left(\frac{3}{32} t - \frac{1}{16} t^3\right) \cos t\right] \varepsilon^2 + \cdots \tag{1.174}$$

By induction, ε^k is multiplied by a polynomial in $t, \cos t, \sin t$ of degree $2k$ in t. We see that for the expansion to be convergent, or even useful, we need $t^{2k}\varepsilon^k \to 0$ that is $t^2\varepsilon < \delta$, where δ needs to be relatively small.

In a region where $t < \delta\varepsilon^{-1/2}$ with δ small enough, we can set up a contractive mapping argument to justify the expansion, which will turn out to be convergent indeed. We leave this as an exercise as well.

Note also that in the time interval $0 < t \ll \sqrt{\varepsilon}$ we have $l = l_0 + O(\sqrt{\varepsilon}$, that is, the length does not change much; this region is not very interesting. A proper treatment of this problem will have to wait until we study adiabatic invariants.

The problem can be viewed as a regular perturbation problem, in the original scaling, but the interval of time over which this regime is relevant is too short.

Now, when $t \sim \varepsilon^{-1/2}$ it is natural to take $t\sqrt{\varepsilon} = \tau$ as a new variable, $q(t) = Q(\tau)$ that will not be necessarily small. The equation for Q reads.

$$\varepsilon\ddot{Q} + \frac{Q}{1+\sqrt{\varepsilon}\tau} = 0 \tag{1.175}$$

Now the limit $\varepsilon \to 0$ is singular: in this limit equation (1.175) would become $\frac{Q}{1+\sqrt{\varepsilon}\tau} = 0$; here, as in the case of $P_2(x;\varepsilon)$ we lose most solutions. Furthermore, the surviving solution $Q = 0$ is not very interesting, and it does not satisfy the initial condition. We need to do something else, in this case WKB, which we introduce in §1.6h.1 below.

1.6c More about regular and irregular singularities of ODEs. Some simple examples.

Consider instead the equation

$$\frac{dg}{dz} - z^{-2}g(z) + z^{-1} = 0 \tag{1.176}$$

The point $z = 0$ is a singular point of (1.176), in fact an irregular singular point; there are no analytic solutions near zero. Again, by dominant balance as $z \to 0^+$, we find that the only consistent leading order terms are $-z^{-2}g$ and z^{-1}. By this balance, the iteration supposed to give the solution is

$$g^{[n+1]} = z + z^2(g^{[n]})';\; g^{[0]} = z \tag{1.177}$$

The iteration (1.177) is well defined, and it is solved by the sequence of polynomials $(g_n)_{n\in\mathbb{N}}$, where $g_n(z) = z + \sum_{k=2}^{n+1}(k-1)!z^k$. The sequence of polynomials has no limit, and we generate the "solution"

$$g(z) \quad \text{“ = ”} \quad z + \sum_{k=2}^{\infty}(k-1)!z^k, \tag{1.178}$$

As we will see, this series with zero radius of convergent is the asymptotic expansion of some solution (in fact, of all of them!) as $z \to 0^+$. In fact, it carries much more information than this. This differential equation is singularly perturbed in essentially the same sense of our polynomial toy models: the equation satisfied by the dominant terms, here $z^{-2}g = z^{-1}$ has only one solution, whereas a first order ODE has a one-parameter family of them. The "dominant equation" has fewer solutions than the full equation. Let us compare the two ODEs:

$$(A)\quad \frac{dg}{dz} - g(z) = z; \quad \text{and } (B)\quad \frac{dg}{dz} - z^{-2}g(z) = -z^{-1} \tag{1.179}$$

The iteration obtained by dominant balance in (A) is

$$(g')^{[n+1]} = g^{[n]} + z \;\Leftrightarrow (g)^{[n+1]} = \int_0^z g^{[n]}(s)ds + z^2/2 + C \tag{1.180}$$

and we are iterating on the integral, a regularizing operator, instead of the derivative. Substituting a power series in z in (1.179), and equating the coefficients of z^k for $k \geqslant 2$, we get $(k+1)c_{k+1} = c_k$ in (A), and $c_{k+2} = (k+1)c_{k+1}$ in (B). We see that the contribution of the derivative to terms of the expansion is essentially multiplication by k while the singular term, when present, shifts the index affected by the k multiplication. From the recurrence relations we obtain that the coefficients c_k in (A) are proportional to $1/k!$ while in (B), they are proportional to $(k-1)!$.

Finally, let's see how the expansion (1.178) relate to the solutions of (1.176). In this example, we can write down the exact solution of the equation as

$$g(z) = Ce^{-1/z} - e^{-1/z}\int_1^z s^{-1}e^{1/s}ds \tag{1.181}$$

The change of variables $s = 1/t, z = 1/x$ brings (1.181) to the form

$$g(1/x) = Ce^{-x} + e^{-x}\int_1^x t^{-1}e^t dt = e^{-x}\left(C + \int_{-\infty}^1 t^{-1}e^t\right)$$
$$+ e^{-x}\int_{-\infty}^x t^{-1}e^t dt =: C_2e^{-x} + e^{-x}\int_{-\infty}^x t^{-1}e^t dt = \int_0^\infty \frac{e^{-xu}}{1-u}du + C_2e^{-x}, \tag{1.182}$$

where we used the change of variable $t \to x(1-u)$ and the contour of integration avoids $t = 0$ (*i.e.* $u = 1$). Watson's lemma shows that $g(z) \sim z + \sum_{k=2}^\infty \Gamma(k)z^k$. What we see is that the formal power series solution is, in this case as well as in (1.167), the Maclaurin series as $z \to 0^+$ of some solution (*all* of them in (1.176)). The fact that formal solutions are asymptotic to actual ones is true in much wider generality, as we will see in the sequel.

To get actual solutions, for now we remember not to place the highest derivative on the right side in an iteration scheme.

Note 1.183 Singularly perturbed linear ODEs cannot be expected to have a complete set of solutions with asymptotic behavior described by powers or combination of powers and logs. Consider for instance

$$y'' + A(z)y' + B(z)y = 0 \tag{1.184}$$

with A and B having poles at $z = 0$. Assume two independent solutions have asymptotic behavior $y = z^a(1+o(1))$ and $y = z^b(1+o(1))$ near $z = 0$. Solving for A and B, the leading order asymptotic behavior[13] is found to be

$$A = \frac{1-a-b}{z}(1+o(1))\,; \quad B(z) = \frac{ab}{z^2}(1+o(1)) \tag{1.185}$$

This implies that $z = 0$ is a regular singular point, which constitute only special cases. Indeed, for a general linear homogeneous ODE of any order with meromorphic coefficients, it may be proved that at an irregular singular point, the asymptotic behavior of some (or all solutions) will involve exponentials (cf. [20] eq. (2.4) p.143) possibly multiplied by power series; when this is the case, the series are generically divergent.

Exercise 1.186 Prove that an n-th order linear homogeneous ODE with meromorphic coefficients can have a full set of independent solutions behaving like z^{p_j} to leading order, then the singular point is regular and the solutions are given by convergent series in (possibly noninteger powers) of z.

1.6d Choice of the norm

The choice of a norm is sometimes crucially important in proving asymptotic results. Consider another very simple model,

$$y' - y - y/x^2 - 1/x = 0 \ (x \in \mathbb{R}^+) \tag{1.187}$$

where we want understand the behavior of solutions for large $x > 0$. Of course, (1.187) can be easily solved in closed form. We will not use the explicit solution since we plan to understand some qualitative features about the behavior of solutions of ODEs and, in general, closed form solutions do not exist.

The dominant balances for (1.187) are easy to determine, since, as $x \to \infty$ the third term is always dominated by the second one and cannot influence the leading order balance. Thus $y \sim 1/x$ is one consistent balance, as one can check, and the other one is $y' \sim y$. We look at the latter one. With the intent of estimating away the small term y/x^2 through a contractive mapping argument, we place it on the right hand side of the equation. Written in an integral form, (1.187) reads

$$y = y(x_0)e^{x-x_0} + e^x \int_{x_0}^x \frac{e^{-s}ds}{s} + e^x \int_{x_0}^x \frac{y(s)e^{-s}}{s^2} =: Ce^x + e^x \int_{x_0}^x \frac{e^{-s}ds}{s} + \mathcal{A}[y] \tag{1.188}$$

In principle, the value of $x_0 > 0$ is immaterial, since a change in x_0 can be compensated by a change in C. However, the smaller x_0 is, the larger will the

[13] The differentiability of the asymptotics can be assured using the differential equation

kernel of the integral ($s^{-2}e^{-s}$) be, so we'll choose a large enough x_0 its size to be determined later.

What norm should we choose? In general, the best norm is one that reflects the actual behavior of the solution. Of course we do not know it exactly at this stage, but we can rely roughly on the dominant balance equation, which suggests $y \sim Ce^x$.

Based on this we construct a Banach space based on the norm

$$\|f\| = \sup_{x \geqslant x_0} |e^{-x} f(x)| \tag{1.189}$$

and choose a ball in this space where we plan to apply the contractive mapping principle:

$$B = \{f : \|f\| < M\} \tag{1.190}$$

for some $M > 0$. To show contractivity of the map, the norm of the linear operator $\mathcal{A}$ (see §2.15) in expression of L should be < 1:

$$\sup \|\mathcal{A}y\| \leqslant \|y\| \sup_{x \geqslant x_0} |e^{-x}\mathcal{A}[e^x]| \leqslant x_0^{-1}\|y\| \tag{1.191}$$

and contractivity is ensured, with a small contractivity factor, if x_0 is large.

The value of M cannot be arbitrarily small: since the norm of $\mathcal{A}$ is small, the leading behavior of y comes from the terms independent of y on the right side of the equation, that is the first two terms, thus certainly $M > |C|$. By direct calculation it is seen that any $M > |C| + 1$ suffices when x_0 is chosen sufficiently large.

Let us now try more generally a norm of the form

$$\|f\| = \sup_{x \geqslant x_0} |e^{-\nu x} f(x)| \tag{1.192}$$

Simple estimates show that for the contractivity of L we need $\nu \geqslant 1$. This is to be expected, since if L were contractive for $\nu < 1$ there would exist solutions that grow slower than e^x, and this is inconsistent with (1.188). We also see that, for large ν and x_0, the norm of $\mathcal{A}$ is $O(\nu^{-1}x_0^{-2})$. The *contractivity factor depends on the norm chosen.* Generally, speaking, relaxing the conditions imposed through the norm makes the operator more contractive, at the expense of course of having poorer control on the solution.

Exercise 1.193 *Estimate the norms of Ce^x, $\mathcal{A}[y]$ and $\mathcal{A}_1[y]$ in the norm* (1.192) *for ν in the range $(-\infty, \infty)$.*

1.6e Choice of limits of integration

In view of the fact that $y(x) \sim Ce^x$ for large x, one can write the general integral form of the equation as

$$y = Ce^x + \int_\infty^x \frac{e^{x-s}ds}{s} + e^x \int_\infty^x s^{-2}y(s)e^{-s}ds =: Ce^x + \int_\infty^x \frac{e^{x-s}ds}{s} + \mathcal{A}_1[y] \tag{1.194}$$

The choice $x_0 = \infty$ in (1.194) ensures that the maximum of the integrand is reached at the variable point of integration (think at the connection with Watson's lemma). Such a choice is especially important in problems in which the solution decreases instead of increasing. See also Exercise 1.193.

Changing the sign in (1.187) and adding a nonlinearity,

$$y' + y - y/x^2 - 1/x + y^2 = 0 \ (x \in \mathbb{R}^+) \tag{1.195}$$

we now see that only the balance $y = 1/x$ is consistent. Writing the integral equation in a for similar to that for (1.187),

$$y = y(x_0)e^{-(x-x_0)} + e^{-x}\int_{x_0}^{x} \frac{e^s ds}{s} + e^{-x}\int_{x_0}^{x} \frac{y(s)e^s}{s^2} + e^{-x}\int_{x_0}^{x} y(s)^2 e^s ds$$
$$=: Ce^{-x} + e^{-x}\int_{x_0}^{x} \frac{e^s ds}{s} + \mathcal{A}[y] \tag{1.196}$$

we see that the behavior $1/x$ in the region $x > x_0$ cannot be enforced by a norm, in any easy way (try!). Instead, this is possible by choosing one limit of integration to be $+\infty e^{i\varphi}$, $\varphi = \pi/2 + \varepsilon$, as indicated before:

$$y = Ce^{-x} + e^{-x}\int_{\infty e^{i\varphi}}^{x} \frac{e^s ds}{s} + e^{-x}\int_{\infty e^{i\varphi}}^{x} \frac{y(s)e^s}{s^2} + e^{-x}\int_{\infty e^{i\varphi}}^{x} y(s)^2 e^s ds$$
$$=: Ce^{-x} + e^{-x}\int_{\infty e^{i\varphi}}^{x} \frac{e^s ds}{s} + \mathcal{A}[y] \tag{1.197}$$

1.6f An irregular singular point of a nonlinear equation

Consider Abel's equation

$$y' = y^3 + x \tag{1.198}$$

in the limit $x \to +\infty$. We first find the asymptotic behavior of solutions formally, and then justify the argument. We use again dominant balance. As x becomes large, y, y', or both need to become large if the equation (1.198) is to hold. Assume first that the balance is between y' and x and that $y^3 \ll x$. If $y' \sim x$ then we have $y \sim x^2/2$ and $y^3 \sim x^6/8$, and this is inconsistent since it would imply $x^8/8 = O(x)$. Now, if we assume $x \ll y^3$ then the balance would be $y' \approx y^3$, implying $y \sim \pm\frac{1}{\sqrt{2(x_0-x)}}$ but this is small for $x - x_0 \gg 1$ except when there is a sequence of singularities $x_0 \to \infty$ and it is not possible to assume $x - x_0$ to be uniformly large; this situation is analyzed in a later chapter (§**??**). Therefore, with those exceptions, we are led to a contradiction in assuming $x \ll y^3$. We have one possibility left: $y = \alpha x^{1/3}(1 + o(1))$, where $\alpha^3 = -1$, which assuming differentiability implies $y' = O(x^{-2/3})$ which is now consistent. We substitute

$$y = \alpha x^{1/3}(1 + v(x)) \tag{1.199}$$

with the expectation that v will be small. This part of the analysis does not have to be rigorous. If the balance is mistaken, the final integral reformulation of the problem would simply fail to be contractive. For definiteness, we choose $\alpha = e^{i\pi/3}$, the analysis being similar for other cubic roots of -1. We get

$$\alpha x^{1/3} v' + 3xv + 3xv^2 + xv^3 + \frac{\alpha}{3} x^{-2/3} + \frac{\alpha}{3} x^{-2/3} v = 0 \tag{1.200}$$

To find the dominant balance, we first eliminate the terms that *cannot* participate in the leading order balance: $3xv^2 + xv^3 + \frac{\alpha}{3}x^{-2/3}v$ are dominated by $3xv$ if indeed v is small. We are again left with three terms that can participate to the leading order balance. We note two things: it is often the case that, after a substitution based on the leading behavior and smaller corrections, of the form (1.199), the new equation is of course more involved, but many terms are easily excluded from the dominant balance. Also, the size of the nonlinear terms relative to the linear ones decreases. By elimination we see that to leading order only two terms can be of comparable size, $3xv$ and $\frac{\alpha}{3}x^{-2/3}$ giving

$$v \sim -\frac{\alpha}{9} x^{-5/3} \tag{1.201}$$

Based on the discussion in the previous section, we know that although v' does not participate in the leading order balance, for a rigorous analysis, we have to we keep it on the left side and try to rewrite (1.200) in a suitable integral form. We hence place the formally largest term(s) containing v *and* v' on the left side and the smaller terms as well as the terms not depending on v on the right side:

$$\alpha x^{1/3} v' + 3xv = h(x, v(x)); \;\; -h(x, v(x)) := 3xv^2 + xv^3 + \frac{\alpha}{3} x^{-2/3} + \frac{\alpha}{3} x^{-2/3} v \tag{1.202}$$

We treat (1.202) as a linear inhomogeneous equation, and solve it thinking for the moment that h is given.

This leads to

$$v = \mathcal{N}(v);$$
$$\mathcal{N}(v) := Ce^{-\frac{9}{5\alpha}x^{5/3}} + \frac{1}{\alpha} e^{-\frac{9}{5\alpha}x^{5/3}} \int_{x_0}^{x} e^{\frac{9}{5\alpha}s^{5/3}} s^{-1/3} h(s, v(s)) ds \tag{1.203}$$

Since the domain of interest in this problem is $x \geqslant x_0$ we chose the limits of integration in such a way that the integrand is maximal when $s = x$: if $x \to +\infty$, then $x^{-1/3} e^{\frac{9}{5\alpha}x^{5/3}} \to \infty$, and our choice corresponds indeed to this prescription.

We note that $h(s, 0) = \frac{\alpha}{3}x^{-2/3}$ and by integration by parts, we conclude that its contribution to the integral in (2.49) to the leading order is $-\frac{\alpha}{9}x^{-5/3}$ (consistent with (1.201)). The asymptotic behavior of the contribution of

$h(s,0)$ for large x can be also determined simply from L'Hospital:

$$\frac{\int_a^x e^{bs^m}/s^n ds}{e^{bx^m}/x^M} \underset{x\to+\infty}{\sim} \frac{1}{bm} \text{ if } M = m+n-1 \tag{1.204}$$

or by using Watson's lemma after suitable changes of variables. Thus, it is natural to choose x_0 large enough and introduce the the Banach space $\mathcal{B} = \{f : \|f\| < \infty\}$ where

$$\|f\| := \sup\left\{|x^{5/3} f(x)| : |x| > x_0, \frac{5}{3}\arg x - \arg\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\} \tag{1.205}$$

Since the arguments apply more generally to a complex sector, we have extended our domain accordingly, instead of restricting the analysis to the real positive line. Within $\mathcal{B}$, we consider a ball

$$B_1 := \left\{f \in \mathcal{B} : \|f\| \leqslant \tfrac{2}{3}\right\} \tag{1.206}$$

whose size is large enough to include inhomogeneous terms resulting from integration of $h(s,0)$ in (2.49) since

$$\left\| \frac{1}{\alpha} e^{-\frac{9}{5\alpha}x^{5/3}} \int_{x_0}^x e^{\frac{9}{5\alpha}s^{5/3}} s^{-1/3} h(s,0)ds \right\| \leqslant \left\| \frac{\alpha}{9} x^{-5/3} \right\| + o(x^{-5/3}) < \frac{2}{3} \tag{1.207}$$

for large enough x_0.

Lemma 1.208 *For given C, if x_0 is large enough, then the operator $\mathcal{N}$ is contractive in B_1 and thus* (2.49) *(as well as* (1.202)*) has a unique solution there.*

PROOF We leave this as an exercise. □

Exercise 1.209 Find an asymptotic power series solution as $z \to \infty$ of the Painlevé equation P_1,

$$y'' = y^2 + z \tag{1.210}$$

and prove that there are actual solutions asymptotic to it as $z \to \infty$.

1.6g Integral reformulations in PDEs: an example

The *free* w equation is $u_{tt} - c^2 u_{xx} = 0$; c can be scaled out, by changing variables to $\tilde{x} = x/c$; without loss of generality we can then assume $c = 1$. Usually, the equation comes with initial conditions (at, say, $t = 0$):

$$u_{tt} - u_{xx} = 0; \;\; u(x,0) = f(x), \;\;\; u_t(x,0) = g(x) \tag{1.211}$$

When $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, the change of variable $\xi = x - t$, $\eta = x + t$ leads to the well-known D'Alembert solution

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds \tag{1.212}$$

Without smoothness of f and g, (1.212) is interpreted as a *weak solution*, one which satisfies the equation in the sense of distributions. For simplicity, in the sequel we will assume that there is enough smoothness to work in a space of functions. In the same way we can solve the wave equation with a source,

$$u_{tt} - u_{xx} = S(x,t);\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{1.213}$$

to obtain

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) \\ + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} S(y,s)dyds \tag{1.214}$$

The wave equation with potential arises naturally in a number of physical problems, ranging from electrodynamics to the wave evolution in the presence of a black hole. It reads

$$u_{tt} - u_{xx} + V(x)u(x,t) = 0;\ u(x,u) = f(x), \quad u_t(x,0) = g(x) \tag{1.215}$$

Clearly, at least for general V we cannot expect to solve (1.215) in closed form.

Here we assume that $V \in L^\infty(\mathbb{R})$ and f, g are in $L^1(\mathbb{R})$ and show that (1.215) has a global solution $u(\cdot,t) \in L^1(\mathbb{R})$ and $\|u(\cdot,t)\|_{L^1}$ grows at most exponentially in t. That exponential growth is possible for some potentials can be seen in the following way. Looking for a solution of the PDE in (1.215) of the form $u(x,t) = e^{\lambda t}U(x)$ we obtain

$$-U'' + V(x)U = -\lambda^2 U \text{ with initial condition } u(x,0) = U(x), u_t(x,0) = \lambda U(x) \tag{1.216}$$

Eq. (1.216) is the time-independent Schrödinger equation; in that setting it is natural to assume that V decays as $x \to \infty$. An L^2 solution of (1.216) for $\lambda \neq 0$ is called a *bound state* of the quantum *Hamiltonian* $-\frac{d^2}{dx^2} + V(x)$, and for many potentials of interest these do exist (for instance for any large enough square well).

We can use (1.217) to rewrite (1.215) in integral form,

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) \\ + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds - \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} V(y)u(y,s)dyds =: \mathcal{A}[u](x,t) \tag{1.217}$$

Proposition 1.218 *Assume the initial conditions* $f(x) = u(t, 0)$ *and* $g(x) = u_t(x, 0)$ *are in* $L^1(\mathbb{R})$ *and* $V \in L^\infty(\mathbb{R})$. *Then, if* $\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}}$ *we have* $\sup_{t>0} e^{-\nu t}\|u(t, \cdot)\|_{L_1} < \infty$.

PROOF We write the Duhamel formula as

$$u = \mathcal{A}u;\ \mathcal{A}u := \frac{f(x-t) + f(x+t)}{2}$$
$$+ \tfrac{1}{2} \int_{-\infty}^{\infty} I_t(y-x)g(y)dy - \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} u(y, s)V(y)I_{t-s}(y-x))dyds \quad (1.219)$$

where $I_a = \chi_{[-a,a]}$, the characteristic function of the interval $[-a, a]$. Consider the Banach space

$$\mathcal{B} = \{u : |||u|||_\nu := \sup_{t\in\mathbb{R}^+} e^{-\nu t}\|u(t, \cdot)\|_1 < \infty\};\quad (\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}}) \qquad (1.220)$$

Applying Fubini to integrate first in x, we see that $\| \int_{-\infty}^{\infty} \chi_t(y-x)g(y)dy\|_1 \leqslant 2t\|g\|_1$ and (since by definition $\|u(\cdot, s)\|_1 \leqslant \|u\|_\nu e^{\nu s}$)

$$\sup_{t>0} e^{-\nu t} \left\| \int_0^t \int_{-\infty}^{\infty} u(y, s)V(y)\chi_{t-s}(y-x)dydt \right\|_1$$
$$\leqslant \|V\|_\infty \|u\|_\nu \sup_{t>0} e^{-\nu t} \int_0^t 2(t-s)e^{\nu s} ds \leqslant 2\|V\|_\infty \nu^{-2}\|u\|_\nu \quad (1.221)$$

Using (1.221) we see that $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is contractive. Also, assuming f, g and V are smooth, the solution is seen to be smooth too: since $u \in L^1$, Duhamel's formula shows that it is continuous; then, as usual, using continuity we derive differentiability, and inductively, we see that u is smooth.

$\Box$

Note 1.222 This is one of the cases mentioned before in which exponential growth is possible, but the rate of growth cannot be determined "in closed form" for an arbitrary V, and we settle for an *overestimate.*

Exercise 1.223 Complete the details by showing that this result implies global existence of a solution of (1.215).

Exercise 1.224 (i) Assume $V \in L^2(\mathbb{R})$. Prove a similar result with $\|u\|$ given by $\sup_{t\geqslant 0} e^{-\nu t} \left(\int_{-\infty}^{\infty} |u(x, t)|^2 dx\right)^{1/2}$. Use this result to estimate the largest possible eigenvalue of V.

1.6h Singularly perturbed differential equations with respect to a parameter

1.6h.1 Heuristics and formal expansions approach

Consider first the very simple equation

$$\varepsilon^2 \frac{d^2y}{dx^2} + y = 0; \quad \varepsilon \ll 1 \tag{1.225}$$

which can be of course solved in closed form, which we will do after we explore some qualitative features. The limit $\varepsilon \to 0$ is singular: taking $\varepsilon = 0$ in (1.225) leaves us with $y = 0$. Most solutions of (1.225) are lost by formally setting $\varepsilon = 0$. This is one of the indications that an equation is singularly perturbed. Also, if by rescaling variables the parameter can be scaled out (for (1.225) $x = t\varepsilon$ achieves this), then as $\varepsilon \to 0$, any fixed $x \neq 0$, t approaches ∞, an irregular singular point of the equation. Further, we note that the ε dependence at zero is not analytic: this is seen of course by solving the equation, or, having more general equations in mind, by attempting to find solutions as convergent series in ε: there are no nonzero ones.

Similarly, the equation

$$\frac{d^2y}{dx^2} - a^2 y = 0 \tag{1.226}$$

is singularly perturbed fixed for any fixed $a \neq 0$ as $x \to \infty$, since the change of variable $x = 1/z$ brings it to

$$z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz} - a^2 y(z) = 0 \tag{1.227}$$

and we see that for small z the coefficients of the derivatives on the left side of the equation vanish at $z = 0$, and if we ignored these terms we would be once more left with a scalar equation, $y = 0$.

The eigenvalue problem for the one-dimensional Schrödinger equation

$$-\hbar^2 \psi'' + V(x)\psi = E\psi \tag{1.228}$$

is singularly perturbed when the Planck constant $\hbar \to 0$ (its physical value is $\approx 6.626068 \times 10^{-34} m^2 kg/s$), if $x \to \infty$ or both. Here ψ is the wave function and it has the physical interpretation that $|\psi(x)|^2$ is probability density function for a particle and the total probability is one: $\|\psi\|_2 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. For a typical potential V going to zero as $x \to \infty$, Eq. (1.228) is also singularly perturbed when $x \to \infty$ for fixed $\hbar$. Indeed, taking $x = 1/z$ we get

$$-\hbar^2 \left(z^4 \frac{d^2\psi}{dz^2} + 2z^3 \frac{d\psi}{dz} \right) + V(1/z)\psi = E\psi \tag{1.229}$$

and for $z = 0$ we are left with the scalar equation $E\psi = 0$. The solution $\psi = 0$ is not physically acceptable, as it violates $\|\psi\|_2 = 1$. The limit $\hbar \to 0$ is also singular, for the same reason.

Let us look first at $x \to \infty$. We can analyze (1.229) using dominant balance, but the form (1.228) is algebraically simpler. It is clear that $V(x)\psi$ cannot be part of the dominant balance for large x, since it is necessarily much smaller than $E\psi$. We are left with the equation $-\hbar^2\psi'' \sim E\varphi$ which can be solved exactly. Pretending for a moment that we did not have a closed form solution for the dominant equation and we tried a leading behavior of the solution as $x \to \infty$ in the form $\psi \sim x^a$, we would get

$$V(x) - E + \frac{\hbar^2 a(1-a)}{x^2} = 0 \tag{1.230}$$

in which case no balance is possible. Therefore behavior cannot be power like. The exact solution of the leading order equation is

$$\psi \sim Ce^{\pm\hbar^{-1}x\sqrt{-E}} \tag{1.231}$$

We will see that (1.231) is not quite right unless $V(x)$ decays faster than $\frac{1}{x}$. Corrections to the exponent $\hbar^{-1}\sqrt{-E}x$ may be in the form $o(x)$ and $e^{o(x)} \neq O(1)$. Indeed if $V(x) \sim \frac{a}{x}$, this correction is proportional to $\ln x$, say $\alpha \ln x$, which on exponentiating results in $x^\alpha e^{\pm\hbar^{-1}x\sqrt{-E}} \not\sim Ce^{\pm\hbar^{-1}x\sqrt{-E}}$. This is another feature characteristic for singularly perturbed equations: more than one term may be needed in the initial ansatz for the exponent to correctly represent the leading order behavior of the solution. This and other reasons will make the WKB approach below almost necessary.

For now we note that for $E < 0$ the leading order formal solution is either exponentially large or exponentially small when $x \to \infty$. Given that ψ must be in L^2, any solution with exponentially large behavior either at $x = \pm\infty$ has to be ruled out. The solution that behaves like $x^\alpha e^{-\hbar^{-1}x\sqrt{-E}}$ as $x \to +\infty$ does not necessarily have the same behavior as $x \to -\infty$ since in the the intermediate regime $x = O(1)$, the asymptotics is invalid. Determining the asymptotic behavior of a solution at $+\infty$ in terms of its behavior at $-\infty$ is referred to as a connection problem. Requiring that the solution decays exponentially both for $x \to \pm\infty$ selects special values of E, the eigenvalues of the problem. While solutions ψ do not have power-like behavior for large x, $\log\psi$ does. An exponential substitution, $y = e^{w(x)}$ is suggested, and this *WKB ansatz* is very helpful in singularly perturbed equations.

We will proceed formally first, and then prove a result for (1.228). So, consider again (1.228) and substitute $\psi(x) = e^{w(x)}$. After dividing by $e^{w(x)}$ we get

$$-\hbar^2(w'' + w'^2) = E - V(x) \tag{1.232}$$

or, with $w' = f$, we get the first order nonlinear ODE

$$-\hbar^2(f' + f^2) = E - V(x) \tag{1.233}$$

We analyze (2.338) by dominant balance. We first assume for simplicity that $E > V(x)$ for all x; a similar argument works if $E < V(x)$ for all x, with

$\sqrt{V(x)-E}$ replacing $i\sqrt{E-V(x)}$. Near points x_t where $E=V(x_t)$, called *turning points*, the dominant ballance changes and we will study these points separately.

Note 1.234 In a WKB ansatz, we have $w'' \ll {w'}^2$. Later we will prove a more general result on this. In the present example, it is easy to see that the only consistent dominant balance, say for fixed x and small ε, is between $-\hbar^2 f^2$ and $E-V(x)$, and with this balance w'' is much smaller than the other terms (unless $V(x)$ is very close to E; points where $V(x)=E$ are called turning points and will be discussed separately). For now we simply remark that

$$\frac{w''}{(w')^2} = -\frac{1}{\alpha}(1+o(1)) \Rightarrow w = \alpha \log(x)(1+o(1))$$

on exponentiating would give $\psi \sim x^\alpha e^{o(\ln x)}$ for large x, a power-like balance that we ruled out.

According to Note 1.234 we place w'' on the right side of the equation, treated as being relatively small. With $f=w'$, (1.233) implies

$$f = \pm\frac{i}{\hbar}\sqrt{E-V(x)+\hbar^2 f'} \tag{1.235}$$

where we choose one sign at a time, say plus for now, and we expand (1.235), by the usual Picard-like asymptotic iterations,

$$f^{[n+1]} = \frac{i}{\hbar}\sqrt{E-V(x)+\hbar^2 {f^{[n]}}'} \tag{1.236}$$

with $f^{[-1]}=0$.

The fact that the highest order derivative is on the right side of the iteration strongly indicates that the expansion thus obtained is divergent.

To get the first few terms in the expansion, it is more convenient to expand the rhs of (1.236) to a few orders in $\hbar$. To three orders we get

$$f^{[n+1]} = \tfrac{i}{\hbar}\sqrt{E-V(x)} + \frac{i\hbar {f^{[n]}}'}{2\sqrt{E-V(x)}} - \frac{i({f^{[n]}}')^2\hbar^3}{8(E-V(x))^{3/2}} + \cdots \tag{1.237}$$

In this way we get (assuming V is twice differentiable) the following results when expanded out in powers of $\hbar$ to the order presented.

$$\begin{aligned}
f^{[0]} &= \tfrac{i}{\hbar}\sqrt{E-V(x)}\\
f^{[1]} &= \tfrac{i}{\hbar}\sqrt{E-V(x)} + \frac{1}{4}\frac{V'}{E-V} + O(\hbar)\\
f^{[2]} &= \tfrac{i}{\hbar}\sqrt{E-V(x)} + \frac{1}{4}\frac{V'}{E-V} + \hbar\frac{\frac{5i}{32}{V'}^2 + \frac{i}{8}V''(E-V))}{(E-V(x))^{5/2}} + O(\hbar^2)\\
f^{[3]} &= f^{[2]} - \hbar^2\left(\frac{15}{64}\frac{{V'}^3}{(V-E)^4} - \frac{9}{32}\frac{V'V''}{(V-E)^3} + \frac{1}{16}\frac{V'''}{(V-E)^2}\right) + O(\hbar^3)
\end{aligned} \tag{1.238}$$

In terms of w this gives

$$w^{[0]} = \tfrac{i}{\hbar}\int_{x_0}^{x} \sqrt{E - V(s)}ds + C \tag{1.239}$$

$$w^{[1]} = \tfrac{i}{\hbar}\int_{x_0}^{x} \sqrt{E - V(s)}ds - \frac{1}{4}\ln(E - V(x)) + C + O(\hbar) \tag{1.240}$$

and indeed we see that a log term is present in the exponent. Finally, returning to ψ we get the (formal for now) solution

$$\psi = C_1(E - V(x))^{-1/4} e^{\frac{i}{\hbar}\int_{x_0}^{x}\sqrt{E-V(s)}ds}\,(1 + o(1)) \tag{1.241}$$

If $V - E$ has a zero, the expansion (1.238) makes no sense in a small neighborhood of the zero since terms that appear latter in the iteration become progressively more singular near the zero. A question that arises naturally is how close can we get to the zero with the expansion in $\hbar$?

We notice that $\hbar(E - V)^{-3/2} \ll 1$ ensures that $f^{[0]} - f^{[1]} \ll f^{[0]}$ and $f^{[1]} - f^{[2]} \ll f^{[1]}$. This can be shown to be the case for all orders of the expansion, and stems from the simple fact that, to obtain this expansion we assumed that $\hbar^2 f' \ll E - V$. If this is indeed the case, if we approach a zero of $E - V$ where $V' \neq 0$, then f' is expected to be of order $(E - V)^{-1/2}/\hbar$ as seen by differentiating (1.238). The procedure has a chance to be legitimate if

$$\hbar^2 \frac{1}{\hbar\sqrt{V - E}} \ll (E - V) \Rightarrow E - V \gg \hbar^{2/3}\ , \tag{1.242}$$

which is the same condition $\hbar(E - V(x))^{-3/2} \ll 1$ that ensures that the iteration (1.238) is properly ordered. This shows that the region where $E - V \not\gg \hbar^{2/3}$ has to be dealt with in a different way. This is the turning point region and we will analyze separately. We also note the coefficients of higher order powers of $\hbar$ contain, as $V - E \to 0$ a leading term and corrections of order $(V - E)$ as compared to the leading term. This shows the formation of a new type of expansion, to carry through the region where $V - E$ is small.

1.6h.2 PDEs and formal WKB

In the context of PDEs, the Cauchy-Kowalewski theorem[14] guarantees local existence of solutions of initial value problems of the form

$$\partial_t^k h = F\left(x, t, \partial_t^j \partial_x^\alpha h\right),\ \text{where } j < k \text{ and } |\alpha| + j \leqslant k \tag{1.243}$$

where $x \in \mathbb{R}^n$ and the multi-index $\alpha = (\alpha_1, \alpha_1, ..\alpha_n)$ is in $(\mathbb{N} \cup \{0\})^n$, $|\alpha| = \sum_{j=1}^n \alpha_j$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} \cdot \partial_{x_n}^{\alpha_n}$. We use initial conditions

$$\partial_t^j h(x, 0) = f_j(x), \qquad 0 \leqslant j < k \tag{1.244}$$

[14] In our context, we look for solutions as power series in t with functions of x as coefficients rather than expanding in both x and t.

where all the functions involved except for h itself are assumed to be analytic. Note that the allowed order of the spatial derivatives is at most the same as that of the time derivative. This is the case, for instance, of the one-dimensional wave equation

$$u_{tt} = u_{xx};\ u(t,0) = u_0(x);\ u_t(t,0) = v_0(x)$$

where $x \in \mathbb{R}$. Assuming analyticity of u_0, v_0, and pretending, once more, we did not have a closed form solution, a power series ansatz

$$u(t,x) = \sum_{k=0}^{\infty} f_k(x)t^k \tag{1.245}$$

yields

$$f_{k+2} = \frac{f_k''}{(k+1)(k+2)} \Rightarrow f_k = \frac{u_0^{(2k)}}{(2k)!}\ (k \text{ even}) \text{ and } f_k = \frac{v_0^{(2k)}}{(2k)!}\ (k \text{ odd}) \tag{1.246}$$

Analyticity implies that $\left|u_0^{(2k)}\right| \lesssim (2k)!C^k$; similar bounds hold for v_0. and thus $\sum_{k=0}^{\infty} u_k(x)t^k$ converges. In this case, substituting these f_k in the series, we get the exact solution (no surprise)

$$\tfrac{1}{2}\big(u_0(x+t) + u_0(x-t) + V_0(x+y) - V_0(x-t)\big);\ \frac{dV_0}{dt} = v_0 \tag{1.247}$$

By contrast, in the heat equation

$$h_t = h_{xx} \tag{1.248}$$

the highest derivative in t is of lower order than the highest derivative in x (the principal symbol is parabolic), and not of the form (1.243). The same power series ansatz yields

$$h_{k+1} = \frac{h_k''}{k+1} \Rightarrow h_k = \frac{h_0^{(2k)}}{k!} \sim k! \text{ as } k \to \infty \tag{1.249}$$

and, by the same estimate as above, we see that for analytic initial which are not entire, the now *formal* series solution (1.245) is factorially divergent. The effect of parabolicity is very similar to that of a singular perturbation: the derivatives on the right side of the recurrence are of too high order for the formal series to converge. Much as in the case of ODEs (cf. Note 1.183) factorial behavior is associated with the potential presence of solutions behaving exponentially, and this suggests a WKB substitution:

$$u = e^W \tag{1.250}$$

leading to

$$W_{xx} + W_x^2 = W_t \tag{1.251}$$

Under the ansatz that we used before, $|W_{xx}| \ll W_x^2$, we obtain the leading order equation

$$W_x^2 = W_t \tag{1.252}$$

which can be made quasi-linear by taking an x derivative. With $V(x,t) = W_x(x,t)$ we get

$$2VV_x = V_t \tag{1.253}$$

with the general solution gotten by characteristics is

$$F(V) + 2tV + x = 0 \quad (F \text{ general}) \tag{1.254}$$

For example, with $F = 0$ we get $V(x,t) = -\frac{x}{2t} \Rightarrow W = -\frac{x^2}{4t}$. Substituting $W(x,t) = -\frac{x^2}{4t} + \delta(x,t)$ in (1.250) into (1.251) we get

$$\delta_x^2 - \delta_t + \delta_{xx} - \frac{x}{t}\delta_x - \frac{1}{2t} = 0 \tag{1.255}$$

with one of the consistent balances, between the second and the last term giving

$$\delta(x,t) = -\frac{1}{2}\ln t \tag{1.256}$$

The δ in (1.256) solves the full equation (1.255) and we fortuitously don't need further correction terms because $\exp\left[-\frac{x^2}{4t} - \frac{1}{2}\log t\right]$ is an exact solution to heat equation. Such cancellation cannot be expected for more general equations. In terms of h, this exact solution is simply the heat kernel,

$$h = t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \tag{1.257}$$

We can try WKB on the wave equation also, since the growth of solutions of PDEs depends on the initial conditions, and could be exponential. With $u = e^W$ we get

$$W_{xx} + W_x^2 = W_t^2 + W_{tt} \tag{1.258}$$

With the ansatz $|W_{xx}| \ll W_x^2, |W_{tt}| \ll W_t^2$ we get

$$W_x = \pm W_t, \quad W_\pm = F(x \pm t) \tag{1.259}$$

Both W_+ and W_- solve the full equation (1.258) (again, this cannot be expected in general).

We will return to asymptotics of PDEs after we have developed adequate tools.

1.6h.3 Proof of existence of a solution of (1.228) in the form (1.241)

One way of proving the expansion is to return to (1.233) where we substitute for $f(x) = f^{[j]}(x) + \delta(x),\ j \geqslant 1$ [15]. Here we choose $j = 1$; assuming that the regularity of V allows for calculating higher order terms which involve higher derivatives of V as seen in (1.238) taking $j = j_1 > 1$ would allow for proving asymptoticity of the expansion with $j_1 + 1$ terms.

$$f(x) = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{1}{4}\frac{V'}{E - V} + \delta(x) \tag{1.260}$$

The equation for $\delta(x)$ is

$$\hbar\delta' + 2i\sqrt{E - V(x)}\delta + \frac{\hbar V'}{2(E - V(x))}\delta = -\hbar g - \hbar\delta^2$$
$$\text{where}\ \ g(x) := \frac{5}{16}\left(\frac{V'}{E - V(x)}\right)^2 + \frac{V''}{4[E - V(x)]} \tag{1.261}$$

(as discussed earlier, the highest derivative cannot be treated as a perturbation in a rigorous argument).

We then write the equation in integral form. Let $J = \frac{2i}{\hbar}\int_{x_0}^{x}\sqrt{E - V(s)}ds$ and $\mu(x) = (E - V(x))^{-1/2}$. Using the integrating factor for the left side of 1.261, we get

$$\delta(x) = -\frac{e^{-J(x)}}{\mu(x)}\int_{\infty}^{x}\mu(s)e^{J(s)}g(s)ds - \frac{e^{-J(x)}}{\mu(x)}\int_{\infty}^{x}e^{J(s)}\mu(s)\delta^2(s)ds$$
$$:= \delta_0 + \mathcal{N}\delta \tag{1.262}$$

To prove a rigorous result we need some assumptions.

Assumption 1.263 For simplicity we let $V : \mathbb{R} \to \mathbb{R}$, $V \in C^2(\mathbb{R})$, assume that V is $O(1/x^{1+\varepsilon})$ for large x and that this estimate can be differentiated: $V' = O(1/x^{2+\varepsilon})$ and $V'' = O(1/x^{3+\varepsilon})$. We work on an interval, say $[x_0, \infty)$, where $E - V(x) > a > 0$. We note that under these assumptions we have $g(x) = O(x^{-3-\varepsilon})$ for large x.

We introduce the Banach space

$$\mathcal{B} = \{\delta : [x_0, \infty) \to \mathbb{C} \mid \|\delta\| := \sup_{x \geqslant x_0}|x^{1+\varepsilon}\delta(x)| < \infty\} \tag{1.264}$$

We first prove that the term

$$\delta_0 := \frac{e^{-J(x)}}{\mu(x)}\int_{x}^{\infty}\mu(s)e^{J(s)}g(s)ds \tag{1.265}$$

[15] The minimum j needed depends on the problem; in some settings, $j = 0$ suffices. As a rule, the more terms we pull out, the more contractive the operator becomes, at the expense of the algebra getting more involved .

in (1.262) decays as $\hbar \to 0$ or as $x_0 \to +\infty$.

Lemma 1.266 *Under Assumption 1.263, we have* $\lim_{\hbar\to 0} \|\delta_0\| = 0$*. Furthermore, for any* $\hbar \neq 0$*,* $\lim_{x_0\to\infty} \|\delta_0\| \to 0$*.*

PROOF Let

$$F(x) := x^{1+\varepsilon} \int_x^\infty \mu(s) e^{J(s)} g(s) ds \tag{1.267}$$

Since $|x^{1+\varepsilon} e^{J(s)} g(s)\mu(s)| \leqslant s^{1+\varepsilon}\mu(s)|g(s)| \in L^1$, (1.267) implies

$$\lim_{x\to\infty} F(x) = 0 \tag{1.268}$$

Using the fact that $\frac{1}{\mu(x)} e^{-J(x)}$ is bounded, (1.264), (1.262) and (1.268) imply $\lim_{x_0\to\infty} \|\delta_0\| = 0$ for any $\hbar \neq 0$. Now, consider the case of $\hbar \to 0$ with $x_0 > 0$ fixed.

We claim that for any $\varepsilon > 0$, there exists $\hbar_0$ such that $|f(x)| \leqslant \varepsilon$ for any x if $|\hbar| \leqslant \hbar_0$.

First, from (1.268), it follows that for large enough M and $x > M$ we have $|F(x)| \leqslant \varepsilon$.

Let now $t = t(x) =: \hbar J/(2i)$. Note that $t : [x_0, \infty] \to [0, \infty]$ is increasing since $t'(x) = \sqrt{E - V(x)} = 1/\mu(x)$. We change to the variable t in (1.267). Since

$$F(x) = x^{1+\varepsilon} \int_t^\infty e^{2it/\hbar} g(s(t)) \mu^2(s(t)) dt$$

the Riemann-Lebesgue lemma implies that for $x \in [x_0, M]$ we have $\lim_{\hbar\to 0} f(x) = 0$. Since f is uniformly continuous on $[x_0, M]$, convergence is uniform in x, *i.e.* there exists $\hbar_0$ so that $|\hbar| \leqslant \hbar_0$ implies $|F(x)| \leqslant \varepsilon$ for any x. Therefore, $\lim_{\hbar\to 0} \|\delta_0\| = 0$ since $e^{-J(x)}/\mu(x)$ is bounded. ☐

Theorem 1.269 *Under Assumption 1.263, if* x_0 *is large enough or* $\hbar$ *is small enough, then two linearly independent solutions of (1.228) for* $x \in (x_0, \infty)$*,* $\psi = \psi_1$ *and* $\psi = \psi_2$*, satisfy*

$$\psi_1(x) = [E - V(x)]^{-1/4} \exp\left[\frac{i}{\hbar} \int_{x_0}^x [E - V(t)]^{1/2}\, dt\right] \{1 + o(1)\} \tag{1.270}$$

$$\psi_2(x) = [E - V(x)]^{-1/4} \exp\left[-\frac{i}{\hbar} \int_{x_0}^x [E - V(t)]^{1/2}\, dt\right] \{1 + o(1)\} \tag{1.271}$$

PROOF We only prove the result for ψ_1 since the proof for ψ_2 is the same after changing the sign of i. Since $\psi_1 = e^W$, $W' = f^{[1]} + \delta$ with $f^{[1]}$ defined in (1.238). it is enough to show that (1.262) has a solution in a ball where $\|\delta\|$ is small:

$$B_\varepsilon = \{\delta \in \mathcal{B} | \|\delta\| \leqslant \varepsilon\} \tag{1.272}$$

Using Lemma 1.266, we see that for any $\varepsilon > 0$, if x_0 is large enough or $\hbar$ is small enough, then $\|\delta_0\| \leqslant \frac{1}{2}\varepsilon$. Now, for any $\delta \in B_\varepsilon$, we have

$$|x^{1+\varepsilon}\mathcal{N}[\delta]| \leqslant \|\delta\|^2 x^{1+\varepsilon} \int_x^\infty \frac{\mu(s)}{\mu(x)} s^{-2-2\varepsilon} ds \leqslant C\|\delta\|^2 \leqslant C\varepsilon^2,$$

where C is independent of ε. Choosing $\varepsilon < \frac{1}{2C}$, this implies that $\|\delta_0 + \mathcal{N}[\delta]\| < \varepsilon$ and $\|\mathcal{N}[\delta_1] - \mathcal{N}[\delta_2]\| \leqslant 2C\varepsilon\|\delta_1 - \delta_2\| < \|\delta_1 - \delta_2\|$. Therefore, the contraction mapping theorem implies that there exists a unique solution in B_ε. ☐

If, as mentioned at the beginning of the section we took $j = j_1 > 1$ instead, then the remainder g in the map (1.283) will be of higher order in $\hbar$. With this change, contractivity is proved in the same way, to obtain an asymptotic expansion with $j_1 + 1$ terms.

Remark 1.273 (i) No decay assumption on V is necessary for Theorem 1.269 to apply for x in a fixed ($\hbar$−independent interval $[a, b]$. (ii) The assumption $x_0 > 0$ in Theorem 1.269 is not needed. To allow for $x_0 < 0$ the proof is largely the same. Assuming $V(x) = O(|x|^{-1-\varepsilon})$ as $x \to -\infty$, we would instead use the norm $\|\delta\| = \sup_{x\in(x_0,\infty)} |1 + |x|)^{1+\varepsilon}|\delta(x)|$.

1.6i The case $V(x) - E \geqslant a > 0$

Assumption. $V(x) - E \geqslant a > 0,\ V \in C^2$ for $x \in [x_0, x_1)$ where x_1 can be $+\infty$

Repeating the WKB procedure in §1.6h.1 we obtain

$$f^{[2]} = \pm\hbar^{-1}\sqrt{V-E} - \frac{1}{4}\frac{V'}{V-E} \pm \hbar\left[\frac{V''}{8(V-E)^{3/2}} - \frac{5}{32}\frac{(V')^2}{(V-E)^{5/2}}\right] \quad (1.274)$$

which formally leads to

$$\psi_\pm = C_1 [V(x) - E]^{-1/4} \exp\left[\pm\frac{1}{\hbar}\int_{x_0}^x \sqrt{V(t) - E}dt\right][1 + o(1)] \ \text{ for } x \in [x_0, x_1) \quad (1.275)$$

either for $\hbar \to 0$ or $x \to +\infty$ (if $x_1 = +\infty$). The precise result is given below:

Theorem 1.276 *For $V \in C^2(x_0, \infty)$, and $V(x) - E \geqslant a > 0$, as $x \to +\infty$ (assuming $x_1 = \infty$) or as $\hbar \to 0^+$, two independent solutions of (1.228) are given by $\psi = \psi_1$ and $\psi = \psi_2$, where*

$$\psi_1(x) = [V(x) - E]^{-1/4} \exp\left\{\frac{1}{\hbar}\int_{x_0}^x [V(t) - E]^{1/2}\, dt\right\}\{1 + o(1)\} \quad (1.277)$$

$$\psi_2(x) = [V(x) - E]^{-1/4} \exp\left\{-\frac{1}{\hbar}\int_{x_0}^x [V(t) - E]^{1/2}\, dt\right\}\{1 + o(1)\} \quad (1.278)$$

PROOF We carry out the proof first for ψ_1. Substituting

$$W' = \frac{1}{\hbar}\sqrt{V(x)-E} - \frac{V'}{4[V(x)-E]} + \delta$$

where $\psi = e^W$, and inverting the linear part of the first order ODE for δ, we obtain the integral equation

$$\delta = -\frac{e^{-J(x)}}{\mu(x)}\int_{x_0}^{x} g(s)\mu(s)e^{J(s)}ds - \frac{e^{-J(x)}}{\mu(x)}\int_{x_0}^{x} e^{J(s)}\mu(s)\delta^2(s)ds =: \delta_0 + \mathcal{N}\delta \ , \tag{1.279}$$

where g as before is given by (1.261), but now $J(x) = \frac{2}{\hbar}\int_{x_0}^{x}\sqrt{V(t)-E}dt$ and $\mu = (V(x)-E)^{-1/2}$. The contraction mapping theorem shows existence of a unique solution to (1.279) in a ball of size $2\|\delta_0\|_\infty$. We note that δ_0 can be made arbitrarily small by choosing a small enough $\hbar$, as it is seen using Laplace's method. In the proof for ψ_2, we substitute $W' = -\sqrt{V(x)-E} - \frac{V'}{4[V(x)-E]} + \delta$, where $\psi = \psi_2 = e^W$. We obtain the same form of integral equation (1.279) except that the sign of $J(x)$ is switched and the limits of integration are x_1 and x. In both cases it is crucial to choose the limits of integration so that $e^{-J(x)+J(s)} \leqslant 1$ throughout the interval of integration and is maximal at $s = x$. By Laplace's method, these conditions and the presence of $1/\hbar$ in the exponent ensure that the integrals go to zero as $\hbar \to 0$ ensuring contractivity for small $\hbar$. □

Exercise 1.280 Complete the details of the proof of Theorem 1.276.

1.6i.1 Turning points

In the previous subsection we assumed that $E - V$ is bounded below. This assumption is in fact necessary, otherwise the asymptotic behavior of the solutions is different. If we examine the procedure used to derive (1.237) from (1.236), we see that the expansion is only valid if $\hbar^2 f^{[n]'} \ll E - V(x)$, that is, to have $f \approx f^{[0]}$ we need $\hbar(E - V(x))^{-1/2} \ll E - V(x)$, that is, $E - V(x) \gg \hbar^{2/3}$. Something else must be done when the latter condition fails.

In our assumption V is smooth. Generically, near a zero of $V(x) - E$, also referred to as a turning point, $V(x) = \alpha(x - x_t) + O(x - x_t)^2$, where $\alpha \neq 0$. Without loss of generality we can take $x_t = 0$ and $\alpha = -1$ through translation and scaling. The region where our WKB does not hold is given by $|x| \lesssim \hbar^{2/3}$. It is natural to change variables to $t = x/\hbar^{2/3}$ in (1.228); we get, after dividing by $\hbar^{2/3}$,

$$-\psi''(t) - t\psi(t) = \hbar^{2/3}t^2\varphi_1(x(t))\psi(t) \tag{1.281}$$

where $\varphi_1(x) = x^{-2}[E - V(x) - x]$. To leading order in small $\hbar$, ψ satisfies $-\psi_0''(t) - t\psi_0(t) = 0$ with the general solution

$$\psi_0(t) = C_1\mathrm{Ai}(-t) + C_2\mathrm{Bi}(-t) \tag{1.282}$$

Since the right hand side of (1.282) is a regular perturbation in $\hbar^{2/3}$ for t in any finite interval, we can obtain higher order corrections in $\hbar$ as usual.

1.7 Borderline region: $x \gg \hbar^{2/3}$

Assume a turning point at $x = 0$, *i.e.* , $E = V(0)$ and that $E - V(x) > 0$ for $x > 0$. Then, for $x > x_0 > 0$, independent of $\hbar$, Theorem 1.269 applies. We now write a mapping for an interval (a, x_0) where a is allowed to depend on $\hbar$:

$$\delta(x) = -\frac{e^{-J(x)}}{\mu(x)} \int_a^x e^{J(s)} \mu(s) g(s) ds - \frac{e^{-J(x)}}{\mu(x)} \int_a^x e^{J(s)} \mu(s) \delta(s)^2 ds := \delta_0 + \mathcal{N}\delta \tag{1.283}$$

The reasoning is similar to that in §1.6h.3. We choose a as small as possible, while still allowing the right side of (1.283) to be contractive. For this to be the case, a so that $\delta^2 \ll g$; when this is possible, as shown at the end of the argument, the results of Theorem 1.269 extend to the interval (a, x_0). To determine what this condition entails, we use dominant balance in (1.261): $\delta \ll \hbar|gx^{-1/2}| \ll \hbar|x|^{-5/2}$, and thus $\delta^2 \ll g$ implies $\hbar^2|x|^{-5} \ll \hbar|x|^{-2}$, that is $|x| \gg \hbar^{2/3}$. For contractivity we need, as in §1.6h.3, $|\delta_1 + \delta_1| \ll 1$ which for $\delta_1, \delta_2 = O\left(\hbar x^{-5/2}\right)$ holds if $x \gg \hbar^{2/5}$. This condition is more stringent than $|x| \gg \hbar^{2/3}$. We then choose $a = \nu\hbar^{2/3}$ with ν sufficiently large, and with this, the map is contractive on (a, x_0). We leave the details as an exercise.

1.7a Inner region: Rigorous analysis

$$-\psi'' - t\psi = -\hbar^{2/3} t^2 \varphi_1(\hbar^{2/3} t)\psi := f(t) \tag{1.284}$$

which can be transformed into an integral equation in the usual way,

$$\psi(t) = \pi \mathrm{Ai}(-t) \int^t f(s) \mathrm{Bi}(-s)\psi(s) ds - \pi \mathrm{Bi}(-t) \int^t f(s) \mathrm{Ai}(-s)\psi(s) ds \\ + C_1 \mathrm{Ai}(-t) + C_2 \mathrm{Bi}(-t) \tag{1.285}$$

where Ai, Bi are the Airy functions, with the integral representations:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3 - zt} dt \tag{1.286}$$

$$\mathrm{Bi}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3 - zt} dt + \frac{1}{2\pi} \int_{-\infty}^{\infty e^{-\pi i/3}} e^{\frac{1}{3}t^3 - zt} dt \tag{1.287}$$

The integral representations allow us to derive the *global* behavior at ∞, that is, the asymptotic expansion in any direction towards infinity, with explicit constants. With $\zeta = \frac{2}{3}|t|^{\frac{3}{2}}$ we have

$$\mathrm{Ai}(-t) \sim \frac{1}{2\sqrt{\pi}}|t|^{-1/4}e^{-\zeta}; \quad \mathrm{Bi}(-t) \sim \frac{1}{\sqrt{\pi}}|t|^{-1/4}e^{\zeta}; \quad t \to -\infty \tag{1.288}$$

[1] and

$$\mathrm{Ai}(-t) \sim \frac{1}{\pi^{1/2}t^{1/4}}\sin(\zeta + \tfrac{\pi}{4}), \ \mathrm{Bi}(-t) \sim \frac{1}{\pi^{1/2}t^{1/4}}\cos(\zeta + \tfrac{\pi}{4}) \ (t \to \infty) \tag{1.289}$$

as $t \to -\infty$. As usual, we have to choose the limits of integration in (1.285) so that the maximum point of the integrand is at the variable point of integration. We note that we cannot quite choose infinity as an upper limit since the Airy-type behavior was derived in the inner region $|x| \ll \hbar^{2/3}$ and in general is not expected to be the same outside. We will choose as large a t-interval $(-M_1, M_2)$, possibly depending on $\hbar$ for which the leading order behavior $\psi \sim C_1 Ai(-t) + C_2 Bi(-t)$ can be shown. We rewrite (1.284) in the integral form

$$\psi(t) = \pi\mathrm{Ai}(-t)\int_0^t f(s)\mathrm{Bi}(-s)\psi(s)ds - \pi\mathrm{Bi}(-t)\int_{-M_1}^t f(s)\mathrm{Ai}(-s)\psi(s)ds$$
$$+ C_1\mathrm{Ai}(-t) + C_2\mathrm{Bi}(-t) = J\psi + \psi_0 \tag{1.290}$$

Next, to control the norm of J, for large M_1 the estimate

$$|t|^{-1/4}e^{-\frac{2}{3}|t|^{3/2}}\hbar^{2/3}\int_0^{|t|} s^2 s^{-1/4}e^{\frac{2}{3}s^{3/2}}ds \lesssim \hbar^{2/3}M_1, \ (t \to -\infty) \tag{1.291}$$

follows from Watson's Lemma after the change of variable $p = 1 - s^{3/2}/|t|^{3/2}$, and similarly

$$|t|^{-1/4}e^{\frac{2}{3}|t|^{3/2}}\hbar^{3/2}\int_{|t|}^{M_1} s^2 s^{-1/4}e^{-\frac{2}{3}s^{3/2}}ds \lesssim \hbar^{2/3}|t| \lesssim \hbar^{2/3}M_1 \tag{1.292}$$

The right sides of (1.293) and (1.292) are small if $M_1 \ll \hbar^{-2/3}$. For $t \to +\infty$, estimating crudely $|\sin|, |\cos|$ by one, we get

$$t^{-1/4}\hbar^{2/3}\int_0^t |s^2 s^{-1/4}|ds \lesssim \hbar^{2/3}t^{5/2} \lesssim \hbar^{2/3}M_2^{5/2}, \tag{1.293}$$

which is small for $M_2 \ll \hbar^{-4/15}$. We now work in the sup norm on $[-M_1, M_2]$ and obtain, in the usual way, the following result

Proposition 1.294 *If $\hbar$ is small enough, then J defined in Eq. (1.290) is contractive in $L^\infty(-M_1, M_2)$ when $\hbar^{2/3}M_1$ and $\hbar^{4/15}M_2$ are small enough.*

We leave the details as an exercise. We see that the region of contractivity for $t < 0$ simply requires $|x| \ll 1$. On the other hand, the same is true for $t > 0$, with the price of making the argument quite a bit more involved.

Note 1.295 The contractivity of the map for $x < 0$ only requires $|x| \ll 1$. However, the norm used, L^∞ does not allow for controlling *the asymptotic behavior of solutions as t becomes large.* In particular, we would like to understand for what range of (large, negative) t does the solution of (1.284) have the behavior described by Airy function asymptotics, (1.288). The behavior (1.288) does not follow from our arguments, and in fact it is not even correct if $|t| \gg \hbar^{-4/15}$ as we will see in §1.7b.

1.7b Matching region

Let's analyze the behavior of solutions in the region $1 \ll |t| \ll \hbar^{-4/15}$. We will only analyze $t < 0$, as for $t > 0$ the analysis is similar (in fact, slightly simpler).

We first write $t = -u$ to make the analysis clearer. We get

$$-\psi'' + u\psi = -\hbar^{2/3}u^2\varphi_1(-\hbar^{2/3}u)\psi \tag{1.296}$$

We next bring (1.284) to a form that is best suited for looking at large t, a process called *normalization.* In the region where solutions have Airy-like asymptotic behavior, roughly $u^{-1/4}e^{\pm\frac{2}{3}u^{3/2}}$, we change variables so that the leading behavior is of the form e^s. A way to do this is simply by rescaling the dependent and independent variables, $\psi(u) = u^{-1/4}g(\frac{2}{3}u^{3/2})$.

With $s = \frac{2}{3}u^{3/2}$, this leads to the equation

$$g'' - g = -\frac{5}{36}s^{-2}g(s) + \frac{18^{1/3}}{2}\hbar\varphi_1(s)s^{2/3}g(s) = F(s)g(s) \tag{1.297}$$

where φ_1 is bounded. Choosing s_0 large enough, we write (2.47) in the integral form:

$$g = Ae^s + Be^{-s} + \frac{1}{2}\left(e^s\int_M^s F(v)e^{-v}g(v)dv - e^{-s}\int_{s_0}^s F(v)e^v g(v)dv\right) \tag{1.298}$$

where M will be "large but not too large" so that two solutions with asymptotic behavior e^s and e^{-s} respectively exist for $s \in [s_0, M]$.

We now look for a solution with the behavior $g(s) = e^{-s}$ for large s. The adapted norm to measure this type of behavior is $\|g\| = \sup_{s>s_0}|g(s)e^s|$. We should take $A = 0$ in (1.298), since the norm of e^s is very large, of order e^{2M}. To check for the contractivity of the map in this norm, we use the fact that, by the definition of the norm, $|g(v)| \leqslant \|g\|e^{-v}$. For the first integral we have

$$e^s\left|e^s\int_M^s F(v)e^{-v}g(v)dv\right| \lesssim \|g\|e^{2s}\int_M^s(\hbar^{2/3}v^{2/3} + v^{-2})e^{-2v}dv$$
$$\lesssim \|g\|(\hbar^{2/3}s^{2/3} + s^{-2}) \lesssim \|g\|(\hbar^{2/3}M^{2/3} + s_0^{-2}) \tag{1.299}$$

where we used Watson's lemma. In order for the norm of this part of the operator to be less than one, we need s_0 to be large, which we assumed already, and, once more, $|x| \lesssim 1$.

For the second integral, we see that the exponential in the definition of the norm cancels the exponential which was already in the integrand and we get

$$e^s \left| e^{-s} \int_{s_0}^{s} F(v) e^v g(v) dv \right| \lesssim \|g\| \int_{s_0}^{s} (\hbar^{2/3} v^{2/3} + v^{-2}) ds \lesssim \|g\| \hbar^{2/3} s^{5/3} + s_0^{-1} \lesssim x\hbar^{-1}|x|^{5/2} + s_0^{-1} \quad (1.300)$$

which can be made small if s_0 is large, as before, and if $|x| \lesssim \hbar^{2/5}$. The mapping is now contractive in a smaller region– the one that we have obtained before in the oscillatory regime.

Exercise 1.301 *Complete the details of the analysis, and do a similar analysis for the behavior* e^s *(where now the norm would be* $\|g\| = \sup_s |e^{-s} g(s)|$*). Show the existence of solutions of* (1.284) *with the behavior of the Airy functions Ai and Bi, cf.* (1.288) *in the region* $|x| \lesssim \hbar^{2/5}$.

Note now that, when approaching $x = 0$ from the outer region, we have $E - V(x) = ax + o(x^2)$ where, by scaling we chose $a = 1$; then $i\hbar^{-1} \int \sqrt{E - V(x)} = i\hbar^{-1} \frac{2}{3}(x^{3/2} + O(x^{5/2}))$ and

$$(E - V(x))^{-1/4} e^{i\hbar^{-1} \int \sqrt{E-V(x)}} = x^{-1/4} e^{i\hbar^{-1} \frac{2}{3}(x^{3/2} + O(x^{5/2}))} \quad (1.302)$$

and by switching to the variable $t = \hbar^{-2/3}x$ we get the behavior of a linear combination of Ai and Bi in the oscillatory region. Similarly, changing i to $-i$ in the analysis above we get a linearly independent solution, with the behavior given by a different combination of Ai and Bi. This was to be expected since we are, after all, dealing with the same equation in the inner and outer region, up to these changes of variables, and the behaviors should correspond to each other.

Matching means simply finding the concrete values of the constants so that an outer solution equals an inner one.

We note that there is a difference between the oscillatory outer region and the one with growing/decaying exponential behavior. If only the decaying exponential is present in the outer solution, the matching is straightforward: it corresponds simply to the solution with the behavior Ai in the inner region (Bi should not be present since it *grows* exponentially). But if the outer solution has both growing and decaying components, matching becomes more delicate since the small exponential is masked by the larger one to all orders of an asymptotic expansion in $\hbar$ and finding the correspondence between constants cannot be done by classical asymptotic means. One has to go to the complex domain if the potential is analytic or use exponential asymptotic tools.

1.8 Recovering actual solutions from formal ones

Consider the simple ODE

$$y' = y + 1/x \tag{1.303}$$

(1.303) has an irregular singularity at infinity. If we look for formal asymptotic series solutions $\tilde{y} = \sum_{k\geqslant 0} c_k x^{-k}$ we get $c_0 = 0$, $c_k = (-1)^k(k-1)!$, that is

$$\tilde{y} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}k!}{x^{k+1}} \tag{1.304}$$

This series has empty domain of convergence. Nonetheless, we can do the following. Writing

$$k! = \int_0^\infty e^{-t}t^k dt \Rightarrow \frac{(-1)^{k+1}k!}{x^{k+1}} = \int_0^\infty e^{-px}(-1)^{k+1}p^k dp \tag{1.305}$$

and inserting (1.305) into (1.306), we get

$$\tilde{y} = \sum_{k=0}^{\infty} \int_0^\infty (-1)^{k+1}p^k e^{-px} dp \tag{1.306}$$

This following step requires serious justification, but for now we formally interchange summation and integration,

$$\tilde{y} = \int_0^\infty e^{-px}\sum_{k=0}^{\infty}(-1)^{k+1}p^k dp \overset{\text{formally}}{=} -\int_0^\infty \frac{e^{-px}}{1+p}dp = e^x\mathrm{Ei}(-x) \tag{1.307}$$

where

$$Ei(x) = \mathrm{PV}\int_{-\infty}^{x} \frac{e^t}{t}dt$$

and PV stands for the Cauchy principal part: if the path of integration crosses zero, then the integral is defined as the limit as $\varepsilon \to 0$ of $\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{x}$. The use of PV in the definition of Ei is perhaps one of the oldest and simplest forms of more general forms of regularization, the "medianizations" of Écalle.

If our sole purpose was to solve (1.303) we could bypass the intermediate steps and any need for justification, and simply check that the function we obtained at the end, $e^x\mathrm{Ei}(-x)$, satisfies the ODE. For the general solution of (1.303), we just add Ce^x, the solution of the associated homogeneous equation, to $e^x\mathrm{Ei}(-x)$.

Of course however, (1.303) is very simple and we could have solved it by variation of constants or other elementary means. The questions are (1) Can we extend this to a much more general procedure, applicable to generic ODEs near irregular singularities? (the answer is yes) and (2) Can we justify the formal steps that led from (1.306) to the function in (1.307)? (the answer is yes again). We leave these issues for later, now we simply note that there is another way to interpret the operations that led to "summing" the divergent series: (1) we took the formal inverse Laplace transform of the series, that is, term-by-term; (indeed $\mathcal{L}^{-1}x^{-k-1} = p^k/k!$, (2) we summed the geometric series $\sum_{k=0}^{\infty}(-p)^k = (1+p)^{-1}$, and, since the radius of convergence of this geometric series is one, we extended $(1+p)^{-1}$ analytically $(1+p)^{-1}$ on $\mathbb{R}^+$, and (3) we took the Laplace transform of $\mathcal{L}$ the result. Since $\mathcal{L}\mathcal{L}^{-1} = I$ the identity, and at a formal level what we did is just that, $\mathcal{L}\mathcal{L}^{-1}$, we expect that if $\tilde{y}$ satisfied an ODE, so will the $\mathcal{L}\mathcal{L}^{-1}\tilde{y}$. This is the route we will take in justifying this procedure.

We also note that the formal series $\tilde{y}$ is divergent since it is obtained by repeatedly differentiating a function which is not entire: the iterative asymptotic process leading to $\tilde{y}$ is $y^{[n+1]} = -1/x + \partial_x y^{[n]}$. The inverse Laplace transform is a Fourier transform in the imaginary direction, and the Fourier transform is the unitary operator that diagonalizes differentiation. After a form of Fourier transform, repeated differentiation becomes repeated multiplication by the "symbol" of the differential operator, denoted by p here. This can only lead to geometric behavior of the terms of the formal series, something we know much more about: this is dealt with by analytic function theory.

Finally, and this is another important point, in this and many problems, applying the inverse Laplace transform has a regularizing effect. Indeed, the formal solution $\sum_{k=0}^{\infty}(-1)^k k! x^{-k-1}$ becomes, after applying $\mathcal{L}^{-1}$, $\sum_{k=0}^{\infty}(-p)^k$ which is convergent. Whatever problem the new series is a solution of, that new problem is expected to have at mot a regular singularity, given this convergence. Indeed, assuming first that y has an inverse Laplace transform, to be confirmed later and taking$\mathcal{L}^{-1}$ in (1.303) we get, with $\mathcal{L}^{-1}y = Y$,

$$(p+1)Y = 1 \tag{1.308}$$

an ordinary equation with meromorphic solutions. We now note that $y = \mathcal{L}Y$ is inverse Laplace transformable indeed and $\mathcal{L}^{-1}y = Y$, and that $\mathcal{L}Y$ solves the equation. A more general rigorous approach will based, in broad lines, on this approach.

The same can be dome in the context of PDEs. Let's take the heat equation,

$$h_t = h_{xx}; \quad \text{with } h(0,x) = \frac{1}{1+x^2} \tag{1.309}$$

Since the equation if parabolic, the Cauchy-Kowalesky does not apply. In fat, looking for power series solutions

$$h = \sum_{k=0}^{\infty} H_k(x) t^k \tag{1.310}$$

we obtain the recurrence

$$H_{k+1}(x) = \frac{H_k''(x)}{k+1}; \quad H_0(x) = \frac{1}{1+x^2} = \mathrm{Re}\left(\frac{1}{1+ix}\right) \tag{1.311}$$

where we wrote the initial condition in a way that facilitates taking high order derivatives. We get for H_k,

$$H_{k+1} = \frac{H_k''}{k+1} \Rightarrow H_k = \frac{H_0^{(2k)}(x)}{k!} = (-1)^k \frac{(2k)!}{k!}\mathrm{Re}\left((1+ix)^{-2k-1}\right) \tag{1.312}$$

and (1.312) shows that, with the given initial condition, (1.313) is divergent.
Denoting $t = 1/T$ we write

$$\tilde{h} = T\sum_{k=0}^{\infty} H_k(x)T^{-k-1} = T\sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{k!}\mathrm{Re}\left((1+ix)^{-2k-1}\right)T^{-k-1} \tag{1.313}$$

and we apply to the sum in (1.313) the procedure we used in (1.307), (1.306), (1.305), with $x = T$. We get

$$\begin{aligned} T\sum_{k=0}^{\infty}&(-1)^k \frac{(2k)!}{k!}\mathrm{Re}\left((1+ix)^{-2k-1}\right)T^{-k-1} \\ &= t^{-1}\int_0^{\infty} e^{-\frac{p}{t}}\mathrm{Re}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!(1+ix)^{-2k-1}p^k}{k!^2}dp\right\} \\ &= t^{-1}\int_0^{\infty} e^{-\frac{p}{t}}F(p,x)dp: \\ F(p,x)) &= \mathrm{Re}\sqrt{\frac{1}{\xi^2+4p}} \quad \text{where } \xi = (1+ix) \end{aligned} \tag{1.314}$$

where we used the binomial series representation of $(1+4z)^{-1/2}$.

Chapter 2

Borel summation: an introduction

2.1 The Borel transform $\mathcal{B}$

The Borel transform is defined on formal power series in the reciprocal of a variable, say x, with values in the space of formal power series in a dual variable, that we will often denote by p. By definition,

$$\mathcal{B}x^{-s} = \frac{p^{s-1}}{\Gamma(s)}, \quad \operatorname{Re} s > 0 \tag{2.1}$$

defined for p in $\mathbb{C}$ (or, if s is not an integer, on the universal covering of $\mathbb{C} \setminus \{0\}$)[1]. The Borel transform is similar to a (formal) inverse Laplace transform $\mathcal{L}^{-1}$, except that $\mathcal{L}^{-1}$ vanishes in the left half plane:

$$\mathcal{L}^{-1}x^{-s} = \begin{cases} p^{s-1}/\Gamma(s) & \text{for } \operatorname{Re} p > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \operatorname{Re}(s) > 0 \tag{2.2}$$

For a power series $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k-1}$, the Borel transform is applied, by definition, term-by-term:

$$\mathcal{B}\sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}} = \sum_{k=1}^{\infty} \frac{c_k}{k!} p^k \tag{2.3}$$

More generally, we can allow for noninteger power series. If for instance $0 < \operatorname{Re}(s_k) < \operatorname{Re}(s_{k+1})$ for all $k \in \mathbb{N}$, then we define

$$\mathcal{B}\sum_{k=1}^{\infty} \frac{c_k}{x^{s_k}} = \sum_{k=1}^{\infty} \frac{c_k}{\Gamma(s_k)} p^{s_k-1} \tag{2.4}$$

Because the k-th coefficient of $\mathcal{B}\{\tilde{f}\}$ is smaller by a factor $k!$ than the corresponding coefficient of $\tilde{f}$, $\mathcal{B}\{\tilde{f}\}$ may converge even if $\tilde{f}$ does not. Note also that $\mathcal{LB}$ is *formally* $\mathcal{LL}^{-1}$, the identity operator. These two facts account for the central role played by $\mathcal{LB}$ in summability of factorially divergent series.

[1] This consists of classes of curves in $\mathbb{C} \setminus \{0\}$, where two curves are equivalent if they can be continuously deformed into each other without crossing 0.

If $\mathcal{B}\{\tilde{f}\}$ *converges* to a function$f$ which is Laplace transformable, and we apply $\mathcal{L}$ to f, we effectively get an identity-like operator from series to functions.

2.2 Definition of Borel summation and basic properties

We define Borel summation of integer power series, but the definition extends to more general series, see Note 2.9 below.

Borel summation is relative to a direction; see §2.2a. The same formal series $\tilde{f}$ may yield different functions by Borel summation in different directions.

Borel summation along $\mathbb{R}^+$ consists of three operations, assuming (2) and (3) are possible:

1. Borel transform, $\tilde{f} \mapsto \mathcal{B}\{\tilde{f}\}$.

2. Convergent summation of the series $\mathcal{B}\{\tilde{f}\}$ and analytic continuation along $\mathbb{R}^+$ (denote the continuation by F and by $\mathcal{D}$ an open set in $\mathbb{C}$ containing $\mathbb{R}^+ \cup \{0\}$ where F is analytic).

3. Laplace transform, $F \mapsto \int_0^\infty F(p)e^{-px}\mathrm{d}p =: \mathcal{LB}\{\tilde{f}\}$, which requires exponential bounds on F, defined in some half-plane $\mathrm{Re}\,(x) > x_0$.

Note 2.5 If AC denotes analytic continuation and S applied to a formal series is its sum, wherever the sum converges, then Borel summation is the composition $\mathcal{L}\circ\mathrm{AC}\circ\mathrm{S}\circ\mathcal{B}$. Whenever AC$\circ$S exists, it is clearly an isomorphism between formal series and analytic functions.

Definition 2.6 The *domain* of Borel summation along $\mathbb{R}^+$ is the subspace $S_{\mathcal{B}}$ of series for which the conditions for 2. and 3. above are met. For step 3 we can require that for some constants C_F, ν_F we have $|F(p)| \leqslant C_F e^{\nu_F p}$. Or we can require that $\|F\|_\nu < \infty$ where, for $\nu > 0$ we define

$$\|F\|_\nu := \int_0^\infty e^{-\nu p}|F(p)|dp \tag{2.7}$$

If $\tilde{f}$ is Borel summable, then the Borel sum of $\tilde{f}$, denoted by $\mathcal{LB}\tilde{f}$, is defined to be $\mathcal{L}F$.

2.2.1 Extensions

Remark 2.8 Borel summation of series starting with *finitely many* powers of x with positive real part is defined by

$$\mathcal{LB}\sum_{k\in\mathbb{N}} c_k x^{-s_k} = \sum_{k:\mathrm{Re}\,s_k\leqslant 0} c_k x^{-s_k} + \mathcal{LB}\sum_{k:\mathrm{Re}\,s_k>0}^{\infty} c_k x^{-s_k}$$

assuming $\mathrm{Re}\,(s_k) < \mathrm{Re}\,(s_{k+1})$ for all $k \in \mathbb{N}$. In case some of the s_j are noninteger, the definition of $\mathcal{LB}$ is essentially the same, replacing analyticity at zero with ramified analyticity. (This is needed since $\int_0^\infty p^s e^{-xp}dp$ exists, as a classical integral, only if $\mathrm{Re}\, s > -1$.)

Note 2.9 Series of the form

$$\tilde{f} = \sum_{k_i \geqslant 0} c_{k_1 k_2 \dots k_m} x^{-\beta_1 k_1 - \dots - \beta_m k_m - 1} =: \sum_{\mathbf{k} \geqslant 0} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\boldsymbol{\beta} - \mathbf{1}}$$

with $\mathrm{Re}\,(\beta_j) > 0$ frequently arise as formal solutions of differential systems. Here $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$ and $c_{\mathbf{k}} = c_{k_1 k_2 \dots k_m}$. We define

$$\mathcal{B} \sum_{\mathbf{k} \geqslant 0} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\boldsymbol{\beta} - \mathbf{1}} = \sum_{\mathbf{k} \geqslant 0} c_{\mathbf{k}} p^{\mathbf{k}\cdot\boldsymbol{\beta}} / \Gamma(\mathbf{k}\cdot\boldsymbol{\beta} + 1) \tag{2.10}$$

*

2.2a Borel summation along other directions

Borel summation along other directions in $\mathbb{C}$ is most easily defined by changes of variables. We say that a power series in inverse powers of x, $\tilde{f} = \tilde{f}(x)$, is Borel summable as $x \to \infty e^{i\theta}$ or Borel summable along the ray $e^{i\theta}$ if $\tilde{f}(ye^{i\theta})$ is Borel summable for y along $\mathbb{R}^+$. We write $\mathcal{LB}_\theta[\tilde{f}(x)] = \mathcal{LB}[\tilde{f}(ye^{i\theta})]$.

In general, $\mathcal{LB}_\theta$ depends nontrivially on θ. We can take as an illustration the formal series

$$\tilde{f}_1 = \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} \tag{2.11}$$

that we have examined before. A straightforward calculation shows that the Borel sum of $\tilde{f}_1$ in the direction θ is

$$\mathcal{LB}_\theta \tilde{f}_1 = \mathcal{LB} \sum_{k=0}^{\infty} \frac{k!}{(ye^{i\theta})^{k+1}} = \int_0^\infty \frac{e^{-yp}dp}{e^{i\theta} - p} = \int_0^{\infty e^{-i\theta}} \frac{e^{-xp}dp}{1-p} \tag{2.12}$$

This is well defined if $\theta \neq 0 \pmod{2\pi}$.

Taking a $\theta \in (0, 2\pi)$ we note that by the residue theorem (and Jordan's lemma, allowing us to deform the contour of the improper integral in (2.12)) we have

$$[\mathcal{LB}_\theta - \mathcal{LB}_{-\theta}]\, \tilde{f}_1 = 2\pi i e^{-x} \tag{2.13}$$

that is, the Borel sums of $\tilde{f}_1$ on the two sides of $\mathbb{R}^+$ differ by an exponentially small term. This is a manifestation of the Stokes phenomenon. Divergent expansions are generally associated with different behaviors as the singular point (here the point at infinity) is approached from different directions. Note that the contour in $\mathcal{LB}_\theta$ can be rotated to $-\pi/2$ (and beyond) and Watson's Lemma applies to it as $x \to \mathrm{i}\infty$. Thus $\mathcal{LB}_\theta$ has an asymptotic series along

$i\mathbb{R}^+$, but, from (2.13), $\mathcal{LB}_{-\theta}$ does not (because of the presence of the oscillatory term $2\pi i e^{-x}$. The *classical* asymptotic behavior of $\mathcal{LB}_{-\theta}$ changes as the *antistokes line* $i\mathbb{R}$ is crossed. The exponential is born at $\pi/2$ before that, but is *hidden beyond all orders in a whole quadrant.*

We also have the following simple general result.

Proposition 2.14 (Divergence implies Stokes phenomena) Assume f is analytic in $\mathcal{D} = \{x : |x| > R$ and $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$, $z = x^{-1}$ as $x \to \infty e^{i\theta}$ for some θ, where $\tilde{f}$ has zero radius of convergence. Then, for any $m \in \mathbb{Z}^+$, $x^{-m} f$ is unbounded in a neighborhood of infinity. In particular, $f \sim \tilde{f}$ only holds when ∞ is approached along some, but not all, curves.

PROOF Let $g(z) = f(1/z)$. Then g is analytic in a punctured neighborhood of 0. By standard complex analysis, if gz^m is bounded for some $m > 0$ then 0 is a pole, and the meromorphic Taylor expansion at zero converges (recall that asymptotic series are unique, and a meromorphic expansion is also an asymptotic expansion). ∎

Exercise 2.15 The function

$$g(x) = e^{-x^2} \int_0^x e^{t^2} dt \tag{2.16}$$

is entire. Use the saddle point method or reduction to Watson's lemma to study the behavior of g as $x \to \infty e^{i\theta}$ for $\theta \in [0, 2\pi)$ and study the change in the asymptotic expansion as θ changes.

Note 2.17 A *function* f is sometimes called Borel summable (by slight abuse of language), if it is analytic and suitably decaying in a half-plane (say $\mathbb{H}$), and its inverse Laplace transform F is analytic in a neighborhood of $\mathbb{R}^+ \cup \{0\}$. Such functions are clearly into a one-to-one correspondence with their asymptotic series. Indeed, if the asymptotic series of f and g coincide, then $\int_0^{\infty} e^{-xp} H(p) dp = h = f - g \to 0$ in the sense of power series, and by Watson's lemma H is zero with all derivatives at zero, and by analyticity it vanishes identically.

2.2b Limitations of classical Borel summation; BE summation

The need of extending Borel summation arises because the domain of definition of Borel summation is not wide enough. We see that $\tilde{f}_1$ in (2.11) is not summable along $\mathbb{R}^+$. Yet there is nothing singular in the behavior of $f_1 := \mathcal{LB}_\theta \tilde{f}_1$, for any $\theta \neq 0$ and then analytically continued to $\theta = 0$. In view of (2.13), no particular direction can be naturally chosen for summation along $\mathbb{R}^+$.

While in applications it is usually possible to deform the contour of integration of $\mathcal{L}$ in the complex plane to avoid going through singularities, of course there is no *single* ray of integration that would allow for summation of general series $\tilde{f}$ when $\mathcal{B}\tilde{f}$ has singularities in $\mathbb{C}$ (this is the more interesting case).

If the ray of integration were chosen in an $\tilde{f}$–dependent fashion, then the resulting operator would not have good algebraic properties, in particular it would fail to commute with complex conjugation and even be linear.

To overcome the limitations of classical Borel summation, Écalle has found general weights which do not depend on the origin of the formal series, such that, replacing the Laplace transform along a *singular rays* with weighted averages of Laplace transforms of these continuations, all the algebraic properties of Borel summation are preserved, and its domain is vastly widened. The fact that such averages exist is nontrivial, though the expression of the averaging weights is relatively simple.

Multiplicativity of the summation operator is the main difficulty that is overcome by these special averages. Perhaps surprisingly, convolution *does not* commute in general with analytic continuation along curves passing between singularities! As we shall see, convolution is the Borel image of multiplication. Naive averaging would not allow for addressing complicated problems such as nonlinear equations.

A simplified form of medianization, the balanced average, which works for generic ODEs (but not in the generality of Écalle's averages) is discussed in §**??**.

Another difficulty is the possibility of super-exponential growth of the Borel transform.

Example 2.18 If we substitute $x = t^2$ in $\tilde{f}_1$ and take the Borel transform from t to p we get

$$\mathcal{B}\sum_{k=0}^{\infty}\frac{k!}{t^{2k+2}} = \sum_{k=0}^{\infty}\frac{k!p^{2k+1}}{\Gamma(2k+2)} = \sqrt{\pi}e^{p^2/4}\mathrm{erf}(p/2) \tag{2.19}$$

where

$$\mathrm{erf}(p) = 2\pi^{-1/2}\int_0^p e^{-s^2}ds \sim 1 - e^{-p^2}p^{-1}\pi^{-1/2}(1+o(1)) \text{ as } p\to+\infty \tag{2.20}$$

(a way to calculate the second sum in (2.19) is to note that it satisfies the ODE $y' = 1 + \frac{1}{2}py$). Whereas in (2.12) we can deform the contour of integration to give a meaning to the integral when $\theta = 0$, there is nothing obvious we can do to Laplace transform $e^{p^2/4}\mathrm{erf}(p/2)$ which grows like $e^{p^2/4}$ when $p \to \infty$.

Super-exponential growth, as it turns out, can be generically dealt with by changes of variable, here essentially by undoing the $x \mapsto t$ transformation. Though non-generic in the case of ODEs, there are cases however when mixtures of different factorial rates of divergence in the same series, preclude this simple fix (an artificial example is, say, $\tilde{f}(t)[\tilde{f}(t^2) - \tilde{f}(t^3)]^2$). It is clear that we cannot hope to disentangle all such combinations of series with different rates of divergence.

Acceleration and multisummation (the latter considered independently, from a cohomological point of view by Ramis; see also §**??**), universal processes too, were introduced by Écalle to deal with this problem in many contexts. Essentially BE summation is Borel summation, supplemented by averaging and acceleration when needed.

More generally we can allow for expansions containing exponentials by defining $\mathcal{LB}\exp(ax) = \exp(ax)$.

These generalizations of Borel summation which allow for the analysis of quite complicated functions are known as Borel-Écalle summation, or BE summation.

Definition 2.21 (Inverse Laplace space convolution) *If $F, G \in L^1_{loc}$, then*

$$(F * G)(p) := \int_0^p F(s)G(p-s)ds \tag{2.22}$$

If F and G are exponentially bounded, then so is $F * G$. Indeed, if $|F_1| \leqslant C_1 e^{\nu_1 p}$ and $|F_2| \leqslant C_2 e^{\nu_2 p}$, then

$$|F_1 * F_2| \leqslant C_1 C_2 e^{\nu_3 p}$$

where $\nu_3 = \max\{\nu_1, \nu_2\} + 1$. The norm $\|f\|_\nu$ in (2.7) is particularly useful.

Lemma 2.23 1. $L^1_\nu := \{f : \|f\|_\nu < \infty\}$ forms a Banach space.

2. Convolution is continuous in $\|\cdot\|_\nu$, namely

$$\|F * G\|_\nu \leqslant \|F\|_\nu \|G\|_\nu$$

3. We have $L^1_\nu \subset L^1_{\nu'}$ if $\nu' > \nu$, and $\mathcal{L}(F * G) = \mathcal{L}(F)\mathcal{L}(G)$.

4.

$$\|F\|_\nu \to 0 \quad \text{as } \nu \to \infty \tag{2.24}$$

PROOF

1. Since $L^1_\nu = \{e^{\nu p} f : f \in L^1\}$, the conclusion is obvious.
2. Note that

$$\int_0^\infty e^{-\nu p} \left| \int_0^p F(s)G(p-s)ds \right| dp \leqslant \int_0^\infty e^{-\nu s} e^{-\nu(p-s)} \int_0^p |F(s)||G(p-s)|dsdp$$
$$= \int_0^\infty \int_0^\infty e^{-\nu s}|F(s)|e^{-\nu\tau}|G(\tau)|d\tau ds = \|F\|_\nu \|G\|_\nu \tag{2.25}$$

by Fubini.

3. This follows by monotonicity:

$$\int_0^\infty e^{-\nu' p}|F(p)|dp = \int_0^\infty e^{-(\nu'-\nu)p}e^{-\nu p}|F(p)|dp \leqslant \int_0^\infty e^{-\nu p}|F(p)|dp \quad (2.26)$$

4. This follows from the middle representation above: by dominated convergence, the integral goes to zero as $\nu' - \nu \to \infty$. ∎

Lemma 2.27 The space of functions which are in $L^1([0,a))$ for any $a > 0$ and real-analytic on $[0,\infty)$ is closed under convolution.

PROOF This simply follows from the rewriting

$$\int_0^p f(s)g(p-s)ds = p\int_0^1 f(pt)g(p(1-t))dt \quad (2.28)$$

∎

The previous lemma implies that $\mathcal{LB}(\tilde{f}\tilde{g}) = \mathcal{LB}(\tilde{f})\mathcal{LB}(\tilde{g})$.

2.3 Borel summation as an isomorphism

We start with a discussion about the relation between convergent power series and their sums. Rarely does one make a distinction between a convergent series as a formal object and its sum as an actual function. Some mention of this distinction is made when we need to be careful about the radius of convergence, such as in writing $(1-p)^{-1} = \sum_{k=0}^\infty p^k$ if $|p| < 1$. The right side of this equality is already interpreted as the sum of the underling formal series. We can understand this if we look at the properties of S, the operator that associates to a formal convergent series its sum. If we restrict series to one sided Taylor series (such as expansions of meromorphic functions at zero), S is a differential algebra isomorphism, that is, it commutes with multiplication, division, differentiation, etc. For instance $\mathsf{S}(\tilde{f}'\tilde{g}') = \mathsf{S}(\tilde{f})'\mathsf{S}(\tilde{g})'$. If a linear or nonlinear ODE $\mathcal{N}(y, y', ..., y^{(n)}, z) = 0$ with analytic coefficients is solved by a formal series $\tilde{f}$ whose coefficients grow at most geometrically, then, by the isomorphism, $\mathsf{S}\mathcal{N}(\tilde{f}, \tilde{f}', ..., \tilde{f}^{(n)}, z) = 0$ iff $\mathcal{N}(\mathsf{S}\tilde{f}, \mathsf{S}\tilde{f}', ..., \mathsf{S}\tilde{f}^{(n)}, z) = 0$, that is, iff $\mathsf{S}\tilde{f}$ is an actual solution. This is true in several variables as well, in solving PDEs for instance.

Borel summable series are in a similar relation with their associated functions obtained by Borel summation.

Let $S_{\mathcal{B}}$ be the set of Borel summable series along some direction, say along $\mathbb{R}^+$.

Proposition 2.29 (i) $\mathcal{LB}\{S_{\mathcal{B}}\}$ is linear and commutes with multiplication, division (when it exists) and differentiation.[2]

(ii) Let S_c be the space of convergent series. Then $S_c \subset S_{\mathcal{B}}$ and $\mathcal{LB}_{|S_c} = \mathsf{S}$, the usual summation.

(iii) In addition, for $\tilde{f} \in S_B$, $\mathcal{LB}\{\tilde{f}\} \sim \tilde{f}$ as $|x| \to \infty$, $\mathrm{Re}\,(x) > 0$.

Note 2.30 (ii) above implies that $S_{\mathcal{B}}$ and $\mathcal{LB}$ are proper extensions of S_c and S: Borel summable series have the same algebraic properties as convergent series and their sums have properties similar to those of analytic functions.

PROOF of Proposition 2.29 Some of the properties such as linearity of $\mathcal{LB}$ and commutation with differentiation are straightforward and we leave them as an exercise; so is commutation with multiplication: it stems from the fact that $\mathcal{L}(F * G) = \mathcal{L}(F)\mathcal{L}(G)$ following by a calculation similar to (2.25). For multiplication and division, we need to look more closely at convolutions. The only nontrivial part is to show that if $\tilde{f}$ is a Borel summable series, then so is $1/\tilde{f}$. We have $f = Cx^m(1+s)$ for some m where s is a small series, that is a series only involving negative powers of x. We naturally define $1/\tilde{f} = x^m/(1+s)$ and

$$\frac{1}{1+s} = 1 - s + s^2 - s^3 \cdots \,. \tag{2.31}$$

First, note that this infinite series is well defined formally. Indeed, assuming for simplicity that $s = \sum_{k=0}^{\infty} c_k x^{-k-1}$, the coefficient of x^{-k} for any fixed k is collected only from the terms s^m with $j \geqslant k$. This is because the highest power of x in s^{k+j} is $-k-j$. Straightforward algebra shows that

$$(1+s)(1 - s + s^2 \cdots) = 1$$

We want to show that

$$1 - s + s^2 - s^3 + \cdots \tag{2.32}$$

is Borel summable, or that

$$s_1 = -s + s^2 - s^3 + \cdots \tag{2.33}$$

is Borel summable.

Now,

$$s_1 = -s + s^2 - s^3 + \cdots = \sum_{k>1} C_k x^{-k} \tag{2.34}$$

where C_k is the coefficients of x^{-k} in the finite sum $-s+s^2-s^3+...+(-1)^k s^k$. Let $\mathcal{B}s = H$. We examine $\mathcal{B}s_1$, or, in fact the function series

$$S = -H + H * H - H^{*3} + \cdots \tag{2.35}$$

[2] $\mathcal{LB}\{S_{\mathcal{B}}\}$ is in fact a *differential algebra isomorphism.*

where H^{*n} is the self-convolution of H n times. Each term of the series is analytic, by Lemma 2.27. Let K be an arbitrary compact subset of $\mathcal{D}$ (cf. the definition of Borel summation). If $\max_{p\in K}|H(p)| = m$, then it is easy to see that

$$|H^{*n}| \leqslant m^n 1^{*n} = m^n \frac{p^{n-1}}{(n-1)!} \tag{2.36}$$

Thus the function series in (2.35) is absolutely and uniformly convergent in K and the limit is analytic. Let now ν be large enough so that $\|H\|_\nu < 1$ (see Lemma 2.23, 4.). Then the series in (2.35) is norm convergent, thus an element of L^1_ν.

Exercise 2.37 *Check that* $(1+\mathcal{L}H)(1+\mathcal{L}S) = 1$.

It remains to show that the asymptotic expansion of $\mathcal{L}(F * G)$ is indeed the product of the asymptotic series of $\mathcal{L}F$ and $\mathcal{L}G$. This is, up to a change of variable, a consequence of Lemma 1.31.

(ii) Since $\tilde{f}_1 = \tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k-1}$ is convergent, then $|c_k| \leqslant CR^k$ for some C, R and $F(p) = \sum_{k=0}^{\infty} c_k p^k/k!$ is entire, $|F(p)| \leqslant \sum_{k=0}^{\infty} CR^k p^k/k! = Ce^{Rp}$ and thus F is Laplace transformable for $|x| > R$. By dominated convergence we have for $|x| > R$,

$$\mathcal{L}\{\sum_{k=0}^{\infty} c_k p^k/k!\} = \lim_{N\to\infty} \mathcal{L}\{\sum_{k=0}^{N} c_k p^k/k!\} = \sum_{k=0}^{\infty} c_k x^{-k-1} = f(x)$$

(iii) This part follows simply from Watson's lemma. □

Note 2.38 The domain of manifest analyticity of $\mathcal{LB}\tilde{f}$ where $\tilde{f}$ is convergent may exceed the domain of convergence of $\tilde{f}$. For instance, if

$$\mathcal{LB}\sum_{k=1}^{\infty}(-1)^{k+1}x^{-k} = \int_0^{\infty} e^{-xp}e^{-p}dp \tag{2.39}$$

clearly analytic for $\mathrm{Re}\,(x) > -1$.

2.3.1 Analytic functions of Borel summable series

Proposition 2.40 *Assume A is an analytic function in the disk of radius ρ centered at the origin, $a_k = A^{(k)}(0)/k!$, and $\tilde{s} = \sum s_k x^{-k}$ is a small series which is Borel summable along $\mathbb{R}^+$. Then the formal power series obtained by reexpanding*

$$\sum a_k \tilde{s}^k$$

in powers of x is Borel summable along $\mathbb{R}^+$.

PROOF Let $S = \mathcal{B}s$ and choose ν to be large enough so that $\|S\|_\nu < \rho$ in L^1_ν. Then

$$\|F\|_\nu := \|A(*S)\|_\nu := \|\sum_{k=0}^{\infty} a_k S^{*k}\|_\nu \leqslant \sum_{k=0}^{\infty} a_k \|S\|_\nu^k \leqslant \sum_{k=0}^{\infty} a_k \rho^k < \infty \quad (2.41)$$

thus $A(*S) \in L^1_\nu$. Similarly, $A(*S)$ is in $L^1_\nu([0,a))$ and in $\mathcal{A}_{K,\nu}([0,a))$ for any a. ∎

Note 2.42 (i) To ensure Borel summability of a series, the rule of thumb is that we first change the independent variable so that the new series has factorial divergence, with power of factorial one. In generic applications, this ensures that the Borel transform will be convergent, but will not ensure that the growth is at most like $e^{\nu|p|}$ for some ν, the maximal growth allowed for Laplace transformability, see Example 2.18.

(ii) *If* Borel summability succeeds, then, by deforming the contour of integration past a singularity of the Borel transform, we collect exponentially small terms, as in (2.13). Those exponentially small terms, since they come from residues or from contours surrounding branch points are of the form $Ce^{-p_0 x}$ where p_0 is the position of the singularity.

(iii) When we are dealing with linear equations (e.g., linear ODEs), we note that the sum of two solutions is a solution. Assume that we get one solution by Borel summation in the direction θ, $y = \mathcal{LB}_\theta Y$ and that $\mathcal{LB}_\theta Y \neq \mathcal{LB}_{-\theta} Y$. Then, by linearity, the function

$$y_2 = \mathcal{LB}_\theta Y - \mathcal{LB}_{-\theta} Y \quad (2.43)$$

is also a solution of the equation. Thus, by the discussion in (ii), $y_2 \sim e^{-p_0 x(1+o(1))}$ for some p_0. A similar estimate holds in the nonlinear case as we will see though in this case the sum of solutions is not a solution.

(iv) In view of (iii), if we are solving linear ODEs using Borel summation tools, we first need to normalize the equation, by changes of dependent and independent variables, so that the behavior of solutions as $x \to \infty$ of the associated homogeneous equation is of the form $e^{Ax+O(1)}$ (the exponent is to leading order linear in x). For instance, for the Airy equation discussed next, solutions behave like $e^{\pm 2/3 x^{3/2}}$; for Borel summation, we should switch to the variable $t = x^{3/2}$. Normalization through (iv) ensures (i) while being usually much simpler to find.

(v) Still when solving equations, instead of finding a formal series first and then summing it is more convenient to take the Borel transform (formal inverse Laplace) of (2.49). Now: if we obtain a solution of the transformed equation in the form of a function which is ramified analytic near the origin and analytic and bounded along $\mathbb{R}^+$ (or along some other direction θ), we will have proved Borel summability (in the direction $-\theta$).

Note 2.44 (On regularization) Since the Borel transform maps factorially divergent expansions into convergent ones, it is natural that the equations satisfied by the transformed series have milder singularities, and are, in general, simpler than the original ones. For example, our prototypical equation $y' + y = 1/x$ is transformed by $\mathcal{B}$ to the trivial equation $(1-p)Y = 1$ while the Bessel equation becomes the first order ODE (2.56). Sometimes we end up with explicit representations of the solutions and most often with integral representations which, even if not explicit, allow for a detailed study of the asymptotic behavior of solutions.

2.4 Some examples

In the following we will derive convenient representations for a number of special functions. As discussed in §1.2, special functions are distinguished by the existence of integral representations allowing for detailed global description, in particular for calculating explicitly connection constants, giving the precise relation between the behavior of a given function in various directions towards infinity. From the point of view of a Borel summation approach, the fact that irregular singularities become regular ones and the fact that these special functions solve typically second order differential equations implies that the Borel transform satisfies first or second order equations with simpler singularities, allowing for explicit solutions.

2.4.2 The Airy equation

Let us look again at the Airy equation,

$$y'' - xy = 0 \tag{2.45}$$

Here, the behavior of solutions at infinity, that we have already obtained by WKB is

$$y \sim Cx^{-\frac{1}{4}} e^{-\frac{2}{3}x^{3/2}} \tag{2.46}$$

We use the transformation $y(x) = g(\frac{2}{3}\, x^{\frac{3}{2}})$ to achieve the normalization described in Note 2.42 (iv), and get

$$g'' + \frac{1}{3t} g' - g = 0 \tag{2.47}$$

In view of (2.46) we have

$$g(t) \sim Ct^{-\frac{1}{6}} e^{\pm t} \tag{2.48}$$

To eliminate the exponential behavior of one solution, say of the decaying one, we substitute $g = he^{-t}$, and get

$$h'' - \left(2 - \frac{1}{3t}\right) h' - \frac{1}{3t} h = 0 \tag{2.49}$$

To obtain a second solution, we can resort to the substitution $g = he^t$, or we can rely on the Stokes phenomenon to obtain it from the one above, as we will do in §2.5. Now h behaves like a small power series, which we would Borel sum. We apply the strategy outlined in Note 2.42 (v), and apply $\mathcal{B}$ to (2.49). We get

$$p(2+p)H' + \frac{5}{3}(1+p)H = 0 \tag{2.50}$$

with the solution

$$H = Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} \tag{2.51}$$

and thus

$$h(t) = \mathcal{L}\left(Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}\right) \tag{2.52}$$

and, comparing the asymptotic expansion obtained from (2.52) with that of Airy functions (1.289) to identify the solution we h we get

$$\mathrm{Ai}(x) = \frac{3^{-\frac{1}{6}}\exp(-\frac{2}{3}x^{3/2})}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{6})}\int_0^\infty e^{-\frac{2}{3}x^{\frac{3}{2}}p}p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}dp \tag{2.53}$$

from which it is easy to derive the global behavior of Airy functions. We note that $\mathrm{Ai}(x)$ is entire, yet the fact that the integrand in (2.53) has a singularity at $p = -2$ entails a change of asymptotic behavior similar to that discussed after (2.13) (Stokes phenomenon) when the contour of integration crosses $\mathbb{R}^-$. We return to this in §2.5.

2.4.3 Modified Bessel functions

The equation for modified Bessel functions is

$$t^2y'' + ty' - (\nu^2 + t^2)y = 0 \tag{2.54}$$

By WKB, we obtain $y \sim Ce^{-t}t^{-1/2}$.

The substitution $y(t) = t^\nu e^{-t}h(t)$ into (2.54) (we chose the power t^ν to eliminate the quadratic terms in t in the equation and simplify the Borel transform) leads to

$$h'' - \left(2 - \frac{2\nu+1}{t}\right)h' - \frac{2\nu+1}{t}h = 0 \tag{2.55}$$

with inverse Laplace transform

$$p(p+2)H' + (1 - 2\nu + p(1-2\nu))H = 0 \Rightarrow \; H = Cp^{\nu-\frac{1}{2}}(2+p)^{\nu-\frac{1}{2}} \tag{2.56}$$

If $\nu = -1/2$, (2.55) becomes elementary, with solutions $e^{\pm x}$. Also, in the original form (2.54) the sign of ν does not matter. So it is enough to consider the case where $\mathrm{Re}\,\nu > -1/2$ or $\nu = -1/2 + ai, a \neq 0$.

If $\mathrm{Re}\,\nu > -1/2$ we can take the Laplace transform of H; if $\nu = -1/2+ai, a \neq 0$, the integral is a convergent improper integral. Using Watson's lemma, we see that

$$h = \mathcal{L}H \sim C2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})t^{-\nu-\frac{1}{2}}; \quad (t \to \infty) \tag{2.57}$$

while

$$K_\nu(t) \sim \sqrt{\frac{\pi}{2t}}e^{-t} \quad (t \to \infty) \tag{2.58}$$

Comparing (2.57) to (2.58), we see that

$$K_\nu(t) = e^{-t}\frac{t^\nu\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty p^{\nu-\frac{1}{2}}(2+p)^{\nu-\frac{1}{2}}e^{-tp}dp; \ \mathrm{Re}\,(\nu) > -1/2 \tag{2.59}$$

2.4a Note on using equivalent normalizations to obtain identities

2.4a.1 A simple hypergeometric function

We know, by Corollary 1.53 that the Laplace transform is injective. If we obtain the representation of a function as a Laplace transform $\int_0^\infty e^{-xp}F(p)dp$, and F is analytic at zero, then this F is unique. Uniqueness also easily follows if F has a prescribed form of Puiseux series at zero (such as, say, $p^a A_1(p)+A_2(p)$ with A_1, A_2 analytic). On the other hand, one can make some changes of variables first, that do not change the factorial rate of divergence of the formal solution , find the Laplace transform and undo the changes of variables. Comparing the two representations and using the aforementioned uniqueness is a rich source of identities.

Had we chosen the substitution $g = t \cdot t^{-1/6}e^{-t}h_1(t)$ in (2.47) ($t^{-1/6}$ corresponds to the prefactor $x^{-1/4}$ in (2.48) while a positive power of t is required for h_1 to decay and have an inverse Laplace transform) we would have obtained instead an equation for $h_1(t)$ whose Borel transform is

$$(p^2-1)H_1'' + 4pH_1' + \frac{77}{36}H_1 = 0 \tag{2.60}$$

clearly this equation is more complicated, and the substitution *without* $t^{-1/6}$ is better *in this regard*. However, (2.60) has a general solution in terms of hypergeometric functions. Indeed, the substitution $H_1(p) = G(-p/2)$ in (2.60) leads to

$$q(1-q)G'' + (1-2q)G - \frac{5}{36}G = 0 \tag{2.61}$$

Substituting $G(q) = q^a\sum_{j=0}^\infty b_jq^j$ into (2.61) we see that the indicial equation is simply $a^2 = 0$, a degenerate case in Frobenius theory, which implies that there is an analytic solution at zero and a second one behaving like $\ln q$ for small q. On the other hand (2.61) is a hypergeometric equation; in its most general form the hypergeometric equation is:

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \tag{2.62}$$

Comparing the two equations, we see that they coincide up to the change of variables $(x, y) \to (q, G)$ if $c = 1$, $a + b + 1 = 2, ab = 5/36$; solving, either $a = 5/6, b = 1/6, c = 1$ or $b = 5/6, a = 1/6, c = 1$. However, as it is obvious from the equation, a and b are interchangeable, so in reality we only obtain one independent solution in this way:

$$G_1(q) = {}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; 1; q) \tag{2.63}$$

which is analytic at zero, as is seen from the definition of the hypergeometric function as a series. This is the solution we are looking for, since applying Watson's Lemma to the follwing representation

$$g(t) = t^{5/6} e^{-t} h_1(t) = t^{5/6} e^{-t} \int_0^\infty e^{-pt} G_1(-p/2) dp \tag{2.64}$$

we obtain the expected large t asymptotic behavior $g(t) \sim Ct^{-1/6} e^{-t}$ (recall that $\mathrm{Ai}(x) = g(x^{3/2})$). If we added to G_1 any multiple of a second, linearly independent solution of (2.61) in (2.64), it would contribute with a logarithmic behavior at $q = 0$ which would result in a different large t asymptotics for $g(t)$. The hypergeometric series also gives

$${}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; 1; q) = 1 + \frac{5}{36} q + ... \tag{2.65}$$

Thus, returning to H_1 in (2.60), we have

$$H_1(p) = {}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; 1; -p/2) \tag{2.66}$$

Now we compare with (2.52). Using $\Gamma(5/6)t^{-5/6} = \mathcal{L}p^{-1/6}$, $\Gamma(1/6)t^{-1/6} = \mathcal{L}p^{-5/6}$, $\mathcal{L}f\mathcal{L}g = \mathcal{L}(f * g)$ and $\Gamma(1/6)\Gamma(1 - 1/6) = 2\pi$ we get, after changes of variables, the identity

$${}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; 1; q) = \frac{1}{2\pi} \int_0^1 u^{-5/6} (1-u)^{-1/6} (1-qu)^{-5/6} du \tag{2.67}$$

We will return to this procedure to obtain a representation for the general hypergeometric function in§2.4c. Next however we look at another simple example.

2.4a.2 The quintic equation $\frac{\tau^5}{5} + \tau = p$

This equation, as is well known, is not solvable by radicals, but the solution can be expressed in terms of generalized hypergeometric functions; this fact is known, but a relatively simple derivation of this is possible through Borel transform (The reader is assumed to have familiarity with hypergeometric functions ${}_4F_3$).

Consider the "higher order Airy" integrals

$$A_5(x) := \int_{\infty e^{2\pi k i/5}}^{\infty e^{2\pi(k+1)i/5}} e^{-t^5/5 - xt} dt \tag{2.68}$$

for $k = 0, \cdots, 4$. Direct verification shows that each of these five integrals satisfy satisfies the following differential equation

$$y^{(4)} + xy = 0 \tag{2.69}$$

and that they form a basis in the space of solutions. After changing variables

$$t = x^{1/4}\tau, \ \tau + \frac{\tau^5}{5} = p \tag{2.70}$$

in (2.68), we obtain

$$A_5(x) = x^{1/4} \int_{C_k} e^{-pz} \frac{d\tau}{dp} dp \ , \ \text{where } z = x^{5/4} \tag{2.71}$$

$\tau = \tau(p)$ is the inversion of the relation (2.70) and C_k is a path parallel to the real p axis that circles around the branch point singularity $p_k = i^k \frac{4}{5} e^{i\pi/4}$ of $\tau(p)$ in the positive sense. With the change of variables $y(x) = x^{1/5} v(z)$, $z = x^{5/4}$, and taking the Borel transform of the resulting ODE from in z, one obtains a fourth order linear ODE solvable by generalized hypergeometric functions we obtain an alternative representation

$$A_5(x) = x^{1/4} \int_{C_k} e^{-pz} V(p) dp \tag{2.72}$$

where V has four free constants. This gives rise to the identity $\tau'(p) = V(p)$. Integrating and choosing the constants so that the series for A_5 in p for small p matches $\tau(p) = p - p^5/5 + O(p^9)$ obtained from (2.68) we obtain

$$\tau_1(p) = p \, {}_4F_3 \left(\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right], \left[\frac{1}{2}, \frac{3}{4}, \frac{5}{4} \right], -\left(\frac{25}{16} \right)^2 p^4 \right) \tag{2.73}$$

Similar expressions can be obtained for the other four roots of the quintic as discussed in the Appendix.

2.4b Whittaker functions

The equation

$$y'' - \left(\frac{1}{4} - \frac{\kappa}{z} - \frac{\frac{1}{4} - \mu^2}{z^2} \right) y = 0 \tag{2.74}$$

has as solutions the Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$, [1], eq.13.1.31; for large positive z, $M_{\kappa,\mu}(z) \sim e^{z/2(1+o(1))}$ and $W_{\kappa,\mu}(z) \sim e^{-z/2(1+o(1))}$. To analyze W, we make the substitution $y(z) = z^{\mu+1/2} e^{-z/2} g(z)$ in (2.74). The power is chosen to eliminate the term in z^{-2} and simplify the Borel transform. For M we would substitute $y(z) = z^{\mu+1/2} e^{z/2} g(z)$. We obtain

$$g'' - \left(1 - \frac{2\mu + 1}{z} \right) g' + \frac{\kappa - \mu - \frac{1}{2}}{z} g = 0 \tag{2.75}$$

The Borel transform of (2.75) is

$$p(p+1)G' + \left[(1-2\mu)p + \kappa - \mu + \tfrac{1}{2}\right] G = 0 \tag{2.76}$$

with general solution

$$G = Cp^{-\kappa+\mu-1/2}(p+1)^{\mu-1/2+\kappa} \tag{2.77}$$

giving

$$W_{\kappa,\mu}(z) = Cz^{\mu+1/2}e^{-z/2}\int_0^\infty p^{\mu-\kappa-1/2}(p+1)^{\mu-1/2+\kappa}e^{-zp}dp \tag{2.78}$$

which holds for $\mathrm{Re}\,(\mu-\kappa-1/2) > -1$; otherwise W can be defined by analytic continuation in μ. Using Watson's lemma in (2.78) we get

$$W_{\kappa,\mu}(z) \sim C\Gamma(\mu+\tfrac{1}{2}-\kappa)z^{\kappa}e^{-z/2}(1+o(1));\ \mathrm{Re}\,z\to\infty \tag{2.79}$$

Since, by the definition of Whittaker functions, $W_{\kappa,\mu}(z)z^{\kappa}e^{z/2}\to 1$ as $z\to +\infty$,[1], we get $C = 1/\Gamma(\mu-\kappa+\frac{1}{2})$, or

$$W_{\kappa,\mu}(z) = \frac{z^{\mu+1/2}e^{-z/2}}{\Gamma(\mu-\kappa+\frac{1}{2})}\int_0^\infty p^{\mu-\kappa-1/2}(p+1)^{\mu-1/2+\kappa}e^{-zp}dp \tag{2.80}$$

(for $\mathrm{Re}\,(\mu-\kappa-1/2) > -1$).

Parabolic cylinder functions, occurring naturally in double nondegenerate turning point asymptotics solve the equation

$$y'' - \left(\tfrac{1}{4}x^2 + a\right)y = 0 \tag{2.81}$$

whose general solution is

$$x^{-1/2}\left(C_1M_{-\frac{a}{2},\frac{1}{4}}(x^2/2) + C_2W_{-\frac{a}{2},\frac{1}{4}}(x^2/2)\right) \tag{2.82}$$

from which we can obtain an integral representation for the solutions of (2.81).

2.4c Hypergeometric functions

Hypergeomtric functions solve Fuchsian equations, with only regular singular points on the Riemann sphere. Thus all series expansions are convergent. Nonetheless, we can obtain integral representations for them by relating them to other functions, such as Whittaker, in the following way.

The general hypergeometric equation has the form

$$p(p-1)F'' + [(a+b+1)p-c]F' + abF = 0;\quad F = {}_2F_1(a,b;c,p) \tag{2.83}$$

Taking the Laplace transform of (2.83) we get

$$x^2 f'' + x(x - a - b + 3)f' + [(2 - c)z + (1 - a)(1 - b)]f = 0 \tag{2.84}$$

to obtain a simpler, close form solution in Borel plane we try to eliminate the quadratic terms in x by a substitution of the form $f = x^d g$. Trying a general d we see that this works if $d \in \{a - 1, b - 1\}$. With $f = x^{a-1}g$ we get

$$zg'' + (z + a - b + 1)g' + (a - c + 1)g = 0 \tag{2.85}$$

with Borel transform

$$p(p - 1)G' + [(b - a + 1)p + a - c]G = 0 \tag{2.86}$$

with solution

$$G = Cp^{a-c}(p - 1)^{c-b-1} \tag{2.87}$$

The more general substitution $y(z) = z^{\beta+3/2}e^{-z/2}g_2$ in (2.74) and inverse Laplace transform leads to the equation

$$p(p + 1)G_2'' + [(1 - 2\beta)p + \tfrac{1}{2} + \kappa - \beta]G_2' + (\beta^2 - \mu^2)G_2 = 0 \tag{2.88}$$

Noting that

$$\mathcal{B}x^{a-1} = \frac{p^{-a}}{\Gamma(1 - a)} \tag{2.89}$$

and that

$$f = x^{a-1}g \Rightarrow F = C\frac{p^{-a}}{*}p^{c-a}G \tag{2.90}$$

we get that

$$F = Cp^{-a} * p^{a-c}(p - 1)^{-b-1-c} = \int_0^p (p - s)^{-a}s^{a-c}(1 - s)^{-b-1+c}ds = p^{-a+a-c} \tag{2.91}$$

with one solution ${}_2F_1(a, b; c; p + 1)$ where $a = -\mu - \beta$, $b = \mu - \beta$ and $c = \frac{1}{2} - \beta - \kappa$. These equations are nondegenerate and μ, κ, β can be expressed as a function of a, b, c. Comparing with the substitution $y(z) = z^{\mu+1/2}e^{-z/2}g(z)$ in §2.4b we see that $g_2 = z^{\mu-1-\beta}g$; inverse Laplace transforming this relation we get, with $C = 1/\Gamma(\mu - \kappa + 1/2)$,

$$\begin{aligned}\Gamma(1 - b)G_2(p) = \Gamma(\beta - \mu + 1)G_2(p) = p^{\beta-\mu} * G(p) = \\ C\int_0^p (p - s)^{\beta-\mu}s^{-\kappa+\mu-\frac{1}{2}}(s + 1)^{\mu-\frac{1}{2}+\kappa}ds \\ = Cp^{\beta-\kappa+\frac{1}{2}}\int_0^1 t^{-\kappa+\mu-1/2}(1 - t)^{\beta-\mu}(1 + pt)^{\mu-1/2+\kappa}dt \\ = \frac{p^{c-a-b}}{\Gamma(c - a)}\int_0^1 t^{c-a-1}(1 - t)^{-b}(1 + pt)^{b-c}dt,\end{aligned} \tag{2.92}$$

from which we get an integral representation

$$ {}_2F_1(a,b;c,p+1) = \frac{p^{c-a-b}}{\Gamma(1-b)\Gamma(c-a)} \int_0^1 t^{c-a-1}(1-t)^{-b}(1+pt)^{b-c}dt \ , \quad (2.93) $$

valid for $\operatorname{Re} b < 1$, $\operatorname{Re} c > \operatorname{Re} a$.

2.4d The Gamma function

The function $f(x) = \log\Gamma(x)$ satisfies the recurrence $f(x+1) - f(x) = \log x$. To get a recurrence for which the formal solution decreases in the right half plane to be able to take the Borel transform, we first subtract out the large terms in the asymptotic expansion of $f(x)$ for $x \to \infty$. A simple way to do that is to resort to the Euler-Maclaurin summation formula, see [8, 21] and §2.17, a systematic method of obtaining a sequence of approximations of sums by integrals. Let $g(x) = f(x) - (x\log x - x - \frac{1}{2}\log x)$.
The recurrence satisfied by g is

$$ g(x+1) - g(x) = q(x) = 1 - \left(\frac{1}{2} + x\right)\ln\left(1 + \frac{1}{x}\right) = -\frac{1}{12x^2} + \frac{1}{12x^3} + ... $$

First notice that $p^2Q(p) = \mathcal{L}^{-1}[q''](x)$, where $Q(p) = \mathcal{L}^{-1}[q](p)$. Therefore, $\mathcal{L}^{-1}q = p^{-2}\mathcal{L}^{-1}q''$ with

$$ q'' = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{(x+1)^2} + \frac{1}{x^2}\right) \Rightarrow \mathcal{L}^{-1}q'' = 1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p} $$

Thus, with $\mathcal{L}^{-1}[g](p) := G(p) = Q(p)/(e^{-p} - 1)$ we obtain

$$ (e^{-p} - 1)G(p) = \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2} $$

$$ g(x) = \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2(e^{-p} - 1)} e^{-xp}dp $$

(It is easy to check that the integrand is analytic at zero; its Taylor series is $\frac{1}{12} - \frac{1}{720}p^2 + O(p^3)$.)

The integral is well defined, and it follows that

$$ f(x) = C + x(\log x - 1) - \frac{1}{2}\log x + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2(e^{-p} - 1)} e^{-xp}dp $$

solves our recurrence. The constant $C = \frac{1}{2}\ln(2\pi)$ is most easily obtained by comparing with Stirling's formula (1.91) for large x and we thus get the identity

$$\log\Gamma(x) = x(\log x - 1) - \frac{1}{2}\log x + \frac{1}{2}\log(2\pi) + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)} e^{-xp} dp \tag{2.94}$$

which holds with x replaced by $z \in \mathbb{C}$ as well.

This represents, as it will be clear from the definitions, the Borel summed version of Stirling's formula.

2.5 Stokes phenomena

Near an essential singularity, the behavior of an analytic function depends on the direction of approach of the singularity. For instance, e^x is decreasing in the left half plane and growing in the right half plane, and a transition occurs as $i\mathbb{R}$ is crossed. The Stokes phenomenon describes more subtle phenomena. The Stokes phenomenon relates to the fact that the solution that is asymptotic to one fixed formal solution is generally different in different sectors at infinity. One of the simplest examples, that we have already explored is provided by

$$\int_0^\infty \frac{e^{-px} dx}{p+1} \tag{2.95}$$

We illustrate the Stokes phenomenon in the following analysis of the solutions of the Airy equation; their change in behavior at $+\infty$ versus $-\infty$ is very important in turning point problems, as we saw.

2.5a The Airy equation

To leading order, we see from (2.53) that

$$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}\, z^{\frac{1}{4}}} \; z \to +\infty \tag{2.96}$$

We work for now with the variable $t = \frac{2}{3}x^{\frac{3}{2}}$, and write $f(t) = \pi^{\frac{1}{2}}3^{\frac{1}{6}}\Gamma(\frac{1}{6})e^t \mathrm{Ai}(x(t))$. Then, using (2.53), we obtain

$$f(t) = \int_0^\infty e^{-tp} p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} dp \tag{2.97}$$

If we analytically continue t anticlockwise, the p contour is rotated homotopically clockwise by the same angle, to keep tp real and positive and have in the process the integral presented in a form suitable for Watson's lemma.

A rotation in the p-plane by more than $-\pi$ requires crossing the negative real line where the integrand has a branch point. With J_- the integral along a ray of angle $-\pi+0$ and J_+ the same integral along a ray of angle $-\pi-0$, we have

$$J_- = J_+ - (J_+ - J_-) = J_+ + f_2(t);\ \ f_2(t) := -\oint_{\mathcal{C}} e^{-tp} p^{-\frac{5}{6}} (2+p)^{-\frac{5}{6}} dp \quad (2.98)$$

where $\mathcal{C}$ is a curve starting at $\infty e^{-(\pi-0)i}$, goes around $p=-2$ and then to $\infty e^{-(\pi+0)i}$, see Fig. 2.1.

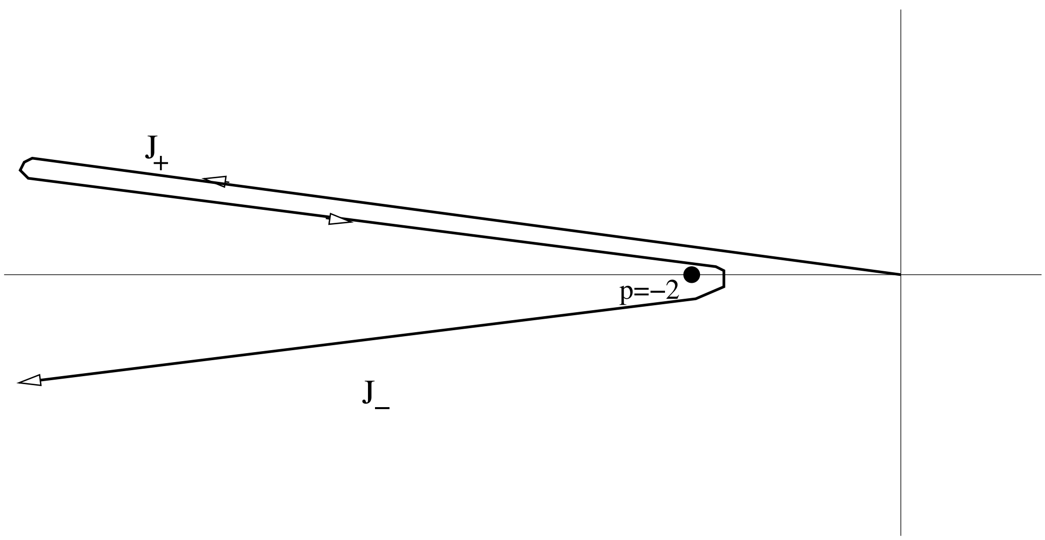

FIGURE 2.1: Deformation of contour for (2.53).

Reasoning as in Note 2.42 (iii), the function

$$f_2(t) = -\oint_{\mathcal{C}} e^{-tp} p^{-\frac{5}{6}} (2+p)^{-\frac{5}{6}} dp \quad (2.99)$$

provides, after multiplying it by e^{-t} and changing variable back to x, a linearly independent solution of the Airy equation. $\mathbb{R}^-$ is a Stokes line, and the fact that by crossing it we collected this new term is the Stokes phenomenon.

We note that the contour in f_2 can be deformed all the way to $p=-2$, where we have an integrable singularity.

In the part of the integral f_2 which is above $\mathbb{R}^-$ and to the left of $\mathrm{Re}\,(p) = -2$ we have by construction $\arg p = -\pi - 0$, $\arg(2+p) = \pi + 0$, and thus $p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} = |p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$. In the part below $\mathbb{R}^-$ we have $\arg p = -\pi + 0$,

$\arg(2+p) = -\pi + 0$ and thus $p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} = e^{\pi i/3}|p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$. Noting that $1 - e^{5\pi i/3} = e^{\pi i/3}$, and changing variables to $s = e^{i\pi}p$, we get

$$f_2(t) = e^{\pi i/3}\int_2^\infty s^{-\frac{5}{6}}(s-2)^{-\frac{5}{6}}e^{ts}ds = e^{\pi i/3}e^{2t}\int_0^\infty s^{-\frac{5}{6}}(s+2)^{-\frac{5}{6}}e^{ts}ds \tag{2.100}$$

with $\arg(s) = 0$. In J_+, thought of an integral along the upper side of $\mathbb{R}^-$, $p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}$ equals $e^{5\pi i/6}|p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$ from $\operatorname{Re} p = 0$ to $\operatorname{Re} p = -2$ and, as before, $|p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$ to the left of $\operatorname{Re}(p) = -2$. Changing variables to $s = e^{i\pi}p$, this is the same as the integral below $\mathbb{R}^-$

$$J_+ = -e^{5\pi i/6}\int_0^{\infty - i0} s^{-\frac{5}{6}}(2-s)^{-\frac{5}{6}}e^{ps}ds \tag{2.101}$$

with the natural choice of the argument, that is starting with $\arg s = 0$, $\arg(2-s) = 0$ for small $s > 0$. With this choice, we note that $\arg(2-s) = \pi$ for $s > 2$ in the integrand in (2.101). Thus,

$$f(t) = -e^{5\pi i/6}\int_0^{\infty e^{-0i}} s^{-\frac{5}{6}}(2-s)^{-\frac{5}{6}}e^{ts}ds + e^{\pi i/3}e^{2t}\int_0^\infty s^{-\frac{5}{6}}(s+2)^{-\frac{5}{6}}e^{ts}ds \tag{2.102}$$

for $\arg(t) > \pi$. By the change of variables, if p rotates clockwise, so does s.

To reach $\arg x = \pi$, *i.e.* $\arg t = \frac{3\pi}{2}$, we rotate further s by $-\pi/2$. Then, for $x \in \mathbb{R}^-$, after the change of variable $s = e^{-i\pi/2}u$, we get

$$\begin{aligned}\pi^{\frac{1}{2}}3^{\frac{1}{6}}\Gamma(\tfrac{1}{6})\operatorname{Ai}(-|x|) &\\ &= e^{-\frac{\pi i}{4}}e^{i|t|}\int_0^\infty u^{-\frac{5}{6}}(2+iu)^{-\frac{5}{6}}e^{-u|t|}du \\ &+ e^{\frac{\pi i}{4}}e^{-i|t|}\int_0^\infty u^{-\frac{5}{6}}(2-iu)^{-\frac{5}{6}}e^{-u|t|}du; \quad t = \tfrac{2}{3}x^{\frac{3}{2}}\end{aligned} \tag{2.103}$$

which is real-valued as expected.

For large $|x|$, applying Watson's Lemma to (2.103), one obtains the asymptotic behavior

$$\begin{aligned}\operatorname{Ai}(-|x|) = \frac{1}{2\sqrt{\pi}}|x|^{-1/4}e^{i\frac{2}{3}|x|^{2/3} - i\frac{\pi}{4}}\left[1 + O\left(|x|^{-3/2}\right)\right] &\\ + \frac{1}{2\sqrt{\pi}}|x|^{-1/4}e^{-i\frac{2}{3}|x|^{2/3} + i\frac{\pi}{4}}\left[1 + O\left(|x|^{-3/2}\right)\right]&\end{aligned} \tag{2.104}$$

Equation (2.104) gives the *connection formula* for Ai: It provides the behavior at $-\infty$, given the behavior at $+\infty$, with *explicit constants*. We see that the behavior at $-\infty$ differs *even formally* from (2.96). It is usually not possible to obtain in closed form the connection constants for more complicated ODEs, even linear ones. The asymptotic behavior of one solution in various sectors can be obtained, up to the value of some constants, in quite general linear ODEs. The possibility of finding the constants explicitly problems distinguishes *special functions* from mere solutions of linear ODEs, and is almost always linked to the existence of underlying integral representations. More generally, for nonlinear integrable systems, solving connection problems can be linked to the existence of Riemann-Hilbert reformulations.

Note 2.105 When we will study more general equations, we will see that the fact that the Borel transform of the normalized Airy equation has two singularities in the Borel plane results in the existence of two linearly independent solutions of any equation with coefficients analytic at infinity having Ai as a solution. Thus, there can be no "simpler" equation with these analytic properties which as Ai as a solution.

2.5b Nonlinear Stokes phenomena

When a differential equation is nonlinear, typically, there are infinitely many singularities in the p plane, as we will see in more detail later.

For now let us take a reverse-engineered example. The function $y(x) = e^{-x}\mathrm{Ei}(x)$ satisfies the model equation $y' + y = 1/x$ that we studied before, and thus $v = y/(1-y)$ satisfies the nonlinear equation

$$v' + \left(1 - \frac{2}{x}\right) v + v^2 = \frac{v^2}{x} + \frac{1}{x} \tag{2.106}$$

We know that the asymptotic series of $y(x)$ is Borel summable in any direction other that $\mathbb{R}^+$, and thus, by the proof of Proposition 2.29 so is v. On the other hand with $v = \mathcal{L}V, y = \mathcal{L}Y$ we have

$$v = y + y^2 + y^3 + \cdots \Rightarrow V = Y + Y * Y + \cdots \tag{2.107}$$

where $Y(p) = 1/(1-p)$. Then

$$(Y * Y)(p) = \int_0^p \frac{ds}{(1-s)[1-(p-s)]} = -\frac{2\ln(1-p)}{2-p}$$

which is singular at $p = 1$ *and* $p = 2$. Likewise, Y^{*3} introduces a singularity at $p = 3$ and so on, and without going through a rigorous proof for now, we claim that the last term in (2.107) is singular (on a Riemann surface) exactly at $p \in \mathbb{N}$. To illustrate this on an even more simplified model, note that

$$\delta_a(p) * \delta_b(p) = \delta_{a+b}(p)$$

where $\delta_a(p)$ is the Dirac delta distribution concentrated at $p = a$; the equality is very checked seen by Laplace transforming both sides above.

What would be the Stokes phenomenon for v? If J_+ (J_-) is the Borel sum of the asymptotic series of y for $\arg x = 0_+$ ($\arg x = 0_-$, resp.), then

$$1+v_+ = \frac{1}{1-J_+} = \frac{1}{1-J_- + 2\pi i e^{-x}} = \frac{1}{1-J_-} - \frac{2\pi i e^{-x}}{(1-J_-)^2} + \frac{(2\pi i)^2 e^{-2x}}{(1-J_-)^3} - \cdots \tag{2.108}$$

since, by Watson's lemma J_+ and J_- are of order $1/x \gg \exp(-x)$. On the other hand, in the proof of Proposition 2.29 we showed that, if S is a Borel summable small series then so is the series (of series) $1+S+S^2+... = 1/(1-S)$. Summarizing,

$$v_+ = -1 + \sum_{k=0}^{\infty} C^k y_k e^{-kx}, \quad C = -2\pi i \tag{2.109}$$

where y_k are Borel summable. We can check that in fact (2.109) is a solution of (2.106) for any C, for instance by recalling that $v = y/(1-y)$ and noting that $e^{-x}\mathrm{Ei}(x) + Ce^{-x}$ is a solution of $y' + y = 1/x$. The infinite sum in (2.109) is a *Borel summed transseries* and is the prototypical form of a solution of a nonlinear first order ODE, after *normalization* (meaning bringing the equation to a canonical form by changes of dependent and independent variables). A similar form, except in vector presentation describes solutions of generic systems of normalized nonlinear ODEs.

Exercise 2.110 Show that $1 - J_+$ has infinitely many zeros in any sector of the form $\arg x \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$ using the integral representation of J_+, cf. (2.95).

The phenomenon of formation of arrays of singularities is quite general, and is the effect of pile-up of exponentials near antistokes lines, the lines where the exponentials become oscillatory. To the right of the poles, the leading behavior of v is $1/x$ and to the left it is 1.

2.5c The Gamma function

Difference equations also produce infinitely many singularities in p plane. Let us look at (2.94). Consider the part

$$g(x) = \int_0^{\infty} \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right) e^{-p}}{p^2(e^{-p} - 1)} e^{-xp} dp \tag{2.111}$$

We analytically continue (2.111) to complex x by choosing the integration path to be the ray $\arg p = \theta = -\arg x$ (except when $x \in i\mathbb{R}$ when we would take $\arg p = \theta = -\arg x + i0$ instead) thus ensuring px to be real and positive. When x crosses the negative imaginary axis and moves to the third quadrant, $\arg p$ crosses $\frac{\pi}{2}$ and in the process residues at poles $p = 2in\pi$ for $n \in \mathbb{N}$ have to be collected; this gives

$$
\begin{aligned}
g(x) = \int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp + 2\pi i \sum_{j=1}^{\infty} \mathrm{Res} G(p)e^{-xp}|_{p=2j\pi i} \\
= \int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp + \sum_{j=1}^{\infty} \frac{1}{je^{2xj\pi i}} \\
= \int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp - \ln(1-\exp(-2x\pi i)), \quad (2.112)
\end{aligned}
$$

We note that both the integral and the sum are convergent when $\arg x = -\theta = -\pi/2 - \varepsilon$ for $\varepsilon \in \left(0, \frac{\pi}{2}\right]$ if x is not an integer. Then from (2.94) we get

$$
\Gamma(x) = \frac{1}{1-\exp(-2x\pi i)} \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \exp\left(\int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp \right) \quad (2.113)
$$

Since $\varepsilon \in \left(0, \frac{\pi}{2}\right]$, it is manifest from (2.113) that $\Gamma(x)$ is analytic for $\arg x \in \left(-\pi, -\frac{\pi}{2}\right)$ while analyticity for $\arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is manifest in (2.111). When $\arg x = -\frac{\pi}{2}$, we skirt the singularities of the integrand on $\arg p = \frac{\pi}{2}$ by choosing instead integration along $\arg p = \frac{\pi}{2} - \varepsilon$ to conclude analyticity of $\Gamma(x)$ on the negative imaginary axis. Setting $\theta = \pi$ in (2.113), it is clear that $\Gamma(x)$ is real valued and meromorphic on $e^{-i\pi}\mathbb{R}^+$, with simple poles at negative integers. Since $\Gamma(x)$ is real valued on $\mathbb{R}$, by the Schwartz reflection principle, $\Gamma(x)$ is analytic in the lower half plane and $\Gamma(x)$ is singular only at simple poles at negative integers. The poles may be seen to originate in the infinite sum on the second line of (2.112), in a way similar to the one mentioned in the preceding section. Since Watson's Lemma may be applied to $\int_0^{\infty e^{i\theta}} G(p)e^{-px}$, Stirling's formula holds for large $|x|$ when $\arg\ x \in (-\pi, 0]$ since $e^{-2x\pi i} \ll 1$ in this regime. Using again the Schwartz reflection principle, the same is true for $\arg\ x \in [0, \pi)$, and the only exception is $\mathbb{R}^-$.

Furthermore, taking $x = -y$ with $y \notin \mathbb{N}$ and then setting $\theta = \pi$ in the expression for $\Gamma(-x)$, we get, by straightforward algebra from (2.113) $\Gamma(-x)\Gamma(x) = -\dfrac{\pi}{x\sin(\pi x)}$ from which it follows that

$$
\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)} \quad (2.114)
$$

We can, especially by analogy with the ODE case, think of the reflection formula (2.114) as the connection formula for the Gamma function.

2.6 Analysis of convolution equations

2.6a Properties of the p plane convolution, Definition 2.21

We will study convolution equations along $\mathbb{R}^+$, in spaces of functions which are locally integrable and exponentially bounded at infinity, or in sectorial domains in $\mathbb{C}$, typically containing a neighborhood of the origin or in compact subsets of such sectorial domains. In all these cases, convolution is well-defined; see also cf. Lemma 2.27 for analyticity properties.

Lemma 2.115 *In the settings above, functions form a commutative algebra with respect to convolution and addition.*

PROOF The needed properties can be checked directly, but it is convenient to use injectivity of the Laplace transform, see Corollary 1.53 and the fact that $\mathcal{L}[f*g] = (\mathcal{L}f)(\mathcal{L}g)$. For function defined in (pre)compact subsets of $\mathbb{C}$ we can extend them with zero to take the Laplace transform.

For instance convolution is associative since

$$\mathcal{L}[f*(g*h)] = \mathcal{L}[f]\mathcal{L}[g*h] = \mathcal{L}[f]\mathcal{L}[g]\mathcal{L}[h] = \mathcal{L}[(f*g)*h] \tag{2.116}$$

and by injectivity of $\mathcal{L}$, we get

$$f*(g*h) = (f*g)*h \tag{2.117}$$

$\square$

Similarly, it is commutative and distributive.

2.6b Banach convolution algebras

We now define a family of suitable norms so that the Banach spaces induced by these norms are Banach algebras [3]. without a unit element with respect to addition and convolution (defined to be the multiplication in this algebra).

Some spaces arise naturally and are well suited for the study of convolution equations.

1. Let $\nu \in \mathbb{R}$ and define $L^1_\nu := \{f : \mathbb{R}^+ \to \mathbb{C} : f(p)e^{-\nu p} \in L^1(\mathbb{R}^+)\}$; then the norm $\|f\|_\nu$ is defined as $\|f(p)e^{-\nu p}\|_1$ where $\|\cdot\|_1$ denotes the L^1 norm; equivalently, $\|f\|_\nu = [\mathcal{L}(|f|)](\nu)$. Lemma 2.23 shows that L^1_ν is a Banach algebra with respect to convolution without a unit element.

[3] A Banach algebra is a Banach space of functions endowed with multiplication (*) which is distributive, associative and continuous in the Banach norm, continuity meaning $\|F*G\| \leqslant \|F\|\|G\|$. A unit "1" is an element s.t. $1*x = x*1 = x$.

Exercise 2.118 Show that there is indeed no L^1_ν unit with respect to convolution, that is no u s.t. $u * f = f$ for all $f \in L^1_\nu$.

We see that the norm $\|\cdot\|_\nu$ is the Laplace transform of $|f|$ evaluated at large argument ν, and it is, in this sense, a Borel dual of the sup norm in the original space— since for $\operatorname{Re} x > \nu$, $|\mathcal{L}[f](x)| \leqslant \int_0^\infty e^{-\nu p}|f(p)|dp$.

2. *The space* $L^1_\nu(\mathbb{R}^+e^{i\varphi})$. By definition $f \in L^1_\nu(\mathbb{R}^+e^{i\varphi})$ if $f_\varphi(t) := f(te^{i\varphi}) \in L^1_\nu$. Convolution along $\mathbb{R}^+e^{i\varphi}$ can be expressed directly as

$$(f*g)(|p|e^{i\varphi}) = \int_0^{|p|e^{i\varphi}} f(s)g(|p|e^{i\varphi}-s)ds = \\ e^{i\varphi}\int_0^{|p|} f(te^{i\varphi})g(e^{i\varphi}(|p|-t))dt = e^{i\varphi}(f_\varphi * g_\varphi)(|p|) \quad (2.119)$$

It is clear that $L^1_\nu(\mathbb{R}^+e^{i\varphi})$ is also a Banach algebra.

3. Similarly, we say that $f \in L^1_\nu(S)$ where $S = \{te^{i\varphi} : t \in \mathbb{R}^+, \varphi \in (a,b)\}$ if $f \in L^1_\nu(\mathbb{R}^+e^{i\varphi})$ for all $\varphi \in (a,b)$. We define $\|f\|_{\nu,S} = \sup_{\varphi\in(a,b)} \|f\|_{L^1_\nu(\mathbb{R}^+e^{i\varphi})}$. The space $L^1_\nu(S) = \{f : \|f\|_{\nu,S} < \infty\}$ is also a Banach algebra.

4. The L^1_ν spaces can be restricted to an initial interval along a ray, or a compact subset of S, restricting the norm to an appropriate set. For instance,

$$L^1_\nu([0,1]) = \left\{ f : \int_0^1 e^{-\nu s}|f(s)|ds < \infty \right\} \quad (2.120)$$

These spaces are Banach algebras as well. Obviously, if $A \subset B$, $L^1_\nu(B)$ is naturally embedded (cf. footnote 5 on p. 94) in $L^1_\nu(A)$. Note also that $f \in L^1_\nu([0,1]$ if f extended by zero on $[1,\infty)$ is in $L^1_\nu(\mathbb{R}^+)$, and through this mapping many properties of $L^1_\nu(\mathbb{R}^+)$ carry over to $L^1_\nu([0,1]$.

5. Another important space is $\mathcal{A}_\nu(\mathcal{E})$, the space of analytic functions in a star-shaped[4] neighborhood $\mathcal{E}$ of a disk $\{p : |p| \leqslant a\}$ in the norm ($\nu \in \mathbb{R}^+$; $R = \frac{1}{2}\mathrm{diam}(\mathcal{E})$)

$$\|f\| = R \sup_{p\in\mathcal{E}} \left| e^{-\nu|p|} f(p) \right|$$

Note. This norm is topologically equivalent with the sup norm (convergent sequences are the same), but better behaved for finding exponential bounds.

[4]Containing every point p together with the segment linking it to 0.

6. We can allow the star-shaped domain $\mathcal{E}$ to be non-compact by taking the following norm:

$$\|f\| = \frac{\pi}{2} \sup_{\mathcal{E}} \left| (|p^2 + 1| e^{-\nu|p|} f(p) \right|$$

Indeed, if $|f| \leqslant \|f\| \frac{(}{2}/\pi) e^{-\nu|p|}(|p|^2+1)^{-1}$ and $|g| \leqslant \|g\| \frac{(}{2}/\pi) e^{-\nu|p|}(|p|^2+1)^{-1}$, then $f * g$ is bounded by

$$\frac{\|f\|\|g\|}{\pi^2} \int_0^{|p|} \frac{2^2 e^{\nu|p|} \pi^{-2} ds}{[(|p|-s)^2+1](s^2+1)} \leqslant \frac{\|f\|\|g\| e^{\nu|p|} 2^2}{\pi^2} \frac{\arctan |p|}{|p|^2+1} \leqslant \|f\|\|g\| \quad (2.121)$$

7. The subspace of functions $\mathcal{A}_{0\nu}(\mathcal{E})$ which vanish at zero.

Proposition 2.122 (Deformation of contour) *Assume that* $f \in L^1_\nu(S)$, $\|f\|_\nu < M$ *where* S *is a sector* $\{p : \arg(p) \in [a,b] \subset (-\pi/2, \pi/2)\}$ *and that* f *is analytic in* S. *Then* $\int_0^{\infty e^{i\theta}} f(p) e^{-\nu p} dp$ *does not depend on* $\theta \in [a,b]$.

PROOF Let $\varepsilon > 0$. We have

$$\int_0^{\infty e^{i\theta}} e^{-(\nu+\varepsilon)p} f(p) dp = \int_0^{\infty e^{i\theta}} e^{-\varepsilon p} \frac{d}{dp} \int_0^p e^{-\nu t} f(t) dt$$
$$= \varepsilon \int_0^{\infty e^{i\theta}} e^{-\varepsilon p} \int_0^p e^{-\nu t} f(t) dt dp = \varepsilon \int_0^{\infty e^{i\theta'}} e^{-\varepsilon p} \int_0^p e^{-\nu t} f(t) dt dp$$
$$= \int_0^{\infty e^{i\theta'}} e^{-(\nu+\varepsilon)p} f(p) dp \quad (2.123)$$

for all $\theta, \theta' \in [a,b]$ by Jordan's Lemma, where we use the fact that $\int_0^p e^{-\nu t} f(t) dt$ is analytic and uniformly bounded by M. Taking $\varepsilon \to 0$ and applying dominated convergence, the result follows. ∎

Alternatively, the result above follows from the following

Proposition 2.124 *Assume* $F \in L^1_\nu$ *and is analytic in a sector* $S = \{p : \arg p \in (a,b)\}$. *Then* $e^{-\nu|p|}|F(p)|$ *is bounded as* $p \to \infty$ *in any subsector* $S' = \{p : \arg p \in (a', b')\}$, $[a', b'] \subset (a,b)$.

PROOF Indeed,

$$e^{-|p|\nu} \left| \int_0^{|p|} |Y|(|t|) d|t| \right| \leqslant \left| \int_0^{|p|} e^{-|t|\nu} |Y|(|t|) d|t| \right| \leqslant \|Y\|_\nu$$

Thus, the antiderivative of Y is uniformly exponentially bounded in S by $\|Y\|_\nu$. It is easy to see using Cauchy's integral formula that Y is bounded in S' by $\nu'\|Y\|_\nu e^{\nu|p|}$ for some $\nu' \geqslant \nu$ (which can be chosen arbitrarily close to ν if p is large). ◻

Proposition 2.125 *The space $\mathcal{A}_{R;\nu}$ is a Banach algebra with respect to convolution.*

PROOF For analyticity, see Lemma 2.27 . To estimate the norm of convolution we write, with $P = |p|$,

$$\left|Re^{-\nu P}\int_0^p f(s)g(p-s)ds\right| = \left|Re^{-\nu P}\int_0^P f(te^{i\varphi})g((P-t)e^{i\varphi})dt\right|$$
$$= \left|R^{-1}\int_0^P Rf(te^{i\varphi})e^{-\nu t}Rg((P-t)e^{i\varphi})e^{-\nu(P-t)}dt\right|$$
$$\leqslant R^{-1}\|f\|\|g\|\int_0^R dt = \|f\|\|g\| \tag{2.126}$$

◻

Note that $\mathcal{A}_{R;\nu} \subset L^1_\nu(\mathcal{E})$.
(6) Finally, we note that the space $\mathcal{A}_{R,\nu;0}(\mathcal{E}) = \{f \in \mathcal{A}_{R,\nu}(\mathcal{E}) : f(0) = 0\}$ is a closed subalgebra of $\mathcal{A}_{R,\nu}$.

Remark 2.127 *If f is a bounded function, then*

$$\|fg\| \leqslant \|g\|\|f\|_\infty \tag{2.128}$$

in L^1_ν. The same holds if f is holomorphic in $\mathcal{E}$, with sup *now over $\mathcal{E}$, for the spaces $\mathcal{A}_{R;\nu}$ and $\mathcal{A}_{R,\nu;0}$.*

2.6c Spaces of sequences of functions

In Borel summing more general expansions (transseries), it is convenient to look at sequences of vector-valued functions belonging to one or more of the spaces introduced before. For instance, in the scalar case, when the transseries is given by $\tilde{y} = \sum_{j=0}^\infty \tilde{y}_j z^j$ with $z = e^{-\lambda x}$, we have

$$y^m = \sum_{j=0}^\infty z^j \sum_{k_1+k_2+\cdots+k_m=j} y_{k_1}y_{k_2}\cdots y_{k_m} \tag{2.129}$$

It is convenient to represent y as a vector

$$\mathsf{y} = \{y_j\}_{j\geqslant 0} \;\; (\mathsf{y}_j := y_j) \tag{2.130}$$

and introduce the product of sequences fg by

$$(\mathsf{fg})_j = \sum_{j_1+j_2=j} \mathsf{f}_{j_1}\mathsf{g}_{j_2} \tag{2.131}$$

To Borel transform a transseries we look at the sequence of Borel transforms of each $\mathsf{y}_j =: y_j$ above. Thus the Borel dual of y is the sequence

$$\mathsf{Y} = \{Y_k\}_{k\geqslant 0}\,; \tag{2.132}$$

where $\mathsf{Y}_j = \mathcal{B}\mathsf{y}_j$. For $\mu > 0$ we define

$$L^1_{\nu,\mu} = \{\mathsf{Y} \in (L^1_\nu)^{\mathbb{N}\cup\{0\}} : \sum_{k\geqslant 0} \mu^{-k}\|Y_k\|_\nu < \infty\} \tag{2.133}$$

and introduce the following convolution on $L^1_{\nu,\boldsymbol{\mu}}$

$$(\mathsf{F}\boldsymbol{*}\mathsf{G})_k = \sum_{j=0}^{k} \mathsf{F}_j * \mathsf{G}_{k-j} \tag{2.134}$$

which, as we see, is a double convolution: in p through $*$ and a discrete one in the index.

Exercise 2.135 *Show that*

$$\|\mathsf{F}\boldsymbol{*}\mathsf{G}\|_{\nu,\mu} \leqslant \|\mathsf{F}\|_{\nu,\mu}\|\mathsf{G}\|_{\nu,\mu} \tag{2.136}$$

and $(L^1_{\nu,\mu}, +, \boldsymbol{*}, \|\,\|_{\nu,\mu})$ *is a Banach algebra. Show that the subspace* $L^1_{\nu,\mu;n} := \{\mathsf{y} \in L^1_{\nu,\mu} : \mathsf{y}_0 = \cdots = \mathsf{y}_{n-1} = 0\}$ *is closed, and thus a Banach algebra too.*

In the vectorial case we have, in general, m exponentials $e^{-\lambda_1 x}, \cdots, e^{-\lambda_m x}$ and the solution is vector valued, with values in say $\mathbb{C}^n$. We then define sequences $\{\mathbf{y}\}_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^m}$ where $\mathsf{y}_{\mathbf{k}} \in \mathbb{C}^n$. When writing the Borel transform of the transseries solution, once more we do so componentwise, for each $\mathbf{k}$ separately. Furthermore, the nonlinear terms in the differential equation are of the form $\mathbf{g}^{\mathbf{l}} = g_1^{l_1}\cdots g_n^{l_n}$ which are *scalar*. See also [44], §2.1.3.

2.7 Focusing spaces and algebras

An important property of the norms (1)–(4) and (6) in §2.6 is that for any f we have $\|f\|_\nu \to 0$ as $\nu \to \infty$. This is used to control nonlinear terms: for large enough ν they become negligibly small.

A family of norms $\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **focusing** if for any f with $\|f\|_{\nu_0} < \infty$ for some ν_0 we have

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{2.137}$$

($\downarrow$ means monotonically decreasing to the limit, $\uparrow$ means increasing).

Let $\mathcal{V}$ be a linear space of functions and $\{\|\|_\nu\}$ a family of norms satisfying (2.137) and (2.128) for any uniformly continuous $f \in V$. For each ν we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{V} : \|f\|_\nu < \infty\}$. For $\alpha < \beta$, (2.137) shows $\mathcal{B}_\alpha$ is naturally embedded in $\mathcal{B}_\beta$.[5] Let $\mathcal{F} \subset \mathcal{V}$ be the inductive limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \bigcup_{\nu>0} \mathcal{B}_\nu \tag{2.138}$$

where a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focusing space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focusing algebra**.

Examples. The spaces $\bigcup_{\nu>0} L^1_\nu$ and $\bigcup_{\nu>0} \mathcal{A}_{R;\nu;0}$ and $L^1_{\nu,\mu}$ are focusing algebras. The last space is focusing as $\nu \to \infty$ and $\mu \to \infty$.

An extension to distributions, very useful in studying singular convolution equations, is the space of staircase distributions $\mathcal{D}'_{m,\nu}$; see [44].

Remark 2.139 The following simple observation is useful when we want to show that one solution has a number of different properties: analyticity, boundedness, etc. Let f be defined on $S_1 \cup S_2$, and assume that the equation $f(x) = 0$ has a unique solution x_1 in S_1, a unique solution x_2 in S_2 and a unique solution x_3 in $S_1 \cap S_2$. Then $x_1 = x_2 = x_3 \in S_1 \cap S_2$. Of course, we can equivalently analyze the equation f in $S_1 \cap S_2$ to start with but when we are dealing with more sets when only various combinations of S_i intersect non-emptily, the first approach is more economical.

2.8 Borel summation analysis of nonlinear ODEs

We select a number of relatively simple yet illustrative examples and send for the general theory to [44].

Consider first the ODE

$$y' = x^{-2} - y + y^3 \tag{2.140}$$

[5]That is, we can naturally identify $\mathcal{B}_\alpha$ with a subset of $\mathcal{B}_\beta$ which is isomorphic to it.

This equation was chosen so that it is already in normalized form (more about this later) and it is not solvable by any known methods (in fact, it is nonintegrable in the sense of Painlevé, see §2.10); yet the analysis of this relatively simple problem illustrates the main aspects of the transseries analysis for a generic system of nonlinear ODEs [44]. The extension of focusing algebras techniques to transseries and contour integral representations (as will be seen later) simplifies the proofs in [44] and avoids using special distributions. Here, to avoid notational complications we just illustrate the approach on this particular nonlinear ODE, but the extension to general nonlinear systems of ODEs is relatively straightforward. We will point out where the differences occur, and defer to the appendix the necessary adaptations.

We can look for formal power series solutions in the usual way, by inserting a series with unknown coefficients and identifying them, or by iteration. Estimating the coefficients, one would see that the series is divergent, and in this sense, the point at infinity is an irregular singular point. Since we will not base the analysis on the formal series but rather on the *Borel transform of the equation*, and, furthermore, the formal series and its asymptotic properties will emerge as a byproduct, we skip this step now.

As in the linear cases, the logic will be that we apply formally the inverse Laplace transform (Borel transform) to the equation, find a solution of the transformed (Borel plane) equation and then show that the solution of this new equation, when Laplace transformed, results in a solution of the ODE.

Since we have a Banach algebra structure in Borel plane, differential equations become effectively algebraic equations (with convolution $*$ acting as multiplication), which is much easier to deal with. In our case, the formal inverse Laplace of (2.140) is

$$-pY + Y = p + Y^{*3}; \Leftrightarrow Y = \frac{p}{1-p} + \frac{1}{1-p}Y^{*3} := \mathcal{N}(Y) \tag{2.141}$$

where $\mathcal{L}^{-1}y = Y$ and $Y^{*3} = Y * Y * Y$.

Definition 2.142 *Let $[a,b] \subset (0, 2\pi)$, and $S^+ = \{p : \arg(p) \in (a,b)\}$, $S_K^+ = \{p \in S^+ : |p| < K\}$, $\mathbb{D}_\alpha = \{p : |p| < \alpha < 1\}$. Similarly for $[a,b] \subset (-2\pi, 0)$, and $S^- = \{p : \arg(p) \in (a,b)\}$, $S_K^- = \{p \in S^- : |p| < K\}$.*

Proposition 2.143 Let S be a star-shaped subset of a closed set s.t. the distance to $[1,\infty]$ is $a > 0$ and $\mathcal{F}$ a focusing algebra of functions in S closed under convolution, containing $\frac{p}{1-p}$. For large enough ν, (2.141) has a unique solution Y_0^+ in S.[6]

[6] S could be, for instance, any of the spaces in Definition 2.142, and the proof would go through for large enough ν.

PROOF By the focusing property, for large enough ν we have

$$\left\| \frac{p}{1-p} \right\|_\nu < \varepsilon/2 \tag{2.144}$$

Let $\mathfrak{B}$ be the ball of radius ε in the norm ν and F be a function in $\mathfrak{B}$. Then,

$$\|\mathcal{N}(F)\|_\nu \leqslant \left\| \frac{p}{1-p} \right\|_\nu + \max_{p\in S} \left| \frac{1}{p-1} \right| \|F\|_\nu^3 = \varepsilon/2 + c\varepsilon^3 \leqslant \varepsilon \tag{2.145}$$

if ε is small enough (that is, if ν is large). Furthermore, for large ν, $\mathcal{N}$ is contractive in $\mathfrak{B}$ for we have, for small ε,

$$\|\mathcal{N}(F_1) - \mathcal{N}(F_2)\|_\nu \lesssim \|F_1^{*3} - F_2^{*3}\|_\nu = \|(F_1 - F_2) * (F_1^{*2} + F_1 * F_2 + F_2^{*2})\|_\nu \lesssim 3\varepsilon^2 \|(F_1 - F_2)\|_\nu \tag{2.146}$$

$\square$

Corollary 2.147 There is a unique solution to the convolution equation (2.141) in each of following spaces $\mathcal{A}_{\nu,0}(S_K^+ \cup \mathbb{D}_\varepsilon)$, $L_\nu^1(S^+)$, $\mathcal{A}_{\nu,0}(S_K^+ \cup \mathbb{D}_\varepsilon)$, (for large enough ν) and the solution belongs to their intersection.

PROOF

In the following, the subscript 0 means the functions in the set vanish at zero.

We have the following embeddings: $\mathcal{A}_{\nu,0}(S_K^+ \cup \mathbb{D}_\varepsilon) \subset L_\nu^1(S^+)$ (extending the elements of $\mathcal{A}_{\nu,0}$ by zero) and $\mathcal{A}_{\nu,0}(S_K^+ \cup \mathbb{D}_\varepsilon) \subset \mathcal{A}_{\nu,0}(\mathbb{D}_\varepsilon)$. Thus, there exists a unique solution Y_0 of (2.141), the same for all these spaces. $\square$

Thus Y is analytic in S^+ and belongs to $L_\nu^1(S^+)$, in particular it is Laplace transformable. The Laplace transform is a solution of (2.140) as it is easy to check.

It also follows that the formal power series solution $\tilde{y}$ of (2.140) is Borel summable in any sector not containing $\mathbb{R}^+$, which is a Stokes ray. We have, indeed, $\mathcal{B}\tilde{y} = Y$ (check!).

2.8a Borel summation of the transseries solution

Let $\tilde{y}_0$ the asymptotic series of $\mathcal{L}Y_0$. Looking for a transseries solution,

$$\tilde{y} = \tilde{y}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{y}_k \tag{2.148}$$

we insert (2.148) in (2.140) and equate the coefficients of e^{-kx}; this results in the system of equations

$$\tilde{y}_0' = -\tilde{y}_0 + x^{-2} + \tilde{y}_0^3 \tag{2.149}$$

$$\tilde{y}_k' + (1 - k - 3\tilde{y}_0^2)\tilde{y}_k = 3\tilde{y}_0 \sum_{j=1}^{k-1} \tilde{y}_j \tilde{y}_{k-j} + \sum_{|j|=k; j_i \geqslant 1} \tilde{y}_{j_1}\tilde{y}_{j_2}\tilde{y}_{j_3}; \quad k \geqslant 1 \tag{2.150}$$

where, as usual, $|j| = j_1 + j_2 + j_3$ and $\sum_{\emptyset} = 0$. The equation for $\tilde{y}_1$ is linear and homogeneous:

$$\tilde{y}_1' = 3\tilde{y}_0^2 \tilde{y}_1 \tag{2.151}$$

For a general nonlinear system, instead of (2.151) one obtains a system of linear equations. Thus

$$\tilde{y}_1 = Ce^{\tilde{s}}; \quad \tilde{s} := \int_{\infty}^{x} 3\tilde{y}_0{}^2(t)dt \tag{2.152}$$

Since $\tilde{s} = O(x^{-3})$, by Proposition 2.29 and Proposition 2.40, $e^{\tilde{s}}$ is Borel summable in $\mathbb{C} \setminus \mathbb{R}^+$. We note that $\tilde{y}_1 = C(1 + o(1))$ and we cannot take the inverse Laplace transform of $\tilde{y}_1$ directly. But the series $x^{-1}\tilde{y}_1$ is Borel summable (say to $\check{\Phi}_1$) see Proposition 2.40. It is convenient[7] to make the substitution $\tilde{y}_k = x^k \tilde{\varphi}_k$ We get

$$\tilde{\varphi}_k' + (1-k-3\tilde{\varphi}_0^2 + kx^{-1})\tilde{\varphi}_k = 3\tilde{\varphi}_0 \sum_{j=1}^{k-1} \tilde{\varphi}_j \tilde{\varphi}_{k-j} + \sum_{j_1+j_2+j_3=k; j_i \geqslant 1} \tilde{\varphi}_{j_1}\tilde{\varphi}_{j_2}\tilde{\varphi}_{j_3} \tag{2.153}$$

where clearly $\tilde{\varphi}_0 = \tilde{y}_0$, $\tilde{\varphi}_1 = x^{-1}\tilde{y}_1$, with $\tilde{y}_1$ given in (2.152). We choose $\Phi_1 = C\mathcal{B}x^{-1}e^{\tilde{s}}$, define for a choice of sign,

$$\begin{aligned} \boldsymbol{Y}_0 = (Y_0^{\pm}, 0, ..., 0, ...); \quad \boldsymbol{Y}_1 = (0, \Phi_1^{\pm}, ..., 0, ...), 1 = (1, 0, ...); \\ \boldsymbol{\Phi} = (0, 0, \Phi_2, \Phi_3, ...); \quad \hat{k}\boldsymbol{\Phi} = (0, 0, 2\Phi_2, 3\Phi_3, 4\Phi_4, \cdots) \end{aligned} \tag{2.154}$$

and, after Borel transform, we get

$$-p\boldsymbol{\Phi} + (1 - \hat{k})\boldsymbol{\Phi} = \mathbf{G}_0 + (-\hat{k}1 + \mathbf{G}_1)*\boldsymbol{\Phi} + \mathbf{G}_2*\boldsymbol{\Phi}^{*2} + \boldsymbol{\Phi}^{*3} \tag{2.155}$$

where

$$\mathbf{G}_0 = 3\boldsymbol{Y}_0*\boldsymbol{Y}_1^{*2} + \boldsymbol{Y}_1^{*3}; \ \mathbf{G}_1 = 3(\boldsymbol{Y}_0^{*2} + 2\boldsymbol{Y}_0*\boldsymbol{Y}_1 + \boldsymbol{Y}_1^{*2}); \ \mathbf{G}_2 = 3(\boldsymbol{Y}_0 + \boldsymbol{Y}_1) \tag{2.156}$$

We treat (2.156) as an equation in $L^1_{\mu,\nu;0^2} \subset L^1_{\mu,\nu}$, the subspace of sequences $\{\Phi_j\}_{j\in\mathbb{N}}$, $\Phi_0 = \Phi_1 = 0$ (and similar subspaces of other focusing algebras).

[7] In this problem, $\tilde{y}_1 = 1 + \tilde{v}_1$ would suffice to ensure $O(x^{-2})$ decay of $(\tilde{v}_1, \tilde{y}_2, ...)$.

Note 2.157 (i) It is important to subtract out Y_1, as we have, since its equation allows for a free constant and no contractive mapping argument would work unless the constant C is specified.

(ii) One can show check inductively that

$$\tilde{\varphi}_{2k+1} = O(x^{-2k-1}); \quad \tilde{\varphi}_{2k} = O(x^{-2k-2}); \ \forall \mathbb{N} \ni k \geqslant 1 \tag{2.158}$$

or

$$\tilde{y}_{2k+1} = O(1); \quad \tilde{y}_{2k} = O(x^{-2}); \ \forall \mathbb{N} \ni k \geqslant 1 \tag{2.159}$$

where the constants implicit in (2.161) and (2.159) can be calculated in closed form for any given k.

Proposition 2.160 (i) For μ, ν large enough, eq. (2.156) is contractive in $L^1_{\nu,\mu;0^2}(S^+)$, $\mathcal{A}_{\nu,\mu,0,0^2}(S^+_K \cup \mathbb{D}_\varepsilon)$ and $\mathcal{A}_{\nu,\mu,0^2}(\mathbb{D}_\varepsilon)$. Thus (2.156) has a unique solution $\mathbf{\Phi}^+$ in this space. Similarly, it has a unique solution in these spaces. Likewise, there is a unique solution $\mathbf{\Phi}^-$ in the corresponding spaces in the lower half-plane[8].

(ii) Thus there is a ν large enough so that for all k

$$\varphi_k^-(x) = \int_0^{\infty e^{-i\arg(x)}} e^{-xp}\Phi_k^+(p)dp \tag{2.161}$$

exist for $|x| > \nu$. The functions $\varphi_k^-(x)$ are analytic in x for $\arg(x) \in (-2\pi - \pi/2, \pi/2)$. The similarly obtained $\varphi_k^+(x)$ by Laplace transforming Φ_k^- along a ray in the fourth quadrant are analytic in x, $\arg(x) \in (-\pi/2, 2\pi + \pi/2)$.

(iii) The function series

$$y^+(x; C_+) = \sum_{k=0}^{\infty} C_+^k e^{-kx} x^k \varphi_k^+(x) \tag{2.162}$$

and

$$y^-(x; C_-) = \sum_{k=0}^{\infty} C_-^k e^{-kx} x^k \varphi_k^-(x) \tag{2.163}$$

converge for sufficiently large $\mathrm{Re}\, x$, $\arg(x) \in (-\pi/2, \pi/2)$ and solve (2.140). (See also Proposition 2.165 below.). In fact, $|\varphi_k^-(x)| \leqslant \mu^k e^{-(\mathrm{Re}\,(x)-\nu)}$ if $\mathrm{Re}\,(x)$ is large enough.

Note. The solution cannot be written in the form (2.162) or (2.163) in a sector of opening more than π centered on $\mathbb{R}^+$ because the exponentials would become large and convergence is not ensured anymore. Growing exponentials implies, generically, blow-up of the actual solutions. All the φ_k however are well behaved.

[8] Like Y_0, the functions Φ_k are analytic for $|p| < 1$, but generally have branch points at $1, 2, \ldots$

Exercise 2.164 (*) *Prove Proposition 2.160.*

Proposition 2.165 *Any solution of (2.140) which is $o(1)$ as $x \to +\infty$ can be written in the form (2.162) or, equally well, in the form (2.163).*

Note 2.166 The proof of uniqueness of small solutions in other directions in the right half plane is very similar; in the left half plane, the fixed limit of integration x_0 in the integral form of the equation (see (2.168)) in the proof would need to be taken $-\infty$.

PROOF The method we use here can be extended relatively straightforwardly to general systems, see [44].

The idea is the following: we show that for any $o(1)$ solution y there exists a y_C s.t. $y - y_C = o(e^{-x})$, and this is incompatible with the differential equation unless $y - y_C = 0$.

Let $y_0 := y^+$ be the solution of (2.140) of the form (2.162) with $C = 0$. Let y be another solution which is $o(1)$ as $x \to +\infty$ and let $\delta = y - y^+$. We have

$$\delta' = -\delta + 3y_0^2\delta + 3y_0\delta^2 + \delta^3 \tag{2.167}$$

We first choose x_0 s.t. $\sup\{|\delta(x)|, e^{-x}, |y_0| : x \geqslant x_0\} \leqslant \varepsilon < \frac{1}{4}$ and then write the equation in integral form ($\delta =: \delta_C$):

$$\delta_C = Ce^{-x} + e^{-x}\int_{x_0}^{x} e^s \left(3y_0(s)^2\delta_C(s) + 3y_0\delta_C(s)^2 + \delta_C(s)^3\right) ds \tag{2.168}$$

where $|Ce^{-x_0}| = |\delta_C(x_0)| \leqslant \varepsilon$ by assumption. Then, with x_0 large enough, eq. (2.168) is contractive on $[x_0, \infty)$ in a ball of radius 2ε in the norm $\sup_{x \geqslant x_0} |\delta(x)|$. This is shown in the usual way.
Let

$$M(x) := \sup_{s \in [x_0, x)} |\delta_C(s)e^s|$$

By direct estimates, using (2.168) and $|\delta| \leqslant 2\varepsilon$ for $x \geqslant x_0$, we get for large x

$$M(s) \leqslant |C| + O(\varepsilon)M$$

Solving for M we get

$$M(x) \leqslant 2|C| \tag{2.169}$$

for large x. Thus we can write $\int_{x_0}^{x} = \int_{x_0}^{\infty} + \int_{\infty}^{x}$ and the integral equation becomes

$$\delta_C = C_1e^{-x} + e^{-x}\int_{\infty}^{x} e^s \left(3y_0(s)^2\delta_C(s) + 3y_0\delta_C(s)^2 + \delta_C(s)^3\right) ds \tag{2.170}$$

for some C_1. Define now the norm $\sup_{x \geqslant x_1} |\delta(x)|e^x$ and the Banach space $\{\delta : \|\delta\| < \infty\}$ and finally the ball $B = \{\delta : \|\delta\| < 2C_1\}$. Then, eq. (2.168) is contractive on $[x_1,\infty)$ in B, if $x_1 > x_0$ is large enough. In particular, if $C_1 = 0$ then this unique solution is zero.

Since the exponentially weighted spaces are contained in the L^∞ ones, it follows that there is a unique solution to (2.168) and $e^x\delta_C = C_1(1+o(1))$ for large x. On the other hand, the solution y_{C_1} given by (2.162) with $C_+ = C_1$ also satisfies $y_{C_1} - y_+ = C_1 e^{-x}(1+o(1))$. It follows that $\delta_1 = y - y_{C_1} = o(e^{-x})$ and satisfies (2.168), with C_1 replaced by some $c \in \mathbb{R}$.

By the same arguments as above, $\delta_1 = ce^{-x}(1+o(1))$. But $ce^{-x}(1+o(1)) = o(e^{-x})$ implies $c = 0$ and thus $\delta_1 \equiv 0$.

∎

2.8a.1 Finding the asymptotics of δ_C by linearization[21]

We could also reason as follows. Choose again x_0 as for (2.168). Note that the sought-for δ (which we will still denote δ_C) is also the unique solution of the linear equation (with $\delta_C = y - y_0$, where y is the fixed $o(1)$ solution treated as "known")

$$\delta' = -\delta + 3y_0^2\delta + 3y_0\delta_C\delta + \delta_C^2\delta;\ \delta(x_0) = \delta_C(x_0) \tag{2.171}$$

We have

$$(\ln\delta)' = -1 + 3y_0^2 + 3y_0\delta_C + \delta_C^2 = -1 + r(x) = -1 + o(1) \tag{2.172}$$

implying by integration

$$\delta = \delta_C(x_0)e^{-(x-x_0)+o(1)(x-x_0)} \Rightarrow \delta_C = \delta_C(x_0)e^{-(x-x_0)+o(1)(x-x_0)} \tag{2.173}$$

since we know $\delta = \delta_C$. Using the estimate (2.173) to bound $r(x)$ (instead of $o(1)$) we get

$$\ln\delta = -x + O(1/x^3) + C_1 \tag{2.174}$$

that is,

$$\delta = C_1 e^{-x}(1 + O(1/x^3))$$

Now we estimate as above $\delta_1 = y_{C_1} - y$, which should be by construction $o(e^{-x})$, the proof is finished as before (by redoing the calculation above with y_{C_1} replacing y_0).

2.8b Analytic structure of Y_0 along $\mathbb{R}^+$

The approach sketched in this section is simple, but relies substantially on the ODE origin of the convolution equations; it would not necessarily extend, say to PDEs.

A different, complete proof, that uses the differential equation only minimally is given in [44]; for extensions to PDEs see, e.g. [42] and references therein.

*

By Proposition 2.143, $Y = Y_0^+$ is analytic in any region of the form $\mathbb{D} \cup S_K^+$. We now sketch a proof that Y_0 has analytic continuation along curves that do not pass through the positive integers.

For this purpose we use (2.162) and (2.163) in order to derive the behavior of Y. It is a way of exploiting what Écalle has discovered in more generality, *bridge equations*. We start with exploring a non-generic possibility (which does not happen for our sample equation).

2.8b.1 Case I $y_0^+ = y_0^- =: y_0$

Since $y_0^+ = y_0^- = y$ we have

$$y = \int_0^{\infty e^{ia'}} Y^+(p)e^{-px}dp = \int_0^{\infty e^{-ia'}} Y^-(p)e^{-px}dp \tag{2.175}$$

where we take $a' \in (a, \pi/2)$ with a defined in Proposition 2.143.

We first show Y is analytic in a sector or opening a centered on $\mathbb{R}^+$. From (2.175) it follows that y is analytic and $O(x^{-2})$ in

$$S_x := \{x : |x| > x_0 := \nu m, \arg(x) \in (-\pi/2 - a', \pi/2 + a')\};$$
$$m^{-1} := \min\{\sin(a'/2), \cos(a'/2)\} \tag{2.176}$$

Now, by Proposition 1.56 (ii), Note 1.61 and (2.176) $Y = \mathcal{L}_\alpha^{-1}(y)$, the inverse Laplace transform of y with integration path along a line orthogonal on $\mathbb{R}e^{-i\alpha}$ exists for any $\alpha \in (-a', a')$ and does not depend on α, and is analytic in a sector $S_{a'}$ of opening a' centered on $\mathbb{R}^+e^{-i\alpha}$. At the same time, from (2.175) $\mathcal{L}_\alpha^{-1}(y)$ equals Y^+ in a small sector just above the real line and equals $\mathcal{L}_{-\alpha}^{-1}(y) = Y^-$ in a small sector just below the real line, then $Y = Y^+ = Y^-$ in any domain in *the union* of the domains of analyticity of Y, Y^+, Y^-, which is $\mathbb{C}$. The monodromy theorem implies that Y is entire. By Proposition 1.56 (iii) $|Y(p)|$ is bounded by $e^{c|p|}$ for some c in $S_{a'}$. Thus Y is entire and, by Propositions 2.147 and 2.124 uniformly bounded by $Ce^{\nu'|p|}$ for large ν, for some $\nu' \geqslant \max\{c, \nu|\}$ and all $p \in \mathbb{C}$.

It is an easy Cauchy formula exercise to show that, if $|Y| < Ce^{\nu|p|}$ then $|Y^k(0)| < C\sqrt{k}\nu^k/k!$ (see the appendix §2.18). Applying Laplace transform to the Taylor series of $Y(p)$ term by term, which is justified by the dominated convergence theorem, it follows that the series $\tilde{y}_0$ is convergent, and by basic ODE theory, or by the isomorphism of usual summation, it converges to a solution y.

We arrived at the following result:

Proposition 2.177 *The formal series $\tilde{y}_0$ is convergent iff (nongenerically) $y_+ = y_-$ and factorially divergent otherwise.*

PROOF The proof is contained in the analysis preceding the Proposition, except for the factorial divergence, which follows from the fact that $Y_+ \neq Y_-$ implies that the Taylor series of Y at $p = 0$ has finite radius of convergence and applying Watson's lemma to $\int_0^{\infty e^{i\theta}} Y(p)e^{-px}dp$, one obtains a divergent series $\tilde{y}_0$. ∎

2.8c The analytic continuations of Y_0

By Proposition 2.165, y_+ can be represented in the form (2.163). Thus, there exists a constant S (called *Stokes multiplier*) such that

$$y^+ = y^- + \sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^-(x) \tag{2.178}$$

implying

$$y^+ - y^- = Se^{-x} x\varphi_1^-(x) + O(x^2 e^{-2x}) \tag{2.179}$$

If $S = 0$ the analysis in §2.8b.1 applies, Y_0 is entire and y_0 is analytic at infinity. So we assume $S \neq 0$. More generally than (2.178), for any C_+ there is C_- such that we have

$$y^+ + \sum_{k=1}^{\infty} C_+^k e^{-kx} x^k \varphi_k^+(x) = y^- + \sum_{k=1}^{\infty} C_-^k e^{-kx} x^k \varphi_k^-(x) \tag{2.180}$$

from which we obtain

$$y^+ - y^- = (C_- - C_+)e^{-x} x\varphi_1^-(x) + O(x^2 e^{-2x}) \tag{2.181}$$

Comparing with (2.179) we get, after multiplication with e^x,

$$(C_- - C_+ - S)x\varphi_1^-(x) = O(x^2 e^{-x}) \tag{2.182}$$

implying

$$C_- - C_+ = S \tag{2.183}$$

since $x\varphi^-(x) \to 1$ as $x \to \infty$.

Recalling that $\varphi_0^{\pm} = y_0^{\pm}$, and denoting $C = C_+$ (2.180) we get the identity

$$\sum_{k=0}^{\infty} C^k e^{-kx} x^k \varphi_k^+(x) = \sum_{k=0}^{\infty} (C+S)^k e^{-kx} x^k \varphi_k^-(x) \tag{2.184}$$

which holds for any C ! From the convergence of $\mathbf{\Phi}$ in the space $L_{\mu,\nu}$ for large μ, ν, it follows that the expansions (2.184) are analytic in C if x is chosen large enough. This means we get a series of identities by taking the derivatives of (2.184) at $C = 0$. We get

$$\varphi_k^+ - \varphi_k^- = \sum_{j=1}^{\infty} \binom{k+j}{j} S^j x^j e^{-jx} \varphi_{k+j}^- \tag{2.185}$$

2.8d Local analytic structure of Y_0, Φ_k along $\mathbb{R}^+$

Definition 2.186 We introduce the notation $\oint_{a;c}^{\infty}$ to denote an integral along a contour encircling $a + \mathbb{R}^+$ in a counter-clockwise manner, with a minimal distance $0 < c < 1$ from this set; $\oint_{a;c}^{\infty e^{\pm i\varphi}}$ denote a similar integral, except that it approaches $\infty e^{i\varphi}$ on the upper side of $\mathbb{R}^+$ and $\infty e^{-i\varphi}$ on the lower-side. Finally, for $\varphi \neq 0$, $\oint_{a;c}^{\infty e^{i\varphi}}$ denotes an integral encircling $a + e^{i\varphi}\mathbb{R}^+$ in a positive direction, such that the points $a + l$ for all $l \in \mathbb{Z}^+$ are in the exterior of the curve, and the curve is at least $0 < c < 1$ away from the set $\{a + e^{i\varphi}\mathbb{R}^+, a + l : l \in \mathbb{Z}^+\}$.

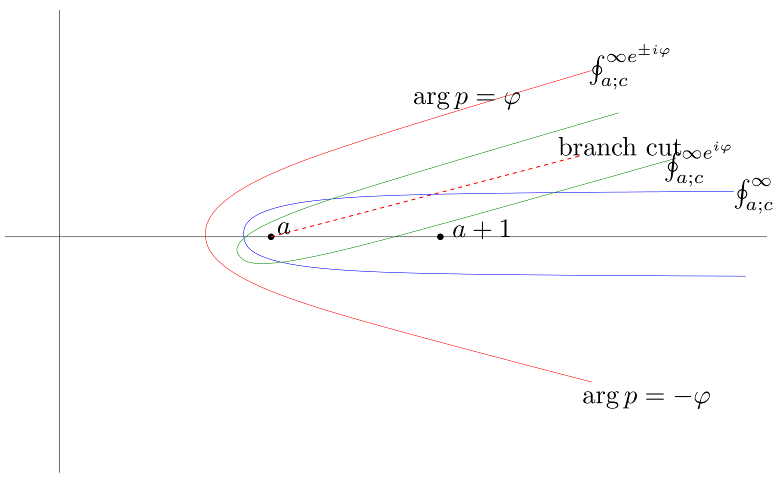

FIGURE 2.2: Various curves around a; the notation for the integrals, taken along the positive direction of each curve.

We take $\nu' > \nu$, and define

$$H_0 = e^{-\nu' p} Y_0 \text{ and } u = x - \nu' \text{ (we take } \mathrm{Re}\,(u) > 0) \tag{2.187}$$

Noting that $y_- = \mathcal{L}Y^+$ and $y_+ = \mathcal{L}Y^-$ and that in $\mathbb{D}$ we have $Y^+ = Y^- = Y_0$ and Y_0 is analytic there, we may write (2.178) as

$$\oint_{1;c}^{\infty e^{\pm i\alpha}} H_0 e^{-up} dp = \sum_{k=1}^{N} S^k e^{-kx} x^k \varphi_k^-(x) + a_N(u), \tag{2.188}$$

where

$$a_N(u) = \sum_{k=N+1}^{\infty} S^k e^{-k\nu'} e^{-ku} (\nu' + u)^k \varphi_k^-(\nu' + u) = O\left(u^{N+1} e^{-(N+1)u}\right) \tag{2.189}$$

Now, we will introduce an artificial singularity at zero of the integrand, to get a simple expression for the terms of the sum in (2.188) by integration by parts, as follows. For $\alpha \in (a, \pi/2)$ we have

$$S^k x^k e^{-kx} \varphi_k^-(x) = S^k x^k e^{-kx} \int_0^{\infty e^{i\alpha}} \Phi_k^+(p) e^{-px} dp$$
$$= \frac{S^k}{2\pi i} x^k e^{-kx} \oint_{0;c}^{\infty e^{i\alpha}} \Phi_k^+(p) e^{-px} \ln p dp = \frac{S^k e^{-kx}}{2\pi i} \oint_{0;c}^{\infty e^{i\alpha}} \left[\Phi_k^+(p) \ln p\right]^{(k)} e^{-px} dp$$
$$= \frac{S^k e^{-ku-k\nu'}}{2\pi i} \oint_{0;c}^{\infty e^{i\alpha}} \left[\Phi_k^+(p) \ln p\right]^{(k)} e^{-\nu' p} e^{-pu} dp \quad (2.190)$$

Now, since for large ν', $\left[\Phi_k^+(p) \ln p\right]^{(k)} e^{-\nu' p}$ and H_0 are in L^1 along the given contour of integration, we can take the Laplace transform $f \mapsto \int_0^\infty f(u) e^{zu} du$, where $\mathrm{Re}\, z < 0$, (*note the choice of sign*) on both sides of (2.188), using (2.190) and interchanging the (u, p) order of variables by Fubini we get

$$\oint_{1;c}^{\infty e^{\pm i\alpha}} \frac{H_0(p)}{p - z} dp = \sum_{k=1}^{N} \frac{S^k e^{-k\nu'}}{2\pi i} \oint_{0;c}^{\infty e^{i\alpha}} \frac{\left[\Phi_k^+(p) \ln p\right]^{(k)}}{p - z + k} e^{-\nu' p} dp - A_N(z) \quad (2.191)$$

We note that $a_N(u)$ is analytic for $u > 0$, and it is $O(u^{N+1} e^{-(N+1)u})$. Therefore, $A_N(z) = \int_0^\infty e^{zy} a_N(u) du$ is analytic for $\mathrm{Re}\ z < N + 1$.

We start with large negative $\mathrm{Im}\, z$. The right side of (2.191) is analytic for $\mathrm{Re}\,(z) < N + 1$, except at the points $z = k$: this is clear since $\Phi_k^+(p) \ln p$ are analytic inside the contour except for a possibly branched singularity at zero, and the contour of integration can be deformed to accommodate for the variation of z, except if z approaches k, for some $k = 1, 2, ..., N$.

To see what happens in a neighborhood of $z = k$, let

$$G_k(z) := G(z) = \oint_{0;c}^{\infty e^{i\alpha}} \frac{\left[\Phi_k^+(p) \ln p\right]^{(k)}}{p - z + k} e^{-\nu' p} dp \quad (2.192)$$

Lemma 2.193 *The function G is analytic on the first sheet of the Riemann surface of* $\ln(z - k)$.

PROOF G is manifestly analytic when z is in the exterior of the contour in (2.192). The analytic continuation of $G(z)$ for z inside the loop can be found through contour deformation, crossing z and in the process collecting a residue at $p = z - k$, see Fig. 2.3, implying

$$\frac{S^k e^{-k\nu'}}{2\pi i} G(z) = S^k e^{-\nu' z} \left[\Phi_k^+(z-k)\ln(z-k)\right]^{(k)} + \frac{S^k e^{-k\nu'}}{2\pi i} \oint_{0;c}^{\infty e^{i\alpha}} \frac{\left[\Phi_k^+(p)\ln p\right]^{(k)}}{p - z + k} e^{-\nu' p} dp \quad (2.194)$$

where the last integral is along the deformed contour in Fig. 2.3, where part of the curve is replaced by an arccircle having z **inside** the new contour, in the process collecting a residue (from the small circle in the figure) The integral along the deformed contour is manifestly analytic for z near k. ∎

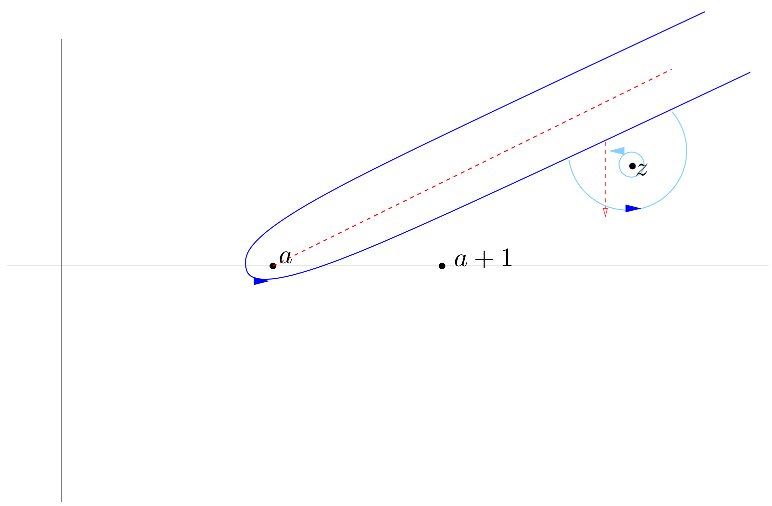

FIGURE 2.3: Deformation of contour $\int_{a;c}^{\infty e^{i\varphi}}$ to bring z in the interior of the (new) curve. The contour integral around z is replaced by its value in terms of the residue at z. The integral on the deformed contour is manifestly analytic when z is in its interior **even at points where the integrand is singular in the interior of the curve.**

We can likewise find the analytic continuation of $\oint_{1;c}^{\infty e^{i\alpha}} \frac{H_0(p)dp}{p-z}$ to z inside the contour by contour deformation and collecting a residue,

$$\oint_{1;c}^{\infty e^{i\alpha}} \frac{H_0(p)dp}{p-z} = \oint_{1;c'}^{\infty e^{i\alpha}} \frac{H_0(p)dp}{p-z} + 2\pi i H_0(z) \quad (2.195)$$

where on the right side z is now in the interior of the curve, and the right side integral is manifestly analytic near $z = k$. Equating the two sides of

(2.191) after analytic continuation inside the loop, we obtain[9] in a (ramified) neighborhood of $z=k$, as $z\uparrow\mathbb{R}^+$. We obtain

Proposition 2.196

$$H_0^-(z)=Y_0^-(z)=\frac{S^k}{2\pi i}\Big[\Phi_k^+(z-k)\ln(z-k)\Big]^{(k)}+\tilde{A}_k(z)\text{ near }z=k \quad (2.197)$$

where $\tilde{A}_k(z)$ is analytic *in a neighborhood of* $z=k$ *and along the ray* $\{z=k+te^{i\alpha}:t\in\mathbb{R}^+\}$*, see Fig. 2.3*, manifestly so, because of the analyticity of all Φ_k^+.

Thus, the only singularities of Y_0 on the first Riemann sheet are at $p=k$, and the singular structure is given in (2.199) below. It follows from (2.161) and the fact that $\varphi_k(x)=O(x^{-k-2})$ for even k and $\varphi_k(x)=O(x^{-k})$ for odd k for large x that $\Phi_k^+(p)=p^{k-\sigma}B_k(p)$ with $\sigma=1$ if k is odd and $\sigma=-1$ if k is even. Thus,

$$Y_0^-(z)=\frac{S^k}{2\pi i}\Big[(z-k)^{k-\sigma}B_k(z-k)\ln(z-k)\Big]^{(k)}+A_{k;1}(z) \quad (2.198)$$

where B_k and $A_{k;1}$ are analytic near $z=k$, or, finally,

$$Y_0^-(z)=\frac{S^k}{2\pi i}\Big[(z-k)^{1-\sigma}B_{k;2}(z-k)\ln(z-k)\Big]'+A_{k;2}(z) \quad (2.199)$$

$B_{k;2}$ and $A_{k;2}$ are analytic near $z=k$. In particular, Y_0^+ is analytic on the Riemann surface of a punctured neighborhood of $p=k$.

Similarly, from (2.185) we get, near $z=k$,

$$\Phi_j^-(z)=\frac{S^k}{2\pi i}\binom{j+k}{j}\Big[\Phi_{j+k}^+(z-k)\ln(z-k)\Big]^{(k)}+A_{k;j}(z) \quad (2.200)$$

with $A_{k;j}(z)$ some functions analytic for $|z-k|<1$. They are also analytic when $z-k$ is inside the whole contour of integration of Φ_k.

Note 2.201 In fact by rotating further z and crossing the other side of the integration contour, the new residue collected cancels the old one. Thus, we have the following proposition:

Proposition 2.202 Y_0 is analytic on the universal covering of $\mathbb{C}\setminus\mathbb{Z}^+$.

[9]We quote here results for $Y_0^-(z)$ since we approach $z=\mathbb{R}^+$ from $\operatorname{Im} z<0$. Obviously similar results can be found for $Y_0^+(z)$ by approaching $\mathbb{R}^+$ from above.

Remark 2.203 In the model differential equation, the transseries involved powers of $x^{\beta}e^{-x}$ for integer β. This is generally not the case. When β is not an integer, we can get similar results without the introduction of log in (2.190) in the following way. Assume $\mathrm{Re}\, k\beta < 0$ (if not, then $\mathrm{Re}\, a > 0$ where $a = -N + k\beta$ for some $N \in \mathbb{N}$ and we write $x^{k\beta}\mathcal{L}F = x^N const.x^N \mathcal{L}[p^{-a-1} * F]$). Then, $\oint_{0;c}^{\infty e^{i\varphi}} (p^{-a-1} * F(p))e^{-px}dp$ is a a constant nonzero multiple of $\int_0^{\infty e^{i\alpha}} F(p)dp$.

Note 2.204 We see that the formal series $\tilde{y}_0$ generates, at least in principle, the full transseries and the one-parameter family of small solutions of (2.140). Indeed

$$e^{-kx}x^k\varphi_k^-(x) = x^k e^{-kx}\oint_{0;c}^{\infty e^{i\alpha}} \Phi_k^+(p)e^{-px}\ln p\ dp$$
$$= \frac{1}{2\pi i}\oint_{k;c}^{\infty e^{i\alpha}} \left[\Phi_k^+(p-k)\ln(p-k)\right]^{(k)} e^{-px}dp = S^{-k}\oint_{k;c}^{\infty e^{i\alpha}} Y_0^{-k+}(p)e^{-px}dp \tag{2.205}$$

since the analytic part of Y^{-k+} does not contribute, obviously, to a loop integral such as the one on the right side of (2.205). Therefore,

$$y^- + \sum_{k=1}^{\infty} C_-^k e^{-kx}x^k\varphi_k^-(x) = \int_0^{\infty e^{i\alpha}} Y_0^+(p)e^{-px}dp + \sum_{k=1}^{\infty} c^k \oint_{k;c}^{\infty e^{i\alpha}} Y_0^{-k+}(p)e^{-px}dp \tag{2.206}$$

where $c = C_-/S$.

Note 2.207 It would not be hard to show for (2.140) that a solution that behaves like $\tilde{y}_0$ as $x \to i\infty$ will have the asymptotic behavior $Se^{-x}(1+o(1))$ close to the negative imaginary line, in a narrow region where $x^{-1} \ll e^{-x} \ll 1$. The value of the Stokes multiplier S is crucial to completely solving the *connection problem, here:* given the behavior at $i\infty$ find the behavior at $-i\infty$. There are several very efficient ways to determine this important parameter numerically, but a closed form expression for it essentially is only known *in integrable systems* to be discussed in the next section.

Note 2.208 Thus, the whole information about the small solutions of (2.140) is contained in $\tilde{y}_0$. The singularities of Y_0 are determined also by formal analysis of $\tilde{y}_k$, which in turn can be determined from $\tilde{y}_0$ *up to the same constant* S.

2.9 Spontaneous singularities near antistokes lines: a preview

Let us return to the toy model $y' = y^2 + 1$. We pretend of course that we do not know the exact solution. If we are looking for solutions having an asymptotic expansion at ∞, we can try dominant balance. Discarding y' we get to leading order $y = \pm i$, then as usual, write $y = \pm i + s(x)$ and expect s to be small for the asymptotic iteration to work. In this case though we get $s(x) = C_{\pm} e^{\pm 2ix}$ which is of the same order of magnitude as the leading term, i, so this balance fails. It is easy to see that other balances don't work either. But we also see that one choice of sign in $y = \pm i + s(x)$ *would work* in any direction of x other than $\arg(x) = 0$, as then one of the two exponentials would be decaying. So let us first analyze the solutions in a different direction, one for which indeed $\arg x \neq 0$, say $x = it$ with $t \in \mathbb{R}$. The substitution $y(x) = ig(it)$ leads to

$$g' = g^2 - 1 \tag{2.209}$$

and with $g(t) = -1 + s(t)$ we get

$$s' = -2s + s^2 \tag{2.210}$$

and the balance is meaningful as to leading order $s = Ce^{-2t}$ and $s^2 \ll s$. We calculate the transseries by successive iterations as usual,

$$s'^{[n]} = -2s^{[n]} + {s^{[n-1]}}^2 \tag{2.211}$$

starting with $s^{[0]} = 0$, and get

$$s = -2\sum_{k=0}^{\infty}(-\xi)^{k+1};\ \xi = Ce^{-2t} \tag{2.212}$$

which is a meaningful transseries as long as ξ is small ($|\xi| < 1$, the maximal disk of analyticity of the series above). By Abel's theorem, there must exist singular points on $|\xi| = 1$, and in this case $\xi = -1$ is a singularity (the only one here).

Remark 2.213 *Note that one singularity in ξ translates into infinitely many singularity of g. Indeed, $\xi = -1$ means $t = (2k+1)\pi i, k \in \mathbb{Z}$, and singularities occur on the antistokes line, the line along which the exponential in the transseries is purely oscillatory.*

Of course, summing the series (2.212) explicitly

$$g = \frac{1-\xi}{1+\xi} \tag{2.214}$$

and reverting the changes of variables, we recover the familiar tan solution of the toy problem.

This singularity formation mechanism is very general. Take a typical transseries (say, for simplicity in one dimension):

$$\tilde{y} = \tilde{y}_0 + \sum_{k=1}^{\infty} C^k x^{k\beta} e^{-kx} \tilde{y}_k^+ \tag{2.215}$$

We remember that the $\tilde{y}_k$ are Borel summable in any direction other than $\mathbb{R}^+$, that $\mathcal{LB}\tilde{y}_k^+ = y_k^+$ are analytic in x in $S = (-\pi/2, 2\pi + \pi/2)$ for large enough $|x|$, and that $|y_k| \lesssim \mu^k$, in the Borel summed transseries representation

$$y = y_0 + \sum_{k=1}^{\infty} C^k x^{k\beta} e^{-kx} y_k^+ \tag{2.216}$$

For a linear equation, the transseries consists of y_0 and the term with $k = 1$ only, and (2.216) solves the the associated ODE equation throughout S. For a linear system of order n there would be at most n terms involving exponentials, and the conclusion would be similar.

In the nonlinear case, the convergence of the series in (2.216) is contingent on $\xi = Cx^\beta e^{-x}$ being small enough, which is the case roughly in the right half plane for large enough $|x|$.

Note also and that, after expanding each series $\tilde{y}_k$ in powers of $1/x$, the transseries is a formal function

$$\tilde{y} = \sum_{k,j} c_{kj} \xi^k x^{-j} \tag{2.217}$$

that is, $\tilde{y}$ is a formal expansion in *two* variables, $1/x$ and ξ. If we are interested in what happens when ξ is not small enough relative to powers of $1/x$; then it is natural not to expand in ξ anymore, and write

$$\tilde{y} = \sum_j F_j(\xi) x^{-j} \tag{2.218}$$

where, since ξ is of order one and is rapidly oscillating (because of the presence of e^{-x} in ξ, periodic with period $2\pi i$) the variables ξ and x are practically independent. We then write

$$\tilde{y}' = \xi_x \sum_j F_j'(\xi) x^{-j} - \sum_j j F_j(\xi) x^{-j-1} \tag{2.219}$$

insert in the differential equation and collect the like powers of x (since F_j do not go to zero), solving, order by order for F_j. We illustrate this on $\mathrm{P_I}$ in the next section.

2.9.1 The Painlevé equations $\mathrm{P_I}$

We analyze now a nonlinear problem– $\mathrm{P_I}$, (2.247)– in the region where solutions have poles [22]. For the analysis of forst order equations, see [23]

We use a different normalization of $\mathrm{P_I}$ so that instead of (1.210), the equation is in the form

$$y'' = 6y^2 + z \tag{2.220}$$

For an equation allowing for formal, factorially divergent, power series solutions, the normalized form is the one in which the series are Gevrey-1, see Note 2.42 on p.74. This normalization often works best in studying the general solution as well, see Note 2.400 below.

Looking for a power behavior for large z, we substitute $y = Az^b$ in (2.220), and this gives $A = \pm\sqrt{-1/6}$ and $b = \frac{1}{2}$. This balance is consistent and leads to formal power series solutions $y \sim \pm\sqrt{\frac{-z}{6}}$ for large z.

We will study the family of solutions with $y \sim +\sqrt{\frac{-z}{6}}$ as the opposite sign can be treated similarly. Their transseries can be obtained by determining first the asymptotic power series $\tilde{y}_0$ with leading order $+\sqrt{\frac{-z}{6}}$. Then by linear perturbation theory around it one finds the form of the small exponential, and notices that the exponential is determined up to one multiplicative parameter. We get the transseries solution

$$\tilde{y} = \sqrt{\frac{-z}{6}} \sum_{k=0}^{\infty} \xi^k \tilde{y}_k \tag{2.221}$$

where

$$\xi = \xi(z) = Cx^{-1/2}\mathrm{e}^{-x}; \quad \text{with} \quad x = x(z) = \frac{(-24z)^{5/4}}{30} \tag{2.222}$$

and $\tilde{y}_k$ are power series, in particular

$$\tilde{y}_0 = 1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} + \frac{7^2}{2^8 \cdot 3}\frac{1}{z^5} + \ldots + \frac{\tilde{y}_{0;k}}{(-z)^{5k/2}} + \ldots$$

To normalize the equation, cf. again Note 2.42, the new independent variable x is chosen to be such that the linearized equation around $\tilde{y}_0$ admits exponentially small correction with exponents linear in x. It is also convenient to pull out $\sqrt{\frac{-z}{6}}$ from the dependent variable. We take

$$x = \frac{(-24z)^{5/4}}{30}; \; y(z) = \sqrt{\frac{-z}{6}}\, Y(x)$$

and $\mathrm{P_I}$ becomes

$$Y''(x) - \frac{1}{2}Y^2(x) + \frac{1}{2} = -\frac{1}{x}Y'(x) + \frac{4}{25}\frac{1}{x^2}Y(x) \tag{2.223}$$

which, in fact, coincides with Boutroux's form (cf. [61]). To apply the results in [46] and [44], (2.223) needs to be further normalized and to this end we subtract the $O(1)$ and $O(x^{-1})$ terms of the asymptotic behavior of $Y(x)$ for large x. It is convenient to subtract also the $O(x^{-2})$ term (since the resulting equation becomes simpler). Then the substitution

$$Y(x) = 1 - \frac{4}{25x^2} + h(x)$$

transforms (2.223) to

$$h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \tag{2.224}$$

Finally, the results in [44] and in [46] apply to first order systems of the form

$$\mathbf{y}' + \left(\hat{\Lambda} + \frac{1}{x}\hat{B}\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y});\ \hat{\Lambda} = \mathrm{diag}(\lambda_i),\ \hat{B} = \mathrm{diag}(\beta_i) \tag{2.225}$$

(eq. (1.1) [46]) in where $g = O(x^{-2}, y^2)$, rather than to $n-$th order equations. We then write

$$\begin{pmatrix} h \\ h' \end{pmatrix}' = \begin{pmatrix} 0 \\ \frac{392}{625}x^{-4} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h \\ h' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{x} \end{pmatrix}\begin{pmatrix} h \\ h' \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}h^2 \end{pmatrix} \tag{2.226}$$

Simple algebra shows that the transformation

$$\begin{pmatrix} h \\ h' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 - \frac{1}{2x} & 1 + \frac{1}{2x} \\ -1 - \frac{1}{2x} & 1 - \frac{1}{2x} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.227}$$

brings (2.226) to the normal form (2.225).

Note 2.228 (Results from [46] and [45]) (i) By Theorem (ii) in [46] if $\mathbf{y}$ is a solution of the system (2.225) with $\mathbf{y} = o(x^{-3})$[10] for $x \to \infty,\ x \in e^{-i\varphi}\mathbb{R}$ (for some φ) then $\mathbf{y}$ has a *unique Borel summed transseries*: for some C

$$\mathbf{y}(x; C) = \sum_{k=0}^{\infty} C^k e^{-kx}(\mathcal{L}_\varphi \mathbf{Y}_k)(x) \quad \text{for} \quad x \in e^{-i\varphi}\mathbb{R},\ |x| \text{ large} \tag{2.229}$$

where $\mathbf{Y}_0 = p^3\mathbf{A}_0(p)$ $\mathbf{Y}_k(p) = p^{k/2-1}\mathbf{A}_k(p)$, with $\mathbf{A}_k(p)$ *independent of* C and analytic in $\mathbb{C} \setminus \{\pm 1, \pm 2, \ldots\}$. With $\mathbf{F}_k(p) = \int_0^p \mathbf{Y}_k(s)ds$, all $\mathbf{F}_k(|p|e^{i\varphi})$ are left and right continuous in φ at $\varphi = 0$ and $\varphi = \pi$. There exist ν and M independent of k such that $\sup_{p\in\mathbb{C}\setminus\mathbb{R}} |\mathbf{F}_k(p)e^{-|p|\nu}| \leqslant M^k$. The singularities of $Y_k(p)$ for Painlevé are of the form $(p-1)^{-1/2}$, $\ln(p-2)$, $(p-3)^{1/2}$, etc.

[10] This is the case for (2.224), where $h = O(x^{-4})$

(ii) The constant C in (2.229) depends on the direction φ: $C = C(\arg x)$ is piecewise constant; it can only change at the Stokes rays.

(iii) We have

$$\mathcal{L}_\varphi \mathbf{Y}_k := \mathbf{y}_k \sim \mathbf{c}_k x^{-\frac{k}{2}};\ k \geqslant 1; \mathbf{y}_0 = O(x^{-4}) \ \text{ for } x \to \infty, \ \ x \in e^{-i\varphi}\mathbb{R} \quad (2.230)$$

where, if $C \neq 0$, c_1 is chosen to be 1 by convention, thus fixing C.

(iv) For any $\delta > 0$ there is $b > 0$ so that for all $k \geqslant 0$ and φ in $(-\pi, 0)\cup(0,\pi)$ we have $\int_0^\infty |\mathbf{Y}_k(pe^{-i\varphi})|e^{-bp}dp < \delta^k$ (Proposition 20 in [46]).

Note 2.231 (i) Algebraically, the equation is simpler in variables (h, h') than in $\mathbf{y}$, and it is more convenient to work directly with the second order equation (1.210); the results in [46], [44] and [45] translate easily through the linear substitution (2.227) into results about h and $H := \mathcal{L}^{-1}h$. In particular, (2.233) below holds for solutions $h = o(x^{-3})$, where H_k satisfy all the analyticity properties and, up to constants, bounds satisfied by $\mathbf{Y}_k$.

The following result follows from [45]:

Lemma 2.232 *(i) Assume h solves* (2.224) *and satisfies $h(x) = o(x^{-3})$ as $x \to \infty$ with* $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *Then $h(x) \sim C_+ x^{-1/2}e^{-x}$ as $x \to +i\infty$ for some C_+.*

Furthermore, for such h there is C_- such that $h \sim C_- x^{-1/2}e^{-x}$ as $x \to -i\infty$, and there exists a unique sequence $\{c_k\}_k$ such that $h \sim \frac{392}{625x^4} + \sum_{k=5}^\infty c_k x^{-k}$ as $x \to +\infty$ where the asymptotic expansion is differentiable.

A solution h as above has the Borel summed transseries representations:

$$h(x) = \sum_{k=0}^{\infty} C_\pm^k e^{-kx}(\mathcal{L}_\varphi H_k)(x) \quad \textit{for} \ \pm\varphi \in (0, \pi/2] \qquad (2.233)$$

where $H_k(p) = p^{\frac{k}{2}-1}A_k(p)$ and A_k are analytic in $\mathbb{C}_r := \mathbb{C} \setminus \{\pm 1, \pm 2, \ldots\}$. The functions H_k satisfy bounds of the type in Notes 2.231 and 2.228(iv).

For $\varphi \in (0, \pi)$, the functions $(\mathcal{L}_\varphi H_k)(x)$ are analytic in a sector $(-\pi/2, 3\pi/2)$. However, the exponentials e^{-kx} blow up in the left half plane. The transseries expansion (2.233) cannot hold in the left half plane, and close to the direction where its asymptoticity fails we need to match it to a different expansion. As usual, we first identify the critical quantity, call it ξ, responsible for loss of asymptoticity. In this case, it is $\xi = Ce^{-x}x^{-1/2}$, where we took into account the dominant behavior of $\mathbf{Y}$, (2.230) or equivalently the behavior of $\mathcal{L}_\varphi H_k$.

We then re-expand $\mathbf{Y}$ as a function series in powers of $1/x$ and coefficients depending on ξ. This is similar to a two scale expansion, except that there is no "external" small parameter ε, its role being played by x^{-1}.

Given h as in (2.229), there is a unique constant C_+ with the following properties. The leading behavior of h for large $|x|$ with $\arg x$ close to $\pi/2$ is

$$h \sim H_0(\xi) + \frac{H_1(\xi)}{x} + \frac{H_2(\xi)}{x^2} + \cdots \quad (x \to i\infty \text{ with } |\xi - 12| > \varepsilon,\ |\xi| < M) \qquad (2.234)$$

Formally, these functions are simply obtained as in a two scale expansion, thinking of x and ξ as being practically independent variables. Substituting (2.234) in (2.224) the equation for H_0 is the leading order one, the coefficient of $1/x^0$,

$$\xi^2 H_0'' + \xi H_0' - H_0 - \frac{1}{2} H_0^2 = 0; \qquad H_0(\xi) \sim \xi \text{ as } \xi \to 0 \tag{2.235}$$

The solution to (2.235) for which (2.234) matches to (2.229) in a complex x-region near $x = i\mathbb{R}^+$ where $\xi^j \left(Cx^{-1/2}e^{-x}\right)^j \gg \frac{1}{x}$ for $j \in \mathbb{N}$, corresponds to $H_0(\xi) \to \xi$. After introducing $\eta = \log \xi$ as a variable in (2.235)), multiplying the resulting equation by $\frac{d}{d\eta} H_0$, and integrating in η, using asymptotic condition as $\xi \to 0$, the resulting separable first order equation can be solved explicitly:

$$H_0(\xi) = \frac{144\xi}{(\xi - 12)^2} \tag{2.236}$$

Similarly, the coefficient of $1/x$ gives rise to

$$\xi^2 H_1'' + \xi H_1' - (1 + H_0) H_1 = \xi H_0' - \frac{1}{2} H_0^2 - H_0 \tag{2.237}$$

Since the transseries for $h(x)$ is in the form

$$h(x) = \tilde{h}_0(x) + Cx^{-1/2}e^{-x}\left(1 - \frac{1}{8x} + ..\right) + \left(Cx^{-1/2}e^{-x}\right)^2 \left[\frac{1}{6} + O\left(\frac{1}{x}\right]\right) + .. \tag{2.238}$$

where

$$\tilde{h}_0(x) = -\frac{392}{625x^4} - \frac{6272}{625x^6} + .., \tag{2.239}$$

it follows that matching requires that as $\xi \to 0$, $H_1(\xi) \sim -\frac{\xi}{8} + O(\xi^2)$. Generally for $n \geqslant 4$, $H_n(\xi) = O(1)$ as $\xi \to 0$ sinec matching involves appropriate terms from $\tilde{h}_0(x)$ as well, which is not present for $n < 4$. With the matching condition, solution to (2.237) is given by

$$H_1(\xi) = \frac{210\xi(\xi + 12)}{(\xi - 12)^3} - \frac{\xi(138240 - 180\xi^2 + \xi^3)}{60(\xi - 12)^3} \tag{2.240}$$

and generally one can conclude from induction that

$$H_n(\xi) = \frac{P_n(\xi)}{(\xi - 12)^{n+2}} \tag{2.241}$$

with P_n polynomials of degree $2n + 2$.

There is an equivalent of the non-secularity condition: each H_n contains a free constant which is determined from the equation of H_{n+1} by requiring that $H_{n+1} = O(1)$ for large x

for general constants, one would get $H_{n+1} = O(x)$, $H_{n+1} = O(x^2)$ etc. undermining the asymptoticity of the series). In [45], the validity of (2.234) is proved in a general setting.

The first array of poles beyond $i\mathbb{R}^+$ is located at points $x = p_n$ near the solutions $\tilde{p}_n$ of the equation $\xi(x) = 12$ where H_0 has a pole:

$$p_n = \tilde{p}_n + o(1) = 2n\pi i - \frac{1}{2}\ln(2n\pi i) + \ln C_+ - \ln 12 + o(1), \quad (n \to \infty) \quad (2.242)$$

Rotating x further into the second quadrant, h develops successive arrays of poles separated by distances $O(\ln x)$ of each other as long as $\arg(x) = \pi/2 + o(1)$ [45].

Note 2.243 The array of poles developed near the other edge of the sector of analyticity, for $\arg(x) = -\pi/2 + o(1)$, is obtained by the conjugation symmetry.

2.10 Spontaneous singularities and the Painlevé property

In nonlinear differential equations, the solutions may be singular at points x where the equation is regular. For example, the equation

$$y' = y^2 + 1 \quad (2.244)$$

has a one parameter family of solutions $y(x) = \tan(x + C)$; each solution has infinitely many poles. Since the location of these poles depends on C, thus on the solution itself, these singularities are called *movable* or *spontaneous*. Whether these spontaneous singularities are poles or essential singularities, particularly branch points, is crucial for the integrability of the equation. Written in an implicit form, we have, with $y = y(x)$,

$$\arctan y - x = C \quad (2.245)$$

or, after converting arctan to logs, multiplying by i and exponentiating,

$$\Phi(x, y) = \frac{1 + iy}{1 - iy} e^{-2ix} = c' \quad (2.246)$$

The function Φ is called a conserved quantity. It has the property that for any solution $y(x)$ of the equation there is a constant c s.t. $\Phi(x, y(x)) = c$. More generally, for an ODE of order n, conserved quantities are functions

$\Phi_i(x, \{y^{(j)}\}_{j=0,\dots,n-1})$ s.t. for each solution Φ_i is constant along the solution. The existence of n independent regular enough $-C^\infty$, meromorphic etc., depending on the problem– conserved quantities for general ODEs ($n/2$ suffice for Hamiltonian systems), distinguishes integrable from non-integrable systems. In the simple system above Φ is simply rational in y and entire in x. Note that Φ is real-analytic along $\mathbb{R}$ whereas the solutions are not (if C is real); of course, on $\mathbb{R}$, the form (2.245) has similar properties.

Let us analyze formally for now the isolated singularities of the Painlevé equation $\mathrm{P_I}$,

$$y'' = y^2 + z \tag{2.247}$$

A rigorous analysis of the behavior of solutions of $\mathrm{P_I}$ near an isolated singularity is done in §2.10c.

We look at the local behavior of a solution that blows up, and will find solutions that are meromorphic but not analytic. In a neighborhood of a point where y is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(z - z_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)(z-z_0)^{p-2} = A^2(z-z_0)^{2p}$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(z-z_0)^{-2}$: $y = 6(z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -z_0/10, c_3 = -1/6$ and c_4 is undetermined, thus free. Note that we have two free constants now: the position of the pole, z_0, and c_4, consistent with the fact that the equation is second order. We expect no further free constants in the Taylor series, and this, indeed, can be checked without difficulty.

2.10a The Painlevé property

To address the question whether nonlinear equations can define new functions, Fuchs had the idea that a crucial criterion now known as the *Painlevé property* (PP), is the absence of movable (meaning their position is solution-dependent) essential singularities, primarily branch-points, see [32]. First order equations were classified with respect to the PP by Fuchs, Briot and Bouquet, and Painlevé by 1888, and it was concluded that they give rise to no new functions. Painlevé took this analysis to second order, looking for all equations of the form $u'' = F(t, u, u')$, with F rational in u', algebraic in u, and analytic in t, having the PP [39, 40]. His analysis, revised and completed by Gambier and Fuchs, found some fifty types (types since some have free parameters) with this property and succeeded to solve all but six of them in terms of previously known functions. The remaining six types are now known as the Painlevé equations, and their solutions, the Painlevé *transcendents*, play a fundamental role in many areas of pure and applied mathematics.

2.10b Analysis of a modified $\mathrm{P_I}$ equation

The Painlevé property is very demanding. It is of course beyond the scope of this course to analyze a general second order equation. We can however experiment with a general analytic function $A(z) = \sum_{k=0}^{\infty} a_k z^k$ instead of z:

$$y'' = y^2 + A(z) \tag{2.248}$$

We can see that $A(0)$ can be eliminated by a shift of the independent variable, and $A'(0)$, if nonzero, can be normalized to 1. Looking for singular solutions, the dominant balance is the same as in the beginning of §2.10 and thus the local expansion starts with the same term, $6(z-z_0)^{-2}$. Substituting $y = \sum_{k\geqslant -2} c_k(z-z_0)^k$ in (2.248) and identifying the coefficients, the c_k with $k<4$ can be determined order by order. The coefficient of $(z-z_0)^2$ however does not involve c_k at all; it is

$$-\sum_{k=2}^{\infty} \frac{k(k-1)}{2} a_k z_0^k \tag{2.249}$$

Since (2.249) must vanish, the possibilities, up to linear changes of variables are $A(z) = 0$, $A(z) = 1$ and $A(z) = z$. The first two give the equation of elliptic functions and the third is $\mathrm{P_I}$ itself .

2.10b.1 The Painlevé test, further discussion

S. Kovalewsky searched for cases of the spinning top having the PP. She found a previously unknown integrable case and solved it in terms of hyperelliptic functions. Her work [36], [37], [38] was so outstanding that not only did she receive the 1886 Bordin Prize of the Paris Academy of Sciences, but the associated financial award was almost doubled.

The method pioneered by Kovalevskaya to identify integrable equations using the Painlevé property is now known as the *Painlevé test*, which she combined with Liouville's results on integrability of Hamiltonian systems. As mentioned , the Painlevé equations, as well as others with the PP were subsequently rederived from linear problems. Why this is so often the case is not completely understood.

However, at an informal level, we note that the Painlevé property guarantees some form of integrability of the equation, in the following sense. Consider for simplicity a system of equations,

$$\mathbf{y}' = \mathbf{F}(\mathbf{y}) \tag{2.250}$$

(which we can always assume autonomous by adding the equation $t' = 1$ if needed) with $\mathbf{F}$ is say, meromorphic in $\mathbb{C}^n$. Take a neighborhood $\mathcal{N}$ of a point

$(t_0, \mathbf{y}_0)$ (say, $t_0 = 0$) where $\mathbf{F}$ is analytic and consider the unique analytic solution of the ODE (2.250) with the initial condition $\mathbf{y}(0) = \mathbf{y}_0$. We can think of it as a flow $\mathbf{y}_0 \mapsto \mathbf{y} = \mathbf{\Psi}_t(\mathbf{y}_0)$ where $\mathbf{\Psi}_0(\mathbf{y}_0) = \mathbf{y}_0$, the identity map, I. By the ODE itself, $\mathbf{y}_0 = \mathbf{\Psi}_{-t}(\mathbf{y}(t))$, the solution of the same ODE. Of course, $\mathbf{y}(0)$ is a constant, and we have thus found n conserved quantities. This local property is essentially what the flowbox theorem provides. The word *local* is crucial. A system is integrable if *global* conserved quantities exist. Note again that $\mathbf{C}$ is obtained by solving, with $t \mapsto -t$, the same ODE. What prevents $\mathbf{C}$ to extend to a global conserved quantity? It is the possibility of spontaneous singularities.

Since the solutions of $\mathrm{P_I}$ however are meromorphic, with $\mathbf{y} = (y, y')$, it is always possible, in principle, to extend C_0 and C_1 by solving the equation along a path avoiding the poles. If an equation with movable poles has also some fixed singularities, the solutions still have a common Riemann surface of meromorphicity, and the fixed singularities can be avoided in a way common to all solutions.

On the contrary, movable branch-points have the potential to prevent the existence of well-behaved constants of motions for the following reason. Suppose y_0 satisfies a meromorphic (second order, for concreteness) ODE and $K(t; y, y')$ is a constant of motion. If t_0 is a branch point for y_0, then y_0 can be continued past t_0 by avoiding the singular point, or by going around t_0 any number of times before moving away. This leads to different branches $(y_0)_n$ of y_0, all of them, by simple analytic continuation arguments, solutions of the same ODE. By the definition of $K(t; y, y')$ however, we should have $K(t; (y_0)_n, (y_0)'_n) = K(t; y_0, y'_0)$ for all n, so K assumes the same value on this infinite set of solutions. We can proceed in the same way around other branch points $t_1, t_2, \ldots$ possibly returning to t_0 from time to time. Generically, we expect to generate a family of $(y_0)_{n_1,\ldots,n_j}, (y_0)'_{n_1,\ldots,n_j}$ which is dense in the phase space. This is an expectation, to be proven in specific cases. To see whether an equation falls in this generic class M. Kruskal introduced a test of nonintegrability, the *poly-Painlevé test* which measures indeed whether branching is "dense", meaning in a precise way that the analytic continuations described above are indeed dense in the space of all solutions. See, e.g., [18].

Exercise 2.251 ***Show that the solution of $y' = y^5 - 1$ has no single-valued conserved quantity in $K(t, y)$ in $\mathbb{C}^2$: solve the differential equation implicitly and show that by winding around the five logarithmic singularities of $t(y)$ in suitable ways, $K(t, y_j(t))$ takes the same value on a family of $y_j(t)$ which is dense in $\mathbb{C}$.***

2.10b.2 The list of Painlevé equations

The six classes of Painlevé transcendents, identified by Painlevé (P), Gambier (G) and R. Fuchs (F) are

$$\frac{d^2y}{dt^2} = 6y^2 + t \ (I;P) \tag{2.252}$$

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha \ (II;P) \tag{2.253}$$

$$ty\frac{d^2y}{dt^2} = t\left(\frac{dy}{dt}\right)^2 - y\frac{dy}{dt} + \delta t + \beta y + \alpha y^3 + \gamma t y^4 \ (III;P) \tag{2.254}$$

$$y\frac{d^2y}{dt^2} = \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2 + \beta + 2(t^2-\alpha)y^2 + 4ty^3 + \tfrac{3}{2}y^4 \ (IV;G) \tag{2.255}$$

$$\begin{aligned}\frac{d^2y}{dt^2} &= \left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} \\ &\quad + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1} \ (V;G)\end{aligned} \tag{2.256}$$

$$\begin{aligned}\frac{d^2y}{dt^2} &= \tfrac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right) \ (VI;F)\end{aligned} \tag{2.257}$$

In these equations, $\alpha, \beta, \gamma, \delta$ are arbitrary parameters in $\mathbb{C}$.

Beginning in the 1980's, almost a century after their discovery, these problems were solved, using their striking relation to linear problems[11], by various methods including the powerful techniques of isomonodromic deformation and reduction to Riemann-Hilbert problems [28], [29], [34].

2.10b.3 Linearization of the Painlevé equations

The Painlevé equations are related to linear problems in a number of ways, in some broad sense equivalent: isomonodromic deformations, Lax Pairs, Riemann-Hilbert problems and others, and in view of the connection to solvable linear problems are considered themselves to be solvable. The following is one of the simplest to explain such links [27]. Consider the system of equa-

[11] Some linear problems conducive to Painlevé equations were known already at the beginning of last century. In 1905 Fuchs found a linear isomonodromic problem leading to P_{VI}.

tions:

$$\frac{\partial \mathbf{\Psi}}{\partial \lambda} = \mathbf{A}(t,\lambda)\mathbf{\Psi} \tag{2.258}$$

$$\frac{\partial \mathbf{\Psi}}{\partial t} = \mathbf{B}(t,\lambda)\mathbf{\Psi} \tag{2.259}$$

where in which $\mathbf{A}$ and $\mathbf{B}$ are matrices and (t,λ) are independent variables. Then we have a compatibility equation

$$\frac{\partial^2 \mathbf{\Psi}}{\partial t \partial \lambda} = \frac{\partial^2 \mathbf{\Psi}}{\partial \lambda \partial t} \tag{2.260}$$

or

$$\frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \mathbf{B}}{\partial \lambda} + \mathbf{AB} - \mathbf{BA} = 0 \tag{2.261}$$

The equation P_I is the compatibility condition for

$$\begin{aligned} \mathbf{A}(t,\lambda) &= (4\lambda^4 + 2y^2 + t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i(4\lambda^2 y + 2y^2 + t)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &\quad - \left(2\lambda y' + \frac{1}{2\lambda}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \mathbf{B}(t,\lambda) &= \left(\lambda + \frac{y}{\lambda}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{iy}{\lambda}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned} \tag{2.262}$$

while for P_{II}, we have

$$\begin{aligned} \mathbf{A}(t,\lambda) &= -i(4\lambda^2 + 2y^2 + t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - 2y'\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \left(4\lambda y - \frac{\alpha}{\lambda}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \mathbf{B}(t,\lambda) &= \begin{pmatrix} -i\lambda & y \\ y & i\lambda \end{pmatrix} \end{aligned} \tag{2.263}$$

2.10c Rigorous analysis of the meromorphic expansion for P_I

Substituting $y(x) = 6(x-x_0)^{-2} + \delta(x)$, with $\delta(x) = o((x-x_0)^{-2})$ and taking $x = x_0 + z$ we obtain

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{2.264}$$

To find the dominant balance we note that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (2.264) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximated by a linear equation. We then rewrite (2.264) as

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{2.265}$$

which we convert into an integral equation. The indicial equation for the Euler equation corresponding to the left side of (2.265) is $r^2 - r - 12 = 0$ with solutions $4, -3$. We get

$$\delta = \frac{D}{z^3} - \frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \quad (2.266)$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$. The occurrence of a "disallowed" freedom, $D/z^3 \gg \delta$ in this case is related to the phenomenon of negative resonances, quite common in Painlevé analysis; see [31] for a discussion. Now, C is arbitrary. To find δ formally, we would simply iterate (2.266) as usual: we first take $\delta = 0$ on the right side and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $\delta^2 = \delta_0^2$ and compute δ_1 from (2.266) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + ... \quad (2.267)$$

To prove convergence of the expansion, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z s u^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \quad (2.268)$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ε such that this is a contractive equation for u in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \varepsilon$. We conclude that δ is analytic and that y is meromorphic near $x = x_0$.

Note. The Painlevé property discussed in requires that y is globally meromorphic, and we did *not* prove this. That indeed y is globally meromorphic is in fact true, but the proof is delicate (see e.g. [1]). Generic equations fail even the local Painlevé property. For instance, for the simpler, autonomous, equation

$$f'' + f' - f^2 = 0 \quad (2.269)$$

the same analysis yields a local behavior starting with a double pole, $f \sim -6z^{-2}$. Further terms in a local *power series* expansion are:

$$f = \frac{6}{z^2} - \frac{6}{5z} - \frac{1}{50} - \frac{z}{250} - \frac{7z^2}{5000} - \frac{79}{75000}z^3 + \text{undefined} \quad (2.270)$$

that is, no coefficient of z^{-4} works. More terms have to be pulled out for a contractive mapping approach to work. We take

$$f = \frac{6}{z^2} - \frac{6}{5z} - \frac{1}{50} - \frac{z}{250} - \frac{7z^2}{5000} - \frac{79z^3}{75000} + \delta(z)$$

proceeding as above and integrating by parts the δ' term we get, after some work,

$$\delta = \frac{18z^4 \ln z}{21875} + Cz^4 + z^5 P(z) + \mathcal{N}(z, \delta(z), \delta(z)^2) \tag{2.271}$$

where $P(z)$ is a polynomial of third degree and $\mathcal{N}$ is a contractive operator in the space of functions which are $O(z^4 \ln z)$. If we pull out fewer terms from f a contraction mapping argument may not work (at least not in a naive way). We see that a log term is generated in the process.

Note 2.272 Eq. (2.269) does not have the Painlevé property. The log terms generate infinitely many solutions by analytic continuation around one singular point, and suggests the equation is not integrable.

2.10d The Painlevé property for PDEs

We briefly discuss the Weiss-Tabor-Carnevale (WTC) method [83] which adapts Painlevé analysis to PDEs. Remarkably, when applied to Burgers' equation and to KdV, their method leads naturally to the solution of these equations, through the Cole-Hopf transformation and Lax pairs respectively. We base the presentation on their aforementioned paper.

Meromorphic functions of several variables are locally ratios P/Q of analytic functions, and singularities occur when the denominator vanishes,

$$Q(z_1, ..., z_n) = 0 \tag{2.273}$$

which is a manifold of complex dimension $N-1$ (in particular, singularities are not isolated anymore). The WTC test requires that in a neighborhood the manifold described by (2.273) the solution u of the given PDE be single-valued as well, that is, for some analytic functions $u_j(z_1, ..., z_n)$ we have

$$u = Q^{-m} \sum_{j=0}^{\infty} u_j Q^j; \quad \text{for some } m \in \mathbb{N} \tag{2.274}$$

Direct substitution of (2.274) into the PDE determines the compatible values of α and defines recursively the u_j, $j \geqslant 0$. In the resulting expression, we assume that $Q^{j+1} \ll Q^j$ and therefore treat it like an asymptotic series in powers of Q.

The first example is Burgers' equation

$$u_t + uu_x = \sigma u_{xx} \tag{2.275}$$

As in the case of the Painlevé test, the analysis has to be done carefully, but there is no need for rigor, as we are simply dealing with a practical criterion. Substituting (2.274) into (2.275) and using the assumption of analyticity suggests $m = 1$, since otherwise the most negative power of Q would be unbalanced. With $m = 1$, after the aforementioned substitution, the most negative power of Q is -3; the equation for its coefficient to vanish is

$$u_0 = -2\sigma Q_x \tag{2.276}$$

We now take $u = u_0 Q^{-1} + u_1$ where u_0 is given by (2.276) and require that the coefficient of Q^{-2}. This gives

$$Q_t + u_1 Q_x = \sigma Q_{xx} \tag{2.277}$$

For the coefficient of Q^{-1} we get

$$u_1 Q_{xx} + u_{1x} Q_x + Q_{xt} - \sigma Q_{xxx} = 0 \tag{2.278}$$

We note that this equation does not contain u_2; this is similar to the equation for c_4 in the case of Painlevé. The equation is either satisfied, or the expansion fails. Thus, the third order term is resonant and if (2.278) holds, then u_2 is free. On the other hand, we see that the left side of (2.278) is just the x derivative of (2.277), and thus (2.278) indeed holds:

$$\partial_x (u_1 Q_x + Q_t - \sigma Q_{xx}) = 0 \tag{2.279}$$

The general recurrence for u_j is of the form

$$(j+1)(j-2)\sigma \varphi_x^2 u_j = F\left(\{u_k\}_{k<j}; Q_t, \{\partial_x^k Q\}\right) \tag{2.280}$$

where we see two resonances, the negative one corresponding to the freedom in choosing Q and at $j = 2$ we get the identity (2.279). If we modify Burgers' equation by adding, say, $au(t,x)$ to its left side, we get instead of (2.279)

$$\partial_x (u_1 Q_x + Q_t - \sigma Q_{xx} + aQ) = 0 \tag{2.281}$$

and this, for $a \neq 0$, combined with (2.278) implies $Q_x = 0$, and (2.276) would give $u_0 = 0$, and then (2.277) gives $Q_t = 0$ and since Q has a zero, we would have $Q \equiv 0$, a contradiction. For this modified Burgers' equation, a meromorphic expansion (2.274) and the formal Painlevé property fail.

Let's return to $a = 0$; as mentioned, the equation (2.279) for u_2 is automatically satisfied, u_2 is free, and (2.280) implies that all u_j for $j \geqslant 3$ are uniquely determined. There is a meromorphic local expansion in a neighborhood of the singular manifold, and we could check that under suitable analyticity assumptions on Q it actually converges.

If we set $u_1 = u_2 = 0$ we can check that the $u_j = 0$ for $j \geqslant 3$ is consistent. With this choice, (2.277) implies

$$Q_t = \sigma Q_{xx}; \quad u(t,x) = -2\sigma Q_x / Q \tag{2.282}$$

which is the Cole-Hopf transform, see [17], [33], [84], mapping (2.275) to the heat equation and providing the closed form solution of Burgers' equation!

Choosing instead $u_1 = Q$ we get

$$Q_t + QQ_x = \sigma Q_{xx} \text{ and } u = -2\sigma Q_x/Q + Q \text{ imply } u_t + uu_x = \sigma u_{xx} \tag{2.283}$$

which is the Bäcklund transformation for Burgers', discovered by Fokas [30].

2.10d.1 KdV

The KdV equation reads

$$u_t + uu_x + \sigma u_{xxx} = 0 \tag{2.284}$$

the consistent power of Q is -2. Inserting $u = Q^{-2}\sum_{k=0}^{\infty} Q^k u_k$, the most negative power of Q is Q^{-5}; setting the coefficients of Q^{-5} and Q^{-4} to zero, we get

$$u_0 = -12\sigma Q_x^2; \quad u_1 = 12\sigma Q_{xx} \tag{2.285}$$

Setting the coefficient of Q^{-3} to zero gives

$$Q_x Q_t + Q_x^2 u_2 + 4\sigma Q_x Q_{xxx} - 3\sigma Q_{xx}^2 = 0 \tag{2.286}$$

while the similar equation for the coefficient of Q^{-2} can be rewritten as

$$Q_{tx} + u_2 Q_{xx} - u_3 Q_x^2 + \sigma Q_{xxxx} = 0 \tag{2.287}$$

while the equation for the coefficient of Q^{-1} is equivalent to

$$\partial_x(Q_{tx} + u_2 Q_{xx} - u_3 Q_x^2 + \sigma Q_{xxxx}) = 0 \tag{2.288}$$

The equation above corresponds to a resonance, as u_4 does not participate; we see that for KdV it is automatically satisfied. The resonances for the u_j are at $j = -1, 4, 6$, and the resonant equation at $j = 6$ is longer and we omit it.

Looking again for truncated series, in this case

$$u_j = 0 \text{ for all } j > 2 \tag{2.289}$$

it can be checked that u given by

$$u = -12\sigma Q_x^2/Q^2 + 12\sigma Q_{xx}/Q + u_2 = 12\sigma(\ln Q)_{xx} + u_2 \tag{2.290}$$

satisfies (2.284) if u_2 satisfies KdV:

$$u_{2t} + u_2 u_{2x} + \sigma u_{2xxx} \tag{2.291}$$

which is a Bäcklund transformation for (2.284).

With $u_3 = 0$, (2.287) becomes

$$Q_{tx} + u_2 Q_{xx} + \sigma Q_{xxxx} = 0 \tag{2.292}$$

Solving (2.286) for Q_t and differentiating with respect to t we get

$$Q_{xt} = 2VV_t = -2VV_x u_2 - V^2 u_{2x} - 8\sigma V V_{xxx} \tag{2.293}$$

where we substituted $Q_x = V^2$. Taking the ansatz

$$6\sigma V_{xx} + u_2 V = \lambda V \tag{2.294}$$

in (2.293) KdV is linearized, in a Lax pair form, to

$$\begin{aligned} &6\sigma V_{xx} + u_2 V = \lambda V \\ &2V_t + u_2 V_x + \lambda V_x + 2\sigma V_{xxx} \end{aligned} \tag{2.295}$$

and now, if u_2 and V satisfy (2.295), then u defined in (2.290) satisfies (2.284).

2.11 Gevrey classes, least term truncation and Borel summation

Let $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k}$ be a formal power series, with power-one of factorial divergence, and let f be a function asymptotic to it. The definition (1.3) provides estimates of the value of $f(x)$ for large x, within $o(x^{-N})$, $N \in \mathbb{N}$, which are, as we have seen, insufficient to determine a unique f associated to $\tilde{f}$. Simply widening the sector in which (1.3) is required cannot change this situation since, for instance, $\exp(-x^{1/m})$ is beyond all orders of $\tilde{f}$ in a sector of angle almost $m\pi$.

If, however, by truncating the power series at some suitable $N(x)$ instead of a fixed N, we can sometimes achieve exponentially good approximations in a sector of width more than π, then uniqueness is ensured, as this exercise shows:

Exercise 2.296 Assume f is analytic for $|z| > z_0$ in a sector S of opening more than π and that $|f(z)| \leqslant Ce^{-a|z|}$ $(a > 0)$ in S. Show that f is identically zero. Does the conclusion hold if $e^{-a|z|}$ is replaced by $e^{-a\sqrt{|z|}}$?

(This can be shown using Phragmèn-Lindelöf's principle. Without it, take a suitable inverse Laplace transform F of f, show that F is analytic near zero and $F^{(n)}(0) = 0$ and use Proposition 1.56).

This leads us to the notion of Gevrey asymptotics.

Gevrey asymptotics.

$$\tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k}, \quad x \to \infty$$

is by definition Gevrey of order $1/m$, or Gevrey-$(1/m)$ if

$$|c_k| \leqslant C_1 C_2^k (k!)^m$$

for some C_1, C_2 [7]. There is an immediate generalization to noninteger power series.

Remark 2.297 The Gevrey order of the series $\sum_k (k!)^r x^{-k}$, where $r > 0$, is the same as that of $\sum_k (rk)! x^{-k}$. Indeed, we have, by Stirling's formula,

$$const^{-k} \leqslant (rk)!/(k!)^r \leqslant const^k$$

Taking $x = y^m$ and $\tilde{g}(y) = \tilde{f}(x)$, then $\tilde{g}$ is Gevrey-1 and we will focus on this case. Also, the corresponding classification for series in z, $z \to 0$ is obtained by taking $z = 1/x$.

*

Definition 2.298 *Let $\tilde{f}$ be Gevrey-one. A function f is* Gevrey-one asymptotic *to $\tilde{f}$ as $x \to \infty$ in a sector S if for some C_1, C_2, C_5, all $x \in S$ with $|x| > C_5$ and all N we have*

$$|f(x) - \tilde{f}^{[N]}| \leqslant C_1 C_2^{N+1} |x|^{-N-1} (N+1)! \tag{2.299}$$

i.e., if the error $f - \tilde{f}^{[N]}$ is, up to powers of a constant, of the same size as the first omitted term in $\tilde{f}$.

Note the *uniformity requirement* in N and x; this plays a crucial role.

Remark 2.300 (Exponential accuracy) *If $\tilde{f}$ is Gevrey-one and the function f is Gevrey-one asymptotic to $\tilde{f}$, then f can be approximated by $\tilde{f}$ with exponential precision in the following way. Let $N = \lfloor |x/C_2| \rfloor$ ($\lfloor \cdot \rfloor$ is the integer part); then for any $C > C_2$ we have*

$$f(x) - \tilde{f}^{[N]}(x) = O(x^{1/2} e^{-|x|/C}), \qquad (|x| \text{ large}) \tag{2.301}$$

Indeed, letting $|x| = NC_2 + \varepsilon$ with $\varepsilon \in [0,1)$ and applying Stirling's formula we have

$$N!(N+1)C_2^N |NC_2 + \varepsilon|^{-N-1} = O(x^{1/2} e^{-|x|/C_2}) \ \square$$

Note 2.302 *Optimal truncation*, or least term truncation, see e.g., [26], is in a sense a refined version of Gevrey asymptotics. It requires *optimal constants* in addition to an improved form of Rel. (2.299). In this way the imprecision of approximation of f by $\tilde{f}$ turns out to be smaller than the largest of the exponentially small corrections allowed by the problem where the series originated. Thus the cases in which uniqueness is ensured are more numerous. Often, optimal truncation means stopping near the least term of the series, and this is why this procedure is also known as *summation to the least term.*

2.11a Connection between Gevrey asymptotics and Borel summation

The following theorem goes back to Watson [59].

Theorem 2.303 *Let $\tilde{f} = \sum_{k=2}^{\infty} c_k x^{-k}$ be a Gevrey-one series and assume the function f is analytic for large x in $S_{\pi+} = \{x : |\arg(x)| < \pi/2 + \delta\}$ for some $\delta > 0$ and Gevrey-one asymptotic to $\tilde{f}$ in $S_{\pi+}$ as in (2.299). Then*

(i) f is unique.

(ii) $\mathcal{B}(\tilde{f})$ is analytic (at $p = 0$ and) in the sector $S_\delta = \{p : \arg(p) \in (-\delta, \delta)\}$, and Laplace transformable in any closed subsector.

(iii) $\tilde{f}$ is Borel summable in any direction $e^{i\theta}\mathbb{R}^+$ with $|\theta| < \delta$ and $f = \mathcal{L}\mathcal{B}_\theta \tilde{f}$.

(iv) Conversely, if $\tilde{f}$ is Borel summable along any ray in the sector S_δ given by $|\arg(x)| < \delta$, and if $\mathcal{B}\tilde{f}$ is uniformly bounded by $e^{\nu|p|}$ in any closed subsector of S_δ, then f is Gevrey-1 with respect to its asymptotic series $\tilde{f}$ in the sector $|\arg(x)| \leqslant \pi/2 + \delta$.

Note. In particular, when the assumptions of the theorem are met, Borel summability follows using only *asymptotic estimates*.

The Nevanlinna-Sokal theorem [77] weakens the conditions sufficient for Borel summability, requiring essentially estimates in a half-plane only. It was originally formulated for expansions at zero, essentially as follows:

Theorem 2.304 (Nevanlinna-Sokal) Let f be analytic in $C_R = \{z : \mathrm{Re}\,(1/z) > R^{-1}\}$ and satisfy the estimates

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z) \tag{2.305}$$

with

$$|R_N(z)| \leqslant A\sigma^N N!|z|^N \tag{2.306}$$

uniformly in N and in $z \in C_R$. Then $B(t) = \sum_{n=0}^{\infty} a_n t^n/n!$ converges for $|t| < 1/\sigma$ and has analytic continuation to the strip-like region $S_\sigma = \{t : \mathrm{dist}(t, \mathbb{R}^+) < 1/\sigma\}$, satisfying the bound

$$|B(t)| \leqslant K \exp(|t|/R) \tag{2.307}$$

uniformly in every $S_{\sigma'}$ with $\sigma' > \sigma$. Furthermore, f can be represented by the absolutely convergent integral

$$f(z) = z^{-1} \int_0^\infty e^{-t/z} B(t) dt \tag{2.308}$$

for any $z \in C_R$. Conversely, if $B(t)$ is a function analytic in $S_{\sigma''}$ $(\sigma'' < \sigma)$ and there satisfying (2.307), then the function f defined by (2.308) is analytic in C_R, and satisfies (2.305) and (2.306) [with $a_n = B^{(n)}(t)|_{t=0}$] uniformly in every $C_{R'}$ with $R' < R$.

Note 2.309 Let us point out first a possible pitfall in proving Theorem 2.303. Inverse Laplace transformability of f and analyticity away from zero in some sector follow immediately from the assumptions. What does not follow immediately is analyticity of $\mathcal{L}^{-1}f$ at zero. On the other hand, $\mathcal{B}\tilde{f}$ clearly converges to an analytic function near $p = 0$. But there is no guarantee that $\mathcal{B}\tilde{f}$ has anything to do with $\mathcal{L}^{-1}f$! This is where Gevrey estimates enter.

PROOF of Theorem 2.303

(i) Uniqueness clearly follows once we prove (ii) and (iii).

(ii) and (iii) By a simple change of variables we arrange $C_1 = C_2 = 1$. The series $\tilde{F}_1 = \mathcal{B}\tilde{f}$ is convergent for $|p| < 1$ and defines an analytic function, F_1. By Proposition 1.56, the function $F = \mathcal{L}^{-1}f$ is analytic for $|p| > 0, |\arg(p)| < \delta$, and $F(p)$ is analytic and uniformly bounded by $e^{\nu|p|}$ if $\nu > C_5$

and $|\arg(p)| < \delta_1 < \delta$. We now show that F is analytic for $|p| < 1$. (A different proof is seen in §2.11a.1.) Taking p real, $p \in [0, 1)$ we obtain in view of (2.299) that

$$\begin{aligned} |F(p) - \tilde{F}^{[N-1]}(p)| &\leqslant \int_{-i\infty+N}^{i\infty+N} d|s| \left| f(s) - \tilde{f}^{[N-1]}(s) \right| e^{\mathrm{Re}\,(ps)} \\ &\leqslant N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{|x + iN|^N} = N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2 + N^2)^{N/2}} \\ &= \frac{N! e^{pN}}{N^{N-1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{(\xi^2 + 1)^{N/2}} \leqslant C N^{3/2} e^{(p-1)N} \to 0 \text{ as } N \to \infty \end{aligned} \tag{2.310}$$

for $0 \leqslant p < 1$. Thus $\tilde{F}^{[N-1]}(p)$ converges. Furthermore, the limit, which by definition is F_1, is seen in (2.310) to equal F, the inverse Laplace transform of f on $[0, 1)$. Since F and F_1 are analytic in a neighborhood of $(0, 1)$, $F = F_1$ wherever *either* of them is analytic[12]. The domain of analyticity of F is thus, by (ii), $\{p : |p| < 1\} \cup \{p : |p| > 0, |\arg(p)| < \delta\}$.

(iv) Let $|\varphi| < \delta$. We have, by integration by parts,

$$f(x) - \tilde{f}^{[N-1]}(x) = x^{-N} \mathcal{L} \frac{d^N}{dp^N} F \tag{2.311}$$

On the other hand, F is analytic in S_a, some $a = a(\varphi)-$neighborhood of the sector $\{p : |\arg(p)| < |\varphi|\}$. Estimating Cauchy's formula on a radius-$a(\varphi)$ circle around the point p with $|\arg(p)| < |\varphi|$ we get, for some ν,

$$|F^{(N)}(p)| \leqslant N! a(\varphi)^{-N} \|F(p) e^{-\nu \mathrm{Re}\ p}\|_{\infty, S_a} e^{\nu \mathrm{Re}\ p}$$

Thus, by (2.311), with θ, $|\theta| \leqslant |\varphi|$, chosen so that $\gamma = \cos(\theta - \arg(x))$ is maximal we have

$$\begin{aligned} \left| f(x) - \tilde{f}^{[N]} \right| &= \left| x^{-N} \int_0^{\infty \exp(-i\theta)} F^{(N)}(p) e^{-px} dp \right| \\ &\leqslant const N! a^{-N} |x|^{-N} \|F e^{-\nu|p|}\|_{\infty; S_a} \int_0^{\infty} e^{-p|x|\gamma + \nu|p| + \nu a} dp \\ &= const. N! a^{-N} \gamma^{-1} |x|^{-N-1} \|F e^{-\nu\gamma|p|}\|_{\infty; S_a} \end{aligned} \tag{2.312}$$

for large enough x.

∎

[12]Here and elsewhere we identify a function with its analytic continuation.

2.11a.1 Sketch of the proof of Theorem 2.304

We can assume that $f(0) = f'(0) = 0$ since subtracting out a finite number of terms of the asymptotic expansion does not change the problem. Then, we take to $x = 1/z$ (essentially, to bring the problem to our standard setting). Let

$$F = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(1/x)e^{px}dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(x)e^{px}dx$$

We now want to show analyticity in S_σ of F. That, combined with the proof of Theorem 2.303 completes the argument.

We have

$$f(1/x) = \sum_{j=2}^{N-1}\frac{a_j}{x^j} + R_N(x)$$

and thus,

$$F(p) = \sum_{j=2}^{N-1}\frac{a_j p^{j-1}}{(j-1)!} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R_N(1/x)e^{px}dx$$

and thus

$$|F^{(N-2)}(p)| = \left|a_{N-1} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{N-2}R_N(1/x)e^{px}dx\right| \leqslant A_2\sigma^N N!; \quad p \in \mathbb{R}^+$$

and thus $|F^{(n)}(p)/n!| \leqslant A_3 n^2\sigma^n$, and the Taylor series of F at any point $p_0 \in \mathbb{R}^+$ converges, by Taylor's theorem, to F, and the radius of convergence is $1/\sigma$. The bounds at infinity follow in the usual way: let $c = R^{-1}$. Since f is analytic for $\operatorname{Re} x > c$ and is uniformly bounded for $\operatorname{Re} x \geqslant c$, we have

$$\left|\int_{c-i\infty}^{c+i\infty} f(1/x)e^{px}dx\right| \leqslant K_1 e^{cp}\int_{-\infty}^{\infty}\frac{dx}{x^2+1} \leqslant K_2 e^{cp} \tag{2.313}$$

for $p \in \mathbb{R}^+$. In the strip, the estimate follows by combining (2.313) with the local Taylor formula.

Note 2.314 As we see, *control over the analytic properties of* $\mathcal{B}\tilde{f}$ *near* $p = 0$ *is essential to Borel summability and, it turns out,* BE *summability.* Certainly, mere inverse Laplace transformability of a function with a given asymptotic series, in however large a sector, does not ensure Borel summability of its series. We know already that for any power series, for instance one that is not Gevrey of finite order, we can find a function f analytic and asymptotic to it in more than a half-plane (in fact, many functions). Then $(\mathcal{L}^{-1}f)(p)$ exists, and is analytic in an open sector in p, origin not necessarily included. Since the series is not Gevrey of finite order, it can't be Borel summable. What goes wrong is the behavior of $\mathcal{L}^{-1}f$ at zero.

2.12 Multiple scales; Adiabatic invariants

We now look at slightly perturbed equations with periodic solutions. This is a represents a class of asymptotic problems in its own right, with many applications from celestial mechanics to the study of oscillators with changing parameters. Interestingly, there are still open problems in this area, see [3]. One of the simplest problems in this category is the slowly changing length $L(t)$.We sketch the derivation of the equations without getting into details (see e.g. [4]). One writes the position in polar coordinates,

$$x(t) = L(t)\sin\theta(t);\ \ y(t) = -L(t)\cos\theta(t) \tag{2.315}$$

writes the kinetic energy $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), U = mgy(t)$, then the Lagrangian $\mathcal{L} = T - U$. The motion is described by the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

which in appropriate units to eliminate m and g reads

$$\ddot{\theta} + \frac{2\dot{L}}{L}\dot{\theta} + \frac{1}{L}\sin\theta = 0 \tag{2.316}$$

and in the approximations $\sin\theta \approx \theta$ we get

$$\ddot{\theta} + \frac{2\dot{L}}{L}\dot{\theta} + \frac{1}{L}\theta = 0 \tag{2.317}$$

The function L is slowly changing; we will take $L(t) = \varphi(\varepsilon t)$. Let's take for simplicity $L(t) = 1 + \varepsilon t$. The equation becomes

$$\ddot{\theta} + \frac{2\varepsilon}{1+\varepsilon t}\dot{\theta} + \frac{1}{1+\varepsilon t}\theta = 0 \tag{2.318}$$

2.12a The problem as a regularly perturbed equation; secular terms

Superficially, this appears to be a regularly perturbed problem. So let us see first what regular perturbation theory gives. We substitute

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots \tag{2.319}$$

in (2.318) and get

$$\theta_0'' + \theta_0 = 0 \tag{2.320}$$

with the general solution $Ae^{it} + Be^{-it}$. By linearity, it suffices to analyze the sequence of equations for θ_j when $\theta_0 = e^{\pm it}$ and by conjugation symmetry, we can reduce the analysis to the case $\theta_0 = e^{it}$ (this choice is simpler than a trigonometric function, say $\theta_0 = \cos t$ since d/dt generates $\sin t$ as well). Then, θ_1 satisfies

$$\theta_1'' + \theta_1 = (t - 2i)e^{it} \tag{2.321}$$

with the general solution

$$\theta_1 = Ae^{it} + Be^{-it} - (\tfrac{1}{4}it + \tfrac{3}{4})te^{it} = -(\tfrac{1}{4}it + \tfrac{3}{4})te^{it} \tag{2.322}$$

where without loss of generality, we took $A = B = 0$ (other combinations would simply be absorbed in a more general, $\varepsilon-$ dependent, initial condition). Next, we get

$$\theta_2'' + \theta_2 = -(\tfrac{i}{4}t^3 + \tfrac{9}{4}t^2 + \tfrac{9}{2}t + \tfrac{3}{2})e^{it} \tag{2.323}$$

Again choosing the free constants to be zero, we get

$$\theta_2 = -(\tfrac{1}{32}t^3 - \tfrac{5i}{16}t^2 - \tfrac{21}{32}t + \tfrac{3i}{32})te^{it} \tag{2.324}$$

By induction we easily see that θ_k grows with t like t^{2k+2}. For the expansion (2.319) to stay asymptotic we need $t^2\varepsilon \ll 1$ that is, $t \ll \varepsilon^{-1/2}$. But this time is too short for anything interesting to happen, since $L = 1 + \varepsilon t$ is at most of order $1 + O(\varepsilon^{1/2})$, barely away from the initial value 1, and then the expansion becomes invalid. The terms in the expansion that are not periodic in t and lead to grow are called "secular" terms (from Latin –temporal as opposed to eternal). There are various ways to eliminate them (compensate for them would be more accurate). In any case, the solution seems to grow with t, but the accuracy does not allow to determine if this is a problem with the expansion or with the solution.

2.12b The Poincaré-Lindstedt method

We consider a choice of "local" time variable adapted to the changing frequency: instead of t we use

$$\tau = t + \sum_{l=1}^{k}\sum_{k=1}^{N} a_{kl}\varepsilon^l t^k \tag{2.325}$$

and write again $\theta = \theta_0(\tau) + \varepsilon\theta_1(\tau) + \varepsilon^2\theta_2(\tau) + \cdots$. We try to determine the a_{kl} together with the θ_j so that the equation is formally satisfied, and so that θ_j contain no secular terms. The leading equation is the same:

$$\theta_0'' + \theta_0 = 0 \Rightarrow \theta_0 = e^{i\tau} \tag{2.326}$$

The equation for θ_1 is now

$$\theta_1'' + \theta_1 = -\left[-6a_{31}\tau^2 + (6ia_{31} - 4a_{21} - 1)\tau + 2i + 2ia_{21} - 2a_{11}\right]e^{i\tau} \tag{2.327}$$

The right side of (2.327) has to vanish, since $e^{i\tau}$ is *resonant* (it is a solution of the homogeneous equation) and its presence would create secular terms; this determines a_{31}, a_{21} and a_{11}. The procedure works order by order, while becoming more and more cumbersome of course. To $o(\varepsilon^2)$ we get

$$\tau = \tau_+(t) = t - (\tfrac{1}{4}t^2 + \tfrac{3i}{4}t)\varepsilon + (\tfrac{1}{8}t^3 - \tfrac{3i}{8}t^2 - \tfrac{3}{32}t)\varepsilon^2 + (-\tfrac{5}{64}t^4 + \tfrac{9}{128}t^2 + \tfrac{3i}{64}t)\varepsilon^3 \tag{2.328}$$

We can proceed similarly with a second solution of (2.318) starting with e^{-it} and get

$$\tau_-(t) = \left(1 - \frac{3i}{4}\varepsilon - \frac{3}{32}\varepsilon^2\right)t - \left(\frac{1}{4}\varepsilon - \frac{3i}{8}\varepsilon^2\right)t^2 + \frac{1}{8}\varepsilon^2 t^3 + o(\varepsilon^2) \tag{2.329}$$

and finally obtain a general (approximate) solution in the form

$$\theta = C_+ e^{i\tau_+(t)} + C_- e^{i\tau_-(t)} \tag{2.330}$$

If carried to all orders, this covers a time interval $t\varepsilon \ll 1$, where the length still changes very little, but show that there is no growth of the solution on this larger time scale.

2.12c Multi-scale analysis

We describe this briefly here (see [8] for more details), but will not elaborate since for many equations coming for applications (such as Hamiltonian systems) there are better approaches. Multi-scale analysis apparently also goes back to Poincaré and Lindstedt and is meant to make the series calculations more systematic. The problem at hand presents *two scales*: the fast one, $t \sim 1$ related to the period of cos and a scale of order $t\varepsilon = \tau \sim 1$ where it appears that the *period starts to change*. We then write, with $\tau(t) = t\varepsilon$,

$$\theta(t) \sim \sum_{k=0}^{\infty} \Theta_k(t, \tau(t))\varepsilon^k \tag{2.331}$$

treat t and τ as if they were independent variables and get a system of PDEs:

$$\partial_t^2 \Theta_0 + (1+\tau)^{-1}\Theta_0 = 0 \tag{2.332}$$

The equation for Θ_1 is

$$\partial_t^2 \Theta_1 + (1+\tau)^{-1}\Theta_1 = -2\partial_{t\tau}^2 \Theta_0 - 2(1+\tau)^{-1}\partial_t \Theta_0 \tag{2.333}$$

with general solution

$$\Theta_0 = F_+(\tau)e^{it(1+\tau)^{-1/2}} + F_-(\tau)e^{-it(1+\tau)^{-1/2}} \tag{2.334}$$

To obtain further terms, one substitutes (say) $\Theta_0 = F_+(\tau)e^{it(1+\tau)^{-1/2}}$ in the equation for Θ_1 and determines $F(\tau)$ to eliminate secular terms. This is not always possible, and the reason is that the method of multiple scales does not allow for substantial changes in oscillation frequency; they have to be addressed by changes of dependent variable if they do, see [8]. Our equation is one that would need to be modified, since it leads to the inconsistent equation

$$\frac{dF}{d\tau} + \left(\frac{1}{2(1+\tau)} - \frac{it}{2(1+\tau)^{3/2}} \right) F = 0 \tag{2.335}$$

which is inconsistent because of the presence of t while F is a function of τ only. Once more, we will be content with (2.334), which we will compare with results gotten by more systematic methods, and send to [8] on how to amend the approach.

2.12d How do the parameters of the motion change when L is doubled?

Of course none of the methods above gives us any information on this; we have some series expansions that are not known to converge, and the question is no simpler than trying to predict the behavior of an analytic function beyond the disk of analyticity, when only estimates on the Taylor coefficients are provided. A representation with wider range of validity is needed.

2.12e WKB

What we saw in (2.330) an (2.329) is that in fact if we use a power series inside an exponential instead of just a ordinary power series, the domain of validity increases. This suggests that there is an underlying WKB setting. Since the problem is linear, WKB is applicable. The variable that needs to be small in (2.330) and (2.329) for the expansions to be valid is the "slow" variable $s = t\varepsilon$. We change thus to this variable, $s = t\varepsilon$, $\theta(t) = v(s)$, and we get

$$\varepsilon^2 v'' + 2\varepsilon^2 \frac{v'}{1+s} + \frac{v}{1+s} \tag{2.336}$$

which is singularly perturbed. Performing first a Liouville transformation $v = hg$ and choosing $h = (1+s)^{-1}$ so that the first derivative term vanishes we get

$$g'' + \varepsilon^{-2} \frac{g}{1+s} = 0 \tag{2.337}$$

where, as usual, we substitute $g = e^w$, $w' = f$, and obtain

$$f = \pm i \sqrt{\varepsilon^{-2}(1+s)^{-1} + f'} \tag{2.338}$$

wherefrom, by iteration, we get

$$w = \pm 2i(1+s)^{1/2}\varepsilon^{-1} + \frac{1}{4}\ln(1+s) \pm \frac{3i\varepsilon}{16\sqrt{1+s}} + \frac{3\varepsilon^2}{64(1+s)} \tag{2.339}$$

and thus, expressing the results in terms of the original θ, we get two asymptotic solutions,

$$\theta_{\pm} = (1+s)^{-3/4} \exp\left(\pm\frac{2i}{\varepsilon}\sqrt{1+s}\right)\left(1+\frac{3i\varepsilon}{16\sqrt{1+s}}+\frac{15\varepsilon^2}{512(1+s)}+\cdots\right) \tag{2.340}$$

At this stage we note that taking

$$\theta = A(\varepsilon)\theta_+; \text{ where } A(\varepsilon) = \exp\left(-\frac{2i}{\varepsilon}-\frac{3i\varepsilon}{16}-\frac{3\varepsilon^2}{64}\right) \tag{2.341}$$

the Taylor series of θ at $\varepsilon = 0$ is

$$\theta = e^{it}\left[1-\left(\frac{i}{4}t+\frac{3}{4}\right)t\varepsilon+\left(-\frac{1}{32}t^3+\frac{5i}{16}t^2+\frac{21}{32}t-\frac{3i}{32}\right)t\varepsilon^2+\cdots\right] \tag{2.342}$$

which is what we get by combining (2.322) and (2.324). Likewise, if we take

$$\varphi = \ln(\theta_+) - \frac{2i}{\varepsilon} - \frac{3i\varepsilon}{16} - \frac{3\varepsilon^2}{64} \tag{2.343}$$

and expand φ in series we get

$$\varphi = it - \left(\frac{it}{4}+\frac{3}{4}\right)t\varepsilon + \left(\frac{it^2}{8}+\frac{3t}{8}-\frac{3i}{32}\right)t\varepsilon^2+\cdots \tag{2.344}$$

which is the expansion (2.328). We see that the regular perturbation expansion containing secular terms and the Poincaré-Lindstedt series simply correspond to various re-expansions of the WKB solutions. The range of validity of the Poincaré-Lindstedt series, if calculated to all orders is $t\varepsilon \ll 1$ since we are expanding $\ln\theta_+$ for small τ. The regular perturbation expansion has an even smaller range of validity, as we are also expanding out the exponential. The term $e^{const.t\varepsilon^2}$ cannot be expanded asymptotically in ε when $t\varepsilon^2 \not\ll 1$. Thus, the very narrow domain of validity of the regular perturbation expansion is explained by the fact that exponential behavior cannot be approximated by power series, in a region where the exponent is large. The fact that the expansion breaks down is only a sign of the mismatch in behavior type, and not an actual change in the solutions.

Note 2.345 The form (2.340) is valid for all $|t\varepsilon$ provided no singularities or turning points are crossed. Here, this simply means $t \in \mathbb{R}^+$.

2.12f The adiabatic invariant

For a pendulum of fixed length, (2.317) takes the form

$$\ddot{\theta} + L^{-1}\theta =: \ddot{\theta} + \omega^2\theta = 0 \tag{2.346}$$

Multiplying by $L^2\dot\theta$ and integrating we get

$$E = \tfrac{1}{2}L^2\dot\theta^2 + L\tfrac{1}{2}\theta^2 = const. \approx \tfrac{1}{2}v^2 + L(1-\cos\theta) \tag{2.347}$$

where E is the total energy. This is a conserved quantity, or an invariant of the motion. What happens when $L = L(t)$? Are there conserved quantities? Clearly, since the general solution is

$$\theta = C_+\theta_+ + C_-\theta_- \tag{2.348}$$

any combination of C_+ and C_- is conserved. We then solve for C_+ and C_- as a function of $\theta, \dot\theta$. Up to numerical constants of no relevance, and to leading order in ε, we have –recall that $\dot\theta$ denotes $\frac{d\theta}{dt}$

$$C_+ = \frac{1}{2}(1+s)^{3/4}(\theta - i\dot\theta\sqrt{1+s})\exp\left(-\frac{2i}{\varepsilon}\sqrt{1+s}\right)[1+O(\varepsilon)] \tag{2.349}$$

$$C_- = \frac{1}{2}(1+s)^{3/4}(\theta + i\dot\theta\sqrt{1+s})\exp\left(\frac{2i}{\varepsilon}\sqrt{1+s}\right)[1+O(\varepsilon)] \tag{2.350}$$

Certainly C_+ and C_- are constant to the order presented, see Note 2.345. Both of them oscillate rapidly on the slow, s, scale. We notice however that C_+C_- does not oscillate, and in fact any constant of motion that does not change rapidly on the s scale is a function of C_+C_-:

$$C_+C_- \sim \frac{1}{4}L(t)^{3/2}\theta^2 + \frac{1}{4}L(t)^{5/2}\dot\theta^2 = \frac{1}{2}L(t)^{1/2}E(t) = \frac{E(t)}{2\omega(t)} = \frac{E_0}{2}(1+o(1)) \tag{2.351}$$

see (2.347). The quantity $E(t)/\omega(t)$ ($\omega(t)$ being the instantaneous frequency) is an *adiabatic invariant*: it is constant *to leading order* along the solutions of (2.317). How does the pendulum behave when $L \to \lambda L$, $\lambda > 1$? Since $E = L\theta_{max}^2/2$, (2.358) implies that the amplitude θ_{max} decreases by a factor of $\lambda^{3/4}$. To find the position at a time t, one needs calculate a few orders in the asymptotic expansion of C_+ and C_- until enough accuracy is obtained to determine $\theta(t)$. This gives the solution of the main *connection problem*, relating two positions after a very long time.

Having obtained the absolute position at time t, for a small number of periods centered at t the behavior of the pendulum is then well approximated by one of length $L(t)$, energy $E_0L(t)^{-1/2}$ and initial location calculated above. We note that a numerical approach would require integration over a very long time, with a high number of digits to avoid accumulation of errors, a demanding task.

2.12g Solution for more general L

We now write the length as $L(\varepsilon t)$; with the change of variable $s = t/\varepsilon$, $\theta(t) = y(s)$,(2.317) and $'$ denoting d/ds we get

$$y'' + \frac{2L'}{L}y' + \frac{1}{\varepsilon^2 L}y' = 0 \tag{2.352}$$

which we solve for small ε by the usual WKB substitution $y(s) = \exp(w(s)/\varepsilon)$; this leads to

$$w' = \pm \frac{i}{\varepsilon\sqrt{L}} \sqrt{1 + \varepsilon^2 L \left(w'' + \frac{2L'w'}{L} \right)} \tag{2.353}$$

where we iterate in the usual way,

$$w'^{[n+1]} = \pm \frac{i}{\varepsilon\sqrt{L}} \sqrt{1 + \varepsilon^2 L \left(w''^{[n]} + \frac{2L'w'^{[n]}}{L} \right)} \tag{2.354}$$

and this yields

$$y_\pm = L^{-\frac{3}{4}} \exp\left(\pm \frac{i}{\varepsilon} \int \frac{1}{\sqrt{L(u)}} du \right) \left[1 \mp \frac{3i\varepsilon}{8} \left(\frac{L'}{L^{\frac{1}{2}}} + \frac{3}{4} \int \frac{L'(u)^2}{L(u)^{3/2}} \right) + \cdots \right] \tag{2.355}$$

Taking $y = C_+ y_+ + C_- y_-$, and solving for $C_\pm$ in terms of $y(s) = \theta(t), \varepsilon \frac{d}{ds} y = \dot{\theta}$, we obtain

$$C_+ y_+ = \frac{L^{1/2}}{2i} \dot{\theta} + \left(\frac{1}{2} + \frac{3\varepsilon L'}{8iL} + O(\varepsilon^2) \right) \theta, \tag{2.356}$$

$$C_- y_- = -\frac{L^{1/2}}{2i} \dot{\theta} + \left(\frac{1}{2} - \frac{3\varepsilon L'}{8iL} + O(\varepsilon^2) \right) \theta. \tag{2.357}$$

Since $E(t) = \frac{1}{2} L^2(\varepsilon t) \dot{\theta}^2 + \frac{1}{2} L(\varepsilon t) \theta^2$, $\omega(t) = L^{-1/2}(\varepsilon t)$, (2.356), (2.357) and (2.355) imply

$$C_- C_+ = \frac{1}{2} \frac{E(t)}{\omega(t)} + \frac{3}{20} \frac{dL^{\frac{5}{2}}}{ds} \varepsilon \, \theta\dot{\theta} + O(\varepsilon^2); \tag{2.358}$$

If we define the truncation of $C_- C_+$ to order ε to be K, *i.e.*

$$K(\theta, \dot{\theta}, t) == \frac{1}{2} \frac{E(t)}{\omega(t)} + \frac{3}{20} \frac{dL^{\frac{5}{2}}}{ds} \varepsilon \, \theta\dot{\theta}, \tag{2.359}$$

then using (2.317) we get that variation of K is of order ε^2:

$$\frac{d}{dt} K(\theta, \dot{\theta}, t) = \tfrac{3}{8} L^2 \tfrac{d}{dt} \left(L^{-\frac{1}{2}} \tfrac{dL}{dt} \right) \theta\dot{\theta} \tag{2.360}$$

2.12h Working with action-angle variables (Second choice)

The WKB method allows for a rigorous, precise and uniform asymptotic analysis. A serious limitation of the method is that is does not easily extend to nonlinear problems. We discuss, at an informal level a method that generalizes to nonlinear systems as well. We first look at the pendulum, and then apply the method to the equation P_I.

In the pendulum problem, the angle is periodic, a natural *angle variable*. It is convenient to pass to the angle as an independent variable to eliminate the oscillation. To work with slowly changing we choose one of them to be a constant of motion for the pendulum of fixed length. In that case

$$\theta\ddot{\theta} + \frac{\theta\dot{\theta}}{L} = 0 \Rightarrow \dot{\theta}^2 + L^{-1}\theta^2 = const. \tag{2.361}$$

This is, up to a multiplicative constant, the energy $2E = L^2\dot{\theta}^2 + L\theta^2$. From the point of view of calculations in the variable length case, the quantity

$$S = L\dot{\theta}^2 + \theta^2 \tag{2.362}$$

is slightly simpler. Aiming at analyzing slow variables and at eliminating to leading order the oscillatory part of the evolution, we proceed as follows. We perform a hodograph-like transformation, taking θ to be the independent variable and L and S to be the dependent ones. Formally for now, we have

$$\frac{dS}{d\theta} = -3\dot{L}L^{-\frac{1}{2}}\sqrt{S-\theta^2} \tag{2.363}$$

$$\frac{dL}{d\theta} = \dot{L}L^{\frac{1}{2}}(S-\theta^2)^{-\frac{1}{2}} \tag{2.364}$$

Then we analyze the change in L and S after a complete θ cycle.

It is simpler to describe the procedure when L is analytic. We then evolve θ on a positively oriented loop in $\mathbb{C}\setminus J$ where cut $J = (-a, a)$ contains the interval $[-\sqrt{S}, \sqrt{S}]$. Without analyticity assumptions, the procedure would be to evolve θ from $-\sqrt{S}$ to $\sqrt{S}$ and back to $-\sqrt{S}$ changing the sign of the square root every time it becomes zero, to preserve smoothness of the quantities involved. We write (2.363) in integral form, with θ_i an initial value of θ,

$$S(\theta) = S(\theta_i) - 3\int_{\theta_i}^{\theta}\dot{L}(u)L(u)^{-\frac{1}{2}}\sqrt{S(u)-u^2}\,du \tag{2.365}$$

$$L(\theta) = L(\theta_i) + \int_{\theta_i}^{\theta}\dot{L}(u)L(u)^{\frac{1}{2}}(S(u)-u^2)^{-\frac{1}{2}}\,du \tag{2.366}$$

If θ evolves for say, a loop or less, L, $\dot{L}$ and S are approximately constant, equal to their value at θ_i which we denote by $L_i, \dot{L}_i, S_i$ respectively. This can be shown in a straightforward way by noticing that the right side of (2.361) is contractive mapping in the sup norm, since $\dot{L} = \frac{d}{dt}L(\varepsilon t)$ is small. We omit the straightforward details. To leading order, the integrals can then be calculated explicitly, and the result is

$$S(\theta) - S(\theta_i) = -\frac{3}{2}\dot{L}L^{-\frac{1}{2}}u\sqrt{S-u^2}\Big|_{\theta_i}^{\theta} - \frac{3}{2}\dot{L}L^{-\frac{1}{2}}S\arcsin(u/\sqrt{S})\Big|_{\theta_i}^{\theta} + o(\dot{L})$$

$$L(\theta) - L_i = \dot{L}L^{\frac{1}{2}}\arcsin(u/\sqrt{S})\Big|_{\theta_i}^{\theta} + o(\dot{L}) \tag{2.367}$$

To eliminate to leading order the θ dependence, we can calculate the Poincaré map, that is the change of S and L after one full loop. We denote by S_j and L_j the value of these quantities after j loops. In the complement of the cut, $\sqrt{S-u^2}$ is single-valued, and thus only the arcsin changes, by 2π. We obtain the recurrence

$$S_{n+1} - S_n = -3\dot{L}L^{-\frac{1}{2}}S\pi + o(\dot{L});\ L_{n+1} - L_n = 2\dot{L}L^{\frac{1}{2}}\pi + o(\dot{L}) \tag{2.368}$$

The right side of (2.368) is small and the recurrence is to leading order approximated by a differential equation

$$\frac{dS}{dL} = -\frac{3}{2}\frac{S}{L} \Rightarrow SL^{\frac{3}{2}} = 2EL^{\frac{1}{2}} = \frac{2E}{\omega} = const. \tag{2.369}$$

thus recovering to leading order (2.358). To find the long time behavior of the pendulum with changing length, one calculates the adiabatic invariant with sufficiently many orders as a function of L, after n loops; L is known as a function of t, and after n complete loops the position θ is known (through S) and the initial velocity is zero. The missing part of the evolution, the one from loop n to loop $n+1$ is obtained from the integral system (2.365)

2.12i The Physical Pendulum

The analysis can be carried out without the linearization $\sin\theta \approx \theta$, the calculations now involving elliptic functions. We start from eq. (2.317). If L is fixed, we get a conserved quantity as before, by multiplying with $\dot{\theta}$ and integrating once:

$$\tfrac{1}{2}\dot{\theta}^2 - L^{-1}\cos\theta = H \equiv \int_0^t -\frac{2\dot{L}}{L}\dot{\theta}^2(s)ds + C \tag{2.370}$$

and we now define

$$S = \tfrac{1}{2}L\dot{\theta}^2 - \cos\theta \tag{2.371}$$

We have

$$\dot{\theta} = w;\ \dot{S} = -\tfrac{3}{2}\dot{L}w^2 \tag{2.372}$$

Once more we take θ as the independent variable and we get

$$\frac{dt}{d\theta} = \frac{\sqrt{L}}{\sqrt{2(\cos\theta + S)}} \tag{2.373}$$

$$\frac{dS}{d\theta} = -\tfrac{3}{2}\dot{L}L^{-\frac{1}{2}}\sqrt{2(\cos\theta + S)} \tag{2.374}$$

Changing variable to $u = \cos\theta$, $d\theta = -(1-u^2)^{-\frac{1}{2}}du$, switching to the slow variable L instead of t we get

$$\frac{dS}{du} = \frac{3}{\sqrt{2}}\dot{L}L^{-\frac{1}{2}}\frac{\sqrt{u+S}}{\sqrt{1-u^2}} \tag{2.375}$$

$$\frac{dL}{du} = -\frac{\dot{L}\sqrt{L}}{\sqrt{2}\sqrt{1-u^2}\sqrt{u+S}} \tag{2.376}$$

Looking as before at the Poincaré map, using the fact that L and S are approximately constant in one loop, we get for the variation from loop n to $n+1$ in u,

$$S_{n+1} - S_n = \frac{3}{\sqrt{2}} \dot{L} L^{-\frac{1}{2}} J(S) + o(\dot{L}) \tag{2.377}$$

$$L_{n+1} - L_n = -\dot{L}\sqrt{2L}\frac{dJ}{dS} + o(\dot{L}); \quad J := \oint \frac{\sqrt{u+S}}{\sqrt{1-u^2}} du \tag{2.378}$$

The choice of the loop is such that the evolution is nontrivial. If it encircles just one root of $(u+S)(1-u^2)$, the Riemann surface is two-sheeted, and after two loops nothing changes. Same conclusion if we encircle all three roots: the Riemann surface at infinity is two sheeted as well. The loop will then enclose two roots, chosen in such a way that L changes by a real number.

Once more, $\dot{L}$ is small and the evolution is approximated by a differential equation

$$\frac{dS}{dL} = -\frac{3J(S)}{2LJ'(S)} + o(1) \Rightarrow L^{\frac{3}{2}} J(S) = const.1 + o(1) \tag{2.379}$$

The adiabatic constant $L^{3/2}J$ reduces to E/ω when $S+1$ is small.

2.12i.1 Region of validity of the expansion (2.234)

This expansion is valid in the transseries region and in a domain containing one, essentially vertically aligned, array of poles. There are in fact infinitely many arrays of poles in the fourth

quadrant, and the H_n expansion above fails to be asymptotic after the first array. One can however proceed in a similar manner, finding a new $\tilde{\xi}$, involving the original ξ and $\frac{1}{x}$, to obtain a valid expansion near the second array of poles. However, very much as in a two-scale expansion, this re-expansion method does not work in a sufficiently wide area. In fact, angularly, it only covers a sector of rough width $x^{-1}\ln x$ after which no further matching with expansions of the form (2.234) is possible. Beyond this narrow region we need to do something else.

Since WKB is not suitable for nonlinear equations, we use a method similar to an adiabatic invariant representation used earlier for the nonlinear pendulum with slowly varying length. Note that for large x (2.224) is close to the autonomous Hamiltonian system

$$h'' - h - h^2/2 = 0 \tag{2.380}$$

with conserved "Hamiltonian"

$$s = {h'}^2 - h^2 - h^3/3 \tag{2.381}$$

The solutions of (2.380) are elliptic functions, doubly periodic in $\mathbb{C}$. With $w = u'$ we first rewrite equation (2.224) as a system

$$\frac{du}{dx} = w \tag{2.382}$$

$$\frac{dw}{dx} = u + \frac{u^2}{2} - \frac{w}{x} + \frac{392}{625}\frac{1}{x^4} \tag{2.383}$$

We expect the solutions to be asymptotically periodic for large x, with s to be a slow varying quantity; this is certainly the case in the region where (2.234) holds. It is then natural to take $h =: u$ as an independent angle-like variable and treat s and x as dependent variables. Then, with

$$R(u,s) = \sqrt{u^3/3 + u^2 + s}, \tag{2.384}$$

we transform (2.382), (2.383) into a system for $s(u)$ and $x(u)$:

$$\frac{ds}{du} = -\frac{2w}{x} + \frac{784}{625}\frac{1}{x^4} = -\frac{2R(u,s)}{x} + \frac{784}{625}\frac{1}{x^4} \tag{2.385}$$

$$\frac{dx}{du} = \frac{1}{w} = \frac{1}{R(u,s)} \tag{2.386}$$

Note 2.387 Given an initial condition $s(u_I), x(u_I)$ such that the right side of (2.385), (2.386) is analytic, the system $\{(2.385), (2.386)\}$ admits a locally analytic solution $x(u), s(u)$. The function $x(u)$ is analytically invertible by the inverse function theorem since $1/R \neq 0$. Using (2.385) this determines an analytic $s(x)$. From s and u, we define an analytic branch of $w = u'$. The systems $\{(2.382), (2.383)\}$ and $\{(2.385), (2.386)\}$ are then equivalent in any domain in which $u, u', s(u), x(u)$ are analytic.

To obtain a nontrivial evolution in x, we evolve u on a loop around exactly two of the three roots of R, which is the only choice that generates an infinitely-sheeted Riemann surface and avoids x returning to the same value. Indeed, a loop around one square-root branch point would give rise to only two sheets; an evolution around all three corresponds to a loop around infinity, which is also a square-root branch point of R and gives rise to two sheets

(++Ovidiu: The following claim is not easily seen to be true. I assume you purposefully avoided details++) It turns out that there are closed curves $\mathcal{C}$, see Fig. 2.4, similar to the classical *cycles* [35], such that $R(u, s(u))$ does not vanish on $\mathcal{C}$ and $x(u)$ traverses Σ from edge to edge as u travels along $\mathcal{C}$ a number N_m times. More precisely, starting with $u_0 \in \mathcal{C}$ and writing u_n instead of u_0 to denote that u has traveled n times along $\mathcal{C}$, $s_n = s(u_n)$ and $x_n = x(u_n)$, the following hold: (i) $x_0 = x(u_0)$ is close to the first array of poles near $i\mathbb{R}$, $\arg(x_0) = -\pi/2(1 + o(1))$, and $s(u_0)$ is given by (2.381), where $u_0 = h(x_0)$ and corresponding $w_0 = h'(x_0)$ determined from the asymptotic representation $h(x) \sim H_0(\xi) + \frac{H_1(\xi)}{x} + ..$ (ii) for some $N = N_m(x_0)$, x_N is close to the last array of poles, $\arg(x_N) = -\pi(1 + o(1))$. The size of $|x_n|$ is

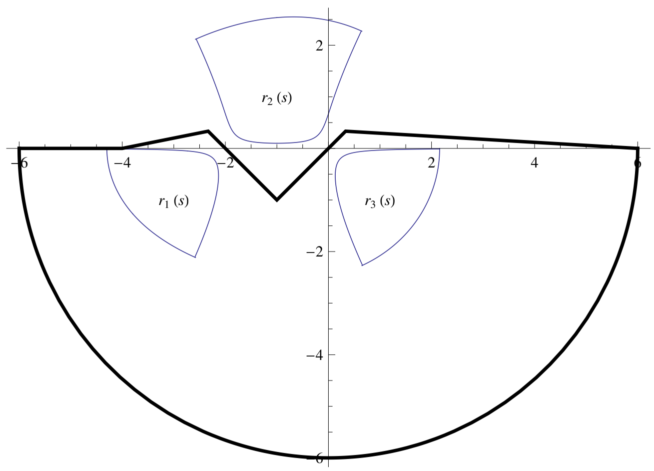

FIGURE 2.4: Regions of the roots of $u^3/3 + u^2 + s$ and the contour $\mathcal{C}$. The r_is are the regions where the three roots of R change as x traverses Σ, and $s = s(x)$ changes accordingly.

of the order $|x_0|$ for all $n \le N$. Two roots of $R(u, s_n), n = 0, 1.., N_m$ are in the interior of $\mathcal{C}$ and a third one is in its exterior. Written in integral form, (2.385) and (2.386) become

$$s(u) = s_n - 2\int_{u_n}^{u} \left(\frac{R(v, s(v))}{x(v)} - \frac{392}{625}\frac{1}{x(v)^4} \right) dv \tag{2.388}$$

$$x(u) = x_n + \int_{u_n}^{u} \frac{1}{R(v, s(v))} dv \tag{2.389}$$

where the integrals are along $\mathcal{C}$.

2.12i.2 The Poincaré map

As in the case of the pendulum, an important ingredient is the Poincaré map for (2.388), (2.389): we look at (s_{n+1}, x_{n+1}) as a function of (x_n, s_n). With the adiabatic invariants analogy in mind, the Poincaré map is used to eliminate the fast evolution. The asymptotic expansions of $s(u)$ and $x(u)$ when u is between u_n and u_{n+1} are straightforward local expansions of (2.388) and (2.389). We denote

$$J(s) = \oint_{\mathcal{C}} R(v, s)\, dv; \quad L(s) = \oint_{\mathcal{C}} \frac{dv}{R(v, s)} \tag{2.390}$$

It is easily checked that

$$J'' + \frac{1}{4}\rho(s) J = 0; \quad \text{where} \quad \rho(s) = \frac{5}{3s\,(3\,s + 4)} \tag{2.391}$$

and, since $J' = L/2$ we get

$$L'' - \frac{\rho'(s)}{\rho(s)}L' + \frac{1}{4}\rho(s)L = 0 \tag{2.392}$$

The points $s = 0$ and $s = -4/3$ are regular singular points of (2.391) (and of (2.392)) and correspond to the values of s for which the polynomial $u^3/3 + u^2 + s$ has repeated roots. Simple asymptotic analysis of (2.388) and (2.389) shows that the Poincaré map satisfies

$$s_{n+1} = s_n - \frac{2J_n}{x_n}(1 + o(1)) \qquad \text{with } J_n = J(s_n) \tag{2.393}$$

$$x_{n+1} = x_n + L_n(1 + o(1)) \qquad \text{with } L_n = L(s_n) \tag{2.394}$$

Here, and in the following *heuristic* outline, $o(1)$ stands for terms which are small for large x_n and large n. The rigorous justification of these estimates is done in [22].

2.12i.3 Solving (2.393) and (2.394); asymptotically conserved quantities

We see from (2.393) that $s_{n+1} - s_n \ll s_n$ and $x_{n+1} - x_n \ll x_n$. Here too it is natural to take a "continuum limit" and approximate $s_{n+1} - s_n$ by ds/dn and $x_{n+1} - x_n$ by dx/dn. We get

$$\frac{ds}{dx} = \frac{ds/dn}{dx/dn} = \frac{-2J(s)}{xL(s)}(1 + o(1)) = -\frac{J(s)}{xJ'(s)}(1 + o(1)) \tag{2.395}$$

which implies, by separation of variables and integration,

$$\mathcal{Q}(x, s) := xJ(s) = x_0 J(s_0)\,(1 + o(1)) \tag{2.396}$$

That is, $\mathcal{Q}$ is asymptotically a constant of motion.

In the case of the pendulum, $L(t)$ is given and one constant of motion suffices: together with the value of L, it provides two independent conditions for a second order equation; for $\mathrm{P_I}$, to fully solve the equation we need to control another quantity; a second (nonautonomous) one is obtained using (2.393) and (2.396) as follows. We write

$$\frac{1}{J^2(s)}\frac{ds}{dn} = -\frac{2}{x_0 J(s_0)}(1 + o(1)) \tag{2.397}$$

Let $\hat{J}$ be a solution of (2.391) with $\hat{J}(0) = 0$, which is independent from the J defined above. Since the first order derivative in (2.391) is missing, the Wronskian $W = \hat{J}'J - J'\hat{J} = \kappa_0$, a constant. Thus $(\hat{J}/J)' = \kappa_0/J^2$ and $1/J^2$ is a perfect derivative.

$$\mathcal{K}(s) := \kappa_0 \int_0^s \frac{ds}{J(s)^2} = \frac{\hat{J}(s)}{J(s)} \tag{2.398}$$

Integrating both sides of (2.397) from 0 to n we get

$$\mathcal{K}(s)-\mathcal{K}(s_0)=-\frac{2n}{\kappa_0\, x_0 J(s_0)}\,(1+o(1)) \Rightarrow \mathcal{K}(s)+\frac{2n}{\kappa_0\, x_0 J(s_0)}=\mathcal{K}(s_0)+o(1) \tag{2.399}$$

for $n=O(x_0)$. $\mathcal{K}$ is in fact a Schwarzian triangle function

.

Note 2.400 As we saw in the analysis leading to (2.234), the singularities of solutions having asymptotic power series behavior in the right half plane are almost periodic, with the same period as the exponential terms in the transseries. While these solutions form a lower dimensional manifold , spontaneous formation of singularities is a "local" process, and it is expected that singularities are produced with roughly the same spatial spacing for all solutions. For this reason, the normalization based on simplifying the exponentials in the transseries is a reasonable choice even in a transseries free region.

2.13 Appendix

In this book we work in $\mathbb{R}^n$ (or $\mathbb{C}$) and we will state the results in this simpler setting. See [74] for general measure spaces. The integrals we use are Lebesgue integrals. A function is in $L^1(S)$ where S is a measurable set if $\int_S |f(x)|dx<\infty$. The Lebesgue measure λ is simply the measure defined first on boxes B by $\lambda(B)=volume(B)$, and then extended to measurable sets by additivity and "continuity" (regularity). A function is measurable if its inverse image of any measurable set is measurable.

2.13a The dominated convergence theorem

Theorem 2.401 (dominated convergence) *Assume $\{f_n\}_{n\in\mathbb{N}}$ is a family of real-valued functions and that $f_n(x)\to f(x)$ for almost all x in S* [13]. *Assume further that for all n $|f_n|\leqslant g$ a.e $[\lambda]$* [13] *, where g is in $L^1(S)$. Then $f\in L^1(S)$ and*

$$\lim_{n\to\infty}\int_S f_n(s)ds \to \int_S f(s)ds \tag{2.402}$$

The Theorem also applies for complex valued functions, when real and imaginary parts have the requisite properties. Furthermore, it is easy to see that

[13] That is, except possibly for a set of measure zero; a set has zero measure if it contained in a union of boxes of arbitrarily small total measure. The notation a.e. $[\lambda]$ simply means for all x except for a zero measure set.

a similar statement holds for more general parametric convergence, that is, if n is replaced by a parameter y in a, say, metric space, under similar assumptions: $|f(y,x)| \leqslant g(x)$ for all (x,y) where g is integrable, and $f(y,x) \to f(x)$ as $y \to y_0$ a.e.$[\lambda]$.

Note 2.403 if K is a compact set in $\mathbb{R}$, then $F \in L^1_\nu(K)$, see (1.45), **iff** $F \in L^1(K)$. Indeed, in this case there exist two positive constants $c \leqslant c_2$ such that $c < e^{-\nu p} < c_2$; the rest is straightforward. Nonetheless, if $F \in L^1([a,b])$, it is still useful to work in $L^1_\nu([a,b])$ $0 \leqslant a < b \in \mathbb{R}$, since $\|F\|_{L^1_\nu([a,b])} \to 0$ as $\nu \to \infty$. Indeed, if $\nu > 0$ we have $|F(p)|e^{-\nu p} \leqslant |F(p)|$ and $|F(p)|e^{-\nu p} \to 0$ on $[a,b]$. Thus Theorem 2.13a applies and $\int_a^b F(p)e^{-xp}dp \to 0$.

2.14 Analyticity and estimates for contour integrals

2.14a Determining singularities in Borel plane from asymptotics of Laplace integrals

Lemma 2.404 *(i) Let H be analytic in the region $\{z : dist(\mathbb{R}^+, z) \in (0,c)\}$ and such that for some ν we have $\sup_{0<|a|<c} \|H(p+ia)\|_\nu < \infty$. Let*

$$h(x) = \oint_{0;c}^{\infty} e^{-px} H(p)dp \quad (\mathrm{Re}\,(x) > \nu) \tag{2.405}$$

Assume further that

$$h(x) = O(e^{-rx}) \text{ as } x \to +\infty \tag{2.406}$$

where $r < c$. Then H is analytic in $\mathbb{D}_r$.

(ii) The same holds in the following other cases:

(a) $\oint_{0;c}^{\infty}$ is replaced by $\oint_{0;c;\varphi}^{\infty}$ and H is analytic inside the curve, except perhaps along $\mathbb{R}^+ e^{i\varphi}$;

(b) $\oint_{0;c}^{\infty}$ is replaced by $\oint_{0;c;\pm\varphi}^{\infty}$ where now the contour surrounds $\mathbb{R}^+$ at distance at least c and approaches ∞ at an angle $\pm\varphi$ and H is analytic inside the curve, except perhaps along $\mathbb{R}^+$.

PROOF Note first that $h_1(x) := h(x+\nu+\varepsilon)$ is analytic in a neighborhood of $(-\varepsilon, \infty)$. This and (2.406) show that $\check{h}_1(q) = \int_0^\infty h_1(x)e^{-qx}dx$ exists, and the integrand in the definition of $\check{h}_1(q)$ satisfies the hypotheses of Fubini's theorem and

$$\check{h}_1(q) = \oint_0^\infty \frac{e^{-\nu p - \varepsilon p} H(p)}{p+q} dp =: \oint_0^\infty \frac{H_1(p)}{p+q} dp \tag{2.407}$$

Note 2.408 Eq. (2.406) implies that the Laplace transform $\check{H} := \int_0^\infty h(x)e^{-qx}dx$ exists and is analytic in the half plane $\mathrm{Re}\, q > -r$.

We start with large $\mathrm{Re}\, q$ and approach the origin. To enter the disk of radius r, q crosses the contour of integration. We bend the contour inward allowing q to approach the origin at a distance $0 < c' < c$ and then pass the contour through q, collecting the residue $2\pi i H(q)$, and then return to the original contour. Thus, for $|q| < r$ we have

$$\check{h}_1(q) = 2\pi i H_1(q) + \oint_{0;c}^{\infty} \frac{H_1(p)}{p+q}dp \tag{2.409}$$

where now q is in $\mathbb{D}_r$. By Note 2.416 $\check{h}_1(q)$ is analytic in $\mathbb{D}_r$ and so is the integral on the right side of (2.417), manifestly so due to the fact that the contour is outside $\mathbb{D}_r$. But then $H_1(q)$ and therefore $H(q)$ is analytic in $\mathbb{D}_r$.

(ii) The proof is very similar to that of (i). ∎

Exercise 2.410 *Adapt the proof above to the weaker condition*

$$h(k) = O(e^{-rk}) \text{ as } \mathbb{N} \ni k \to +\infty \tag{2.411}$$

Hint: consider instead the properties of the generating function $\sum_{k=k_0}^{\infty} h_1(k)z^k$.

Lemma 2.412 *(i) Let H be analytic in the region $\{z : dist(\mathbb{R}^+, z) \in (0, c)\}$ and such that for some ν we have $\sup_{0<|a|<c} \|H(p + ia)\|_\nu < \infty$. Let*

$$h(x) = \oint_{0;c}^{\infty} e^{-px}H(p)dp \quad (\mathrm{Re}\,(x) > \nu) \tag{2.413}$$

where we use the notation $\oint_{a;c}^{\infty}$ for an integral along a contour encircling $\mathbb{R}^+$ counteclockwise, at a distance c of it.

Assume further that

$$h(x) = O(e^{-rx}) \text{ as } x \to +\infty \tag{2.414}$$

where $r < c$. Then H is analytic in $\mathbb{D}_r$.

(ii) The same holds in the following other cases:

(a) $\oint_{0;c}^{\infty}$ is replaced by $\oint_{0;c;\varphi}^{\infty}$ where now the contour surrounds $\mathbb{R}^+$ at distance at least c and approaches ∞ at an angle φ and H is analytic inside the curve, except perhaps for $\mathbb{R}^+e^{i\varphi}$ $\pm\infty$;

(b) $\oint_{0;c}^{\infty}$ is replaced by $\oint_{0;c;\pm\varphi}^{\infty}$ where now the contour surrounds $\mathbb{R}^+$ at distance at least c and approaches ∞ at an angle $\pm\varphi$ and H is analytic inside the curve, except perhaps for $\mathbb{R}^+$.

PROOF Note first that $h_1(x) := h(x+\nu+\varepsilon)$ is analytic in a neighborhood of $(-\varepsilon, \infty)$. This and (2.419) show that $\check{h}_1(q) = \int_0^\infty h_1(x)e^{-qx}dx$ exists, and the integrand in the definition of $\check{h}_1(q)$ satisfies the hypotheses of Fubini's theorem and

$$\check{h}_1(q) = \oint_0^\infty \frac{e^{-\nu p-\varepsilon p}H(p)}{p+q}dp =: \oint_0^\infty \frac{H_1(p)}{p+q}dp \tag{2.415}$$

Note 2.416 Eq. (2.419) implies that the Laplace transform $\check{H} := \int_0^\infty h(x)e^{-qx}dx$ exists and is analytic in the half plane $\mathrm{Re}\, q > -r$.

We start with large $\mathrm{Re}\, q$ and approach the origin. To enter the disk of radius r, q crosses the contour of integration. We bend the contour inward allowing q to approach the origin at a distance $0 < c' < c$ and then pass the contour through q, collecting the residue $2\pi i H(q)$, and then return to the original contour. Thus, for $|q| < r$ we have

$$\check{h}_1(q) = 2\pi i H_1(q) + \oint_{0;c}^\infty \frac{H_1(p)}{p+q}dp \tag{2.417}$$

where now q is in $\mathbb{D}_r$. By Note 2.416 $\check{h}_1(q)$ is analytic in $\mathbb{D}_r$ and so is the integral on the right side of (2.417), manifestly so due to the fact that the contour is outside $\mathbb{D}_r$. But then $H_1(q)$ and therefore $H(q)$ is analytic in $\mathbb{D}_r$.

(ii) The proof is very similar to that of (i). ∎

Exercise 2.418 *Adapt the proof above to the weaker condition*

$$h(k) = O(e^{-rk}) \text{ as } \mathbb{N} \ni k \to +\infty \tag{2.419}$$

Hint: consider instead the properties of the generating function $\sum_{k=k_0}^\infty h_1(k)z^k$.

2.15 Appendix: Banach spaces and the contractive mapping principle

In rigorously proving asymptotic results about *solutions* of various problems, where a closed form solution does not exist or is awkward, the contractive mapping principle is a handy tool. Once an asymptotic expansion solution has been found, if we use a truncated expansion as a quasi-solution, the remainder should be small. As a result, the complete problem becomes one to which the truncation is an exact solution modulo small errors (usually involving the unknown function). Therefore, most often, asymptoticity can be shown rigorously by rewriting this latter equation as a fixed point problem of an operator which is the identity plus a correction of tiny norm. Some general guidelines on how to construct this operator are discussed in §**??**. It is desirable to go through the rigorous proof, whenever possible — this should be straightforward when the asymptotic solution has been correctly found—, one reason being that this quickly signals errors such as omitting important terms, or exiting the region of asymptoticity.

In §2.15.1 we discuss, for completeness, a few basic facts about Banach spaces. There is of course a vast literature on the subject; see e.g. [69].

2.15.1 A brief review of Banach spaces

Familiar examples of Banach spaces are the n-dimensional Euclidian vector spaces $\mathbb{R}^n$. A norm exists in a Banach space, which has the essential properties of a length: scaling, positivity except for the zero vector which has length zero and the triangle inequality (the sum of the lengths of the sides of a triangle is no less than the length of the third one). Once we have a norm, we can define limits, by reducing the notion to that in $\mathbb{R}$: $x_n \to x$ iff $\|x - x_n\| \to 0$. A normed vector space $\mathcal{B}$ is a Banach space if it is complete, that is every sequence with the property $\|x_n - x_m\| \to 0$ uniformly in n, m (a Cauchy sequence) has a limit in $\mathcal{B}$. Note that $\mathbb{R}^n$ can be thought of as the space of functions defined on the set of integers $\{1, 2, ..., n\}$. If we take a space of functions on a domain containing infinitely many points, then the Banach space is usually infinite-dimensional. An example is $L^\infty[0,1]$, the space of bounded functions on $[0,1]$ with the norm $\|f\| = \sup_{[0,1]} |f|$. A function L between two Banach spaces which is linear, $L(x+y) = Lx + Ly$, is bounded (or continuous) if $\|L\| := \sup_{\|x\|=1} \|Lx\| < \infty$. Assume $\mathcal{B}$ is a Banach space and that S is a closed subset of $\mathcal{B}$. In the *induced topology* (i.e., in the same norm), S is a complete normed space.

2.15.2 Fixed point theorem

Assume $\mathcal{M} : S \mapsto \mathcal{B}$ is a (linear or nonlinear) operator with the property that for any $x, y \in S$ we have

$$\|\mathcal{M}(y) - \mathcal{M}(x)\| \leqslant \lambda \|y - x\| \tag{2.420}$$

with $\lambda < 1$. Such operators are called **contractive**. Note that if $\mathcal{M}$ is linear, this just means that the norm of $\mathcal{M}$ is less than one.

Theorem 2.421 *Assume $\mathcal{M} : S \mapsto S$, where S is a closed subset of $\mathcal{B}$ is a contractive mapping. Then the equation*

$$x = \mathcal{M}(x) \tag{2.422}$$

has a unique solution in S.

PROOF Consider the sequence $\{x_j\}_j \in \mathbb{N}$ defined recursively by

$$\begin{aligned} x_0 &= x_0 \in S \\ x_1 &= \mathcal{M}(x_0) \\ &\cdots \\ x_{j+1} &= \mathcal{M}(x_j) \\ &\cdots \end{aligned} \tag{2.423}$$

We see that

$$\|x_{j+2} - x_{j+1}\| = \|\mathcal{M}(x_{j+1}) - \mathcal{M}(x_j)\| \leqslant \lambda \|x_{j+1} - x_j\| \leqslant \cdots \leqslant \lambda^j \|x_1 - x_0\| \tag{2.424}$$

Thus,

$$\|x_{j+p+2} - x_{j+2}\| \leqslant \left(\lambda^{j+p} + \cdots \lambda^j\right) \|x_1 - x_0\| \leqslant \frac{\lambda^j}{1-\lambda} \|x_1 - x_0\| \tag{2.425}$$

and x_j is a Cauchy sequence, and it thus converges, say to x. Since by (2.420) $\mathcal{M}$ is continuous, passing the equation for x_{j+1} in (2.423) to the limit $j \to \infty$ we get

$$x = \mathcal{M}(x) \tag{2.426}$$

that is existence of a solution of (2.422). For uniqueness, note that if x and x' are two solutions of (2.422), by subtracting their equations we get

$$\|x - x'\| = \|\mathcal{M}(x) - \mathcal{M}(x')\| \leqslant \lambda \|x - x'\| \tag{2.427}$$

implying $\|x - x'\| = 0$, since $\lambda < 1$. ☐

Note 2.428 *Note that contractivity and therefore existence of a solution of a fixed point problem depends on the norm. An adapted norm needs to be chosen for this approach to give results.*

Exercise 2.429 *Show that if L is a linear operator from the Banach space $\mathcal{B}$ into itself and $\|L\| < 1$ then $I - L$ is invertible, that is $x - Lx = y$ has always a unique solution $x \in \mathcal{B}$. "Conversely," assuming that $I - L$ is not invertible, then in whatever norm $\|\cdot\|_*$ we choose to make the same $\mathcal{B}$ a Banach space, we must have $\|L\|_* \geqslant 1$ (why?).*

2.15a Fixed points and vector valued analytic functions

A theory of analytic functions with values in a Banach space can be constructed by almost exactly following the usual construction of analytic functions. For the construction to work, we need the usual vector space operations and a topology in which these operations are continuous. A typical setting is that of a Banach algebra. A detailed presentation is found in [51] and [61], but the basic facts are simple enough for the reader to redo the necessary proofs.

2.15b Fréchet derivatives

If $\mathcal{N}$ is an operator in a Banach space, then the Fréchet derivative of $\mathcal{N}$ is a linear operator A with the property

$$\lim_{h\to 0}\frac{\|\mathcal{N}(y+h)-\mathcal{N}(y)-Ah\|}{\|h\|}=0 \tag{2.430}$$

Note 2.431 It is easy to see that , if an operator is linear affine, that is, it is of the form $\mathcal{N}(f)=f_0+Lf$ where L is a linear operator then the Fréchet derivative of $\mathcal{N}$ is L. It is also easy to check that if the derivative of an operator is <1 in a neighborhood of y, then $\mathcal{N}$ is contractive in that neighborhood.

2.16 Solving the quintic*

We seek to determine explicit formulas for the roots $\tau(p)$ of the qunitic equation

$$\frac{\tau^5}{5}+\tau=p \tag{2.432}$$

Locally near $p=0$, we can identify the roots in the following manner: There is one root for which $\tau=0$ when $p=0$ which has a Taylor expansion found through iterating the relation $\tau=p-\frac{\tau^5}{5}$,

$$\tau=\tau_1(p)=p-\frac{p^5}{5}+\frac{p^9}{5}+O(p^{13}) \tag{2.433}$$

For the four other roots, for $j=1,\cdots 4$, with $p=5^{1/4}e^{i(2j-1)\pi/4}q$, we have

$$\tau_{j+1}(p)=5^{1/4}e^{i(2j-1)\pi/4}\left(1-\frac{q}{4}-\frac{5}{32}q^2-\frac{5}{32}q^3-\frac{385}{2048}q^4+O(q^5)\right). \tag{2.434}$$

We want to have general expressions of $\tau=\tau_j(p)$ for any p in terms of hypergeometric function. This is known through other methods, but Borel transform provides a simple derivation. First, we notice on direct substitution and integration by parts

$$y(x)=\int_{\mathcal{C}}e^{-xt-t^5/5}dt\ , \tag{2.435}$$

where C is a path joining $\infty e^{i2k\pi/5}$ to $\infty e^{i2(k+1)\pi/5}$ for $k=0,\cdots,3$ provides for four independent solutions $\{y_{k+1}\}_{k=0}^{3}$ to the fourth order Airy equation

$$y^{(iv)}+xy=0 \tag{2.436}$$

We note that a change of variable in (2.435) leads to the representation

$$y = x^{1/4} \int_{\infty e^{2ik\pi/5}}^{\infty e^{2i(k+1)\pi/5}} \exp\left[-x^{5/4}\left(\tau + \tau^5/5\right)\right] d\tau \ , \tag{2.437}$$

We deform the path to lie along the Steepest descent path, which goes through saddle $\tau_k = i^k e^{i\pi/4}$ Since

$$p = \tau + \tau^{5/5} - \tau_k - \tau_k^5/5 \tag{2.438}$$

is real and monotonically increasing from 0 to ∞ on steepest descent path connecting τ_k to $\infty e^{i(2k+1)\pi/5}$, and decreasing from ∞ to 0 on the steepest descent path connecting $\infty e^{i2k\pi/5}$ to τ_k. This implies

$$y(x) = x^{1/4} \exp\left[-\frac{4}{5}\tau_k z\right] \int_0^\infty e^{-pz} \frac{d\tau^{(1)}}{dp} dp \\ - x^{1/4} \exp\left[-\frac{4}{5}\tau_k z\right] \int_0^\infty e^{-pz} \frac{d\tau^{(2)}}{dp} dp \text{ where } z = x^{5/4} \tag{2.439}$$

where $\tau = \tau^{(1)}(p)$ and $\tau = \tau^{(2)}(p)$ are the inversion of the relation (2.438) along the path connecting τ_k to $\infty e^{i2(k+1)\pi/5}$ and the one connecting τ_k to $\infty e^{i2k\pi/5}$ respectively. Using a complex path C_k that wraps around the branch point $p_k = \frac{4}{5}\tau_k$, (2.439) implies

$$y(x) = x^{1/4} \int_{C_k} e^{-pz} \frac{d\tau}{dp} dp \tag{2.440}$$

Therefore, $x^{-1/4}y(x)$ is amenable to a Borel transform. Using change of variable

$$y = x^{1/4} v(z) \ , \tag{2.441}$$

then $v(z)$ satisfies

$$\frac{d^4 v}{dz^4} + \frac{2}{z}\frac{d^3 v}{dz^3} - \frac{3}{5z^2}\frac{d^2 v}{dz^2} + \frac{3}{5z^3}\frac{dv}{dz} + \left(\frac{256}{625} - \frac{231}{625 z^4}\right) v = 0 \tag{2.442}$$

which on Borel transform and taking four derivatives gives rise to

$$\left(p^4 + \frac{256}{625}\right) V^{(iv)} + 14p^3 V''' + \frac{267}{5} p^2 V'' + 57pV' + \frac{6144}{625} V = 0 \ , \tag{2.443}$$

We note, as expected, the ODE (2.443) has singular point at $p = p_k = \frac{4}{5} i^k e^{i\pi/4}$ the path of integration C_k to obtain $v(z)$ wraps around $p = p_k$. Since

$$v(z) = \int_{C_k} e^{-pz} V(p) dp = \int_{C_k} e^{-pz} \frac{d\tau}{dp} dp \ , \tag{2.444}$$

it follows that for suitable choice of constants any root of the quintic $\tau(p) = C_0 + \int_0^p V(s)ds$. Since the integral of hypergeometric function is again a hypergeometric function, we obtain

$$\begin{aligned}\tau(p) = C_0 + C_1 p \;\; {}_4F_3\left(\left[\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\right],\left[\frac{1}{2},\frac{3}{4},\frac{5}{4}\right],-\frac{625}{256}p^4\right)\\ + C_2 p^3 \;\; {}_4F_3\left(\left[\frac{7}{10},\frac{9}{10},\frac{11}{10},\frac{13}{10}\right],\left[\frac{5}{4},\frac{3}{2},\frac{7}{4}\right],-\frac{625}{256}p^4\right)\\ + C_3 p^2 \;\; {}_4F_3\left(\left[\frac{9}{20},\frac{13}{20},\frac{17}{20},\frac{21}{20}\right],\left[\frac{3}{4},\frac{5}{4},\frac{3}{2}\right],-\frac{625}{256}p^4\right)\\ + C_4 \;\; {}_4F_3\left(\left[-\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{11}{20}\right],\left[\frac{1}{4},\frac{1}{2},\frac{3}{4}\right],-\frac{625}{256}p^4\right)\end{aligned} \tag{2.445}$$

Taylor expanding at $p = 0$, we obtain

$$\tau(p) = (C_0 + C_4) + C_1 p + C_3 p^2 + C_2 p^3 + \frac{77}{2048} C_4 p^4 + O(p^5) \tag{2.446}$$

It follows that if we want to recover the root $\tau = \tau_1(p)$ in (2.433), we must choose $(C_0, C_1, C_2, C_3, C_4) = (0, 1, 0, 0, 0)$ giving rise to

$$\tau_1(p) = p\, {}_4F_3\left(\left[\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\right],\left[\frac{1}{2},\frac{3}{4},\frac{5}{4}\right],-\frac{625}{256}p^4\right) \tag{2.447}$$

The other roots $\tau = \tau_{j+1}(p)$ can similarly be identified by determining $(C_0, C_1, C_2, C_3, C_4)$ so that the Taylor series (2.447) matches (2.434).

2.17 Appendix: The Euler-Maclaurin summation formula

Assume $f(n)$ does not increase too rapidly with n and we want to find the asymptotic behavior of

$$S(n+1) = \sum_{k=k_0}^{n} f(k) \tag{2.448}$$

for large n. We see that $S(k)$ is the solution of the difference equation

$$S(k+1) - S(k) = f(k) \tag{2.449}$$

To be more precise, assume f has a level zero transseries as $n \to \infty$. Then we write $\tilde{S}$ for the transseries of S which we seek at level zero (see p. **??**). Then

$\tilde{S}(k+1) - \tilde{S}(k) = \tilde{S}'(k) + \tilde{S}''(k)/2 + ... + \tilde{S}^{(n)}(k)/k! + ... = \tilde{S}'(k) + L\tilde{S}'(k)$ where

$$L = \sum_{j=2}^{\infty} \frac{1}{j!} \frac{d^{j-1}}{dk^{j-1}} \tag{2.450}$$

is contractive on level zero transseries (check) and thus

$$\tilde{S}'(k) = f(k) - L\tilde{S}'(k) \tag{2.451}$$

has a unique solution,

$$\tilde{S}' = \sum_{j=0}^{\infty} (-1)^j L^j f =: \frac{1}{1+L} f \tag{2.452}$$

(check that there are no transseries solutions of higher level). From the first few terms, or using successive approximations, that is writing $S' = g$ and

$$g_l = f - \frac{1}{2} g_l' - \frac{1}{6} g_l'' - \cdots \tag{2.453}$$

we get

$$\tilde{S}'(k) = f(k) - \frac{1}{2} f'(k) + \frac{1}{12} f''(k) - \frac{1}{720} f^{(4)}(k) + \cdots = \sum_{j=0}^{\infty} C_j f^{(j)}(k) \tag{2.454}$$

We note that to get the coefficient of $f^{(n)}$ correctly, using iteration, we need to keep correspondingly many terms on the right side of (2.453) and iterate $n+1$ times.

In this case, we can find the coefficients explicitly. Indeed, examining the way the C_js are obtained, it is clear that they do not depend on f. Then it suffices to look at some particular f for which the sum can be calculated explicitly; for instance $f(k) = e^{k/n}$ summed from 0 to n. By one of the definitions of the Bernoulli numbers we have

$$\frac{z}{1 - e^{-z}} = \sum_{j=0}^{\infty} (-1)^j \frac{B_j}{j!} z^j \tag{2.455}$$

Exercise 2.456 *Using these identities, determine the coefficients C_j in (2.454).*

Using Exercise 2.456 we get

$$S(k) \sim \int_{k_0}^{k} f(s)ds + \frac{1}{2} f(n) + C + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j!} f^{(2j-1)}(k) \tag{2.457}$$

Rel. (2.457) is called the Euler-Maclaurin sum formula.

Exercise 2.458 (*) *Complete the details of the calculation involving the identification of coefficients in the Euler-Maclaurin sum formula.*

Exercise 2.459 *Find for which values of $a > 0$ the series*

$$\sum_{k=1}^{\infty} \frac{e^{i\sqrt{k}}}{k^a}$$

is convergent.

Exercise 2.460 (*) *Prove the Euler-Maclaurin sum formula in the case f is C^∞ by first looking at the integral $\int_n^{n+1} f(s)ds$ and expanding f in Taylor at $s = n$. Then correct f to get a better approximation, etc.*

That (2.457) gives the correct asymptotic behavior in fairly wide generality is proved, for example, in [41].

We will prove here, under stronger assumptions, a stronger result which implies (2.457). The conditions are often met in applications, after changes of variables, as our examples showed.

Lemma 2.461 *Assume f has a Borel summable expansion at 0^+ (in applications f is often analytic at 0) and $f(z) = O(z^2)$. Then $f(\frac{1}{n}) = \int_0^\infty F(p)e^{-np}dp$, $F(p) = O(p)$ for small p and*

$$\sum_{k=n_0}^{n-1} f(1/k) = \int_0^\infty e^{-np}\frac{F(p)}{e^{-p}-1}dp - \int_0^\infty e^{-n_0 p}\frac{F(p)}{e^{-p}-1}dp \tag{2.462}$$

PROOF We seek a solution of (2.449) in the form $S = C + \int_0^\infty H(p)e^{-kp}dp$, or, in other words we inverse Laplace transform the equation (2.449). We get

$$(e^{-p} - 1)H = F \Rightarrow H(p) = \frac{F(p)}{e^{-p}-1} \tag{2.463}$$

and the conclusion follows by taking the Laplace transform which is well defined since $F(p) = O(p)$, and imposing the initial condition $S(k_0) = 0$. □

For a general analysis of the summability properties of the Euler-Maclaurin formula see [48].

2.18 Taylor coefficients of entire functions of order one

Let F be an entire function of exponential order one, meaning that $p \mapsto |F(p)|e^{-\nu|p|}$ is uniformly bounded in $\mathbb{C}$. By changes of variables we can assume w.l.o.g. that $\nu = 1$ and the uniform bound is 1. We have

$$F^k(0) = \frac{k!}{2\pi i}\oint_C \frac{F(s)}{s^{k+1}}ds \tag{2.464}$$

We take as C a circle of radius k around 0. estimating $|F| \leqslant e^{|k|}$ on this contour we obtain immediately,

$$|F^k(0)| \leqslant \frac{k!e^k}{k^k} \leqslant \sqrt{2\pi k}(1 + o(1/k)), \;\; k \to \infty \tag{2.465}$$

by Stirling's formula.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, pp. 804–806, (1972).

[2] L. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, (1979).

[3] V.I. Arnold *Geometrical Method in the Theory of Ordinary Differential Equations*, Springer-Verlag (1988).

[4] V.I. Arnold *Mathematical Methods of Classical Mechanics*, Springer, (1989).

[5] M. Aschenbrenner and L. van den Dries, Asymptotic differential algebra, in *Analyzable Functions and Applications, pp. 49–85*, Contemp. Math., 373, Amer. Math. Soc., Providence, RI, (2005).

[6] W. Balser, B.L.J. Braaksma, J-P. Ramis and Y. Sibuya, *Asymptotic Anal. 5, no. 1, pp. 27–45*, (1991).

[7] W. Balser, *From Divergent Power Series to Analytic Functions*, Springer-Verlag, (1994).

[8] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, (1978), Springer-Verlag, (1999).

[9] M.V. Berry and C.J. Howls, *Proceedings: Mathematical and Physical Sciences*, vol. 430, No. 1880, pp. 653–668, (1990).

[10] M.V. Berry and C.J. Howls, *Proc. Roy. Soc. London* Ser. A 443, no. 1917, pp. 107–126, (1993).

[11] M.V. Berry, *Proc. R. Soc. Lond.* A 422, pp. 7–21, (1989).

[12] M.V. Berry, *Proc. R. Soc. Lond.* A 430, pp. 653–668, (1990).

[13] M V. Berry, *Proc. Roy. Soc. London* Ser. A 434 no. 1891, pp. 465–472, (1991).

[14] E. Borel, *Leçons sur les Series Divérgentes*, Gauthier-Villars, Paris, (1901).

[15] B.L.J. Braaksma, *Ann. Inst. Fourier*, Grenoble, 42, 3, pp. 517–540, (1992).

[16] A. Cauchy, *Oeuvres Completes d'Augustin Cauchy*, publiées sous la direction scientifique de l'Academie de sciences et sous les auspices du m. le ministre de l'instruction publique. Gauthier-Villars, Paris, (1882–90).

[17] J. D. Cole, *On a quasilinear parabolic equation occurring in aerodynamics*, Quart. Appl. Math., Vol. 9, No. 3, pp. 225236, (1951).

[18] M.D. Kruskal and P.A. Clarkson, *Studies in Applied Mathematics*, 86, pp 87–165, (1992).

[19] E. Hille *Ordinary differential equations in the complex domain*, John Wiley & sons, 1976

[20] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, (1955).

[21] Costin O., Asymptotics and Borel summability, Chapmann & Hall, New York (2009). ++

[22] O. Costin, R.D. Costin and M. Huang *Asymptotically conserved quantities for nonlinear ODEs and connection formulae* preprint, arXiv:1205.0775.

[23] O. Costin, M. Huang and F. Fauvet, *Global Behavior of Solutions of Nonlinear ODEs: First Order Equations* Int Math Res Notices first published online October 31, 2011 doi:10.1093/imrn/rnr203 ++

[24] O. Costin and M.D. Kruskal, *Proc. R. Soc. Lond. A*, 452, pp. 1057–1085, (1996).

[25] O. Costin and M.D. Kruskal, *Comm. Pure Appl. Math.*, 58, no. 6, pp. 723–749, (2005).

[26] O. Costin and M. D. Kruskal, *Proc. R. Soc. Lond. A*, 455, pp. 1931–1956, (1999).

[27] NIST Digital Library of Mathematical Functions http://dlmf.nist.gov/

[28] P A Deift and X Zhou *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation* Ann. of Math. (2) 137 no 2 pp. 295–368 (1993).

[29] H Flashka and A C Newell *Monodromy and spectrum preserving transformations* Commun. Math. Phys. **76** pp. 65–116 (1980).

[30] A.S. Fokas *Invariants, Lie-Bäcklund Operators, and Bäcklund transformations* Ph.D. Thesis (1979).

[31] A. P. Fordy and A. Pickering, Analysing negative resonances in the Painlev test,. Phys. Lett. A 160 347-354 (1991).

[32] L Fuchs *Sur quelques équations différentielles linéaires du second ordre* C. R. Acad. Sci., Paris **141** pp. 555-558 (1905).

[33] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure and Appl. Math., Vol. 3, pp. 201230 (1950).

[34] A R Its, A S Fokas and A A Kapaev *On the asymptotic analysis of the Painlevé equations via the isomonodromy method* Nonlinearity 7 no. 5, pp. 1921–1325 (1994).

[35] A. V. Kitaev, *Elliptic asymptotics of the first and the second Painlev transcendents* Russ. Math. Surv. 49 81, (1994).

[36] S.V. Kovalevskaya *Sur le problème de la rotation d'un corps solide autour d'un point fixe*, Acta Math., **12** H.2, pp. 177–232 (1889). CRM Proc. Lecture Notes, 32, *The Kowalevski property* (Leeds, 2000), 315372, Amer. Math. Soc., Providence, RI, 2002

[37] S.V. Kovalevskaya, *Sur une propriété du système déquations differentielles qui definit la otation d'un corps solide autour d'un point fixe*, Acta Mathematica, vol. 14, pp. 81–93 (1890).

[38] S.V. Kovalevskaya *Mémoire sur un cas particulier du problème d'un corps pesant autour d'un point fixe, où l'integration s'effectue à l'aide de fonctions ultraelliptiques du temps* Mémoires présentés par divers savants à l'Academie des Sciences de l'Institut National de France, Paris 31 pp. 1–62 (1890).

[39] P Painlevé *Mémoire sur les équations différentielles dont l'integrale générale est uniforme* Bull. Soc. Math. France **28** pp 201–261 (1900).

[40] P Painlevé *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'integrale générale est uniforme* Acta Math. **25** pp. 1–85 (1902).

[41] O. Costin and R.D. Costin, SIAM J. Math. Anal. 27, no. 1, pp. 110–134, (1996).

[42] O. Costin and S. Tanveer, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24, no. 5, pp. 795–823, (2007).

[43] O. Costin, *Analyzable functions and applications*, pp. 137–175, Contemp. Math., 373, Amer. Math. Soc., Providence, RI, (2005).

[44] O. Costin, *Duke Math. J.*, 93, No. 2, (1998).

[45] O. Costin and R.D. Costin, *Inventiones Mathematicae*, 145, 3, pp. 425–485, (2001).

[46] O. Costin, *IMRN* 8, pp. 377–417, (1995).

[47] O. Costin, G. Luo and S. Tanveer, *Integral Equation representation and longer time existence solutions of 3-D Navier-Stokes, submitted.* http://www.math.ohio-state.edu/∼tanveer/ictproc.ns.5.pdf (last accessed: August 2008).

[48] O. Costin and S. Garoufalidis *Annales de L'institut Fourier*, vol. 58 no. 3, pp. 893–914, (2008).

[49] E. Delabaere and C.J. Howls, *Duke Math. J.* 112, no. 2, 199–264, (2002).

[50] R.B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, Academic Press, London-New York, (1973).

[51] N. Dunford and J.T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York, (1960).

[52] J. Écalle and F. Menous, *Publicacions Matemàtiques*, vol. 41, pp. 209–222, (1997).

[53] J. Écalle, Preprint 90-36 of Université de Paris-Sud, (1990).

[54] J. Écalle, *Fonctions Resurgentes*, Publications Mathématiques D'Orsay, (1981).

[55] J. Écalle, in *Bifurcations and periodic orbits of vector fields*, NATO ASI Series, vol. 408, (1993).

[56] G.A. Edgar, *Transseries for Beginners*, preprint, (2008). http://www.math.ohio-state.edu/∼ edgar/WG_W08/edgar/transseries.pdf (last accessed: August 2008).

[57] L. Euler, De seriebus divergentibus, *Novi Commentarii academiae scientiarum Petropolitanae* (1754/55) 1760, pp. 205–237, reprinted in *Opera Omnia Series*, I vol. 14, pp. 585–617. Available through The Euler Archive at www.EulerArchive.org (last accessed, August 2008).

[58] E. Goursat, *A Course in Mathematical Analysis, vol. 2: Functions of a Complex Variable & Differential Equations*, Dover, New York, (1959).

[59] C.G. Hardy, *Divergent Series*, Oxford, (1949).

[60] E. Hille *Ordinary Differential Equations in the Complex Domain*, Dover Publications (1997).

[61] E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, AMS Providence, R.I. (1957).

[62] J. van der Hoeven, *Transseries and Real Differential Algebra*, Lecture Notes in Mathematics 1888, Springer, New York, (2006).

[63] A.S.B. Holland, *Introduction to the Theory of Entire Functions*, Academic Press, US, (1973).

[64] I. Kaplansky, *Differential Algebra*, Hermann, Paris, (1957).

[65] T. Kawai and Y. Takei, *Advances in Mathematics*, 203, 2,pp. 636–672 (2006).

[66] D.A. Lutz, M. Miyake and R. Schäfke, *Nagoya Math. J.* 154, 1, (1999).

[67] J. Martinet and J-P. Ramis, *Annales de l'Institut Henri Poincaré (A) Physique théorique*, 54 no. 4, pp. 331–401, (1991).

[68] A.B. Olde Daalhuis, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 454, no. 1968, pp. 1–29, (1998).

[69] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York, (1972).

[70] J-P. Ramis, *Asterisque*, V. 59-60, pp. 173–204, (1978).

[71] J-P. Ramis, *C.R. Acad. Sci. Paris*, tome 301, pp. 99–102, (1985).

[72] J.F. Ritt *Differential Algebra*, American Mathematical Society, New York, (1950).

[73] W. Rudin, *Principles of Mathematical Analysis* p. 110, McGraw-Hill, New York, (1974).

[74] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, (1987).

[75] D. Sauzin, *RIMS Kokyuroku* 1493, pp. 48–117, (31/05/2006). http://arxiv.org/abs/0706.0137v1. (last accessed: August 2008).

[76] Y. Sibuya, *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient*, North-Holland, Amsterdam-New York, (1975).

[77] A. Sokal, *J. Math. Phys.*, 21 pp. 261–263, (1980).

[78] G.G. Stokes *Trans. Camb. Phil. Soc.*, 10, pp. 106–128. Reprinted in *Mathematical and Physical papers by late sir George Gabriel Stokes,* Cambridge University Press, vol. IV, pp. 77–109, (1904).

[79] P. Suppes, *Axiomatic Set Theory*, Dover, (1972).

[80] E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press, USA; 2nd edition, (1976).

[81] J. Weiss, M. Tabor and G. Carnevale, *The Painlevé property for partial differential equations* J. Math. Phys. 24, 522 (1983).

[82] W. Wasow *Asymptotic Expansions for Ordinary Differential Equations*, Interscience Publishers, New York, (1968).

[83] J. Weiss, M. Tabor, and G. Carnevale *The Painlev property for partial differential equations*, J. Math. Phys. 24, 522 (1983).

[84] G.B. Whitham, *Linear and Non-Linear waves*, Wiley, N.Y. (1974).

[85] E. Zermelo, *Mathematische Annalen*, 65, pp. 261–281, (1908).