Notation

$\mathcal{L}$	The Laplace transform, §1.2b
$\mathcal{L}^{-1}$	The inverse Laplace transform, §1.2c
$\mathcal{B}$	The Borel transform, §4.4
p	Usually the variable in Borel plane (after Borel transform)
$\tilde{f}$	A formal expression.
$H(p)$	Borel transform of $h(x)$
$\sim$	Asymptotic to, §1.1a
$\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$	The integers, positive integers, rational numbers, real numbers, complex numbers respectively.

Asymptotics and Borel summability

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Contents

Foreword

The field of asymptotics has evolved substantially in the last thirty years or so and tools have been developed to extract exact solutions from formal expansions. The literature on the subject is still relatively scattered and the bulk of it is found in relatively specialized articles. The present book is intended to provide a self-contained introduction to asymptotic analysis (with some emphasis on tools needed in exponential asymptotics, and on applications which are not part of usual asymptotics books, such as asymptotics of Taylor coefficients), and to explain basic ideas, concepts and methods of generalized Borel summability, transseries and exponential asymptotics. Transseries are first introduced rather heuristically; a complete construction is provided in the Appendix.

To provide a sense on how these latter methods are used in a systematic way, general nonlinear ODEs are near a generic irregular singular point are analyzed in detail. The analysis of difference equations, PDEs and other types of problems, only supperficially touched upon in this book, while certainly providing a number of new challenges, is not radically different in spirit. Mastering the basic techniques for ODEs should provide most of the necssary background to give a smoother access to the many existing articles on other types of problems.

The book assumes standard knowledge of Real and Complex Analysis. Most chapters are suitable for use as a textbook for graduate or advanced undergraduate students.

The book provides complete mathematical rigor, yet it is written so that at a first reading many proofs can be omitted. The included exercises range from simple to more advanced with hints provided at the end of the book. The book has a brief glossary of terms, explaining some notions that might not be part of a usual asymptotics book, and providing references for further reading.

Chapter 1

Introduction

1.1 Expansions and approximations

Classical Asymptotic Analysis studies the limiting behavior of solutions of mathematical problems, when singular points are approached. It shares with analytic function theory the goal of providing a detailed description of functions, and is distinguished from it by the fact that the main focus is on singular behavior. Asymptotic expansions provide better and better approximations as the special points are approached yet they rarely converge to a solution.

Convergent expansions are simpler to understand and we start by looking at a few simple ones to see their power and their limitations.

The local theory of analytic functions is largely a theory of convergent power series. The expansion $-\ln(1-x) = \sum_{k=0}^{\infty} x^k/k$ provides a practical way to calculate the log for small x. Likewise, to calculate $z! := \Gamma(1+z) = \int_0^{\infty} e^{-t}t^z dt$ for small z we can use

$$\ln\Gamma(1+z) = -\gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k) z^k}{k}, \ (|z|<1), \quad \zeta(k) := \sum_{j=1}^{\infty} j^{-k} \qquad (1.1)$$

and $\gamma = 0.5772..$ is the Euler constant. Thus, for small z we have

$$\begin{aligned}\Gamma(1+z) &= \exp(-\gamma z + \pi^2 z^2/12 \cdots) \\ &= \exp(-\gamma z + \sum_{k=2}^{M} (-1)^k \zeta(k) k^{-1} z^k)(1 + O(z^{M+1})) \qquad (1.2)\end{aligned}$$

where, as usual, $f = O(z^j)$ means that $|z^{-j} f| < const$ if z is small.

Exercise 1.3 *Prove formula (1.1); find a bound for* "const" *when* $|z| < 1/2$.

Near $z = 0$ we have a similar convergent expansion of $\Gamma(z)$, though now describing a singular function

$$\Gamma(z) = \Gamma(1+z)/z = z^{-1}\exp(\gamma z + \sum_{k=2}^{M} k^{-1}(-1)^k \zeta(k) z^k k(1 + O(z^{M+1})) \qquad (1.4)$$

This is a perfectly useful way of calculating $\Gamma(z)$ for small z.

Now let us look at a function near an essential singularity, *e.g.* $e^{-1/z}$ near $z = 0$. It has a convergent Laurent expansion

$$e^{-1/z} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!z^j} \tag{1.5}$$

Eq. (1.5) is fundamentally distinct from the first examples. This can be seen by trying to calculate the function from its expansion for say, $z = 10^{-10}$: (1.1) provides the desired value very quickly, while (1.5) is virtually unusable. Mathematically, we see that error bounds as in (1.1) and (1.4) do not hold for (1.5). On the contrary, we have

$$e^{-1/z} - \sum_{j=0}^{M} \frac{1}{j!z^j} \gg z^{-M}, \text{ as } z \to 0 \tag{1.6}$$

where $\gg$ means much larger than. Consequently, (1.5), though convergent, is **antiasymptotic**: the terms of the expansion get larger and larger as $z \to 0$. The exponential needs to be calculated in a different way, and there are certainly many good ways. Surprisingly perhaps, it is the exponential, together with related functions such as $\log, \sin x$ (and powers, since we prefer the notation x to $e^{\ln x}$) are the only ones that we need in order to represent many complicated functions, asymptotically. This fact has been noted already by Hardy, who wrote [12] "No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms". This reflects some important fact about the relation between asymptotic expansions and functions which will be clarified in Chapter 3.

If we need to calculate $\Gamma(x)$ as $x \to +\infty$, the Taylor about a given point, say 1, would not work, since the radius of convergence is 1. Instead we have Stirling's series,

$$\ln(\Gamma(x)) \\ = (x - 1/2)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{M} c_j x^{-2j+1} + O(x^{-2M-1}),\ x \to +\infty \tag{1.7}$$

where $2j(2j-1)c_j = B_{2j}$ and $\{B_{2j}\}_{j\in\mathbb{N}} = \{1/6, -1/30, 1/42...\}$ are Bernoulli numbers. This expansion is *asymptotic* as $x \to \infty$: successive terms get smaller and smaller. Yet, $-x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{\infty} c_j x^{-2j-1}$ cannot converge to $\ln(\Gamma(x)) - (x - 1/2)\ln x$ since $\ln(\Gamma(x))$ is singular at all $n \in -\mathbb{N}$ (why is this relevant?). In fact, S has zero radius of convergence. Nonetheless, truncating the expansion to x^{-5}, we get $\Gamma(6) \approx 120.00000086$ while $\Gamma(6) = 120$.

Unlike asymptotic expansions, convergent but antiasymptotic expansions do not contain manifest, detailed information. Of course, this is not meant to understate the value of convergent representations, nor to advocate for uncontrolled approximations.

1.1a Asymptotic expansions

An asymptotic expansion of a function f at a point t_0, usually dependent on the direction along which t_0 is approached, is a formal series[1] of simpler functions f_k,

$$\tilde{f} = \sum_{k=0}^{\infty} f_k(t) \tag{1.8}$$

in which each successive term is much smaller than its predecessors. For instance if the limiting point is t_0 approached from above along the real line this requirement is written

$$f_{k+1}(t) = o(f_k(t)) \quad \text{or} \quad f_{k+1}(t) \ll f_k(t) \tag{1.9}$$

meaning

$$\lim_{t \to t_0^+} f_{k+1}(t)/f_k(t) = 0 \tag{1.10}$$

We will often use the variable x when the limiting point is $+\infty$ and z when the limiting point is zero.

1.1b Functions asymptotic to an expansion, in the sense of Poincaré

The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits:

Definition 1.11 *A function f is asymptotic to the formal series $\tilde{f}$ as $t \to t_0^+$ if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{1.12}$$

[1] That is, there are no convergence requirements. More precisely, formal series are sequences of functions $\{f_k\}_{k \in \mathbb{N} \cup \{0\}}$, written as infinite sums, with the operations defined as for represented convergent series; see also §1.1c .

We note that condition (1.12) can then be also written in a number of equivalent ways, and this is useful in applications. We have the following simple result.

Proposition 1.13 *If $\tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k(t)$ is an asymptotic series as $t \to t_0^+$ and f is a function asymptotic to it, then the following characterizations are equivalent to each other and to (1.10). (i)*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.14}$$

where $g(t) = O(h(t))$ means $\limsup_{t \to t_0^+} |g(t)/h(t)| < \infty$.

(ii)

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = f_{N+1}(1 + o(1)) \quad (\forall N \in \mathbb{N}) \tag{1.15}$$

(iii) There is a function $a : \mathbb{N} \mapsto \mathbb{N}$ such that $a(N) \geq N$ and

$$f(t) - \sum_{k=0}^{a(N)} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.16}$$

This condition seems strictly weaker, but it is not. It allows us to use less accurate estimates of remainders, provided we can do so to all orders.

PROOF We only show (iii), the others being immediate. Let $N \in \mathbb{N}$. We have

$$\frac{1}{f_{N+1}(t)} \left(f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) \right)$$
$$= \frac{1}{f_{N+1}(t)} \left(f(t) - \sum_{k=0}^{a(N)} \tilde{f}_k(t) \right) + \sum_{j=N+1}^{a(N)} \frac{f_j(t)}{f_{N+1}} = O(1) \tag{1.17}$$

since in the last sum in (1.17) N, and thus the number of terms, is fixed, and thus the sum converges to 1 as $t \to t_0^+$. ∎

Simple examples of asymptotic expansions are

$$\sin z \sim z - \frac{z^3}{6} + ... + \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!} + \cdots \quad (|z| \to 0) \tag{1.18}$$

$$f(z) = \sin z + e^{-\frac{1}{z}} \sim z - \frac{z^3}{6} + ... + \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!} + \cdots (z \to 0^+) \tag{1.19}$$

$$e^{-1/z}\int_1^{1/z}\frac{e^t}{t}dt \sim \sum_{k=0}^{\infty} k!z^{k+1} \quad (z\to 0^+) \tag{1.20}$$

The series on the right side of (1.18) converges to $\sin z$ for any $z\in\mathbb{C}$ and it is asymptotic to it for small $|z|$. The series in the second example converges for any $z\in\mathbb{C}$ but not to f. In the third example the series is nowhere convergent, in short it is a *divergent* series, and it can be obtained by repeated integration by parts:

$$\begin{aligned}\int_1^x \frac{e^t}{t}dt &= \frac{e^x}{x} - e + \int_1^x \frac{e^t}{t^2}dt \\ &= \cdots = \frac{e^x}{x}+\frac{e^x}{x^2}+\frac{2e^x}{x^3}+\cdots+\frac{(n-1)!e^x}{x^n}+C_n+n!\int_1^x\frac{e^t}{t^{n+1}}dt \end{aligned}\tag{1.21}$$

with $C_n = -e\sum_{j=0}^n j!$. For the last term we have

$$\lim_{x\to\infty}\frac{\displaystyle\int_1^x \frac{e^t}{t^{n+1}}dt}{\displaystyle\frac{e^x}{x^{n+1}}} = 1 \tag{1.22}$$

(by L'Hospital) and (1.20) follows.

Note 1.23 *The constant C_n cannot be included in (1.20) using the definition (1.12), since it is smaller than $x^{-N}e^x$ for any N and C_n and its contribution vanishes in any of the limits implicit in (1.12).*

By a similar calculation,

$$f_2=\int_2^x\frac{e^t}{t}dt \sim e^x\tilde{f}_0 = \frac{e^x}{x}+\frac{e^x}{x^2}+\frac{2e^x}{x^3}+\cdots+\frac{n!e^x}{x^{n+1}}+\cdots \ \text{ as } x\to+\infty \tag{1.24}$$

and now, unlike the case of (1.18) versus (1.19) there is no obvious function to prefer, insofar as asymptoticity goes, on the left side of the expansion.

Stirling's formula (1.7) is another example of a divergent asymptotic expansion.

Remark 1.25 *Asymptotic expansions cannot be added, in general. Otherwise, since on the one hand $f_1 - f_2 = \int_1^2 ds e^s/s = 3.0591...$, and on the other hand both f_1 and f_2 are asymptotic to the same expansion, it would follow that* $3.0591... \sim 0$. *This is one reason for considering, for restricted expansions, a weaker asymptoticity condition, see §1.1c .*

Examples of expansions that are *not asymptotic* are (1.5) for small z or

$$\sum_{k=0}^{\infty}\frac{x^{-k}}{k!} + e^{-x} \quad (x\to+\infty) \tag{1.26}$$

(because of the exponential term, this is not an ordered *simple series* satisfying (1.9)). Note however expansion (1.26), *does* satisfies all requirements in the *left* half plane, if we write e^{-x} in the first position.

Remark 1.27 *Sometimes we encounter expansions for large x of the form* $\sin x(1 + a/x + b/x^2 + \cdots)$ *which, while very useful, have to be understood differently and we will discuss this question later. They are not asymptotic expansions in the sense above, since* sin *can vanish. Usually the approximation itself fails near to zeros of the* sin*.*

1.1c Asymptotic power series

A special role is played by power series which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \; z \to 0^+ \tag{1.28}$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$) the series can be centered at t_0 (or $+\infty$, respectively).

Definition 1.29 (Asymptotic power series) *A function possesses an asymptotic* power *series as $z \to 0$ if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \quad \text{as } z \to 0 \tag{1.30}$$

Remark 1.31 *An asymptotic series is* not *an asymptotic expansion in the sense of Definition1.11 and (1.30) is not a special case of (1.14) unless all c_k are nonzero.*

The asymptotic power series at zero in $\mathbb{R}$ of e^{-1/z^2} is the zero series. It is however not true that the asymptotic expansion *of e^{-1/z^2} is zero.*

1.2 Operations with asymptotic power series

Addition and multiplication of asymptotic power series are defined as in the convergent case:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} d_k z^k = \sum_{k=0}^{\infty} (Ac_k + Bd_k) z^k$$

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} d_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j d_{k-j}\right) z^k$$

Remark 1.32 *If the series $\tilde{f}$ is convergent and f is its sum $f = \sum_{k=0}^{\infty} c_k z^k$ (note the ambiguity of the sum notation), then $f \sim \tilde{f}$.*

The proof follows directly from the definition of convergence.

The proof of the following lemma is immediate:

Lemma 1.33 (Algebraic properties of asymptoticity to a power series) *If $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ and $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ then*
(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$
(ii) $fg \sim \tilde{f}\tilde{g}$

Corollary 1.34 (Uniqueness of the asymptotic series to a function) *If $f(z) \sim \sum_{k=0}^{\infty} c_k z^k$ as $z \to 0$ then the c_k are unique.*

PROOF Indeed, if $f \sim \sum_{k=0}^{\infty} c_k z^k$ and $f \sim \sum_{k=0}^{\infty} d_k z^k$, then, by Lemma 1.33 we have $0 \sim \sum_{k=0}^{\infty} (c_k - d_k) z^k$ which implies, inductively, that $c_k = d_k$ for all k. ◻

Algebraic operations with asymptotic series are limited too, division of asymptotic series is not always possible. $e^{-1/z^2} \sim 0$ in $\mathbb{R}$ while $1/\exp(-1/z^2)$ has no asymptotic series at zero.

1.2 .1 Integration and differentiation of asymptotic power series.

Asymptotic relations can be integrated termwise as Proposition 1.35 below shows.

Proposition 1.35 *Assume f is integrable near $z = 0$ and that*

$$f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k$$

Then

$$\int_0^z f(s) ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{c_k z^{k+1}}{k+1}$$

PROOF This follows from the fact that $\int_0^z o(s^n) ds = o(z^{n+1})$ as it can be seen by immediate estimates. ◻

Differentiation is a different issue. Many simple examples show that asymptotic series cannot be freely differentiated. For instance $e^{-1/x^2} \sin e^{1/x^4} \sim 0$ as $x \to 0$ on $\mathbb{R}$, but the derivative is unbounded.

Asymptotic power series of analytic functions can be differentiated if they hold in a region which is not too rapidly shrinking. Such a region is often a sector or strip in $\mathbb{C}$, but can be allowed to be more general.

1.2 .2 Asymptotics in regions in $\mathbb{C}$

Proposition 1.36 *Let $M > 0$ and assume $f(x)$ is analytic in the region $S_a = \{x : |x| > R, |\Im(x)| < a|\Re(x)|^{-M}\}$, and*

$$f(x) \sim \sum_{k=0}^{\infty} c_k x^{-k} \quad \text{as } |x| \to \infty$$

in any subregion of the form $S_{a'}$ with $a' < a$.

Then

$$f'(x) \sim \sum_{k=0}^{\infty} (-kc_k) x^{-k-1}$$

as $|x| \to \infty$ in any subregion of the form $S_{a'}$ with $a' < a$.

PROOF Here, Proposition 1.13 will come in handy. Let $N > M + 2$. By the analyticity and asymptoticity assumptions, there is some constant C so that $|f(x) - \sum_{k=0}^{N} c_k x^{-k}| < C|x|^{-N}$ in $S_{a'}$ $(a' < a)$ (why?). Let $a'' < a'$ and take a circle of radius $(a' - a'')/2|x|^{-M}$ around $x \in S_{a''}$. This circle is contained in $S_{a'}$ if x is large enough (why?).

$$\left| f'(x) - \sum_{k=0}^{N} (-kc_k) x^{-k-1} \right| = \left| \frac{1}{2\pi i} \oint_{\mathcal{C}} (s - x)^{-2} \left(f'(s) - \sum_{k=0}^{N} c_k s^{-k} \right) ds \right|$$
$$\leq \frac{4C}{(a' - a'')^2} |x|^{2M} |x|^{-N} \quad (1.37)$$

and the result follows. ∎

Exercise 1.38 Consider the following integral related to the error function

$$F(z) = e^{z^{-2}} \int_0^z s^{-2} e^{-s^{-2}} ds$$

It is clear that the integral converges at the origin, if the origin is approached through real values (see also the change of variable below); thus we *We define* the integral to $z \in \mathbb{C}$ as being taken on a curve γ with $\gamma'(0) > 0$, and extend F by $F(0) = 0$. The resulting function is analytic in $\mathbb{C} \setminus 0$, see Exercise 3.8 below.

What about the behavior at $z = 0$? It depends on the direction in which 0 is approached! Substituting $z = 1/x$ and $s = 1/t$ we get

$$E(x) = e^{x^2} \int_x^{\infty} e^{-t^2} dt =: \frac{\sqrt{\pi}}{2} e^{x^2} \operatorname{erfc}(x) \quad (1.39)$$

Check that if $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$ and $f'(x) \to L$ as $x \downarrow 0$, then f is differentiable to the right at zero and this derivative equals L. Use this fact, Proposition 1.36 and induction to show that the Taylor series at 0^+ of $F(z)$ is indeed given by (3.7).

1.2a Concrete functions and unique expansions

Prompted by the need to eliminate apparent paradoxes, mathematics was formulated in a precise language with a well defined set of axioms, first by Zermelo [43], [41] within set theory. In this language, a function is defined as *a set of ordered pairs* (x, y) [2] such that *for every x there is only one pair with x as the first element.* All this can be written precisely and it is certainly foundationally satisfactory, since it uses arguably more primitive objects: sets.

A tiny subset of these general functions can arise as unique solutions to well defined problems, however. Indeed, on the one hand it is known that there is no specific way to distinguish two arbitrary functions based on their intrinsic properties alone[3]. On the other hand, a function which is known to be the unique solution to a specific function can a fortiori be distinguished from any other function.

Then most functions just exist in an unknowable realm, and only their collective presence has mathematical consequences. We can usefully restrict the study of functions to those which do arise in specific problems, and hope that they have, in general, better properties than arbitrary ones. For instance, solutions of specific equations, such as systems of linear or nonlinear ODEs or difference equations with meromorphic coefficients, near a regular or singular point, can be described completely in terms of their behavior at such a point (more precisely, they are completely described by their *transseries*, a generalization of series described later).

*

Convegent expansions have been in use for a very long time, as a convenient calculational tool. The error resulting from keeping only a finite number of terms can be made, in principle, as small as we want.

Factorially divergent asymptotic series came later; they were already in use for very precise astronomical calculations calculations at the turn of the 19th century [4]. As the variable, say $1/x$, becomes small, the first few terms of the expansion should provide a good approximation of the function. Taking for instance $x = 100$ and 5 terms in the asymptotic expansion we get the approximation $2.715552711 \cdot 10^{41}$ for both $f_1(100) = 2.715552745 \ldots \cdot 10^{41}$ and $f_2(100) = f_1(100) - 3.05911 \cdots$. However, in using divergent series, there is a threshold in the accuracy of approximation, as it can be seen by comparing (1.20) and (1.24).

[2]Here x, y are themselves sets, and $(x, y) := \{x, \{x, y\}\}$); x is in the domain of the function and y is in its range.

[3]More precisely, in order to select one function out of an arbitrary, *unordered* pair of functions, some form of the *axiom of choice* [41] is needed.

[4]Interestingly, in the early 19th century they were often called "semi-convergent".

The two functions differ by a constant, which is exponentially smaller than each function. The expected relative error cannot be better than exponentially small, at least for one of them. As we shall see, it is exponentially small for each one of them, and slightly smaller for a privileged choice of the constant, when the lower limit of integration is taken to be $-\infty$ (and the improper integral is defined as the Cauchy principal part; this defines the function $\mathrm{Ei}(x)$). This smallest error is achieved by stopping the sum at its *least term*, which can be seen to be $n \approx x$. The error in calculating $\mathrm{Ei}(x)$ is very small, of order $x^{-1/2}$ which is indeed pretty good for a function which grows exponentially as $x \to \infty$. Still, for fixed x, in this calculation there is a buit-in error.

Cauchy [2] , proved that least term truncation in Stirling's formula gives errors of the order of magnitude of the least term.

Stokes refined Cauchy summation to the least term, and discovered the "Stokes phenomenon": if a solution of a linear differential equation is asymptotic to a *divergent* series, it must grow *exponentially* in some other directions.

But a clear definition and a general procedure of summation were absent at the time. Abel, himself the inventor of a number of summation procedures of divergent series, labeled divergent series "an invention of the devil".

Later on, the view of divergent series as somehow linked to specific functions and encoding their properties was later abandoned, together with the notion of a function as a rule. This view was replaced by the rigorous notion of an *asymptotic series*, associated instead to a vast family of functions via the rigorous Poincaré definition 1.11, which is precise and general, but specificity is lost even in simple cases.

*

Can we usefully associate to a divergent series a *unique function* which might call the sum of the series? The answer is yes, in many practical cases, no in general. Exploring this question will carry us through a number of interesting questions.

In [14], Euler considered this question for the formal series $s = 1 - 2 + 6 - 24 + 120 \cdots$, in fact extended to the formal expansion

$$\tilde{f} := \sum_{k=0}^{\infty} k!(-z)^{k+1}, \quad x \to \infty \tag{1.40}$$

Euler notes that $\tilde{f}$ satisfies the equation

$$z^2 y' + y = z \tag{1.41}$$

and concludes that $\tilde{f} = e^{1/z}\mathrm{Ei}(-1/z) + Ce^{1/z}$ and thus, since the series is formally small as $z \to 0^+$, we should have $C = 0$, $\tilde{f} = e^{1/z}\mathrm{Ei}(-1/z)$, and in particular $s = e\mathrm{Ei}(-1)$. What can we make out of this calculation? At the very least, this shows that *if* (1.40) sums to a function, in a way compatible with basic operations and properties, the function can only be $e^{1/z}\mathrm{Ei}(-1/z)$.

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§. 19. Inveſtigemus nunc etiam analytice huius ſeriei valorem, eam vero in latiori ſenſu accipiamus: ſit igitur

$$s = x - 1x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \text{etc.}$$

quae differentiata dabit:

$$\frac{ds}{dx} = 1 - 2x + 6xx - 24x^3 + 120x^4 - \text{etc.} = \frac{x-s}{xx}$$

vnde fit $ds + \frac{s\,dx}{xx} = \frac{dx}{x}$, cuius aequationis, ſi e ſumatur pro numero, cuius logarithmus hyperbolicus eſt $= 1$, integrale erit $e^{-1:x}s = \int \frac{e^{-1:x}dx}{x}$ et $s = e^{1:x} \int \frac{e^{-1:x}dx}{x}$. Caſu ergo quo $x = 1$ erit $1 - 1 + 2 - 6 + 24 - 120 + \text{etc.}$ $= e\int \frac{e^{-1:x}dx}{x}$. Exprimit ergo haec ſeries aream lineae curuae, cuius natura inter abſciſſam x et y hac continetur aequatione $y = \frac{e.e^{-1:x}}{x}$, ſi abſciſſa x ponatur $= 1$: ſeu erit $y = \frac{e}{e^{1:x}x}$. Haec autem curua ita eſt comparata, vt poſito $x = 0$ fiat $y = 0$; ſin autem ſit $x = 1$, erit $y = 1$: medii vero applicatae valores ita ſe habebunt, vt

ſi ſit

FIGURE 1.1: L. Euler, De seriebus divergentibus, *Novi Commentarii Academiae Scientiarum Petropolitanae (1754/55) 1760, p. 220*

On the other hand, the formal expression

$$\sum_{q\in\mathbb{Q}} \frac{1}{x+q} \tag{1.42}$$

cannot have a nontrivial, meaningful sum, since the sum would have any rational number as a period, and would therefore not be Lebesgue measurable [40]. Since it is known that nonmeasurable functions only exist by virtue of some form of the axiom of choice, no definable (such as "the sum of (1.42)") nonmeasurable functions can exist.

A good correspondence between functions and expansions is possible only by carefully restricting both. We will restrict the analysis to functions and expansions arising in differential or difference equations, and some few other concrete problems.

Some elements of Écalle's theory

In the 80's by Écalle found a vast class of functions, closed under usual operations (algebraic ones, differentiation, composition, integration and function inversion) whose properties are, at least in principle, easy to analyze: *the analyzable functions.*

We might fear such a closure is either nonconstructive, or else horrendously intricate. Let's see what happens in the closure using say two operations, reciprocal and integration.

$$\boxed{\int 1 = x} \longrightarrow \boxed{\frac{1}{\cdot}\, x = x^{-1}} \longrightarrow \boxed{\int x^{-1} = \ln x}$$

and $\ln x$ has to be taken as a new primitive object.

$$\longrightarrow \boxed{\frac{1}{\cdot}\, \ln x = \frac{1}{\ln x}} \longrightarrow \boxed{\int \frac{1}{\ln x} \quad \text{nonelementary}} \tag{1.43}$$

and, within functions we would need to include the last integral as yet another primitive object.

Transseries. The way to obtain analyzable functions was in fact to first construct *transseries*, the closure of formal series under operations, a far more manageable task, and then establish a good correspondence between transseries and functions.

Transseries are surprisingly simple. They consists, roughly, in all formally asymptotic expansions, *finitely generated* in terms of powers exponentials and logs in exponentials powers and logs of ordinal length with coefficients which have at most power-of-factorial growth. For instance, as $x \to \infty$, integrations by parts in (1.43) yields

$$\int \frac{1}{\ln x} \quad = x \sum_{k=1}^{\infty} \frac{k!}{(\ln x)^{k+1}}$$

(a divergent expansion). Other examples are:

$$e^{e^x + x^2} + e^{-x} \sum_{k=0}^{\infty} \frac{k!(\ln x)^k}{x^k} + e^{-x \ln x} \sum_{k=-1}^{\infty} \frac{k!^2 2^k}{x^{k/3}} \quad x \to +\infty$$

$$\sum_{k=0}^{\infty} e^{-kx} \left(\sum_{j=0}^{\infty} \frac{c_{kl}}{x^k} \right)$$

Note how the terms are ordered decreasingly, with respect to $\gg$ (far greater than) from left to right. The *generators* in the first and third transseries are $1/x$ and e^{-x}. Transseries contain, order by order, manifest asymptotic information.

Transseries, as constructed by Ecalle, are the closure of series under a number of operations, including

(i) Algebraic operations: addition, multiplication and their inverses.

(ii) Differentiation and integration.

(iii) Composition and functional inversion.

However, operations (i), (ii) and (iii) are far from sufficient; for instance differential equations cannot be solved through (i)–(iii). Indeed, most ODEs *cannot* be solved by *quadratures*, *i.e.* by finite combinations of integrals of simple functions, but by limits of these operations. Limits though are not easily accommodated in the construction. Instead we can allow for

(iv) Solution of fixed point problems of *formally contractive mappings*, see §2.1b .

Operation (iv) was introduced by abstracting from the way problems with a small parameter [5] are solved by successive approximations.

Proposition. Transseries are closed under (i)–(iv).

This will be shown in §4 and §A; it means many problems can be solved within transseries. It is however unlikely that (i)–(iv) encompass all there is needed to solve asymptotic problems and more needs to be understood.

Analyzable functions. To establish a one-to-one isomorphic correspondence between a class of transseries and functions, Écalle also vastly generalized Borel summation .

[5]The small parameter could be the independent variable itself. Infinity is often taken as the special point, one reason being that repeated differentiation of $\exp(1/x)$ is clumsy.

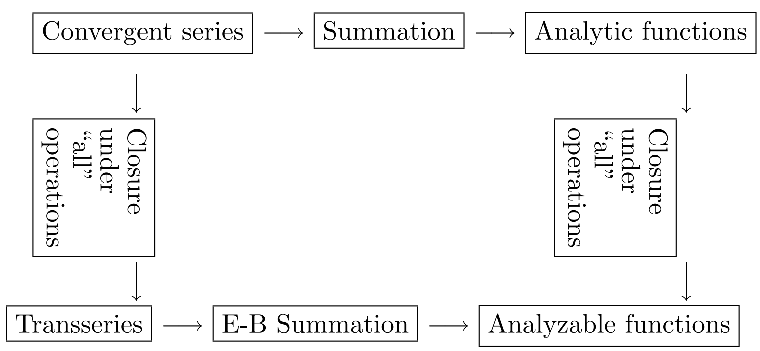
Écalle-Borel (EB) summation extends usual summation, does not depend on how the transseries was obtained, and is a perfect isomorphism between expansions and functions. The sum of an EB summable transseries is, by definition an analyzable function.

EB summable transseries are known to be closed under operations (i)–(iii) but not yet (iv). EB summability has been shown to apply generic systems of linear or nonlinear ODEs, PDEs (including the Schrödinger equation, Navier-Stokes) etc, Quantum Field Theory, KAM and so on. Concrete theorems will be given later.

The representation by transseries is effective, the function associated to a transseries closely following the behavior expressed in the successive, ordered, terms of its transseries.

Determining the transseries of a function is the "analysis" part and transseriable functions are "analyzable ", while the opposite process, reconstruction by EB summation of a function from its transseries is known as "synthesis".

We have the following diagram



This is the only known way to close functions under the listed operations.

Formal and actual solutions.

Few calculational methods have longer history than successive approximation. Suppose ϵ is small and we want to solve the equation $y-y^5=\epsilon$. Looking first for a small solution, we see that $y^5 \ll y$ and then, writing

$$y = \epsilon + y^5 \tag{1.44}$$

as a first approximation, we have

$$y \approx y_1 = \epsilon \tag{1.45}$$

We can use $y \approx y_1$ in (1.44) to improve in accuracy over (1.45):

$$y_2 = \epsilon + \epsilon^5$$

and further

$$y_3 = \epsilon + \left(\epsilon + \epsilon^5\right)^5$$

Repeating this procedure indefinitely, the right side becomes

$$\epsilon + \epsilon^5 + 5\epsilon^9 + 35\epsilon^{13} + 285\epsilon^{17} + 2530\epsilon^{21} + \cdots \tag{1.46}$$

Exercise 1.47 *Show that this series converges for* $|\epsilon| < 4 \cdot 5^{-5/4}$.

The binomial series (1.46) converges to z for $|t| < 1$, and is computationally most useful if t is small.

Regular differential equations can be locally solved much in the same way. Consider the Painlevé equation

$$y'' = y^2 + z$$

near $z = 0$ with $y(0) = y_0$ and $y'(0) = y_1$ small. If y is small like some power of z, then y'' is far larger than y^2 and then, to leading approximation,

$$y'' = z$$

and

$$y = y_0 + y_1 z + \frac{z^2}{2}$$

We can substitute this back into the equation and get a better approximation of the solution, and if we repeat the procedure indefinitely, we get the actual solution of the problem (since, as it follows from the general theory of differential equations, the solution is analytic).

Let us look at the equation

$$f' - f = x^{-1} \tag{1.48}$$

the same as the one Euler used, now in the variable $z = 1/x$, for x large and positive. If f is small like an inverse power of x, then f' should be even smaller, and we can apply again successive approximations to the equation written in the form

$$f = f' - x^{-1} \tag{1.49}$$

To leading order $f \approx f_1 = 1/x$, we then have $f \approx f_2 = 1/x - 1/x^2$ and now if we repeat the procedure indefinitely we get

$$f \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots - \frac{(-1)^n n!}{x^{n+1}} + \cdots \tag{1.50}$$

Something must have gone wrong here. We do not get a solution (in any obvious meaning) to the problem: for no value of x is this series convergent. But note that “not convergent” presupposes a topology, or a notion of convergence (such as pointwise convergence). Is there a good topology under which such wildly divergent expansions converge? The answer is yes, but for now we shall pull a trick out of the hat that allows us to interpret the series. If we write

$$n! = \int_0^\infty e^{-t} t^n dt$$

in (1.50), it becomes

$$\int_0^\infty \sum_{n=0}^\infty e^{-t} t^n x^{-n-1} dt = \int_0^\infty \frac{e^{-np}}{1+p} dp \tag{1.51}$$

provided we can interchange summation and integration, and sum the geometric series to $1/(1+p)$ for all values of p, not only for $|p| < 1$.

Upon closer examination, we see that another way to view the formal calculation leading to (1.51) is to say that we first performed a term-by-term inverse Laplace transform (cf. §2.1a) of the series (the inverse Laplace transform of $n!x^{-n-1}$ being p^n), summed the p-series for small p (to $(1+p)^{-1}$) *analytically continued* this sum on the whole of $\mathbb{R}^+$ and then took the Laplace transform of this result. Up to analytic continuations and ordinary convergent summations, what has been done in fact is the combination Laplace inverse-Laplace transform, which is the identity. In this sense, the emergent function is “equal” to the initial series, and should inherit its formal properties. In particular, (1.51) is a solution of (1.48). The steps we have just described define Borel summation , which applies precisely when the above steps succeed.

What distinguishes the first two examples from the last one? In the first two, the next approximation was obtained from the previous one either by algebraic operations and integration. These processes are regular, and they produce, at least under some restrictions on the variables, convergent expansions. We have, e.g., $\int \cdots \int x = x^n/n!$. But in the last example, we iterated upon differentiation which makes functions “worse and worse”. We have $(1/x)^{(n)} = n!/x^{n+1}$. In §4.6 we will see why the trick above works to interpret such series.

1.2b Laplace transforms

Let $F \in L^1(\mathbb{R})$. Then, by Fubini's theorem and dominated convergence, the Laplace transform

$$\mathcal{L}F := \int_0^\infty e^{-px} F(p) dp \tag{1.52}$$

is well defined and continuous in x in the closed right half plane and analytic in the open RHP (the open right half plane). (Obviously, we could allow $Fe^{-|\alpha|p} \in L^1$ and then $\mathcal{L}F$ would be defined for $\Re x > |\alpha|$.) F is uniquely defined by its Laplace transform, as seen below. The Laplace **convolution** is given by

$$(f * g)(p) = \int_0^p f(s)g(p-s)ds \tag{1.53}$$

and we have

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \tag{1.54}$$

and $\mathcal{L}(pF) = (\mathcal{L}F)'$.

1.2c A Laplace inversion formula

Theorem 1.55 *Assume $c \ge 0$, $f(z)$ is analytic in the closed half plane $H_c := \{z : \Re z \ge c\}$. Assume further that $\sup_{c' \ge c} |f(c'+it)| \le g(t)$ with $g(t) \in L^1(\mathbb{R})$. Let*

$$F(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} f(x)dx =: (\mathcal{L}^{-1}F)(p) \tag{1.56}$$

Then for any $x \in \{z : \Re z > c\}$ we have

$$\mathcal{L}F = \int_0^\infty e^{-px}F(p)dp = f(x) \tag{1.57}$$

PROOF Note that for any $x' = x_1' + iy_1' \in \{z : \Re z > c\}$

$$\int_0^\infty dp \int_{c-i\infty}^{c+i\infty} \left| e^{p(s-x')} f(s) \right| d|s| \le \int_0^\infty dp e^{p(c-x_1')} \|g\|_1 \le \frac{\|g\|_1}{x_1' - c} \tag{1.58}$$

and thus, by Fubini we can interchange the orders of integration:

$$\begin{aligned} U(x') &= \int_0^\infty e^{-px'} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} f(x)dx \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dx f(x)\int_0^\infty dp e^{-px'+px} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{f(x)}{x'-x}dx \end{aligned} \tag{1.59}$$

Since $g \in L^1$ there must exist subsequences $\tau_n, -\tau_n'$ tending to ∞ such that $|g(\tau_n)| \to 0$. Let $x' > \Re x = x_1$ and consider the box $B_n = \{z : \Re z \in [x_1, x'], \Im z \in [-\tau_n', \tau_n]\}$ with positive orientation. We have

$$\int_{B_n} \frac{f(s)}{x'-s} ds = -f(x') \tag{1.60}$$

while, by construction,

$$\lim_{n\to\infty} \int_{B_n} \frac{f(s)}{x'-s} ds = \int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s} ds - \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{x'-s} dx \tag{1.61}$$

On the other hand, by dominated convergence, we have

$$\int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s} ds \to 0 \quad \text{as} \quad x' \to \infty \tag{1.62}$$

□

Chapter 2

Review of some basic analytic tools

2.1 The Phragmén-Lindelöf Theorem

This result is very useful in obtaining information about the size of a function in a sector, when the only information available is on the edges.

Theorem 2.1 (Phragmén-Lindelöf) *Let U be the open sector between two rays from the origin, forming an angle π/β, $\beta > \frac{1}{2}$. Assume f is analytic in U, and continuous on its closure, and for some $C_1, C_2, M > 0$ and $\alpha \in (0, \beta)$ it satisfies the estimates*

$$|f(z)| \le C_1 e^{C_2|z|^\alpha}; \quad z \in U; \quad |f(z)| \le M; \quad z \in \partial U \tag{2.2}$$

Then

$$|f(z)| \le M; \quad z \in U \tag{2.3}$$

PROOF By a rotation we can make $U = \{z : 2|\arg z| < \pi/\beta\}$. Making a cut in the complement of U we can define an analytic branch of the log in U and, with it, an analytic branch of z^β. By taking $g = f(z^{1/\beta})$, we can assume without loss of generality that $\beta = 1$ and $\alpha \in (0, 1)$ and then $U = \{z : |\arg z| < \pi/2\}$. Let $\alpha' \in (\alpha, 1)$ and consider the analytic function

$$e^{-C_2 z^{\alpha'}} f(z) \tag{2.4}$$

Since $|e^{-C_2 z^{\alpha'}}| < 1$ in U (check) and $|e^{-C_2 z^{\alpha'} + C_2 z^\alpha}| \to 0$ as $|z| \to \infty$ on the half circle $|z| = R, \Re z \ge 0$ (check), the usual maximum modulus principle completes the proof. ∎

2.1a Some properties of Laplace transforms

There are many textbooks on integral transforms; we will briefly mention now a few facts about the Laplace transform. We will study Laplace and inverse Laplace transforms in more detail later.

Lemma 2.5 (Uniqueness) *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of x with an accumulation point. Then $F = 0$ a.e.*

We will from now on write $F = 0$ on a set to mean $F = 0$ *a.e.* on that set.

PROOF By analyticity, $\mathcal{L}F = 0$ in the open RHP and by continuity, for $s \in \mathbb{R}$, $\mathcal{L}F(is) = 0 = \hat{\mathcal{F}}F$ where $\hat{\mathcal{F}}F$ is the Fourier transform of F (extended by zero for negative values of p). Since $F \in L^1$ and $0 = \hat{\mathcal{F}}F \in L^1$, by the known Fourier inversion formula [40], $F = 0$. ∎

More however can be said. We can draw interesting conclusions about F just from the rate of decay of $\mathcal{L}F$.

We can apply the Phragmén-Lindelöf theorem to obtain a result on the subexponential behavior of Laplace transforms, which shows in particular that no two different $L^1(\mathbb{R}^+)$ functions, real–analytic on $(0, \infty)$, can have Laplace transforms within exponentially small corrections of each–other.

Proposition 2.6 (Lower bound on decay rates of Laplace transforms) *Assume $F \in L^1(\mathbb{R}^+)$ and for some $\epsilon > 0$ we have*

$$\mathcal{L}F(x) = O(e^{-\epsilon x}) \quad \textit{as} \quad x \to +\infty \tag{2.7}$$

Then $F = 0$ on $[0, \epsilon]$.

PROOF We write

$$\int_0^\infty e^{-px} F(p)dp = \int_0^\epsilon e^{-px} F(p)dp + \int_\epsilon^\infty e^{-px} F(p)dp \tag{2.8}$$

we note that

$$\Big| \int_\epsilon^\infty e^{-px} F(p)dp \Big| \le e^{-\epsilon x} \int_\epsilon^\infty |F(p)|dp \le e^{-p\epsilon} \|F\|_1 = O(e^{-\epsilon x}) \tag{2.9}$$

Therefore

$$g(x) = \int_0^\epsilon e^{-px} F(p)dp = O(e^{-\epsilon x}) \quad \text{as} \quad x \to +\infty \tag{2.10}$$

The function g is entire (prove this!) Let $h(x) = e^{\epsilon x} g(x)$. Then by assumption h is entire and uniformly bounded for $x \in \mathbb{R}$ (since by assumption, for some x_0 and all $x > x_0$ we have $h \le C$ and by continuity $\max |h| < \infty$ on $[0, x_0]$). The function is also manifestly bounded for $x \in i\mathbb{R}$ (by $\|F\|_1$). By Phragmén-Lindelöf (first applied in the first quadrant and then in the fourth quadrant, with $\beta = 2, \alpha = 1$) h is bounded in the closed RHP. Now, for $x = -s < 0$ we have

$$e^{-s\epsilon} \int_0^\epsilon e^{sp} F(p)dp \le \int_0^\epsilon |F(p)| \le \|F\|_1 \tag{2.11}$$

Again by Phragmén-Lindelöf (again applied twice) h is bounded in the closed LHP thus bounded in C, and it is therefore a constant. But, by the Riemann-Lebesgue lemma, $h \to 0$ for $x = is$ when $s \to +\infty$. Thus $h \equiv 0$. But then, with χ_A the characteristic function of A,

$$\int_0^{\epsilon} F(p)e^{-isp}dp = \hat{\mathcal{F}}(\chi_{[0,\epsilon]}F) = 0 \tag{2.12}$$

for all $s \in \mathbb{R}$ entailing the conclusion. ◻

Corollary 2.13 *Assume $F \in L^1$ and $\mathcal{L}F = O(e^{-AX})$ as $x \to +\infty$ for all $A > 0$. Then $F = 0$.*

PROOF This is straightforward. ◻

As we see, uniqueness of the Laplace transform can be reduced to estimates.

2.1b Banach spaces and an essential tool: the contractive mapping principle

We discuss, for completeness, a few basic features of Banach spaces. There is a vast literature on the subject; see e.g. [39]. Familiar examples of Banach spaces are the n-dimensional euclidian vector spaces $\mathbb{R}^n$. A norm exists in a Banach space, which has the essential properties of a length: scaling, positivity except for the zero vector which has length zero and the triangle inequality (the sum of the lengths of the sides of a triangle is no less than the length of the third one). Once we have a norm, we can define limits, by reducing the notion to that in $\mathbb{R}$: $x_n \to x$ iff $\|x - x_n\| \to 0$. A normed vector space $\mathcal{B}$ is a Banach space if it is complete, that is every sequence with the property $\|x_n - x_m| \to 0$ uniformly in n, m (a Cauchy sequence) has a limit in $\mathcal{B}$. Note that $\mathbb{R}^n$ can be thought of the space of functions defined on the set of integers $\{1, 2, ..., n\}$. Then it is clear that there are infinite-dimensional Banach spaces, for instance the space of bounded functions on $[0, 1]$ with the norm $\|f\| = \sup_{[0,1]} |f|$. Infinite dimensional Banach spaces can be different from the finite-dimensional ones:

Exercise 2.14 *Find examples of normed vector spaces which are not complete.*

A function L between two Banach spaces which is linear, $L(x+y) = Lx + Ly$, is bounded (-or continuous) if $\|L\| := \sup_{\|x\|=1} \|Lx\| < \infty$.

In a *complete normed space* the vector structure and associated restrictions on the norm are dropped.

Assume $\mathcal{B}$ is a Banach space and that S is a closed subset of $\mathcal{B}$. In the *induced topology* (i.e, in the same norm), S is a complete normed space.

Throughout the book we will rely on the contractive mapping principle, which is a handy way to obtain solutions of perturbed problems from those of the simpler, unperturbed ones.

Assume $\mathcal{M} : S \mapsto \mathcal{B}$ is a (linear or nonlinear) operator with the property that for any $x, y \in S$ we have

$$\|\mathcal{M}(y) - \mathcal{M}(x)\| \le \lambda \|y - x\| \tag{2.15}$$

with $\lambda < 1$. We call such operators **contractive**. Note that if $\mathcal{N}$ is linear, this just means that the norm of $\mathcal{M}$ is less than one.

Theorem 2.16 *Assume $\mathcal{M} : S \mapsto S$, where S is a closed subset of $\mathcal{B}$ is a contractive mapping. Then the equation*

$$x = \mathcal{M}(x) \tag{2.17}$$

has a unique solution in S.

PROOF Consider the sequence $\{x_j\}_j \in \mathbb{N}$ defined recursively by

$$\begin{aligned} x_0 &= x_0 \in S \\ x_1 &= \mathcal{M}(x_0) \\ &\cdots \\ x_{j+1} &= \mathcal{M}(x_j) \\ &\cdots \end{aligned} \tag{2.18}$$

We see that

$$\|x_{j+2} - x_{j+1}\| = \|\mathcal{M}(x_{j+1}) - \mathcal{M}(x_j)\| \le \lambda \|x_{j+1} - x_j\| \le \cdots \le \lambda^j \|x_1 - x_0\| \tag{2.19}$$

Thus,

$$\|x_{j+p+2} - x_{j+2}\| \le \left(\lambda^{j+p} + \cdots \lambda^j\right) \|x_1 - x_0\| \le \frac{\lambda^j}{1-\lambda} \|x_1 - x_0\| \tag{2.20}$$

and x_j is a Cauchy sequence, and it thus converges, say to x. Since by (2.15) $\mathcal{M}$ is continuous, passing the equation for x_{j+1} in (2.18) to the limit $j \to \infty$ we get the limit we get

$$x = x_0 + \mathcal{M}(x) \tag{2.21}$$

that is existence of a solution of (2.17). For uniqueness, note that if X and Y are two solutions of (2.17), by subtracting their equations we get

$$\|X - Y\| = \|\mathcal{M}(X) - \mathcal{M}(Y)\| \le \lambda \|X - Y\| \tag{2.22}$$

implying $\|X - Y\| = 0$, since $\lambda < 1$. ∎

Note 2.23 *Note that contractivity and therefore existence of a solution of a fixed point problem depends on the norm. An adapted norm needs to be chosen for this approach to give results.*

Exercise 2.24 *Show that if L is a linear operator from the Banach space $\mathcal{B}$ into itself and $\|L\| < 1$ then $I - L$ is invertible, that is $x - Lx = y$ has always a unique solution $x \in \mathcal{B}$. Note thus that assuming that $I - L$ is not invertible, then whatever other norm $\|\cdot\|^*$ which makes $\mathcal{B}$ a Banach space, $\|L\|^* \geq 1$.*

2.1c Fixed points and vector valued analytic functions

A theory of analytic functions from a Banach space to itself can be constructed by almost exactly following the usual construction of analytic functions. For the construction to work, we need the usual vector space operations and a topology in which they are continuous. If multiplication is present –in a Banach algebra setting– multiplication is continuous as well. We can define a derivative in the usual way, by writing $F(f + \epsilon g) = F(f) + \epsilon L_f g + o(\epsilon)$, $\epsilon \in \mathbb{C}$ small, where $L_f =: \partial_f F$ is a linear operator, define an integral in the usual way, as a limit of a sum, or using appropriately generalized measure theory. Cauchy's formula is valid for complex-differentiable (analytic) functions. A detailed presentation is found in [13], but the basic facts are simple enough for the reader to redo the necessary proofs. An immediate recasting of the contractive mapping principle is that

Remark 2.25 *In the context of Theorem 2.16 we have, equivalently: If $\mathcal{N} : S^2 \mapsto S$ is analytic in f and $\|\partial_f \mathcal{N}\| < \lambda < 1$ for f, g in S, then the equation $f = \mathcal{N}(f, g)$, where $\|\partial_f \mathcal{N}\| < \lambda < 1$ in S has a unique fixed point in S.*

Indeed, if $\|h\| = \delta$ we have $h = \delta h_1$ with $\|h_1\| = 1$

$$\|\mathcal{N}(g, f + h) - \mathcal{N}(g, f)\| \leq \int_0^\delta \left\| \frac{\partial \mathcal{N}}{\partial f}\Big(g, f + th_1\Big) \right\| dt \leq \lambda\delta$$

The implicit function theorem could be restated abstractly in a similar setting.

2.1d Choice of $\mathcal{N}$

An equation can be rewritten in a number of equivalent ways. In solving an asymptotic problem, as a rule of thumb the final operator $\mathcal{N}$ should not contain derivatives or other operations poorly behaved with respect to asymptotics, and it should only contain the sought-for solution through formally small corrections. The norms should reflect as well as possible the expected growth/decay tendency of the solution itself and the spaces are spaces where this solution lives.

Note 2.26 The contractive mapping and implicit function results above are trivially equivalent, and the difficulty in proving an asymptotic result virtually never lies here, but in finding the contractive reformulation, and the adequate spaces and norms.

2.1d .1 Applications

2.1d .2 Existence and uniqueness of solutions of differential equations

Assume $G(X,t)$ is a function defined on $\mathcal{B} \times [0,a]$ where a $\mathcal{B}$ is a Banach space and that G is continuous in x and Lipschitz continuous in X, i.e.

$$\|G(X,t) - G(Y,t)\| \le A(t)\|X - Y\| \tag{2.27}$$

for all $t \in [a,b]$ and $X, Y \in \mathcal{B}$ with $A(t)$ continuous. Then $A(t) \le A$ for some A and all $t \in [a,b]$. Then for some $\epsilon > 0$, the differential equation

$$Y'(t) = G(Y(t),t); \quad Y(a) = Y_a \tag{2.28}$$

has a unique solution on the interval $[a, a+\epsilon]$.

PROOF The definition of the derivative is the same as in usual calculus, since we have a well defined notion of limit. Likewise, an Riemann integral of the form

$$\int_{a_1}^{a_2} Y(s)ds \tag{2.29}$$

is defined as in usual calculus as a limit of Riemann sums and it exists if Y is continuous in s. Consider the space of continuous functions $t \mapsto Y(t)$ defined for $t \in [a, a+\epsilon]$ in the norm

$$\|Y(t)\|_\infty = \max_{t\in[a,a+\epsilon]} \|Y(t)\| \tag{2.30}$$

Check that this is a Banach space. Let $\|G(Y_a, a\| = M$. Consider the ball B of radius δ_1 centered at Y_a and the interval $[a, a+\delta_2]$ with δ_2 small. We can choose δ_1 and δ_2 small enough so that $\|G(Y_a + y, a+\epsilon\| \le 2M$ if $\|y\| < \delta_1$ and $\epsilon < \delta_2$.

Consider the auxiliary equation

$$Y(t) = Y_a + \int_a^{a+t} G(Y(s),s)ds =: \mathcal{M}Y \tag{2.31}$$

We first check that the ball B is invariant under $\mathcal{M}$. We have

$$\|\mathcal{M}Y - Y_a\| = \left\| \int_a^s G(Y(s),s)ds \right\| \le |2M\delta_2| \le \delta_1 \tag{2.32}$$

if $\delta_2 < \delta_1/M$.

Now we want to check contractivity. We have

$$\|(\mathcal{M}(X) - \mathcal{M}(Y))\|(t) = \left\| \int_a^{a+t} (G(X(s), s) - G(Y(s), s)) ds \right\|$$
$$\leq \int_a^{a+t} \|G(X(s), s) - G(Y(s), s)\| ds \leq A \int_a^{a+t} \|X(s) - Y(s)\| ds$$
$$\leq A \int_a^{a+t} \|X - Y\|_\infty ds = \delta_2 A \|X - Y\|_\infty \quad (2.33)$$

which is contractive if $A\delta_2 < 1$. We see that the assumptions of the contractive map theorem are met if

$$\delta_2 \leq \left| A^{-1} - \delta_1/M| \right|$$

It is easy to check that the solution to (2.31) solves the original problem. We see that the definition of G can be restricted to a neighborhood of Y_a.

Local existence is all that can be shown in this generality since even an equation as simple as $y' = y^2 + 1$, whose general solution is $\tan(x + \phi)$ has infinitely many poles on $\mathbb{R}$.

□

2.1d .3 Global existence and uniqueness of solutions of linear differential equations

Consider now the equation

$$Y'(t) = L(t)Y; \quad Y(0) = Y_0 \quad (2.34)$$

where $L(t)$ is a uniformly bounded linear operator,

$$\max_{t \in [0,\infty)} \|L(t)\| \leq L \quad (2.35)$$

Then the problem (2.34) has a global solution on $[0, \infty)$.

PROOF Consider the space of continuous functions $Y : [0, \infty) \mapsto \mathcal{B}$ in the norm

$$\|Y\|_{\infty,L} = \sup_{t \in [0,\infty)} e^{-Lt/\lambda} \|Y(t)\| \quad (2.36)$$

with $\lambda < 1$ and the auxiliary equation

$$Y(t) = Y_0 + \int_0^t L(s)Y(s) ds \quad (2.37)$$

which is well defined on $\mathcal{B}$ and is contractive there since

$$e^{-Lt/\lambda} \left| \int_0^t L(s)Y(s) ds \right| \leq L e^{-Lt/\lambda} \int_0^t e^{Ls/\lambda} \|Y\|_{\infty,L} ds = \lambda \|Y\|_{\infty,L} \quad (2.38)$$

□

2.1d .4 Example: A Puiseux series needed in §3.1c , in the asymptotics of the Gamma function

We choose a simple example which can be dealt with in a good number of other ways, yet containing some features of more complicated singular problems. Suppose we need to find the solutions of the equation $x - \ln x = t$ for t (and x) close to 1. The implicit function theorem does not apply to $F(x,t) = x - \ln x - t$ at $(0,0)$. We then attempt to find a simpler equation that approximates well the given one in the singular regime, that is we look for *asymptotic simplification*, and then we try to present the full problem as a perturbation of the approximate one. We write $x = 1 + z, t = 1 + s$, expand the left side in series for small z, and and retain only the first nonzero term. The result is $z^2/2 \approx s$. There are two solutions, thus effectively two different problems when s is small. Keeping all terms, we treat the cubic and higher powers of z as corrections. We look at one choice of sign, the other one being very similar, and write

$$z = \left(2s + \frac{2z^3}{3} - \frac{z^4}{2} + \frac{2z^5}{5} + \cdots\right)^{1/2} = (2s + \epsilon(z))^{1/2} \tag{2.39}$$

where $\epsilon(z)$ is expected to be small. We then have

$$z = \left(2s + O(z^3)\right)^{1/2} = \left(2s + O(s^{3/2})\right)^{1/2} \tag{2.40}$$

hence

$$z = \left(2s + \left[(2s)^{1/2} + O(s^{3/2})\right]^3/3\right)^{1/2} = \left(2s + \frac{4\sqrt{2}}{3}s^{3/2} + O(s^2)\right)^{1/2} \tag{2.41}$$

and further,

$$z = \left(2s + \frac{4\sqrt{2}}{3}s^{3/2} + \frac{2s^2}{3} + O(s^{5/2})\right)^{1/2} = \sqrt{2s} + \frac{2s}{3} + \frac{\sqrt{2}}{18}s^{3/2} - \frac{2s^2}{135} + O(s^{5/2}) \tag{2.42}$$

etc., where in fact the series converges, as shown in the Exercise 2.44 below.

Two ingredients are typically necessary for an iteration about an approximate equation to converge: the equation should be written in such a way that the solution is expected to be unique, and of course, all discarded terms should indeed be small. For the latter condition one can formally check self-consistency: using the approximate solution to calculate the expected correction to the first approximation, the (approximate) correction should indeed be small. According to the expected size of the solution and corrections, we should then be able to write a family of equations equivalent to the original one, which will then be contractive. Here z should be close to $\sqrt{2s}$; we set

$s = w^2/2$ and $z = wZ$ and get

$$Z = \left(1 + \frac{2}{3}wZ^3 - \frac{1}{2}w^2Z^4 + \frac{2}{5}w^3Z^5 + \cdots\right)^{1/2} \tag{2.43}$$

Exercise 2.44 ** Show that if ϵ is small enough, then (2.43) is contractive in the sup norm in a ball of radius ϵ centered at 1 in the space of functions Z analytic in w for $|w| \le \epsilon$. Show thus that z is analytic in $\sqrt{s}$ for small s.*

Once the behavior of the solutions has been clarified, we may sometimes gain in simplicity, or more global information, by returning to the implicit function theorem, but properly applied. Which one is better depends on the problem and on taste. The contraction mapping principle is often more natural, especially when the topology, suggested by the problem itself, is not one of the common ones.

We take $t = \tau^2/2$ and write $z^2/2+(z-\ln(1+z)-z^2/2) =: z^2/2(1+z\phi(z)) = \tau^2/2$ and (differentiating $z\phi$ reintegrating and changing variables) we get

$$z\sqrt{1 - z\phi(z)} = \pm\tau; \quad \phi(z) = \int_0^1 \frac{\sigma^2 d\sigma}{1 + z\sigma} \tag{2.45}$$

with the usual choice of branch for the square root. It is clear that the implicit function theorem applies to the functions $z\sqrt{1 - z\phi(z)} \pm w$ at $(0, 0)$.

The first few terms of the series are easily found from the fixed point equation by repeated iteration, as in §2.1e ,

$$z = \frac{1}{\sqrt{2}}\tau + \frac{1}{12}\tau^2 - \frac{\sqrt{2}}{72}\tau^3 + \frac{13}{4320}\tau^4 + \cdots \tag{2.46}$$

2.1e A nonlinear differential equation

As another example, consider the equation

$$y' + y = x^{-1} + y^3 + xy^5 \tag{2.47}$$

with the restriction $y \to 0$ as $x \to +\infty$. Exact solutions exist for special classes of equations, and (2.47) does not (at least not manifestly) belong to any of them. However, formal asymptotic series solutions, as $x \to \infty$, are usually easy to find. If y is small and power-like, then $y', y^3 \ll y$ and a first approximation is $y_1 \approx 1/x$. Then $y_2 \approx 1/x + y_1^3 + xy_1^5 - y_1'$. A few iterations quickly yield (see Appendix D)

$$y(x) = x^{-1} + x^{-2} + 3\,x^{-3} + 13\,x^{-4} + 69\,x^{-5} + 428\,x^{-6} + O\left(x^{-7}\right) \tag{2.48}$$

To find a contractive mapping reformulation, we have to find what can be discarded in a first approximation. Though the derivative is formally small, as we discussed in §2.1d , it cannot be discarded when a rigorous proof is sought. Since f and $1/x$ are of both formally larger than f' they cannot by discarded either. Thus the approximate equation can only be

$$y' + y = x^{-1} + E(f) \tag{2.49}$$

where the "error term" E is just $f^3 + xf^5$. An equivalent integral equation is obtained by solving (2.49) as though E was known,

$$y = g_0 + \mathcal{N}(y)$$
$$g_0(x) = y(x_0)e^{-(x-x_0)} + e^{-x}\int_{x_0}^{x} \frac{e^s}{s}ds; \;\; \mathcal{N}(y) = e^{-x}\int_{x_0}^{x} e^s\left[y^3(s) + sy^5(s)\right]ds \tag{2.50}$$

say with $x, x_0 \in \mathbb{R}^+$ (a sector in $\mathbb{C}$ can be easily accommodated). Now, the expected behavior of y is, from (2.48) $x^{-1}(1 + o(1)$. We take the norm $\|y\| = \sup_{x \ge x_0} |xy(x)|$ and S the ball $\{y : (x_0, \infty) : \|y\| < a\}$ where $a > 1$ (we have to allow it to be slightly bigger than 1, by (2.48)).

To evaluate the norms of the operators involved in (2.50) we need the following relatively straightforward result.

Lemma 2.51 *For $x > x_0 > m$ we have*

$$e^{-s}\int_{x_0}^{x} e^s s^{-m} \le |1 - m/x_0|^{-1} x^{-m}$$

PROOF In a sense, the proof is by integration by parts: for $x > x_0 > m$ we have

$$e^x x^{-m} \le |1 - m/x_0|^{-1}(e^x x^{-m})'$$

and the result follows by integration.

Exercise 2.52 *(i) Show that, if $a > 1$ and if x_0 is sufficiently large, then $\mathcal{N}$ is well defined on S and contractive there. Thus (2.50) has a unique fixed point in S. How small can you make x_0?*

(ii) A slight variation of this argument can be used to prove the validity of the expansion (2.48). If we write $y = y_N + \delta(x)$ where y_N is the sum of the first N terms of the formal power series of y, then, by construction, y_N will satisfy the equation up to errors of order x^{-N-1}. Write an integral equation for δ and show that δ is indeed $O(x^{-N-1})$.

□

Chapter 3

Review of some results in classical asymptotics

3.0f Asymptotics of integrals: first results

Example: Integration by parts and elementary truncation to the least term. A solution of the differential equation

$$f' - 2xf + 1 = 0 \tag{3.1}$$

is related to the the complementary error function:

$$E(x) = e^{x^2}\int_x^\infty e^{-s^2}ds = \frac{\sqrt{\pi}}{2}e^{x^2}\mathrm{erfc}(x) \tag{3.2}$$

Let us find the asymptotic behavior of $E(x)$ for $x \to +\infty$. One very simple technique is integration by parts, done in a way in which the integrated terms become successively smaller. A decomposition is sought such that in the identity $fdg = d(fg) - gdf$ we have $gdf \ll fdg$ in the region of interest. Note that there is no manifest perfect derivative in the integrand, but we can create a suitable one by writing $e^{-s^2}ds = -(2s)^{-1}d(e^{-s^2})$.

$$\begin{aligned} E(x) &= \frac{1}{2x} - \frac{e^{x^2}}{2}\int_x^\infty \frac{e^{-s^2}}{s^2}ds = \frac{1}{2x} - \frac{1}{4x^3} + \frac{3e^{x^2}}{4}\int_x^\infty \frac{e^{-s^2}}{s^4}ds = ... \\ &= \sum_{k=0}^{m-1}\frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac12)}{x^{2k+1}} + \frac{(-1)^m e^{x^2}\Gamma(m+\frac12)}{\sqrt{\pi}}\int_x^\infty \frac{e^{-s^2}}{s^{2m}}ds \end{aligned} \tag{3.3}$$

On the other hand, we have, by L'Hospital

$$\left(\int_x^\infty \frac{e^{-s^2}}{s^{2m}}ds\right)\left(\frac{e^{-x^2}}{x^{2m+1}}\right)^{-1} \to \frac12 \text{ as } x \to \infty \tag{3.4}$$

and the last term in (3.3) is $O(x^{-2m-1})$ as well. It is also clear that the remainder in (3.3) is alternating and thus

$$\sum_{k=0}^{m-1}\frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac12)}{x^{2k+1}} \le E(x) \le \sum_{k=0}^{m}\frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac12)}{x^{2k+1}} \tag{3.5}$$

if m is even.

Remark 3.6 *In Exercise 1.38, we conclude $F(z)$ has a Taylor series that at zero,*

$$\tilde{F}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2\sqrt{\pi}} \Gamma(k+\frac{1}{2}) z^{2m+1} \tag{3.7}$$

and $F(z)$ is C^∞ on $\mathbb{R}$ and analytic away from zero.

Exercise 3.8 *Show that $z = 0$ is an isolated singularity of $F(z)$. Using Remark 1.32, show that F is unbounded as 0 is approached along some directions in the complex plane.*

Notes. (1) The series (3.7) is not related in any immediate way to the Laurent series of f at 0. Laurent series converge. Think carefully about this distinction and why the positive index coefficients do not coincide.

(2) The rate of convergence of the Laurent series of F is slower as 0 is approached, quickly becoming numerically useless. By contrast, the precision gotten from (3.5) near zero is such that for $z = 10$ the relative error in calculating f is about $5.3 \cdot 10^{-42}\%$ (check) ! However, of course (3.5) is divergent and it cannot be used to calculate *exactly* for any z.

3.0g Laplace's method for linear ODEs with first order polynomial coefficients

Equations of the form

$$\sum_{k=0}^{n} (a_k x + b) y^{(k)} = 0 \tag{3.9}$$

can be solved through explicit integral representations of the form

$$\int_{\mathcal{C}} e^{-xp} F(p) dp \tag{3.10}$$

with F expressible by quadratures and where $\mathcal{C}$ is a contour in $\mathbb{C}$, which has to be chosen subject to the following **conditions**:

- The integral (3.10) should be convergent, together with sufficiently many x-derivatives, and not identically zero.
- The function $e^{-xp}F(p)$ should vanish with sufficiently many derivatives at the endpoints, or more generally, the contributions from the endpoints when integrating by parts should cancel out.

Then it is clear that the equation satisfied by F is first order linear homogeneous, and then it can be solved by quadratures. It is not very difficult to analyze this method in general, but this would be beyond the purpose of this course.

We illustrate the method on Airy's equation

$$y'' = xy \tag{3.11}$$

Under the restrictions above we can check that F satisfies the equation

$$p^2 F = F' \tag{3.12}$$

Then $F = \exp(p^3/3)$ and we get a solution in the form

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{-xp+p^3/3}dp \tag{3.13}$$

along some curve that crosse the real line. It is easy to check the restrictions for $x \in \mathbb{R}^+$, except for the fact that the integral is not identically zero. We can achieve this at the same time as finding the asymptotic behavior of the Ai function.

Solutions of differential or difference equations can be represented in the form

$$F(x) = \int_a^b e^{xg(s)} f(s)ds \tag{3.14}$$

with simpler g and f in wider generality, as it will become clear in later chapters.

3.1 Laplace, stationary phase and saddle point methods

These deal with the behavior for large x of integrals of the form (3.14). We distinguish three particular cases: (1) The case where all parameters are real (dealt with by the so-called Laplace method); (2) The case where everything is real except x which is purely imaginary (stationary phase method) and (3) The case when f and g are analytic (steepest descent method). In this latter case, the integral may also come as a contour integral along some path.

(1) **The Laplace method.** Even when very little regularity can be assumed about the functions, we can still infer something about the large x behavior of (3.14).

Proposition 3.15 *If $f(s) \in L^\infty([a, b])$ then*

$$\lim_{x\to+\infty}\left(\int_a^b e^{xf(s)}ds\right)^{1/x} = e^{\|f\|_\infty}$$

PROOF This is simply the fact that $\|f\|_n \to \|f\|_\infty$. ∎

Note that this does not ensure even a first term in an asymptotic expansion in the sense of (1.12). For that, more regularity is needed.

Heuristics. The intuitive idea is that if x is large and g has a unique absolute maximum, the absolute maximum in s of $\phi(x;s) = \exp(xg(s))$ exceeds, for large x, by a large amount the value of ϕ at any point neighboring point. Then the contribution of the integral outside a tiny neighborhood of the maximum point is negligible. But in a neighborhood of the maximum point, both f and g are very well approximated by their local expansion. For example, assume the absolute maximum is at the left end, $x = 0$ and we have $f(0) \neq 0$ and $g'(0) = -\alpha < 0$. Then,

$$\int_0^a e^{xg(s)} f(s)ds \approx \int_0^a e^{xg(0)-\alpha xs} f(0)ds$$
$$\approx f(0)e^{xg(0)} \int_0^\infty e^{-\alpha xs} ds = f(0)e^{xg(0)} \frac{1}{\alpha x} \quad (3.16)$$

Watson's Lemma, proved in the sequel, is perhaps the ideal way to make the previous argument rigorous, but for the moment we just make the approximate reasoning into a proof following the same line of reasoning.

Proposition 3.17 *(the case when g is maximum at one endpoint). Assume f is continuous on $[a,b]$, $f(a) \neq 0$, g is in $C^1[a,b]$ and $g' < -\alpha < 0$ on $[a,b]$. Then*

$$J_x := \int_a^b f(s)e^{xg(s)}ds = \frac{f(a)e^{xg(a)}}{x|g'(a)|}(1 + o(1)) \quad (x \to +\infty) \quad (3.18)$$

Note: Since the derivative of g enters in the final result, regularity is clearly needed.

PROOF Without loss of generality, we may assume $a = 0$, $b = 1$, $f(0) > 0$. Let ϵ be small enough and choose δ such that if $x < \delta$ we have $|f(x)-f(0)| < \epsilon$ and $|g'(x) - g'(0)| < \epsilon$.
We write

$$\int_0^1 f(s)e^{xg(s)}ds = \int_0^\delta f(s)e^{xg(s)}ds + \int_\delta^1 f(s)e^{xg(s)}ds \quad (3.19)$$

the last integral in (3.19) is bounded by

$$\int_\delta^1 f(s)e^{xg(s)}ds \le \|f\|_\infty e^{xg(0)} e^{x(g(\delta)-g(0))} \quad (3.20)$$

For the middle integral in (3.19) we have

$$\int_0^\delta f(s)e^{xg(s)}ds \le (f(0)+\epsilon)\int_0^\delta e^{x\left[g(0)+(g'(0)+\epsilon)s\right]}ds$$
$$\le -\frac{e^{xg(0)}}{x}\frac{f(0)+\epsilon}{g'(0)+\epsilon}\left[1-e^{x\delta(g'(0)+\epsilon)}\right] \quad (3.21)$$

Combining these estimates, as $x \to \infty$ we thus obtain

$$\limsup_{x\to\infty} xe^{-xg(0)}\int_0^1 f(s)e^{xg(s)}ds \le -\frac{f(0)+\epsilon}{g'(0)+\epsilon} \quad (3.22)$$

A lower bound is obtained in a similar way. Since ϵ is arbitrary, the result follows. ∎

When the maximum of g is reached inside the interval of integration, sharp estimates require more regularity.

Proposition 3.23 *(Interior maximum) Assume $f \in C[-1,1]$, $g \in C^2[-1,1]$ has a unique absolute maximum (say at $x=0$) and that $f(0) \ne 0$ (say $f(0) > 0$) and $g''(0) < 0$. Then*

$$\int_{-1}^1 f(s)e^{xg(s)}ds = \sqrt{\frac{2\pi}{x|g''(0)|}}f(0)e^{xg(0)}(1+o(1)) \quad (x\to+\infty) \quad (3.24)$$

PROOF The proof is similar to the previous one. Let ϵ be small enough and let δ be such that $|s|<\delta$ implies $|g''(s)-g''(0)|<\epsilon$ and also $|f(s)-f(0)|<\epsilon$.
We write

$$\int_{-1}^1 e^{xg(s)}f(s)ds = \int_{-\delta}^\delta e^{xg(s)}f(s)ds + \int_{|s|\ge\delta} e^{xg(s)}f(s)ds \quad (3.25)$$

The last term will not contribute in the limit since by our assumptions for some $\alpha > 0$ and $|s| > \delta$ we have $g(s) - g(0) < -\alpha < 0$ and thus

$$e^{-xg(0)}\sqrt{x}\int_{|s|\ge\delta} e^{xg(s)}f(s)ds \le 2\sqrt{x}\|f\|_\infty e^{-x\alpha} \to 0 \text{ as } x\to\infty \quad (3.26)$$

On the other hand,

$$\int_{-\delta}^{\delta} e^{xg(s)} f(s) ds \le (f(0)+\epsilon) \int_{-\delta}^{\delta} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2} ds$$
$$\le (f(0)+\epsilon) e^{xg(0)} \int_{-\infty}^{\infty} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2} ds = \sqrt{\frac{2\pi}{|g''(0)-\epsilon|}}(f(0)+\epsilon)e^{xg(0)} \tag{3.27}$$

An inequality in the opposite direction is obtained in the same way by noting that

$$\frac{\int_{-a}^{a} e^{-xs^2} ds}{\int_{-\infty}^{\infty} e^{-xs^2} ds} \to 1 \quad \text{as } x \to \infty \tag{3.28}$$

as can be seen by changing variables to $u = sx^{-\frac{1}{2}}$. ☐

With appropriate decay conditions, the interval of integration does not have to be compact. For instance, let $J \subset \mathbb{R}$ be an interval (finite or not) and $[a, b] \subset J$.

Proposition 3.29 *(Interior maximum, noncompact interval) Assume* $f \in C[a, b] \cap L^\infty(J)$, $g \in C^2[a, b]$ *has a unique absolute maximum at* $x = c$ *and that* $f(c) \ne 0$ *and* $g''(c) < 0$.

Assume further that g *is measurable in* J *and* $g(c) - g(s) = \alpha + h(s)$ *where* $\alpha > 0$, $h(s) > 0$ *on* $J \setminus [a, b]$ *and* $e^{-h(s)} \in L^1(J)$. *Then,*

$$\int_A^B f(s) e^{xg(s) ds} = \sqrt{\frac{2\pi}{x|g''(c)|}} f(c) e^{xg(c)} (1 + o(1)) \quad (x \to +\infty) \tag{3.30}$$

PROOF This case reduces to the compact interval case by noting that

$$\left| \sqrt{x} e^{-xg(c)} \int_{J\setminus[a,b]} e^{xg(s)} f(s) ds \right| \le \sqrt{x} \|f\|_\infty e^{-x\alpha} \int_J e^{-xh(s)} ds$$
$$\le Const. \sqrt{x} e^{-x\alpha} \to 0 \text{ as } x \to \infty \tag{3.31}$$

☐

Example. We see that the last proposition applies to the Gamma function by writing

$$n! = \int_0^\infty e^{-t} t^n dt = n^{n+1} \int_0^\infty e^{n(-s+\ln s)} ds \tag{3.32}$$

whence we get Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1+o(1)); \quad n \to +\infty$$

3.1a Watson's Lemma

In view of the wide applicability of Écalle-Borel summability, for instance to formal solutions of generic analytic differential equations as we shall see later, many functions admit representations as Laplace transforms

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-xp}F(p)dp \tag{3.33}$$

The behavior of $\mathcal{L}F$ for large x relates to the behavior for small p of F.

For the error function note that

$$\int_x^\infty e^{-s^2}ds = x\int_1^\infty e^{-x^2u^2}du = \frac{x}{2}e^{-x^2}\int_0^\infty \frac{e^{-x^2p}}{\sqrt{p+1}}dp$$

For the Gamma function, writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ in (3.32) we can make the substitution $t - \ln t = p$ in each integral and obtain (see §3.1c)

$$n! = n^{n+1}e^{-n}\int_0^\infty e^{-np}G(p)dp$$

Lemma 3.34 *Let* $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ *and assume*

$$F(p) \sim p^\beta \quad \text{as } p \to 0^+$$

with $\Re(\beta) > -1$. *Then*

$$\int_0^\infty F(p)e^{-px}dp \sim \Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $U(p) = p^{-\beta}F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose C and a positive so that $|F(p)| < C|p^\beta|$ on $[0,a]$. Since

$$\left|\int_a^\infty F(p)e^{-px}\mathrm{d}p\right| \le e^{-xa}\|F\|_1 \tag{3.35}$$

we have by dominated convergence, and after the change of variable $s = p/|x|$,

$$x^{\beta+1}\int_0^\infty F(p)e^{-px}\mathrm{d}p = e^{i\phi(\beta+1)}\int_0^\infty s^\beta U(s/|x|)\chi_{[0,a]}(s/|x|)e^{-se^{i\phi}}\mathrm{d}s$$
$$+ O(|x|^{\beta+1}e^{-xa}) \to \Gamma(\beta+1) \quad (|x| \to \infty) \tag{3.36}$$

Watson's Lemma

This important tool finds the asymptotic series at infinity of $(\mathcal{L}F)(x)$ provided $F(p)$ a series at zero.

Lemma 3.37 *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants β_i with $\Re(\beta_i) > 0$, $i = 1, 2$. Then, for $a \le \infty$,*

$$f(x) = \int_0^a e^{-xp} F(p)dp \sim \sum_{k=0}^{\infty} c_k \Gamma(k\beta_1 + \beta_2) x^{-k\beta_1 - \beta_2}$$

along any ray ρ in the open right half plane H.

Proof. Induction, using Lemma 3.34. □

Remark 3.38 (i) Clearly, the asymptotic formula holds if $\int_0^{\infty}$ is replaced by $\int_0^a$, $a > 0$, since we can always extend F and the integral with zero for $x > a$.

(ii) The presence of $\Gamma(k\beta_1 + \beta_2)$ makes the large x series often divergent even when F is analytic at zero. However, the asymptotic series of f is still the term-by-term Laplace transform of the series of F at zero, whether a is finite or not or the series converges or not. This freedom also shows that some information is lost.

3.1b Applications. 1. Laplace's method revisited

(i) **Absolute maximum at left endpoint with nonvanishing derivative**.

Proposition 3.39 *Let g be analytic (smooth) on $[a, b]$ where $g' < -\alpha < 0$. Then the problem of finding the large x behavior of F in (3.14) is analytically (resp. smoothly) conjugated to the canonical problem of the large x behavior of*

$$\int_{g(a)}^{g(b)} e^{xs} H(s)ds = e^{xg(a)} \int_0^{g(a)-g(b)} e^{-xu} H(g(a) - u)du \tag{3.40}$$

with $H(s) = f(\varphi(s))\varphi'(s)$

This just means that is, we can reduce to (3.40) after analytic (smooth) changes of variable, and the change is clear, $g(s) = u$, $\varphi = g^{-1}$. The proof of smoothness is immediate, and we leave it to the reader.

Note that we have not required that $f(0) \neq 0$ anymore. If H is smooth and some derivative at zero is nonzero, Watson's lemma clearly provides the asymptotic expansion of the last integral in (3.40). The asymptotic series is dual, as in Lemma 3.37 to the series of H at $g(a)$.

(ii) **Absolute maximum at an interior point with nonvanishing second derivative.**

Proposition 3.41 *Let g be analytic (smooth) on the interval $a \le 0 \le b$ (we assume at least one endpoint is nonzero, otherwise the problem is trivial) where $g'' < -\alpha < 0$ and assume $g(0) = 0$. Then the problem of finding the large x behavior of F in (3.14) is analytically (resp. smoothly) conjugated to the canonical problem of the large x behavior of*

$$\int_{-\sqrt{|g(a)|}}^{\sqrt{|g(b)|}} e^{-xu^2} H(u)du$$
$$= -\frac{1}{2}\int_0^{-|g(a)|} e^{-xv} H(-v^{\frac{1}{2}})v^{-\frac{1}{2}}dv + \frac{1}{2}\int_0^{|g(b)|} e^{-xv} H(v^{\frac{1}{2}})v^{-\frac{1}{2}}dv \quad (3.42)$$

with $H(s) = f(\varphi(s))\varphi'(s)$, $\varphi^2(s) = -g(u)$; Watson's Lemma applies to the last representation. If $g, f \in C^k$, then $\varphi \in C^{k-1}$ and $H \in C^{k-2}$.

PROOF We note that near zero $g = -s^2h(s)$ where $h(0) = 1$. Thus $\sqrt{h}$ is well defined and analytic (smooth) near zero; we choose the usual branch and note that the implicit function theorem applies to the equation $s\sqrt{h}(s) = u$ throughout $[a, b]$. The rest is left to the reader. ∎

Exercise 3.43 *Assume $H \in C^\infty$ and $a > 0$. Show that the asymptotic behavior of*

$$\int_{-a}^{a} e^{-xu^2} H(u)du \quad (3.44)$$

is given by

$$\sum_{l=0}^{\infty} \frac{1}{2l!}\int_{-\infty}^{\infty} H^{(2l)}(0)u^{2l}e^{-xu^2}du = \frac{1}{2}\sum_{l=0}^{\infty}\frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(l+1)}H^{(2l)}(0)x^{-\frac{1}{2}-l} \quad (3.45)$$

(This is a formal series, not expected to converge, in general.) In other words, the classical asymptotic series is obtained by formal expansion of H at the critical point $x = 0$ and termwise integration, extending the limits of integration to infinity and the odd terms do not contribute, by symmetry. The value of a does not enter the formula, so once more, information is lost.

Exercise 3.46 *Generalize (3.24) to the case when $g \in C^4[-1, 1]$ and the first three derivatives vanish at the unique point of absolute maximum, $s = 0$.*

Exercise 3.47 ** Consider the problem (3.18) with f and g smooth and take $a = 0$ for simplicity. Show that the asymptotic expansion of the integral equals the one obtained by the following formal procedure: we expand f and g in Taylor series at zero, replace f in the integral by its Taylor series, keep $ng'(0)$ in the exponent, reexpand $e^{ng''(0)s^2/2!+\cdots}$ in series in s, and integrate the resulting series term by term. The contribution of a term cs^m is $c(g'(0))^{-m-1}m!/x^{-m-1}$.*

Exercise 3.48 (*) *Consider now the inner maximum problem in the form (3.24), with f and g smooth at zero. Formulate and prove a procedure similar to the one in the previous problem. Now the odd terms can be discarded since by symmetry they give a zero contribution. An even power cs^{2m} gives rise to a contribution $c2^{m+1/2}\Gamma(m+1/2)(g''(0))^{-m-1/2}x^{-m-1/2}$.*

Exercise 3.49 (*) *Use Exercise (3.47) to show that the Taylor coefficients of the inverse function ϕ^{-1} can be obtained from the Taylor coefficients of ϕ in the following way. Assume $\phi'(0)=1$. We let $P_n(x)$, a polynomial in x, be the $n-th$ asymptotic coefficient of $e^{y\phi(x/y)}$ as $y\to\infty$. The desired coefficient is $\frac{1}{n!}\int_0^\infty e^{-x}P_{n+1}(x)dx$.*

3.1c 2. The Gamma function

We start from the representation

$$n! = \int_0^\infty t^n e^{-t}dt = n^{n+1}\int_0^\infty e^{-n(s-\ln s)}ds \tag{3.50}$$

We can now use the results in §2.1d .4 and Watson's Lemma to find the behavior of $n!$. With $s=1+z, z-\ln(1+z)=u^2$, $dz=F(u)du$ we have

$$F(u)=\sqrt{2}+\frac{4}{3}u+\frac{\sqrt{2}}{6}u^2-\frac{8}{135}\,u^3+\frac{\sqrt{2}}{216}\,u^4+\frac{8}{2835}\,u^5-\cdots \tag{3.51}$$

Exercise 3.52 (*) *Note the pattern of signs: $++--\cdots$. Show that this pattern continues indefinitely.*

We have, using Exercise 3.43,

$$\int_0^\infty e^{-n(s-\ln s)}ds \sim n^n e^{-n}\sqrt{2}\int_{-\infty}^\infty\left(1+\frac{u^2}{6}+\cdots\right)e^{-nu^2}du \tag{3.53}$$

or

$$n! \sim \sqrt{2\pi n}n^n e^{-n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}+...\right) \tag{3.54}$$

3.1d 3. Asymptotic expansions of differential equations

Consider the differential equation

$$y'-y=x^{-2}-y^3 \tag{3.55}$$

for large x. If f behaves like a power series in inverse powers of x then y' and y^3 are small, and we can proceed as in §2.1e to get, formally,

$$y(x)\sim -x^{-2}+2\,x^{-3}-6\,x^{-4}+24\,x^{-5}-119\,x^{-6}+708\,x^{-7}-4926\,x^{-8}+\cdots \tag{3.56}$$

How do we prove this rigorously? One way is to truncate the series in (3.56) to n terms, say the truncate is y_n, and look for solutions of (3.55) in the form $y(x) = y_n(x) + \delta(x)$. For $\delta(x)$ we write a contractive equation in a space of functions with norm $\sup_{x>x_0} |x^{n+1}\delta(x)|$.

Exercise 3.57 *Carry out the construction above and show that there is a solution with an asymptotic power series starting as in (3.56).*

Alternatively, we can write an integral equation just for y, as in §2.1e and show that it is contractive in a space of functions with norm $\sup_{x>x_0} |x^2 y(x)|$. Then, knowing that it is a contraction, we can iterate the operator a given number of times, with controlled errors. First,

$$e^x \int_x^\infty e^{-s} s^{-2} ds = e^x \frac{1}{x} \int_1^\infty e^{-xs} s^{-2} ds = \frac{1}{x} \int_0^\infty e^{-xs} (1+s)^{-2} ds \\ \sim \frac{1}{x^2} - \frac{2}{x^3} + \frac{6}{x^4} - \frac{24}{x^5} + \cdots \quad (3.58)$$

Then,

$$y(x) = e^x \int_\infty^x e^{-s} s^{-2} ds - e^x \int_\infty^x e^{-s} y(s)^3 ds \quad (3.59)$$

together with contractivity in the chosen norm implies

$$y(x) = e^x \int_\infty^x e^{-s} s^{-2} ds + e^x \int_\infty^x e^{-s} O(s^{-6}) ds \\ = \frac{1}{x^2} - \frac{2}{x^3} + \frac{6}{x^4} - \frac{24}{x^5} + O(x^{-6}) \quad (3.60)$$

We can use (3.60) and (3.58) in (3.59) to obtain the asymptotic expansion of y to $O(x^{-10})$, and by induction, to all orders.

Exercise 3.61 *Based on (3.60) and (3.58) show that y has an asymptotic power series in the open right half plane. In particular, the asymptotic series is differentiable (why?).*

To find the power series of y, we can also note that the asymptotic series must be a formal power series solution of (3.55) (why?). Say we want five terms of the expansion. Then we insert $y = a_2 s^{-2} + a_3 s^{-3} + a_4 s^{-4} + a_5 s^{-5} + a_6 x^{-6}$ in (3.55) and solve for the coefficients. We get

$$\frac{1+a_2}{x^2} + \frac{2a_2+a_3}{x^3} + \frac{3a_3+a_4}{x^4} + \frac{4a_4+a_5}{x^5} + \frac{a_6+5a_5+a_2^3}{x^6} = 0 \quad (3.62)$$

and it follows immediately that

$$a_2 = -1, a_3 = 2, a_4 = -6, a_5 = 24, a_6 = -119 \quad (3.63)$$

Note that the signs alternate! This is true to all orders and has a simple explanation to which we will return later.

*

The integral in (3.13) can be brought to Watson's Lemma setting by simple changes of variables. First we put $p = q\sqrt{x}$ and get

$$\mathrm{Ai}(x) = \frac{1}{2\pi i} x^{1/2} \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{-x^{3/2}(q-q^3/3)} dq \tag{3.64}$$

We see that $(q - q^3/3)' = 1 - q^2 = 0$ iff $q = \pm 1$. We now choose the contour of integration to pass through $q = 1$. It is natural to substitute $q = 1 + z$ and then the integral becomes

$$\mathrm{Ai}(x) = \frac{e^{-\frac{2}{3}x^{3/2}} x^{1/2}}{2\pi i} \Big[\int_{\infty e^{-\pi i/3}}^{0} e^{-x^{3/2}(z^2+z^3/3)} dq + \int_0^{\infty e^{\pi i/3}} e^{-x^{3/2}(z^2+z^3/3)} dq \Big] \tag{3.65}$$

Along each path, the $z^2 + z^3/3 = s$ has a unique well defined solution $z_{1,2}(s)$ where we choose $\arg(z_1) \to \pi/2$, as $s \to 0^+$. As $z_1 \to \infty e^{\pi i/3}$ we have $s \to \infty$ tangent to $\mathbb{R}^+$. We can homotopically deform the contour and write

$$\mathrm{Ai}(x) = \frac{e^{-2/3x^{3/2}} x^{1/2}}{2\pi i} \Big[\int_0^\infty e^{-sx^{3/2}} \frac{dz_1}{ds} ds - \int_0^\infty e^{-sx^{3/2}} \frac{dz_2}{ds} ds \Big] \tag{3.66}$$

where the analysis proceeds as in the Gamma function case, invert $z^2 + z^3/2$ near zero and calculate the expansion to any number of orders.

Exercise 3.67 ** Complete the details of the analysis and show that*

$$Ai(x) = \frac{1}{2\sqrt{\pi} x^{1/4}} e^{-\frac{2}{3}x^{3/2}} (1 + o(1)) \quad (x \to +\infty) \tag{3.68}$$

§2.1d .4.

Again, once we know that an asymptotic series exists, and it is differentiable (by Watson's Lemma), to obtain the first few terms of the asymptotic series it is easier to deal directly with the differential equation, see also [3], pp. 101. We can proceed as follows. The expansion is not a power series, but its logarithmic derivative is. We then substitute $y(x) = e^{w(x)}$ in the equation (a simple instance of the WKB method, discussed later), we get $(w')^2 + w'' = x$, and for a power series we expect $w'' \ll (w')^2$ (check that this would be true if w is a differentiable asymptotic power series), and set the iteration scheme

$$(w')_{n+1} = -\sqrt{x - (w'_{n+1})'}$$

and get

$$w' = -\sqrt{x} - \frac{1}{4x} + \frac{5}{32} x^{-5/2} - \frac{15}{64} x^{-4} + \frac{1105}{2048} x^{-11/2} - \cdots$$

and derive from it

$$y \sim Const. e^{-\frac{2}{3}x^{3/2}} \left(1 - \frac{5}{48}\, x^{-3/2} + \frac{385}{4608}\, x^{-3} - \frac{85085}{663552}\, x^{-9/2} + \cdots \right)$$

and the constant is obtained by comparing to (3.68).

The Bessel equation. This is the equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \tag{3.69}$$

For $\nu = 0$ we get

$$xy'' + y' + xy = 0 \tag{3.70}$$

to which it is easy to apply Laplace's method. We get

$$(p^2 Y)' - pY + Y' = 0 \ \Rightarrow Y = C(p^2+1)^{-1/2} \tag{3.71}$$

We get solutions by taking contours from $+\infty$, around a singularity and back to infinity in

$$\int_C \frac{e^{-xp}}{\sqrt{p^2+1}} dp \tag{3.72}$$

or around both branch points.

Exercise 3.73 * *Find the relations between these integrals (we know that there are exactly two linearly independent solutions to (3.70)).*

To find the asymptotic behavior of an integral starting at $\infty + i - i\epsilon$, going around $x = i$ and then to $\infty + i + i\epsilon$, we note that this integral equals

$$2\int_i^{\infty+i} \frac{e^{-xp}}{\sqrt{p^2+1}} dp = 2e^{-ix} \int_0^\infty \frac{e^{-xs}}{\sqrt{s^2+2is}} ds$$
$$\sim e^{-ix}\sqrt{\pi} \left[\frac{1-i}{\sqrt{x}} + \frac{1}{8}\frac{1+i}{x^{3/2}} - \frac{9}{128}\frac{1-i}{x^{5/2}} + \cdots \right] \tag{3.74}$$

by Watson's lemma.

Exercise 3.75 * *Using the binomial formula, find the general term in the expansion (3.74).*

3.1e 4. Borel-Ritt Lemma

Any asymptotic series at infinity is the asymptotic series in a half plane of some (many in fact) entire functions. First a weaker result.

Proposition 3.76 *Let $\tilde{f}(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series. There exists a function f such that $f(z) \sim \tilde{f}(z)$ as $z \to 0$.*

PROOF

The following elementary line of proof is similar to the technique of optimal truncation of series, a very useful procedure in asymptotics.

By Remark 1.32 we can assume, without loss of generality, that the series has zero radius of convergence.

For every z define $N(z) = \max\{N : \forall\, n \le N,\ |a_n z^{n/2}| \le 2^{-n}$. We have $N(z) < \infty$, otherwise, by Abel's theorem, the series would have nonzero radius of convergence. Noting that for any n we have $n \ln |z| \downarrow -\infty$ as $|z| \downarrow 0$ it follows that $N(z)$ is nonincreasing as $|z|$ decreases and that $N(z) \to \infty$ as $z \to 0$. Consider

$$f(z) = \sum_{j=0}^{N(z)} a_n z^n$$

Let N be given and choose z_0; $|z_0| < 1$ such that $N(z_0) \ge N$. For $|z| < |z_0|$ we have $N(z) \ge N(z_0) \ge N$ and thus

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| = \left| \sum_{n=N+1}^{N(z)} a_n z^n \right| \le \sum_{j=N+1}^{N(z)} |z^{j/2}| 2^{-j} \le |z|^{N/2+1/2}$$

Using now Lemma 1.13, the proof follows. ∎

The function f is certainly not unique. Given a power series there are many functions asymptotic to it. Indeed there are many functions asymptotic to the (identically) zero power series at zero, in any sectorial punctured neighborhood of zero in the complex plane, and even on the Riemann surface of the log on $\mathbb{C} \setminus \{0\}$, e.g. $e^{-x^{-1/n}}$ has this property in a sector of width $2n\pi$.

Lemma 3.77 (Borel-Ritt) *Given a formal power series* $\tilde{f} = \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}$ *there exists an entire function* $f(x)$*, of exponential order one, which is asymptotic to* $\tilde{f}$ *in the right half plane, i.e., if* $\phi \in (-\pi/2, \pi/2)$ *then*

$$f(x) \sim \tilde{f} \text{ as } x = \rho e^{i\phi}, \quad \rho \to +\infty$$

PROOF Let $\tilde{F} = \sum_{k=0}^{\infty} \frac{c_k}{(k-1)!} p^{k-1}$, let $F(p)$ be a function asymptotic to $\tilde{F}$ as in Proposition 3.76. Then clearly the function

$$f(x) = \int_0^1 e^{-xp} F(p)\, dp$$

is entire, bounded by $Const.e^{|x|}$, and, by Watson's Lemma has the desired properties.

□

Exercises.
(1) How can this method be modified to give a function analytic in a sector of opening $2\pi n$ for an arbitrary fixed n (not necessarily entire or with given growth) which is asymptotic to $\tilde{f}$?

(2) Assume F is bounded on $[0,1]$ and has an asymptotic expansion $F(t) \sim \sum_{k=0}^{\infty} c_k t^k$ as $t \to 0^+$. Let $f(x) = \int_0^1 e^{-xp} F(p) dp$

(a) Find necessary and sufficient conditions on F such that $\tilde{f}$, the asymptotic power series of f for large positive x, is a convergent series for $|x| > R > 0$.

(b) Assume that $\tilde{f}$ *converges to* f. Show that f is zero.

(c) Show that in case (a) if F is analytic in a neighborhood of $[0,1]$ then $f = \tilde{f} + e^{-x}\tilde{f}_1$ where $\tilde{f}_1$ is convergent for $|x| > R > 0$.

(3) The width of the sector in Proposition 3.77 cannot be extended to a more than a half plane: Show that if f is entire, of exponential order one, and bounded in a sector of opening exceeding π then it is constant. (This follows immediately from the Phragmen-Lindelöf principle; an alternative proof can be derived from elementary properties of Fourier transforms and contour deformation.) The exponential order has to play a role in the proof: check that the function $\int_0^\infty e^{-px-p^2} dp$ is bounded for $\arg(x) \in (-\frac{3\pi}{4}, \frac{3\pi}{4})$. How wide can such a sector be made?

3.2 Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann–Lebesgue lemma.

Proposition 3.78 *Assume* $f \in L^1[0, 2\pi]$. *Then* $\int_0^{2\pi} e^{ixt} f(t) dt \to 0$ *as* $x \to \pm\infty$.

It is enough to show the result on a set which is dense in L^1. Since trigonometric polynomials are dense in $C[0, 2\pi]$ in the sup norm, and thus in $L^1[0, 2\pi]$, it suffices to look at trigonometric polynomials, thus (by linearity), at e^{ikx} for fixed k; the latter integral can be expressed explicitly and gives

$$\int_0^{2\pi} e^{ixs} e^{iks} ds = O(x^{-1}) \quad \text{for large } x. \qquad \square$$

No rate of decay of the integral in the Proposition follows without further knowledge about the regularity of f. With some regularity we have the following characterization.

Proposition 3.79 *For $\eta \in (0,1]$ let the $C^\eta[0,1]$ be the Hölder continuous functions of order η on $[0,1]$, i.e., the functions with the property that there is some C such that for all $x, x' \in [0,1]$ we have $|f(x)-f(x')| \le C|x-x'|^\eta$.*

(i) We have

$$f \in C^\eta[0,1] \Rightarrow \left|\int_0^1 f(s)e^{ixs}ds\right| \le \frac{1}{2}C\pi^\eta x^{-\eta} + O(x^{-1}) \quad as \quad x \to \infty \tag{3.80}$$

(ii) If $f \in L^1(\mathbb{R})$ and $|x|^\eta f(x) \in L^1(\mathbb{R})$ with $\eta \in (0,1]$, then its Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f(s)e^{-ixs}ds$ is in $C^\eta(\mathbb{R})$.

(iii) Let $f \in L^1(\mathbb{R})$. If $x^n f \in L^1(\mathbb{R})$ with $n \in \mathbb{N}$ then $\hat{f}$ is $n-1$ times differentiable, with the $n-1$th derivative Lipschitz continuous. If $e^{|Ax|}f \in L^1(\mathbb{R})$ then $\hat{f}$ extends analytically in a strip of width $|A|$ centered on $\mathbb{R}$.

PROOF (i) We have as $x \to \infty$ ($\lfloor \cdot \rfloor$ denotes the integer part)

$$\begin{aligned}
&\left|\int_0^1 f(s)e^{ixs}ds\right| = \\
&\left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1 \rfloor}\left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds + \int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1}) \\
&\qquad = \left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1 \rfloor}\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} \big(f(s)-f(s+\pi/x)\big)e^{ixs}ds\right| + O(x^{-1}) \\
&\qquad\qquad \le \sum_{j=0}^{\lfloor \frac{x}{2\pi}-1 \rfloor} C\left(\frac{\pi}{x}\right)^\eta \frac{\pi}{x} \le \frac{1}{2}C\pi^\eta x^{-\eta} + O(x^{-1})
\end{aligned} \tag{3.81}$$

(ii) We see that

$$\left|\frac{\hat{f}(s)-\hat{f}(s')}{(s-s')^\eta}\right| = \left|\int_{-\infty}^{\infty}\frac{e^{ixs}-e^{ixs'}}{x^\eta(s-s')^\eta}x^\eta f(x)dx\right| \le \int_{-\infty}^{\infty}\left|\frac{e^{ixs}-e^{ixs'}}{(xs-xs')^\eta}\right|\Big|x^\eta f(x)\Big|dx \tag{3.82}$$

is bounded. Indeed, by elementary geometry we see that for $|\phi_1-\phi_2| < \pi$ we have

$$|\exp(i\phi_1)-\exp(i\phi_2)| \le |\phi_1-\phi_2| \le |\phi_1-\phi_2|^\eta \tag{3.83}$$

while for $|\phi_1-\phi_2| \ge \pi$ we see that

$$|\exp(i\phi_1)-\exp(i\phi_2)| \le 2 \le 2|\phi_1-\phi_2|^\eta$$

(iii) Follows in the same way as (ii), using dominated convergence. □

Exercise 3.84 *Complete the details of this proof. Show that for any* $\eta \in (0,1]$ *and all* $\phi_{1,2} \in \mathbb{R}$ *we have* $|\exp(i\phi_1) - \exp(i\phi_2)| \le \sqrt{2}|\phi_1 - \phi_2|^{\eta}$.

Exercise 3.85 (*) *(a)* Consider the function f given by the *lacunary trigonometric series* $f(z) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta} e^{ikz}$, $\eta \in (0,1)$. Show that $f \in C^{\eta}[0, 2\pi]$. One way is to write $\phi_{1,2}$ as $a_{1,2}2^{-p}$, use the first inequality in (3.83) to estimate the terms in $f(\phi_1) - f(\phi_2)$ with $n < p$ and the simple bound $2/k^{\eta}$ for $n \ge p$. Then it is seen that $\int_0^{2\pi} e^{-iks} f(s)ds = 2\pi k^{-\eta}$ and the decay of the Fourier transform is exactly given by (3.80).

(b) Use Proposition 3.79 and the result in Exercise 3.85 to show that the function $f(t) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta} t^k$, analytic in the open unit disk, has no analytic continuation across the unit circle, that is, the unit circle is a *barrier of singularities for* f.

Note 3.86 *Dense nondifferentiability is the only way one can get very poor decay, see also Exercise 3.94.*

Notes. In part (i), compactness of the interval is crucial. In fact, the Fourier transform of an $L^2(\mathbb{R}^+)$ entire function may not necessarily decrease pointwise. Indeed, the function $\hat{f}(x) = 1$ on the interval $[n, n+e^{-n^2}]$ for $n \in \mathbb{N}$ and zero otherwise is in $L^1(\mathbb{R})$ and further has the property that $e^{|Ax|}\hat{f} \in L^1(\mathbb{R})$ for any $A \in \mathbb{R}$, and thus $\mathcal{F}^{-1}\hat{f}$ is entire. Thus $\hat{f}$ is the Fourier transform of an entire function, it equals $\mathcal{F}^{-1}\hat{f}$ a.e., and nevertheless it does not decay pointwise as $x \to \infty$. Evidently the issue here is poor behavior of f at infinity, otherwise integration by parts would show decay.
(2) It is worth noting that in Laplace type integrals Watson's Lemma implies that it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral whereas in Fourier-like integrals, the considerably weaker decay (3.80) is optimal.

Proposition 3.87 *Assume* $f \in C^n[a,b]$. *Then we have*

$$\int_a^b e^{ixt} f(t)dt = e^{ixa}\sum_{k=1}^{n} c_k x^{-k} + e^{ixb}\sum_{k=1}^{n} d_k x^{-k} + o(x^{-n})$$

$$= e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + \ldots + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b + o(x^{-n}) \quad (3.88)$$

PROOF This follows by integration by parts and the Riemann-Lebesgue Lemma since

$$\int_a^b e^{ixt} f(t)dt = e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b + \frac{(-1)^n}{(ix)^n}\int_a^b f^{(n)}(t)e^{ixt}dt \quad (3.89)$$

∎

Corollary 3.90 *(1) Assume $f \in C^\infty[0, 2\pi]$ is periodic with period 2π. Then $\int_0^{2\pi} f(t)e^{int} = o(n^{-m})$ for any $m > 0$ as $n \to +\infty, n \in \mathbb{Z}$.*

(2) Assume $f \in C_0^\infty[a, b]$, a smooth function which vanishes with all derivatives at the endpoints; then $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ for any $m > 0$ as $x \to +\infty$.

Exercise 3.91 Show that if f is analytic in a neighborhood of $[a, b]$ but not entire, then both series in (3.88) have zero radius of convergence.

Exercise 3.92 In Corollary 3.90 (2) show that $\limsup_{x\to\infty} e^{\epsilon|x|}|\hat{f}(x)| = \infty$ for any $\epsilon > 0$ unless $f = 0$.

Exercise 3.93 For smooth f, the interior of the interval does not contribute because of cancellations: rework the argument in the proof of Proposition 3.79 under smoothness assumptions. If we write $f(s+\pi/x) = f(s) + f'(s)(\pi/x) + \frac{1}{2}f''(c)(\pi/x)^2$ cancellation is manifest.

Exercise 3.94 Show that if f is piecewise differentiable and the derivative is in L^1, then the Fourier transform is $O(x^{-1})$.

3.2 .1 Oscillatory integrals with monotonic phase

Proposition 3.95 *Let the real valued functions $f \in C^m[a, b]$ and $g \in C^{m+1}[a, b]$ and assume $g' \neq 0$ on $[a, b]$. Then*

$$\int_a^b f(t)e^{ixg(t)}dt = e^{ixg(a)}\sum_{k=1}^m c_k x^{-k} + e^{ixg(b)}\sum_{k=1}^m d_k x^{-k} + o(x^{-m}) \qquad (3.96)$$

as $x \to \pm\infty$, where the coefficients c_k and d_k can be computed by Taylor expanding f and g at the endpoints of the interval of integration.

This essentially follows from Proposition 3.39, since the problem is amenable by smooth transformations to the setting of Proposition 3.87. Carry out the details.

3.3 Stationary phase method

In general, the behavior of oscillatory integrals of the form (3.96) comes from:

- Endpoints
- Stationary points
- Singularities of f or g.

We consider now the case when $g(s)$ has a stationary point inside the interval $[a,b]$. In this case the main contribution to the integral on the lhs of (3.96) comes from a neighborhood of the stationary point of g since around that point the oscillations that make the integral small are less rapid.

We have the following result:

Proposition 3.97 *Assume f,g are real valued $C^\infty[a,b]$ functions and that $g'(c)=0$ $g''(x)\neq 0$ on $[a,b]$. Then for any $m\in\mathbb{N}$ we have*

$$\int_a^b f(s)e^{ixg(s)}ds = e^{ixg(c)}\sum_{k=1}^{2m} c_k x^{-k/2} + e^{ixg(a)}\sum_{k=1}^{m} d_k x^{-k} + e^{ixg(b)}\sum_{k=1}^{m} e_k x^{-k} + o(x^{-m}) \quad (3.98)$$

for large x, where the coefficients of the expansion can be calculated by Taylor expansion around a,b and c of the integrand as follows from the proof. In particular, we have

$$c_1 = \sqrt{\frac{2\pi i}{g''(c)}}f(c)$$

PROOF Again, by smooth changes of variables, the problem is amenable to the problem of the behavior of

$$J = \int_{-a}^{a} H(u)e^{ixu^2}du \quad (3.99)$$

which is given, as we will see in a moment, by

$$J \sim \sum_{k\geq 0}\left(e^{-ixa^2}\int_{-a}^{i\infty e^{i\pi/4}}\frac{H^{(k)}(-a)}{k!}(u+a)^k e^{ixu^2}du\right.$$
$$\left.-e^{ixa^2}\int_{a}^{i\infty e^{i\pi/4}}\frac{H^{(k)}(a)}{k!}(u-a)^k e^{ixu^2}du+\int_{-\infty e^{-i\pi/4}}^{i\infty e^{i\pi/4}}\frac{H^{(2k)}(0)}{2k!}u^{2k}e^{ixu^2}du\right) \tag{3.100}$$

in the sense that J minus a finite number of terms of the series is small on the scale of the last term kept.

For a conveniently small ϵ we break the integral and are left with estimating the three integrals problems

$$J_1=\int_{-a}^{-\epsilon}H(u)e^{ixu^2}du;\quad J_2=\int_{\epsilon}^{a}H(u)e^{ixu^2}du;\quad J_3=\int_{-\epsilon}^{\epsilon}H(u)e^{ixu^2}du$$

By smooth changes of variables, J_1 is turned into

$$\int_{\epsilon^2}^{a^2}H_1(v)e^{ixv}dv \tag{3.101}$$

where H, H_1 are smooth. Proposition 3.87 applies to the integral (3.101); J_3 is treated similarly. For the second integral we write

$$J_2-\sum_{l=0}^{m}\frac{H^{(k)}(0)}{k!}\int_{-\epsilon}^{\epsilon}u^k e^{ixu^2}du$$
$$=\int_{-\epsilon}^{\epsilon}u^{m+1}e^{ixu^2}F(u)du=\int_0^{\epsilon^2}v^{\frac{m-1}{2}}F_1(v)e^{ixv}du \tag{3.102}$$

where F_1 is smooth. We can integrate by parts $m/2$ times in the last integral. Thus, combining the results from the two cases, we see that J has an asymptotic series in powers of $x^{-1/2}$. Since there exists an asymptotic series, we know it is unique. Then, the series of J cannot of course depend on an arbitrarily chosen parameter ϵ. Thus, we do not need to keep any endpoint terms at $\pm\epsilon$: they cancel out. ◻

Note It is easy to see that in the settings of Watson's Lemma and of Propositions 3.87, 3.95 and 3.97 the asymptotic expansions are differentiable, in the sense that the integral transforms are differentiable and their derivative is asymptotic to the formal derivative of the associated expansion.

3.3a Analytic integrands

In this case, contour deformation is used to transform oscillatory exponentials with decaying ones. A classical result in this direction is the following.

Proposition 3.103 (Fourier coefficients of analytic functions) *Assume f is periodic of period 2π, analytic in the strip $\{z : |\Im(z)| < R\}$ and continuous in its closure. Then the Fourier coefficients $c_n(2\pi)^{-1}\int_0^{2\pi} e^{int}f(t)dt$ are $o(e^{-|n|R})$ for large $|n|$. Conversely, if $c_n = o(e^{-|n|R})$, then f is analytic in the given strip.*

PROOF We take $n > 0$ the opposite case being very similar. By analyticity we have

$$\int_0^{2\pi} e^{int}f(t)dt = \int_0^{iR} e^{int}f(t)dt + \int_{iR}^{iR+2\pi} e^{int}f(t)dt - \int_{2\pi}^{2\pi+iR} e^{int}f(t)dt$$

The first and last integrals on the rhs cancel by periodicity while the middle one equals

$$e^{-nR}\int_0^{2\pi} e^{ins}f(s+iR)ds = o(e^{-nR}) \quad \text{as} \quad n \to \infty$$

The converse is straightforward. ◻

3.3b Examples

Example 1. Consider the problem of finding the asymptotic behavior of the integral

$$I(n) = \int_{-\pi}^{\pi} \frac{e^{-int}}{2 - e^{it}}dt := \int_{-\pi}^{\pi} F(t)dt$$

as $n \to \infty$. We see by corollary 3.90 that $J = o(x^{-m})$ for any $m \in \mathbb{N}$. Proposition 3.103 tells us more, namely that the integral is exponentially small. But both methods only give us *upper bounds* for the decay, and no detailed description.

In this simple example however we could simply expand convergently the integrand and use dominated convergence:

$$\int_{-\pi}^{\pi} \frac{e^{-int}}{2 - e^{it}} = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} 2^{-k-1}e^{-it(n-k)} = \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} 2^{-k-1}e^{-it(n-k)} = 2^{-n}\pi$$

In case $n < 0$ we get $I(n) = 0$. If we have $x \notin \mathbb{N}$ instead of n we could try same, but in this case we end up with

$$i(e^{-2\pi ix} - 1)\sum_{k=0}^{\infty} \frac{(-2)^{-k-1}}{x - k}$$

which needs further work to extract an asymptotic behavior in x.

Exercise 3.104 *Make use of the argument below, leading to (3.105) to find the behavior as $y \to +\infty$ of*

$$\sum_{k=0}^{\infty} \frac{a^k}{y+k}; \quad (|a|<1)$$

We can alternatively apply a more general method to estimate the integral, using deformation of contour. The point is to try to express J in terms of integrals along paths of constant phase of e^{-int}. Then Watson's lemma would be applicable. Note that F is analytic in $\mathbb{C} \setminus \{-i\ln 2 + 2k\pi\}_{k\in\mathbb{Z}}$ and meromorphic in $\mathbb{C}$. Furthermore, as $N \to \infty$ we have $F(t-iN) \to 0$ exponentially fast. This allows us to push the contour of integration down, in the following way. We have

$$\oint_C F(t)dt = 2\pi i \operatorname{res}\Big(F(t); t = -i\ln 2\Big) = -\pi 2^{-x}$$

where the contour C of integration is an anticlockwise rectangle with vertices $-\pi, \pi, -iN+\pi, -iN-\pi$ with $N > \ln 2$. As $N \to \infty$ the integral over the segment from $-iN+\pi$ to $-iN-\pi$ goes to zero exponentially fast, and we find that

$$\int_{-\pi}^{\pi} F(t)dt = \int_{-\pi}^{-\pi-i\infty} F(t)dt - \int_{\pi}^{\pi-i\infty} F(t)dt + \pi 2^{-x}$$

$$I(x) = -i(e^{ix\pi} - e^{-ix\pi})\int_0^\infty \frac{e^{-xs}}{2+e^s}ds + \pi 2^{-x} = 2\sin\pi x\int_0^\infty \frac{e^{-xs}}{2+e^s}ds + \pi 2^{-x}$$

Watson's Lemma now applies and we have

$$\int_0^\infty \frac{e^{-xs}}{2+e^s}ds \sim \frac{1}{3x} - \frac{1}{9x^2} - \frac{1}{27x^3} + \frac{1}{27x^4} + \frac{5}{81x^5} - \frac{7}{243x^6} + \ldots$$

and thus

$$I(x) \sim 2\sin\pi x\left(\frac{1}{3x} - \frac{1}{9x^2} - \frac{1}{27x^3} + \frac{1}{27x^4} + \frac{5}{81x^5} - \frac{7}{243x^6} + \ldots\right) \quad (3.105)$$

whenever the prefactor in front of the series is not too small. More generally, the difference between $I(x)$ and the m-th truncate of the expansion is $o(x^{-m})$. Or, the function on the left hand side can be decomposed in two functions using Euler's formula, each of which has a nonvanishing asymptotic expansion. This is the way to interpret similar asymptotic expansions, which often occur in the theory of special functions, when the expansions involve trigonometric functions. But none of these characterizations tells us what happens when the prefactor *is* small. Does the function vanish when $\sin\pi x = 0$? Not for $n > 0$. Another reason to be careful with relations of the type (3.105).

3.3b .1 Note on exponentially small terms

In our case we have more information: if we add the term $\pi 2^{-x}$ to the expansion and write

$$I(x) \sim 2\sin \pi x \left(\frac{1}{3x} - \frac{1}{9x^2} - \frac{1}{27x^3} + \frac{1}{27x^4} + \frac{5}{81x^5} - \frac{7}{243x^6} + ... \right) + \pi 2^{-x} \tag{3.106}$$

then the expansion is valid when $x \to +\infty$ along the positive integers, a rather trivial case since only $\pi 2^{-x}$ survives. But we have trouble interpreting the expansion (3.106) when x is not an integer! The expression (3.106) is not of the form (1.8) nor can we extend the definition to allow for $\pi 2^{-x}$ since 2^{-x} is asymptotically smaller than any term of the series, and no number of limits as in Definition 1.11 would reveal it. We could try to subtract out first the *whole* series preceding the exponential from $I(x)$ to see "what is left", but this subtraction is not well defined either since the series has zero radius of convergence. (The divergence follows from the fact that the term of order k of the series is, by Watson's Lemma, $k!$ times the Maclaurin coefficient of the function $(2+e^s)^{-1}$ and this function is not entire.)

We may nevertheless have the feeling that (3.106) is correct "somehow". Indeed it is, in the sense that (3.106) is the complete transseries of J, as we will see in Chapter 4.

*

3.4 Steepest descent method

Consider the problem of finding the large x behavior of an integral of the form

$$\int_C f(s) e^{xg(s)} ds \tag{3.107}$$

where g is analytic and f is meromorphic in a domain in the complex plane of the contour C and x is a large parameter.

As in the Example 1 on p. 59, the key idea is to use deformation of contour to bring the integral to one which is suitable to the application of the Laplace method. We can assume without loss of generality that x is real and positive.

(A) Let $g = u + iv$ and let us first look at the simple case where C' is a curve such that $v = K$ is constant along it. Then

$$\int_{C'} f(s) e^{xg(s)} ds = e^{xiK} \int_{C'} f(s) e^{xu(s)} ds = e^{xiK} \int_0^1 f(\gamma(t)) e^{xu(\gamma(t))} \gamma'(t) dt$$

is in a form suitable for Laplace's method.

The method of steepest descent consists in using the meromorphicity of f, analyticity of g to deform the contour of integration such that modulo residues, the original integral can be written as a sum of integrals of the type C' mentioned. The name steepest descent comes from the following remark. The lines of v =constant are perpendicular to the direction of ∇v. As a consequence of the Cauchy-Riemann equations we have $\nabla u \cdot \nabla v = 0$ and thus the lines v =constant are lines of steepest variation of u therefore of $|e^{xg(s)}|$. On the other hand, the best way to control the integral is to go along the descent direction. The direction of steepest descent of u is parallel to $-\nabla u$. Thus the steepest descent lines are the integral curves of the ODE system ODEs

$$\dot{x} = -u_x(x,y); \quad \dot{y} = -u_y(x,y) \tag{3.108}$$

We first look at some examples, and then discuss the method in more generality.

Example 2. The Bessel function $J_0(\xi)$ can be written as $\frac{1}{\pi}\mathrm{Re}\, I$, where

$$I = \int_{-\pi/2}^{\pi/2} e^{i\xi \cos t} dt \tag{3.109}$$

Suppose we would like to find the behavior of $J_0(\xi)$ as $\xi \to +\infty$. It is convenient to find the steepest descent lines by plotting the phase portrait of the flow (3.108), which in our case is

$$\dot{x} = -\cos x \sinh y; \quad \dot{y} = -\sin x \cosh y \tag{3.110}$$

and which is easy to analyze by standard ODE means.

$$I = \int_{-\pi/2}^{-\pi/2+i\infty} e^{i\xi \cos t} dt + \int_{\gamma} e^{i\xi \cos t} dt + \int_{\pi/2}^{\pi/2-i\infty} e^{i\xi \cos t} dt \tag{3.111}$$

as shown in the figure.

All the curves involved in this decomposition of I are lines of constant imaginary part of the exponent, and the ordinary Laplace method can be applied to find their asymptotic behavior for $\xi \to +\infty$ (note also that the integral along the curve γ, called Sommerfeld contour, is the only one contributing to J_0, the other two being purely imaginary, as it can be checked by making the changes of variable $t = -\pi/2 \pm is$). Then, the main contribution to the integral comes from the point along γ where the real part of the exponent is maximum, that is $z = 0$. We then expand $\cos t = 1 - t^2/2 + t^4/4! + \cdots$ keep the first two terms in the exponent and expand the rest out:

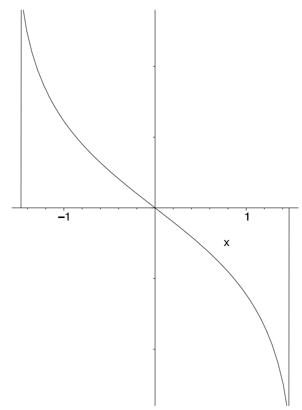


FIGURE 3.1: Relevant contours for J_0

$$\int_\gamma e^{i\xi\cos t}dt \sim e^{ix}\int_\gamma e^{-i\xi t^2/2}(1+i\xi t^4/4!+\cdots)dt$$
$$\sim \int_{\infty e^{3i\pi/4}}^{\infty e^{-i\pi/4}} e^{-i\xi t^2/2}(1+i\xi t^4/4!+\cdots)dt \quad (3.112)$$

and integrate term by term. Justifying this rigorously would amount to redoing parts of the proofs of theorems we have already dealt with. Whenever possible, Watson's Lemma is a shortcut, often providing more information as well. We will use it for (3.109) in Example 4.

*

Example 3. We know by Watson's Lemma that for a function F which has a nontrivial power series at zero, $\mathcal{L}F = \int_0^\infty e^{-xp}F(p)dp$ decreases algebraically as $x \to \infty$. We also know by Proposition 2.6 that regardless of $F \not\equiv 0 \in L^1$, $\mathcal{L}F$ cannot decrease superexponentially. What happens if F has a rapid oscillation near zero? Consider for $x \to +\infty$ the integral

$$I := \int_0^\infty e^{-xp}\cos(1/p)dp \quad (3.113)$$

It is convenient to write

$$I = \Re\int_0^\infty e^{-xp}e^{-i/p}dp = \Re I_1 \quad (3.114)$$

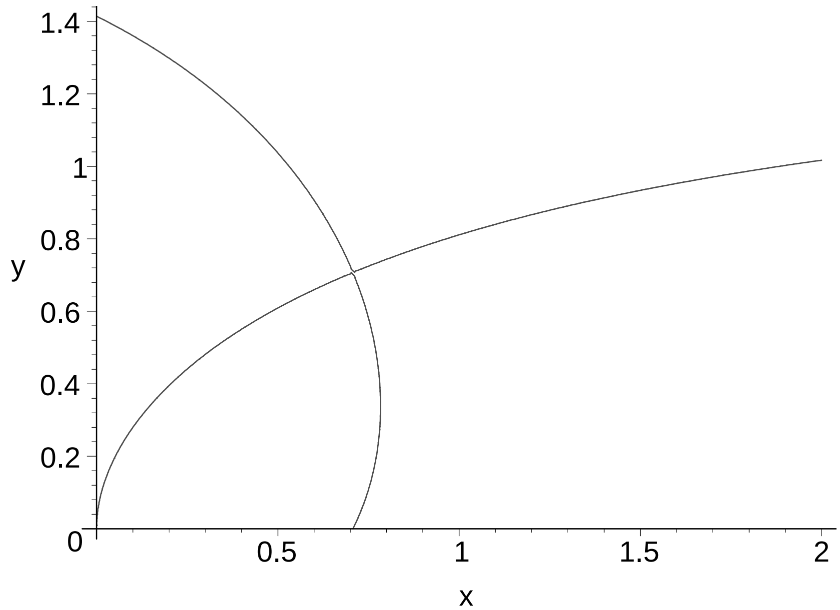


FIGURE 3.2: Constant phase lines for $t + i/t$ passing through the saddle point $t = \sqrt{i}$.

To bring this problem to the steepest descent setting, we make the substitution $p = t/\sqrt{x}$. Then I_1 becomes

$$I_1 = x^{-1/2} \int_0^\infty e^{-\sqrt{x}(t+i/t)} e^{-i/p} dp \tag{3.115}$$

The constant imaginary part lines of interest now are those of the function $t + i/t$. This function has saddle points at $(t + i/t)' = 0$ i.e. $t = \pm\sqrt{i}$. We see that $t = \sqrt{i} = t_0$ is a maximum point for $-\Re g := -\Re(t + i/t)$ and the main contribution to the integral is from this point. We have, near $t = t_0$ $g = g(t_0) + \frac{1}{2} g'' \sqrt{t_0}(t - t_0)^2 + \cdots$ and thus

$$I_1 \sim x^{-1/2} e^{-\sqrt{2}(1+i)\sqrt{x}} \int_{-\infty}^{\infty} \exp\left[\left(-\frac{1}{2} + \frac{i}{2}\right)\sqrt{2x}(t - t_0)^2\right] \tag{3.116}$$

and the behavior of the integral is, roughly, $e^{-\sqrt{x}}$, decaying faster than powers of x but slower than exponentially. The calculation can be justified mimicking the reasoning in Proposition 3.23. But this integral too can be brought to a form suitable for Watson's Lemma, as in (B) below.

Exercise 3.117 *Finish the calculations in this example.*

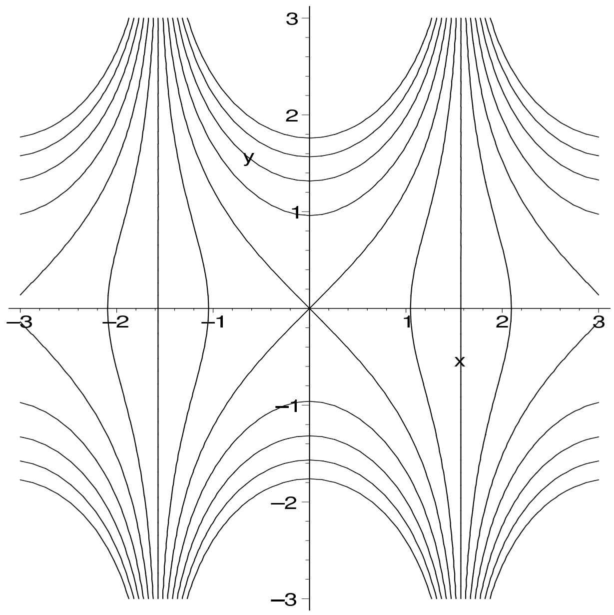


FIGURE 3.3: Steepest descent lines for $\Re[i\cos(x+iy)]$

3.4a Some general remarks about steepest descent lines

Assume for simplicity that g is nonconstant entire and f is meromorphic. We can let the points on the curve $C = (x_0(\tau), y_0(\tau));\ \tau \in [0,1]$ evolve with (3.108) keeping the endpoints fixed. More precisely, at time t consider the curve $t \mapsto C(t) = C_1 \cup C_2 \cup C_3$ where $C_1 = (x(s, x_0(0)), y(s, y_0(0));\ s \in [0,t)$, $C_2 = x(t, x_0(\tau)), y(t, y_0(\tau)),\ \tau \in (0,1))$ and $C_3 = (x(s, x_0(1)), y(s, y_0(1));\ s \in [0,t)$. Clearly, if no poles of f are crossed,

$$\int_C f(s)e^{xg(s)}ds = \int_{C(t)} f(s)e^{xg(s)}ds \tag{3.118}$$

We can see that $z(t, x_0(\tau)) = (x(t, x_0(\tau)), y(t, x_0(\tau)))$ has a limit as $t \to \infty$ on the Riemann sphere, since u is strictly decreasing along the flow:

$$\frac{d}{dt}u\Big(x(t), y(t)\Big) = -u_x^2 - u_y^2 \tag{3.119}$$

There can be no closed curves along which $v = K$ =const. or otherwise we would have $v \equiv K$. since v is harmonic. Thus steepest descent lines extend to infinity. They may pass through *saddle points* of u (and g: $\nabla u = 0 \Rightarrow g' = 0$) where their direction can change non-smoothly. These are equilibrium points of the flow (3.108).

Define $\mathcal{S}$ as the smallest forward invariant set with respect to the evolution (3.108) which contains $(x_0(0), y_0(0))$, all the limits in $\mathbb{C}$ of $z(t, x_0(\tau))$ and the descent lines originating at these points. The set $\mathcal{S}$ is a union of steepest descent curves of u, $\mathcal{S} = \cup_{j=1}^{n} C_j$ and, if s_j are poles of f crossed by the curve $C(t)$ we have, under suitable convergence assumptions[1],

$$\int_C f(s)e^{xg(s)}ds = \sum_{j=1}^{n'\leq n} \int_{C_n} f(s)e^{xg(s)}ds + 2\pi i \sum_j \operatorname{Res}(f(s)e^{xg(s)})_{s=s_j} \quad (3.120)$$

and the situation described in (A) above has been achieved.

One can allow for branch points of f, each of which adds a contributions of the form

$$\int_C \delta f(s)e^{xg(s)}ds$$

where C is a cut starting at the branch point of f, along a line of steepest descent of g, and $\delta f(s)$ is the jump across the cut of f.

3.4b Reduction to Watson's Lemma

It is often more convenient to proceed as follows.

We may assume we are dealing with a simple smooth curve. We assume $g' \neq 0$ at the endpoints (the case of vanishing derivative is illustrated shortly on an example). Then, possibly after an appropriate small deformation of C we have $g' \neq 0$ along the path of integration C and g is invertible in a small enough neighborhood $\mathcal{D}$ of C. We make the change of variable $g(s) = -\tau$ and note that the image of C is smooth and has at most finitely many self-intersections. We can break this curve into piecewise smooth, simple curves. If the pieces are small enough, they are homotopic to straight lines; we get

$$\sum_{n=1}^{N} \int_{c_n}^{c_{n+1}} f\left(s(\tau)\right) e^{-x\tau} \frac{ds}{d\tau} d\tau \quad (3.121)$$

We calculate each integral in the sum separately. Without loss of generality we take $n = 1$, $c_1 = 0$ and $c_2 = i$:

$$I_1 = \int_0^i f(s(\tau))e^{-x\tau}s'(\tau)d\tau \quad (3.122)$$

[1]Convergence assumptions are required, as can be seen by applying the described procedure to very simple integral

$$\int_0^i e^{xe^{-z}}dz$$

The procedure described in (B) is better in many respects.

The lines of steepest descent for I_1 are horizontal, towards $+\infty$. Assuming suitable analyticity and growth conditions and letting $H(\tau) = f(s(\tau))s'(\tau)$ we get I_1 equals

$$I_1 = \int_0^{\infty} e^{-x\tau} H(\tau) d\tau - \int_i^{i+\infty} H(\tau) e^{-x\tau} d\tau \\ - 2\pi i \sum_j \operatorname{Res}\left(H(\tau)e^{-x\tau}\right)_{s=s_j} + \sum_j \int_{d_j}^{d_j+\infty} \delta H(\tau) e^{-x\tau} d\tau \quad (3.123)$$

where the residues come from poles of H in the strip $S = \{x + iy : x > 0, y \in [0,1]\}$, while d_j are branch points of H in S, assumed integrable, and δH denotes the jump of H across the branch cut. If more convenient, one can alternatively subdivide C such that g' is nonzero on the (open) subintervals.

Example 4. In the integral (3.109) we have, using the substitution $\cos t = i\tau$,

$$\int_{-\pi/2}^{\pi/2} e^{i\xi \cos t} dt = 2\int_0^{\pi/2} e^{i\xi \cos t} dt = -2i \int_{-i}^{0} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}} d\tau = 2i \int_0^{\infty} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}} d\tau \\ - 2i \int_{-i}^{-i+\infty} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}} d\tau = 2i \int_0^{\infty} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}} d\tau - 2ie^{i\xi} \int_0^{\infty} \frac{e^{-\xi s}}{\sqrt{-2is+s^2}} ds \quad (3.124)$$

to which Watson's Lemma applies.

Exercise. Find the asymptotic behavior for large x of

$$\int_{-1}^{1} \frac{e^{ixs}}{s^2+1} ds$$

3.5 Asymptotics of Taylor coefficients

There is dual relation between the behavior of the Taylor coefficients of an analytic function and the structure of its singularities in the complex plane. We will study a few examples in which this relationship is exhibited.

Proposition 3.125 *Assume f is analytic in the open disk of radius $R + \epsilon$ with N cuts $z_n = Re^{i\phi_n}$, and in a neighborhood of z_n f has a convergent Puiseux series*

$$f(z) = (z - z_n)^{\beta_1^{[n]}} A_1^{[n]}(z) + \ldots + (z - z_n)^{\beta_m^{[n]}} A_m^{[n]}(z) + A_{m+1}^{[n]}(z)$$

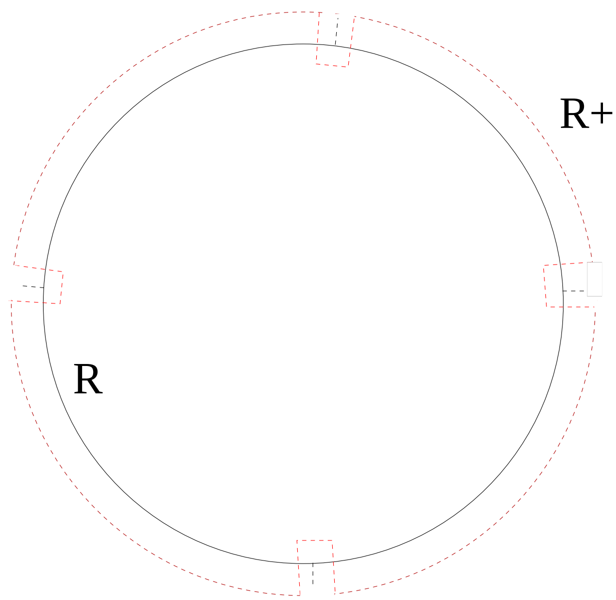


FIGURE 3.4: Deformation of the Cauchy contour.

where $A_1^{[n]}, ..., A_{m+1}^{[n]}$ *are analytic in a neighborhood of* $z = z_n$ *(and we can assume* $\beta_i^{[n]} \notin \mathbb{N} \cup \{0\}$*). With* $c_k = f^{(k)}(0)/k!$*, we have*

$$c_k \sim R^{-k} \sum_{l=1}^{N} e^{-ik\phi_l} \left(k^{-\beta_1^{[l]}-1} \sum_{j=0}^{\infty} \frac{c_{j;1}^{[l]}}{k^j} + ... + k^{-\beta_m^{[l]}-1} \sum_{j=0}^{\infty} \frac{c_{j;m}^{[l]}}{k^j} \right) \tag{3.126}$$

where the coefficients $c_{j;m}^{[n]}$ *can be calculated from the Taylor coefficients of the functions* $A_1^{[n]}, ..., A_m^{[n]}$*, and conversely, this asymptotic expansion determines the functions* $A_1^{[n]}, ..., A_m^{[n]}$*.*

PROOF We have

$$c_k = \frac{1}{2\pi i} \oint \frac{f(s)}{s^{k+1}} ds$$

where the contour is a small circle around the origin. This contour can be deformed, by assumption, to the dotted contour in the figure. The integral around the circle of radius $R + \epsilon$ can be estimated by

$$\frac{1}{2\pi} \left| \oint_{C_{R+\epsilon}} \frac{f(s)}{s^{k+1}} ds \right| \leq \|f\|_\infty (R+\epsilon)^{-k-1} = O((R+\epsilon)^{-k-1})$$

and does not participate in the series (3.126), since it is smaller than R^{-k} times any power of k, as $k \to \infty$. Now the contribution from each singularity is of the form

$$\frac{1}{2\pi} \int_{B_l} \frac{f(s)}{s^{k+1}} ds$$

where B_l is an open dotted box around the branch cut at $Re^{i\phi_l}$ as in the figure, so it is enough to determine the contribution of one of them, say z_1. By the substitution $f_1(z) = f(Re^{i\phi_1} z)$, we reduce ourselves to the case $R = 1$, $\phi = 0$. We omit for simplicity the superscript "[1]".

The integral along B_1 is a sum of integrals of the form

$$\frac{1}{2\pi i} \int_C (s-1)^\beta A(s) s^{-k-1} ds \tag{3.127}$$

We can restrict ourselves to the case when β is not an integer, the other case being calculable by residues.

Assume first that (i) $\Re(\beta) > -1$. We then have

$$\frac{1}{2\pi i} \int_C (s-1)^\beta A(s) s^{-k-1} ds = -e^{\pi i \beta} \frac{\sin(\pi\beta)}{\pi} \int_1^{1+\epsilon} (s-1)^\beta A(s) s^{-k-1} ds \tag{3.128}$$

with the branch choice $\ln(s-1) > 0$ for $s \in (1, \infty)$. It is convenient to change variables to $s = e^u$. The rhs of (3.128) becomes

$$-e^{\pi i \beta} \frac{\sin(\pi\beta)}{\pi} \int_0^{\ln(1+\epsilon)} u^\beta \left(\frac{e^u - 1}{u} \right)^\beta A(e^u) e^{-ku} du \tag{3.129}$$

where $A(e^u)$ and $[u^{-1}(e^u - 1)]^\beta$ are analytic at $u = 0$, the assumptions of Watson's Lemma are satisfied and we thus have

$$\int_C (s-1)^\beta A(s) s^{-k-1} ds \sim k^{-\beta-1} \sum_{j=0}^{\infty} \frac{d_j}{k^j} \tag{3.130}$$

where the d_j can be calculated straightforwardly from the Taylor coefficients of $A[u^{-1}(e^u - 1)]^\beta$. The proof when $\Re\beta \le -1$ is by induction. Assume that (3.130) holds for all for $\Re(\beta) > -m$ with $1 \le m_0 \le m$. One integration by parts gives

$$\int_C (s-1)^\beta A(s)s^{-k-1}ds = \frac{(s-1)^{\beta+1}}{\beta+1}A(s)s^{-k-1}|_C$$
$$-\frac{1}{\beta+1}\int_C (s-1)^{\beta+1}[A(s)s^{-k-1}]'ds = O((R+\epsilon)^{-k-1})$$
$$+k\frac{1+1/k}{\beta+1}\int_C (s-1)^{\beta+1}A(s)s^{-k-2}ds - \frac{1}{\beta+1}\int_C (s-1)^{\beta+1}A'(s)s^{-k-1}ds \tag{3.131}$$

By assumption, (3.130) applies with $\beta+1 \leftrightarrow \beta$ to both integrals in the last sum and the proof is easily completed. ∎

3.6 Singularities of differential equations

We first review briefly some basic notions about singularities of linear differential equations.

3.6a Linear meromorphic differential equations. Regular and irregular singularities

A linear meromorphic m-th order differential equation has the canonical form

$$y^{(m)} + B_{m-1}(x)y^{(m-1)} + ... + B_0(x)y = B(x) \tag{3.132}$$

where the coefficients $B_j(x)$ are meromorphic near x_0. We note first that any equation of the form (3.132) can be brought to a homogeneous meromorphic of order $n = m+1$

$$y^{(n)} + C_{n-1}(x)y^{(n-1)} + ... + C_0(x)y = 0 \tag{3.133}$$

by applying the operator $B(x)\frac{d}{dx}\frac{1}{B(x)}$ to (3.132). We want to look at the possible singularities of the solutions $y(x)$ of this equation. Note first that by the general theory of linear differential equations (or by a simple fixed point argument) if all coefficients are analytic at a point x_0 then the general solution is also analytic. Such a point is called regular point. Solutions of linear ODEs can only be singular because of singularities in the coefficients.

The main distinction is made with respect to the type of local solutions, whether they can be expressed as convergent asymptotic series (regular singularity) or not (irregular one).

Theorem 3.134 *[Frobenius] If near the point $x = x_0$ the coefficients C_{n-j}, $j = 1...n$ can be written as $(x-x_0)^{-j}A_{n-j}(x)$ where A_{n-j} are analytic, then there is a fundamental system of solutions in the form*

$$y_m(x) = (x-x_0)^{r_m} \sum_{j=0}^{N_m} (\ln(x-x_0))^j B_{j;m}(x) \tag{3.135}$$

where $B_{j;m}$ are analytic in an open disk centered at x_0 with radius equal to the distance from x_0 to the first singularity of A_j. The powers r_m are solutions of the indicial *equation*
$r(r-1)\cdots(r-n+1) + A_{n-1}(x_0)r(r-1)\cdots(r-n+2)+\ldots+A_0(x_0) = 0$
Furthermore, logs appear only in the resonant case, when two (or more) characteristic roots r_m differ by an integer.

A straightforward way to prove the theorem is by induction on n. We can take $x_0 = 0$. Let r_M be one of the indicial equation solutions. A transformation of the type $y = x^{r_M}f$ reduces the equation (3.133) to an equation of the same type, but where one characteristic root is zero. One can show there is an analytic solution f_0 of this equation by inserting a power series, identifying the coefficients and estimating the growth of the coefficients. The substitution $f = f_0 \int g(s)ds$ gives an equation for g which is of the same type as (3.133) but of order $n-1$. We will not go into the details of the general case but instead we will illustrate the approach on the simple equation

$$x(x-1)y'' + y = 0 \tag{3.136}$$

around $x = 0$. The indicial equation is $r(r-1) = 0$ (a resonant case). Substituting $y_0 = \sum_{k=0}^{\infty} c_k x^k$ in the equation and identifying the powers of x yields the recurrence

$$c_{k+1} = \frac{k^2-k+1}{k(k+1)}c_k \tag{3.137}$$

with $c_0 = 0$ and c_1 arbitrary. By linearity we may take $c_1 = 1$ and by induction we see that $0 < c_k < 1$. Thus the power series has radius of convergence at least 1. The radius of convergence is in fact exactly one as it can be seen applying the ratio test and using (3.137); the series converges exactly up to the nearest singularity of (3.136).

Exercise 3.138 *What is the asymptotic behavior of c_k as $k \to \infty$?*

We let $y_0 = y_0 \int g(s)ds$ and get for g the equation

$$g' + 2\frac{y_0'}{y_0}g = 0 \tag{3.139}$$

and, by the previous discussion, $2y_0'/y_0 = 2/x + A(x)$ with $A(x)$ is analytic. The point $x = 0$ is a regular singular point of (3.139) and in fact we can check that $g(x) = C_1 x^{-2} B(x)$ with C_1 an arbitrary constant and $B(x)$ analytic at $x = 0$. Thus $\int g(s)ds = C_1(a/x + b\ln(x) + A_1(x)) + C_2$ where $A_1(x)$ is analytic at $x = 0$. Undoing the substitutions we see that we have a fundamental set of solutions in the form $\{y_0(x), B_1(x) + B_2(x)\ln x\}$ where B_1 and B_2 are analytic.

A converse of this theorem also holds, namely

Theorem 3.140 (Fuchs) *If a meromorphic linear differential equation has, at $x = x_0$, a fundamental system of solutions in the form (3.135), then x_0 is a regular singular point of the equation.*

Instead for irregular singularities at least one formal solution contains divergent power series and/or exponentially small (large) terms. The way divergent power series are generated by the higher order of the poles is illustrated below.
Example. Consider the equation

$$y' + x^{-2}y = 1 \tag{3.141}$$

which has an irregular singularity at $x = 0$ since the order of the pole in $C_0 = x^{-2}$ exceeds the order of the equation. Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ we get $c_0 = c_1 = 0$, $c_2 = 1$ and in general the recurrence

$$c_{k+1} = -kc_k$$

whence $c_k = (-1)^k (k-1)!$ and the series has zero radius of convergence. (It is useful to compare this recurrence with the one obtained if x^{-2} is replaced by x^{-1} or by 1.) The associated homogeneous equation $y' + x^{-2}y = 0$ has the general solution $y = Ce^{1/x}$ with an exponential singularity at $x = 0$.

3.6b Singularities of nonlinear differential equations; formal Painlevé property

For nonlinear differential equations, the solutions may be singular at points x where the equation is regular. Indeed, for example, the equation

$$y' = y^2 + 1$$

has a one parameter family of solutions $y(x) = \tan(x + C)$; each solution has infinitely many poles. Since the location of these poles depends on C, thus on the solution itself, these singularities are called *movable* or *spontaneous*. Painlevé studied the problem of finding differential equations, now called equations with the Painlevé property , whose only *movable* singularities are poles[2]. There are no restriction on the behavior at singular points

[2]There is no complete agreement on what the Painlevé property should require and Painlevé himself apparently oscillated among various interpretations; certainly movable branch points are not allowed, but often the property is understood to mean that all solutions are single-valued on a common Riemann surface.

of *the equation.* The solutions of such an equation have a common Riemann surface simple enough we can hope to understand globally.

We also note that the Painlevé property guarantees some form of integrability of the equation, in the following sense. If we denote by $Y(x; x_0; C_1, C_2)$ the solution of the differential equation $y'' = F(x, y, y')$ with initial conditions $y(x_0) = C_1, y'(x_0) = C_2$ we see that $y(x_1) = Y(x_1; x; y(x), y'(x))$ is formally constant along trajectories and so is $y'(x_1) = Y'(x_1; x; y(x), y'(x))$. This gives thus two constants of motion in $\mathbb{C}$ provided the solution Y is well defined almost everywhere in $\mathbb{C}$, i.e., if Y is meromorphic.

On the contrary, "randomly occurring" movable branch-points make the inversion process explained above ill defined.

This does not of course entail that there is no constant of motion. However, the presence of spontaneous branch-points does have the potential to prevent the existence of well-behaved constants of motions for the following reason. Suppose y_0 satisfies a meromorphic (second order, for concreteness) ODE and $K(x; y, y')$ is a constant of motion. If x_0 is a branch point for y_0, then y_0 can be continued past x_0 by avoiding the singular point, or by going around x_0 any number of times before moving away. This leads to different branches $(y_0)_n$ of y_0, all of them, by simple analytic continuation arguments, solutions of the same ODE. By the definition of $K(x; y, y')$ however, we should have $K(x; (y_0)_n, (y_0)'_n) = K(x; y_0, y'_0)$ for all n, so K assumes the same value on this infinite set of solutions. We can proceed in the same way around other branch points $x_1, x_2, ...$ possibly returning to x_0 from time to time. Generically, we expect to generate a family of $(y_0)_{n_1,...n_j}$ which is dense in the phase space. This is an expectation, to be proven in specific cases. To see whether an equation falls in this generic class M. Kruskal introduced a test of nonintegrability, the *poly-Painlevé test* which measures indeed whether branching is "dense". Properly interpreted and justified the Painlevé property measures whether an equation is integrable or not.

Local analysis of Painlevé's equation P1 near a singularity. We write P1 in the form

$$y'' = y^2 + x \tag{3.142}$$

We look at the local behavior of a solution that blows up, and will find solutions that are meromorphic but not analytic. In a neighborhood of a point where y is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Or, we could have instead searched for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)x^{p-2} = A^2p^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(x-x_0)^{-2}$: $y = 6(x - x_0)^{-2} + c_{-1}(x - x_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 =$

$0, c_2 = -x_0/10, c_3 = -1/6$ and c_4 undetermined, thus free. Chosing a c_4, all others are uniquely determined.

To show that there indeed is a convergent such power series solution, we apply a successive correction method. Substituting $y(x) = 6(x-x_0)^{-2} + \delta(x)$ where for consistency we should have $\delta(x) = o((x-x_0)^{-2})$ and taking $x = x_0 + z$ we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{3.143}$$

Note now that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (3.143) is *relatively* small (Thus, to leading order, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximately by a linear equation.) It is then natural to separate out the large terms from the small terms and writing a fixed point equation for the solution based on this separation. We write (3.143) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{3.144}$$

and integrate as if the right side was known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to be *relatively smaller*, by construction this integral equation is expected to be contractive.

The indicial equation for the Euler equation corresponding to the left side of (3.144) is $r^2 - r - 12 = 0$ with solutions $4, -3$. By the method of variation of parameters we thus get

$$\delta = \frac{D}{z^3} - \frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \tag{3.145}$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$; C is arbitrary. To find δ formally, we would simply iterate (3.145) in the following way: We take $r = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $r = \delta_0^2$ and compute δ_1 from (3.145) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + ... \tag{3.146}$$

This series is actually convergent. To see that, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z su^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \quad (3.147)$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ϵ such that this is a contractive equation for u in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \epsilon$. Our conclusion is that δ is analytic and that y is meromorphic near $x = x_0$.

Thus the equation P_I has the local Painlevé property.

Note. The full Painlevé property requires that y is globally meromorphic, and we did *not* prove this. That indeed y is globally meromorphic is in fact true, but the proof is delicate (see e.g. [1]).

Generic equations fail even the local Painlevé property. For instance, for the simpler, autonomous, equation

$$f'' + f' + f^2 = 0 \quad (3.148)$$

the same analysis yields a local behavior starting with a double pole, $f \sim -6z^{-2}$. Taking $f = -6z^{-2} + \delta(z)$ with $\delta = o(z^{-2})$ again leads to a nearly linear equation for δ which can be solved by convergent iteration, using arguments similar to the ones above. The iteration is now (for some $a \neq 0$)

$$\delta = \frac{6}{5z} + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta(s)ds + \frac{z^4}{7}\int_a^z s^{-3}\delta(s)ds \quad (3.149)$$

but now the leading behavior of δ is larger, $\delta = \frac{6}{5z}$. Iterating in the same way as before, we see that this will eventually produce logs in the expansion for δ (it first appears in the second integral, thus in the form $z^4 \ln z$). We get

$$\delta = \frac{6}{5z} + \frac{1}{50} + \frac{z}{250} + \frac{7z^2}{5000} + \frac{79}{75000}z^3 - \frac{117}{2187500}z^4\ln(z) + Cz^4 + ... \quad (3.150)$$

where later terms will contain higher and higher powers of $\ln(z)$. This is effectively a series in powers of z and $\ln z$ a simple example of a transseries, which is convergent as can be straightforwardly shown using the contractive mapping method, as above. In any case, (3.148) does not have the Painlevé property. This log term shows that infinitely many solutions can be obtained just by analytic continuation around one point, and suggests the equation is not integrable.

3.7 Singular perturbations

3.7a Introduction to the WKB method

If the order of the poles of a linear systems are higher than in a regular singularity, Fuchs's theorem shows that a fundamental set of solutions cannot be found in terms of convergent Frobenius series. What is the nature of the solutions then?

We start with a simple example which fails the assumptions of Theorem 3.134 namely,

$$z^2 f' + f = 0 \tag{3.151}$$

The general solution is $f(z) = Ce^{-1/z}$. We see that in a neighborhood of $z = 0$, the solution cannot be expanded in a Taylor or Frobenius series in powers of z (a Laurent series is not of the form (3.135)). More generally, (at least formal) solutions of meromorphic equations that fail the assumptions of Theorem 3.134 can be written in terms of combinations of exponentials and Frobenius-like series (which, this time, may diverge). A formal solution of the equation

$$z^2 f' + f = -z \tag{3.152}$$

is

$$\sum_{k=0}^{\infty} k!(-z)^{k+1} \tag{3.153}$$

Consider now the second order equation

$$z^3 y'' + y = 0 \tag{3.154}$$

Based on the previous example, we may expect solutions roughly behaving like e^{-az^b}. Here $b < 0$, otherwise the exponential can be reexpanded and we are dealing with usual power series solutions. If that is the case, then a substitution of the form $y = e^w$ should bring the equation to one in which power series solutions are to be expected.

In problems depending analytically on a small parameter, internal or external, the dependence of the solution on this parameter may be analytic (*regular perturbation*) or not (*irregular perturbation*). In ordinary differential equations, singular perturbations happen when the small perturbation is such that, in a formal series solution, the highest derivative is formally small. In a formal successive approximation scheme then, small terms, the highest derivative included, should be placed on the rhs and iterated upon. This, as we have seen already in many examples, leads to divergent expansions. Furtthermore, there should exist formal solutions other than power series, since the procedure above obviously yields a space of solutions of dimensionality strictly smaller than the degree of the equation.

An example is the Schrödinger equation

$$-\epsilon^2\psi'' + V(x)\psi - E\psi = 0 \tag{3.155}$$

for small ϵ, which will be studied in more detail later. In an ϵ-power series, ψ'' is subdominant [3]. The leading approximation would be $(V(x) - E)\psi = 0$ or $\psi = 0$ which is not an admissible solution.

Similarly, in

$$x^2 f' + f = x^2 \tag{3.156}$$

the presence of x^2 in front of f' makes f' subdominant if $f \sim x^p$ for some p. In this sense the Airy equation (3.172) below, is also singularly perturbed, at $x = \infty$. It turns out that in many of these problems the behavior of solutions is exponential in the parameter, generically yielding what level one transseries, studied in the sequel, of the form Qe^P where P and Q have algebraic behavior in the parameter. An exponential substitution of the form $f = e^w$ should then make the leading behavior algebraic.

3.7b Singularly perturbed Schrödinger equation. Setting and heuristics

We look at (3.155) under the assumption that $V \in C^\infty(\mathbb{R})$ and would like to understand the behavior of solutions for small ϵ.

3.7b .1 Heuristics

Assume $V \in C^\infty$ and that the equation $V(x_0) = E$ has finitely many solutions.

Applying the WKB transformation $\psi = e^w$ we get

$$-\epsilon^2 {w'}^2 - \epsilon^2 w'' + V(x) - E = -\epsilon^2 {w'}^2 - \epsilon^2 w'' + U(x) = 0 \tag{3.157}$$

where, near an x_0 where

$$U(x_0) \neq 0 \tag{3.158}$$

the only consistent balance[4] is between $-\epsilon^2 {w'}^2$ and $V(x) - E$ with $\epsilon^2 w''$ much smaller than both. For that to happen we need

$$\epsilon^2 U^{-1} h' \ll 1 \quad \text{where} \quad h = w' \tag{3.159}$$

[3]Meaning that it is asymptotically much less than other terms in the equation.

[4]As the parameter, ϵ in our case, gets small, various terms in the equation contribute unevenly. Some become relatively large (the dominant ones) and some are small (the subdominant ones). If no better approach is presented, one tries all possible combinations, and rules out those which lead to conclusions inconsistent with the size assumptions made. The approach roughly described here is known as the method of dominant balance [3]. It is efficient but heuristic and has to be supplemented by rigorous proofs at a later stage of the analysis.

We place the term $\epsilon^2 h'$ on the right side of the equation and set up the iteration scheme

$$h_n^2 = \epsilon^{-2}U - h'_{n-1}; \quad h_{-1} = 0 \tag{3.160}$$

or

$$h_n = \pm\frac{\sqrt{U}}{\epsilon}\sqrt{1 - \frac{\epsilon^2 h'_{n-1}}{U}}; \quad h_{-1} = 0 \tag{3.161}$$

Under the condition (3.159) the square root can be Taylor expanded around 1,

$$h_n = \pm\frac{\sqrt{U}}{\epsilon}\left(1 - \frac{1}{2}\epsilon^2\frac{h'_{n-1}}{U} - \frac{1}{8}\epsilon^4\left(\frac{h'_{n-1}}{U}\right)^2 + \cdots\right) \tag{3.162}$$

We thus have

$$h_0 = \pm\epsilon^{-1}U^{1/2} \tag{3.163}$$

$$h_1 = \pm\epsilon^{-1}U^{1/2}\left(1 \pm \epsilon^2\frac{h'_0}{U}\right) = \pm\epsilon^{-1}U^{1/2} - \frac{1}{4}\frac{U'}{U} \tag{3.164}$$

$$h_2 = \pm\epsilon^{-1}U^{1/2} - \frac{1}{4}\frac{U'}{U} + \epsilon\left(-\frac{5}{32}\frac{(U')^2}{U^{5/2}} + \frac{1}{8}\frac{U''}{U^{3/2}}\right) \tag{3.165}$$

and so on. We can check that the procedure is formally sound if $\epsilon^2 U^{-1} h'_0 \ll 1$ or

$$\epsilon U' U^{-3/2} \ll 1 \tag{3.166}$$

Formally we thus have

$$w = \pm\epsilon^{-1}\int U^{1/2}(s)ds - \frac{1}{4}\ln U + \cdots \tag{3.167}$$

and thus

$$\psi \sim U^{-1/4}e^{\pm\epsilon^{-1}\int U^{1/2}(s)ds} \tag{3.168}$$

If we include the complete series in powers of ϵ in (3.168) we get

$$\psi \sim \exp\left(\pm\epsilon^{-1}\int U^{1/2}(s)ds\right)U^{-1/4}\left(1 + \epsilon F_1(x) + \epsilon^2 F_2(x) + ...\right) \tag{3.169}$$

There are two possibilities compatible with our assumption about x_0, namely $V(x_0) > E$ and $V(x_0) < E$. In the first case there is (formally) an exponentially small solution and an exponentially large one, in the latter two rapidly oscillating ones.

The points where (3.166) fails are called *turning points*. Certainly if $|U(x_1)| > \delta$, then (3.166) holds near x_1, for ϵ small enough (depending on δ). In the opposite direction, assume $U'U^{-3/2} = \phi$ is bounded; integrating from $x_0 + \epsilon$ to x we get $-2(U(x)^{-1/2} + U(x_1)^{-1/2}) = \int \phi(s)ds$, and thus $U(x_0+\epsilon)^{-1/2}$ is uniformly bounded near x_0. For instance if U has a simple root at $x = 0$, the only one that we will consider here (but multiple roots are not substantially more difficult) then condition (3.166) reads

$$x \gg \epsilon^{2/3} \tag{3.170}$$

The region where this condition holds is called *outer* region. In a small region where (3.166) fails, called *inner* region, a different approximation will be sought. We see that $V(x) - E = V'(0)x + x^2h(x) =: \alpha x + x^2h(x)$ where $h(x) \in C^\infty(\mathbb{R})$. We then write

$$-\epsilon^2\psi'' + \alpha x = -x^2h(x)\psi \tag{3.171}$$

and treat the rhs of (3.171) as a small perturbation. The substitution $x = \epsilon^{2/3}t$ makes the leading equation an Airy equation:

$$-\psi'' + \alpha t\psi = -\epsilon^{2/3}t^2h(\epsilon^{2/3}t)\psi \tag{3.172}$$

which is a regularly perturbed equation! For a perturbation method to apply, we merely need that $x^2h(x)\psi$ in (3.171) is much smaller than the lhs, roughly requiring $x \ll 1$. This shows that the inner and outer regions overlap, there is a subregion –*the matching region*– where both expansions apply, and where, by equating them, the free constants in each of them can be linked. In the matching region, *maximal* balance occurs, in that a larger number of terms participate in the dominant balance. Indeed, if we examine (3.157) near $x = 0$, we see that $w'^2 \gg w''$ if $\epsilon^{-2}x \gg \epsilon^{-1}x^{-1/2}$, where we used (3.163). In the transition region, all terms in the middle expression in (3.157) participate equally.

3.7c Outer region. Rigorous analysis

We first look at a region where $U(x)$ is bounded away from zero. We will write $U = F^2$.

Proposition 3.173 *Let $F \in C^\infty(\mathbb{R})$, $F^2 \in \mathbb{R}$, and assume $F(x) \neq 0$ in $[a,b]$. Then for small enough ϵ there exists a fundamental set of solutions of (3.155) in the form*

$$\psi_\pm = \Phi_\pm(x;\epsilon)\exp\left[\pm\epsilon^{-1}\int F(s)ds\right] \tag{3.174}$$

where $\Phi_\pm(x;\epsilon)$ are C^∞ in $\epsilon > 0$.

PROOF We show that there exists a fundamental set of solutions in the form

$$\psi_\pm = \exp\left[\pm\epsilon^{-1}R_\pm(x;\epsilon)\right] \tag{3.175}$$

where $R_\pm(x;\epsilon)$ are C^∞ in ϵ. The proof is by rigorous WKB.

Note first that linear independence is immediate, since for small enough ϵ the ratio of the two solutions cannot be a constant, given their ϵ behavior.

We take $\psi = e^{w/\epsilon}$ and get, as before, to leading order $w' = \pm F$. We look at the plus sign case, the other case being similar. It is then natural to substitute $w' = F + \delta$; we get

$$\delta' + 2\epsilon^{-1}F\delta = -F' - \epsilon^{-1}\delta^2 \tag{3.176}$$

which we transform into an integral equation by treating the rhs as if it was known and integrating the resulting linear inhomogeneous differential equation. Setting $H = \int F$ the result is

$$\delta = -e^{-\frac{2H}{\epsilon}} \int_a^x F'(s) e^{\frac{2H(s)}{\epsilon}} ds - \frac{1}{\epsilon} e^{-\frac{2H}{\epsilon}} \int_a^x \delta^2(s) e^{\frac{2H(s)}{\epsilon}} ds =: J(\delta) =: \delta_0 + N(\delta) \tag{3.177}$$

We assume that $F > 0$ on (a, b), the case $F < 0$ being very similar. The case $F \in i\mathbb{R}$ is not too different either, as we will explain at the end.

Let now $\|F'\|_\infty = A$ in (a, b) and assume also that $\min_{s \in (a,b)} |U(s)| = B^2 > 0$.

Lemma 3.178 *For small ϵ, the operator J is contractive in a ball $\mathcal{B} := \{\delta : \|\delta\|_\infty \le 2AB^{-1}\epsilon\}$*

PROOF i) Preservation of $\mathcal{B}$. We have

$$|\delta_0(x)| \le Ae^{-\frac{2}{\epsilon}H(x)} \int_a^x e^{\frac{2}{\epsilon}H(s)} ds$$

By assumption, H is increasing on (a, b) and $H' \ne 0$ and thus, by the Laplace method, cf. Proposition 3.17, for small ϵ we have (since $H' = \sqrt{U}$),

$$|\delta_0(x)| \le 2Ae^{-\frac{2}{\epsilon}H(x)} \frac{e^{\frac{2}{\epsilon}H(x)}}{\frac{2}{\epsilon}H'(x)} \le \epsilon AB^{-1}$$

Note We need this type of estimates to be uniform in $x \in [a, b]$ as $\epsilon \to 0$. To see that this is the case, we write

$$\int_a^x e^{\frac{2}{\epsilon}H(s)} ds = \int_a^x e^{\frac{2}{\epsilon}H(s)} \frac{2F(s)}{\epsilon} \frac{\epsilon}{2F(s)} ds$$
$$\le \frac{\epsilon}{2B} e^{\frac{2}{\epsilon}H(s)} \Big|_a^x \le \frac{\epsilon}{2B} e^{\frac{2}{\epsilon}H(x)} \tag{3.179}$$

Similarly,

$$\left|\frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x \delta^2(s)e^{\frac{2H(s)}{\epsilon}}ds\right| \le 2\epsilon^2A^2B^{-3}$$

and thus, for small ϵ and $\delta \in \mathcal{B}$ we have

$$J(\delta) \le \epsilon^{-1}AB^{-1} + 2\epsilon^2A^2B^{-3} \le 2\epsilon AB^{-1}$$

ii) *Contractivity.* We have, with $\delta_1, \delta_2 \in \mathcal{B}$, using similarly Laplace's method,

$$|J(\delta_2) - J(\delta_1)| \le \frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x |\delta_2(s) - \delta_1(s)||\delta_2(s) + \delta_1(s)|e^{\frac{2H(s)}{\epsilon}}ds$$
$$\le \frac{2\epsilon A}{B^2}\|\delta_2 - \delta_1\| \quad (3.180)$$

and thus the map is contractive for small enough ϵ.

□

Note. We see that the conditions of preservation of $\mathcal{B}$ and contractivity allow for a dependence of (a, b) on ϵ. Assume for instance $a, b > 0$, $V(x) = E$ has no root in $[a, b + \gamma)$ with $\gamma > 0$, and that a is small. Assume further that $V(0) = E$ is a simple root, $V'(0) = \alpha \ne 0$. Then for some $C > 0$ we have $B \ge Cm^2a^2$ and the condition of contractivity reads

$$\frac{\epsilon^2|\alpha|}{|\alpha|^3} < 1$$

i.e. $a > (\epsilon/|\alpha|)^{2/3}$ and for small enough ϵ this is also enough to ensure preservation of $\mathcal{B}$. We thus find that the equation $\delta = J(\delta)$ has a unique solution and that, furthermore, $\|\delta\| \le const.\epsilon$. Using this information and (3.180) which implies

$$\|J(\delta)\| \le \frac{\epsilon A}{B^2}2AB^{-1}\epsilon$$

we easily get that, for some constants $C_i > 0$ independent on ϵ,

$$|\delta - \delta_0| \le C_1\epsilon|\delta| \le C_1\epsilon|\delta_0| + C_1\epsilon|\delta - \delta_0|$$

and thus

$$|\delta - \delta_0| \le C_2\epsilon|\delta_0|$$

and thus, applying again Laplace's method we get

$$\delta \sim \frac{-\epsilon F'}{2F} \quad (3.181)$$

which gives

$$\psi \sim \exp\left(\pm\epsilon^{-1}\int U^{1/2}(s)ds\right)U^{-1/4}$$

The proof of the C^∞ dependence on ϵ can be done by induction, using (3.181) to estimate δ^2 in the fixed point equation, to get an improved estimate on δ, etc.
In the case $F \in i\mathbb{R}$, the proof is the same, by using the Stationary Phase method instead of the Laplace Method.

□

3.7d Inner region. Rigorous analysis

By rescaling the independent variable we may assume without loss of generality that $\alpha = 1$ in (3.172) which we rewrite as

$$-\psi'' + t\psi = -\epsilon^{2/3}t^2 h_1(\epsilon^{2/3}t)\psi := f(t) \tag{3.182}$$

which can be transformed into an integral equation in the usual way,

$$\psi(t) = -\mathrm{Ai}(t)\int^t f(s)\mathrm{Bi}(s)ds + \mathrm{Bi}(t)\int^t f(s)\mathrm{Ai}(s)ds + C_1\mathrm{Ai}(t) + C_2\mathrm{Bi}(t) \tag{3.183}$$

where Ai, Bi are the Airy functions, with the asymptotic behavior

$$\mathrm{Ai}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{-\frac{2}{3}t^{\frac{3}{2}}};\quad \mathrm{Bi}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}} \tag{3.184}$$

and

$$|t^{-1/4}\mathrm{Ai}(t)| < const.,\quad |t^{-1/4}\mathrm{Bi}(t)| < const. \tag{3.185}$$

as $t \to -\infty$. In view of (3.184) we must be careful in choosing the limits of integration in (3.183). It is important to ensure that the second term does not have a fast growth as $t \to \infty$, and for this purpose we need to integrate from t toward infinity in the associated integral. For that, we ensure that the maximum of the integrand is achieved *at or near the variable endpoint of integration.* Then Laplace's method shows that the leading contribution to the integral comes from the variable endpoint of integration as well, which allows for the opposite exponentials to cancel out. We choose to look at an interval in the original variable $x \in I_M = [-M, M]$ where we shall allow for ϵ-dependence of M. We then write the integral equation with concrete limits in the form below, which we analyze in I_M.

$$\psi(t) = -\text{Ai}(t)\int_0^t f(s)\text{Bi}(s)ds + \text{Bi}(t)\int_M^t f(s)\text{Ai}(s)ds + C_1\text{Ai}(t) + C_2\text{Bi}(t) = J\psi + \psi_0 \quad (3.186)$$

Proposition 3.187 *For some positive const., if ϵ is small enough (3.186) is contractive in the sup norm if $M \le const.\epsilon^{2/5}$.*

PROOF

Using the Laplace method we see that for $t > 0$ we have

$$t^{-1/4}e^{-\frac{2}{3}t^{\frac{3}{2}}}\int_0^t s^{-1/4}e^{\frac{2}{3}s^{\frac{3}{2}}}ds \le const.(|t|+1)^{-1}$$

and also

$$t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}}\int_t^M s^{-1/4}e^{-\frac{2}{3}s^{\frac{3}{2}}}ds \le t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}}\int_t^\infty s^{-1/4}e^{-\frac{2}{3}s^{\frac{3}{2}}}ds \le const.(|t|+1)^{-1} \quad (3.188)$$

and thus for a constant independent of ϵ, using (3.184) we get

$$|J\psi(t)| \le const.\epsilon^{2/3}(|t|+1)^{-1}\sup_{s\in[0,t]}|\psi(s)|$$

for $t > 0$. For $t < 0$ we use (3.185) and get

$$\left|\text{Ai}(t)\int_M^t f(s)\text{Bi}(s)ds\right| \le (1+|t|)^{-1/4}\sup_{s\in[-t,0]}|f(s)|(const. + \int_t^0 s^{-1/4}ds$$

and get for a constant independent of ϵ

$$|J\psi(t)| \le const.\epsilon^{2/3}(1+|t|)^{5/2} \le const.\epsilon^{2/3}(\epsilon^{-2/3}M)^{5/2} < 1$$

We see that for small enough ϵ, the regions where the outer and inner equations are contractive overlap. This allows for performing asymptotic matching in order to relate these two solutions. For instance, from the contractivity argument it follows that

$$\psi = (1-J)^{-1}\psi_0 = \sum_{k=0}^{\infty} J^k\psi_0$$

giving a power series asymptotics in powers of $\epsilon^{2/3}$ for ψ.

∎

3.7e Matching

We may choose for instance $x = const.\epsilon^{1/2}$ for which the inner expansion (in powers of $\epsilon^{2/3}$) and the outer expansion (in powers of ϵ) are valid at the same time. We assume that x lies in the oscillatory region for the Airy functions (the other case is slightly more complicated).

We note that in this region of x the coefficient of ϵ^k of the outer expansion will be large, of order $(U'U^{-3/2})^k \sim \epsilon^{-3k/4}$. A similar estimate holds for the terms of the inner expansion. Both expansions will thus effectively be expansions in $\epsilon^{-1/4}$. Since they represent the same solution, they must agree and thus the coefficients of the two expansions are linked. This determines the constants C_1 and C_2 once the outer solution is prescribed.

3.8 PDE analog

Consider now a a parabolic PDE, say the heat equation.

$$\psi_t = \psi_{xx} \tag{3.189}$$

The fact that the principal symbol is degenerate (there are fewer t than x derivatives) has an effect similar to that of a singular perturbation. If we attempt to solve the PDE by a power series

$$\psi = \sum_{k=0}^{\infty} t^k F_k(x) \tag{3.190}$$

this series will generically have zero radius of convergence. Indeed, the recurrence relation for the coefficients is $F_k = F''_{k-1}/k$ whose solution, $F_k = F_0^{(2k)}/k!$ behaves like $F_k \sim k!$ for large k, if F is analytic but not entire.

Generally, exponential solutions are expected too.[5] If we take $\psi = e^w$ in (3.189) we get

$$w_t = w_x^2 + w_{xx} \tag{3.191}$$

where the assumption of algebraic behavior of w is expected to ensure $w_x^2 \gg w_{xx}$ and so the leading equation is approximately

$$w_t = w_x^2 \tag{3.192}$$

[5]The reason will be better understood after Borel summation methods have been studied. Divergence means that the Borel transform of the formal solution is nontrivial: it has singularities. Upon Laplace transforming it, paths of integration on different sides of the singularities give different results, and the differences are exponentially small.

which can be solved by characteristics. We take $w_x = u$ and get for u the quasilinear equation

$$u_t = 2uu_x \tag{3.193}$$

with a particular solution $u = -x/(2t)$, giving $w = -x^2/(4t)$. We thus take $w = -x^2/(4t) + \delta$ and get for δ the equation

$$\delta_t + \frac{x}{t}\delta_x + \frac{1}{2t} = \delta_x^2 + \delta_{xx} \tag{3.194}$$

where we have separated the relatively small terms to the rhs. We would normally solve the leading equation (the lhs of (3.194)) and continue the process, but for this equation we note that $\delta = -\frac{1}{2}\ln t$ solves not only the leading equation, but the full equation (3.194). Thus

$$w = -\frac{x^2}{4t} - \frac{1}{2}\ln t \tag{3.195}$$

which gives the classical heat kernel

$$\psi = \frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}} \tag{3.196}$$

This exact solvability is of course rather accidental, but a perturbation approach formally works in a more PDE general context.

Chapter 4

Introduction to transseries and analyzable functions

There is, as we have seen an important distinction between asymptotic expansions and asymptotic series. The operator $f \mapsto \mathcal{A}_p(f)$ which associates to f its asymptotic power series is linear as seen in §1.1c . But it has a nontrivial kernel ($\mathcal{A}_p(f) = 0$ for many nonzero functions), and the description through asymptotic power series is fundamentally *incomplete*. There is no unambiguous way to determine a function from its classical asymptotic series alone. On the other hand, the operator $f \mapsto \mathcal{A}(f)$ which associates to f its asymptotic *expansion* has zero kernel, but it is still false that $\mathcal{A}(f) = \mathcal{A}(g)$ implies $f = g$ ($\mathcal{A}$ is *not* linear, see Remark 1.25). The description of a function through its asymptotic *expansion* is also incomplete.

4.1 Analyzable functions and the theory of Écalle: a preview

4.1a Analytic function theory as a toy model of the theory of analyzable functions

Let A denote the set of germs of analytic functions at $z = 0$, let $\mathbb{C}[[z]]$ be the space of formal series in z with complex coefficients, of the form $\sum_{k=0}^{\infty} c_k z^k$, and define $\mathbb{C}_c[[z]]$ as the subspace of series with nonzero radius of convergence.

The Taylor series at zero of a function in A is also its asymptotic series at zero. Moreover, the map $\mathcal{T} : A \mapsto \mathbb{C}_c[[z]]$ is an isomorphism and its inverse $\mathcal{T}^{-1} = \mathcal{S}$ is simply the operator of summation of series in $\mathbb{C}_c[[z]]$. $\mathcal{T}$ and $\mathcal{S}$ commute with most of the useful function operations defined on $\mathcal{A}$, in particular we have, with $\tilde{f}, \tilde{f}_1$ and $\tilde{f}_2$ in $\mathbb{C}_c[[z]]$

$$\begin{aligned}
&(1) && \mathcal{S}\{\alpha\tilde{f}_1+\beta\tilde{f}_2\}=\alpha\mathcal{S}\tilde{f}_1+\beta\mathcal{S}\tilde{f}_2\\
&(2) && \mathcal{S}\{\tilde{f}_1\tilde{f}_2\}=\mathcal{S}\tilde{f}_1\mathcal{S}\tilde{f}_2\\
&(3) && \mathcal{S}\{\tilde{f}^\star\}=\left\{\mathcal{S}\tilde{f}\right\}^\star\\
&(4) && \mathcal{S}\{\tilde{f}\}'=\left\{\mathcal{S}\tilde{f}\right\}';\quad \mathcal{S}\left\{\int_0^x\tilde{f}\right\}=\int_0^x\mathcal{S}\tilde{f}\\
&(5) && \mathcal{S}\{\tilde{f}_1\circ\tilde{f}_2\}=\mathcal{S}\tilde{f}_1\circ\mathcal{S}\tilde{f}_2\\
&(6) && \mathcal{S}1=1
\end{aligned}\tag{4.1}$$

where $\tilde{f}^\star(z)=\overline{\tilde{f}}(\overline{z})$. In fact $\mathcal{T}$ is such a good isomorphism between A and $\mathbb{C}_c[[z]]$, that usually no distinction is made between formal (albeit convergent) expansions and their sums which are actual functions. There does not even exist a notational distinction between a convergent series, as a series, and its sum as a number.

As a consequence of the isomorphism, whenever a problem can be solved in $\mathbb{C}_c[[z]]$, $\mathcal{S}$ provides an actual solution of the same problem. For example, if $\tilde{y}$ is a formal solution of the equation

$$\tilde{y}'=\tilde{y}^2+z\tag{4.2}$$

as a series in powers of z, with nonzero radius of convergence, and we let $y=\mathcal{S}\tilde{y}$ we may write, using (4.1),

$$\left(\tilde{y}'=\tilde{y}^2+z\right)\Leftrightarrow\left(\mathcal{S}\left\{\tilde{y}'\right\}=\mathcal{S}\left\{\ \tilde{y}^2\right\}+z\right)\Leftrightarrow\left(y'=y^2+z\right)$$

i.e. $\tilde{y}$ is a formal solution of (4.2) iff y is an actual solution. The same reasoning would work in most natural problems with analytic coefficients for which solutions $\tilde{y}\in C_C[[z]]$ can be found.

On the other hand, if we return to the example in Remark 1.25, f_1 and f_2 differ by a constant C, coming from the lower limit of integration, and this C is lost in the process of calculating the asymptotic expansion. To have a complete description, clearly we must account for C. It is then natural to try to write instead

$$f_{1,2}\ \text{''}\sim\text{''}\ e^x\tilde{f}+C_{1,2}\tag{4.3}$$

However, Note 1.23 however shows $C_{1,2}$ cannot be defined through (1.12); $C_{1,2}$ they cannot be calculated as $f_{1,2}-e^x\tilde{f}$ since $\tilde{f}$ does not converge. The right side of (4.3) becomes for now a purely formal object, in the sense that it does not connect to an actual function in any obvious way.

It is the task of the theory of analyzable functions to give a general, natural and consistent meaning to expansions such as (4.3) ((4.3) is perhaps the simplest nontrivial instance of a transseries), in such a way that expansions and functions are into a true one-to-one correspondence. An isomorphism like (4.1) holds in much wider generality.

Some ideas of the theory of analyzable functions can be traced back to Euler as seen in §1.2a , Cauchy , Borel who found the first powerful technique to deal with divergent expansions, and by Dingle and Berry who substantially extended optimal truncation methods.

In the early 80's exponential asymptotics became a field of its own, with the a number of major discoveries of Écalle, the theory of transseries and analyzable functions, and a very comprehensive generalization of Borel summation.

Setting of the problem. One operation is clearly missing from both A and $\mathbb{C}_c[[z]]$ namely division, and this severely limits the range of problems that can be solved in either A or $\mathbb{C}_c[[z]]$. The question is then, which spaces $A_1 \supset A$ and $S_1 \supset \mathbb{C}_c[[z]]$ are closed under all function operations, including division, and are such that an extension of $\mathcal{T}$ is an isomorphism between them? (Because of the existence of an isomorphism between A_1 and the formal expansions S_1 the functions in A_1 will be called called, in agreement with Écalle, *formalizable*). Exploring the limits of formalizability is at the core of the modern theory of analyzable functions.

In addition to the obvious theoretical interest, there are many important practical applications. One application of such a theory, for instance for some generic classes of differential systems where it has been worked out, is the possibility of solving problems starting from formal expansions, which are easy to produce (usually algorithmically), and from which the isomorphism produces, constructively, actual solutions.

We start by studying general formal expansions in their own right, to understand their structure and operations with them.

4.1b Formal asymptotic power series

Definition 4.4 *For $x \to \infty$, an asymptotic power series (APS) is a formal structure of the type*

$$\sum_{i\in J} \frac{c_i}{x^{k_i}} \tag{4.5}$$

where J is a set of ordinals and $k_i > k_j$ if $i > j$. We assume that there is no infinite strictly decreasing subsequence $k_{i_1} > k_{i_2} > \ldots$. For simplicity we shall assume $J = \mathbb{N}$ and that there is no accumulation point of the k_i.

In particular, there is a *smallest* power $k_j \in \mathbb{Z}$, possibly negative.

Examples. (1) Integer power series, i.e. series of the form

$$\sum_{k=M}^{\infty} \frac{c_k}{x^k} \tag{4.6}$$

(2) An important instance are the *finitely generated* power series, which are by definition of the form

$$\sum_{k_i \geq M} \frac{c_{k_1,k_2,\ldots,k_n}}{x^{\alpha_1 k_1+\ldots+\alpha_n k_n}} \tag{4.7}$$

for some $M \in \mathbb{Z}$ and $n \in \mathbb{N}$, where $\alpha_1 > 0, ..., \alpha_n > 0$. Its generators are the *monomials* $x^{-\alpha_1}, ..., x^{-\alpha_n}$.

Proposition 4.8 *A series of the form (4.7) is (can be rearranged as) an APS.*

PROOF For the proof we note that for any $L \in \mathbb{Z}$, the set

$$\{(k_1, k_2, ..., k_n) \in \mathbb{Z}^n : k_i \geq M \text{ for } 1 \leq i \leq n \text{ and } L \geq \sum_{i=1}^{n} \alpha_i k_i\}$$

is finite. Indeed, k_i are bounded below, $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i k_i \to \infty$ if at least one of the sequences $\{k_{i_j}\}$ is unbounded.

Exercise 4.9 *As a consequence show that:*
(1)

$$\inf\{\nu : \nu = \alpha_1 k_1 + \cdots + \alpha_n k_n = \nu \textit{ for some } k_1, ..., k_n \geq M\} < \infty$$

(2) The set

$$J := \{\nu : \nu = \alpha_1 k_1 + \cdots + \alpha_n k_n = \nu \textit{ for some } k_1, ..., k_n \geq M\}$$

is countable with no accumulation point. Furthermore J can be linearly ordered

$$\nu_1 < \nu_2 < \cdots < \nu_k < \cdots$$

and all the sets

$$J_i := \{k_1, ..., k_j \geq M : \nu_i = \alpha_1 k_1 + \cdots + \alpha_n k_n\}$$

are finite.
Complete the proof of the proposition.

$\Box$

Thus (4.7) can be written in the form (4.5). In particular we can define the dominance of a series in the following way:

Definition 4.10 (of *Dom***)** *If S is a nonzero APS of the type (4.5) we define $Dom(S)$ to be $c_{i_1} x^{-k_{i_1}}$ where i_1 is the first i in (4.5) for which $c_i \neq 0$. We write $Dom(S) = 0$ iff $S = 0$.*

4.1b .1 Operations with APS

Note 4.11 *The following operations are defined in a natural way and have the usual properties:* $+, -, \times, /$ *differentiation and composition* $S_1 \circ S_2$ *where* S_2 *is a series such that* $k_1 < 0$*. For composition and division, see note after Proposition 4.18. For instance,*

$$\sum_{k=0}^{\infty} \frac{c_j}{x^{\nu_j}} \sum_{l=0}^{\infty} \frac{d_l}{x^{\eta_l}} = \sum_{k,l=0}^{\infty} \frac{c_j d_l}{x^{\nu_j+\eta_l}} \tag{4.12}$$

Exercise 4.13 ** Show that the last series in (4.12) can be written in the form (4.5).*

Exercise 4.14 ** Show that finitely generated power series are closed under the operations mentioned above.*

4.1b .2 Asymptotic order relation

If $C_1, C_2 \neq 0$, we naturally write (remember that $x \to +\infty$ and the definition of $\ll$ in (1.9) and (1.10))

$$C_1 x^p \ll C_2 x^q \quad \text{iff} \quad p < q$$

Definition 4.15 *For two nonzero APSs* S_1*,* S_2 *we write* $S_1 \gg S_2$ *iff* $Dom(S_1) \gg Dom(S_2)$*.*

Proposition 4.16 $Dom(S_1 S_2) = Dom(S_1) Dom(S_2)$*, and if* $Dom(S) \neq const$ *then* $Dom(S') = Dom(S)'$*.*

PROOF Exercise. ◻

Thus we have

Proposition 4.17 *(See note(4.11)).*

(i) $S_1 \ll T$ *and* $S_2 \ll T$ *imply* $S_1 + S_2 \ll T$ *and for any nonzero* S_3 *we have* $S_1 S_3 \ll S_2 S_3$*.*

(ii) $S_1 \gg T_1$ *and* $S_2 \gg T_2$ *imply* $S_1 S_2 \gg T_1 T_2$*.*

(iii) $S \ll T$ *implies* $\frac{1}{S} \gg \frac{1}{T}$*.*

(iv) $S \ll T \ll 1$ *implies* $S' \ll T' \ll 1$ *and* $1 \ll S \ll T$ *implies* $S' \ll T'$ *(prime denotes differentiation). Also,* $s \ll 1 \Rightarrow s' \ll s$ *and* $L \gg 1 \Rightarrow L \gg L' \gg 1$*.* $S' \gg T'$ *and* $T \gg 1$ *implies* $S \gg T$*. Also* $1 \gg S' \gg T'$ *implies* $S \gg T$*.*

(v) There is the following trichotomy for two nonzero APSs : $S \ll T$ *or* $S \gg T$ *or else* $\frac{S}{T} - C \ll 1$ *for some constant* C*.*

PROOF Exercise. ◻

Proposition 4.18 *Any nonzero APS S can be uniquely decomposed in the following way*

$$S = L + C + s$$

where C is a constant and L and s are APS, with the property that L has nonzero coefficients only for positive *powers of x (L is purely large) and s has nonzero coefficients only for* negative *powers of x (s is purely small).*

PROOF Exercise. ∎

Exercise 4.19 ** Show that any nonzero series can be written in then form $S = D(1+s)$ where $D =$Dom(S) and s is a small series.*

Exercise 4.20 *Show that the large part of a series has only finitely many terms.*

Exercise 4.21 ** Show that for any coefficients $a_1, ..., a_m, ...$ and* small *series s the formal expression*

$$1 + a_1 s + a_2 s^2 + \cdots \tag{4.22}$$

defines a formal power series.

Note 4.23 *Let S be a nonzero series and $D = C_1 x^{-\nu_1} =$ Dom(S). We define $1/D = (1/C_1)x^{\nu_1}$ and*

$$\frac{1}{S} = \frac{1}{D}(1 - s + s^2 - s^3 \cdots) \tag{4.24}$$

and more generally

$$S^\beta := C_1^\beta x^{-\nu_1 \beta}\left(1 + \beta s + \frac{1}{2}\beta(\beta-1)s^2 + \cdots\right) \tag{4.25}$$

The composition of two series $S = \sum_{k=0}^{\infty} s_k x^{-\nu_k}$ and L where L is large is defined as

$$S \circ L := \sum_{k=0}^{\infty} s_k L^{-\nu_k} \tag{4.26}$$

Exercise 4.27 ** Show that (4.26) defines a formal power series which can be written in the form (4.5).*

Example

Proposition 4.28 *The differential equation*

$$y' + y = \frac{1}{x} + y^3 \tag{4.29}$$

has a unique solution as an APS which is purely small.

PROOF For the existence part, note that direct substitution of a formal integer power series $y_0 = \sum_{k=1}^{\infty} c_k x^{-k}$ leads to the recurrence relation $c_1 = 1$ and for $k \geq 2$,

$$c_k = (k-1)c_{k-1} + \sum_{k_1+k_2+k_3=k; k_i \geq 1} c_{k_1} c_{k_2} c_{k_3}$$

for which direct induction shows the existence of a solution, and we have

$$y_0 = \frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{12}{x^4} + \frac{60}{x^5} + \cdots$$

For uniqueness assume y_0 and y_1 are APS solutions and let $\delta = y_1 - y_0$. Then δ satisfies

$$\delta' + \delta = 3y_0^2\delta + 3y_0\delta^2 + \delta^3 \tag{4.30}$$

Since by assumption $\delta \ll 1$ we have $\mathrm{Dom}(\delta') \ll \mathrm{Dom}(\delta)$ and similarly $\mathrm{Dom}(3y_0^2\delta + 3y_0\delta^2 + \delta^3) \ll \mathrm{Dom}(\delta)$. But this implies $\mathrm{Dom}(\delta) = 0$ and thus $\delta = 0$.

∎

4.1b .3 The exponential

Proposition 4.31 (the exponential is not a power series.) *The differential equations $f' \pm f = 0$ have no nontrivial solution as an APS.*

PROOF If $\mathrm{Dom}(f) = c = const$ then $f = c + s$ where s is purely small and thus $s' \ll s$. But then the equation $s' \pm (c + s) = 0$ is contradictory. Similarly if $f = s$ where s is purely small then $s' + s = 0$ is impossible by Proposition 4.17. If instead $f \gg 1$ then $f' + f = 0$ is again impossible. ∎

Thus the exponential, which we need since very simple equations generate it, is a new element. We would like it to be compatible with the basic structures of APS and with the asymptotic ordering. Then $e^x \gg 1$ or $e^x \ll 1$ or finally $e^x - c \ll 1$. The last inequality is not consistent if we differentiate it using (iv) of Proposition 4.17. The remaining choices are consistent, and correspond to selecting a sign in $x \to \pm\infty$. Since we chose $x \to +\infty$, we let, *by definition*, $e^x \gg 1$. Compatibility with Proposition 4.17 implies $(e^x)' \gg x'$ and thus $e^x \gg x$. Inductively, $e^x \gg x^n$ for all n.

Proposition 4.32 *If s is a purely small series then the equation $y' = s'y$ (corresponding intuitively to $y = e^s$) has APS solutions of the form $C + s_1$ where s_1 is small. If we choose $C = 1$ then $s_1 = s_{1;1}$ is uniquely defined.*

PROOF Exercise. ∎

Definition 4.33 *We define, according to the previous proposition,* $e^s = 1 + s_{1;1}$*;* $1 + s_{1,;1}$ *is simply the familiar Maclaurin series of the exponential. In general if* $S = L + C + s$ *we write* $e^S = C(1 + s_{1;1})e^L$ *where* e^L *is to be thought of as a primary symbol, subject to the further definitions* $e^{L_1+L_2} = e^{L_1}e^{L_2}$ *and* $(e^L)' = L'e^L$.

4.1b .4 Exponential power series (EPS)

A simple example of EPS is a formal expression of the type

$$\sum_{i,j=1}^{\infty} \frac{c_{ij}}{e^{\lambda_i x} x^{k_j}} \tag{4.34}$$

where λ_i are increasing in i and k_j are increasing in j. Again the usual operations are well defined on EPS (composition is not defined on (4.34) but it would be if more general terms of the form $C(1 + s_{1;1})e^L$ are allowed; we will postpone this until the formal introduction of transseries).

The order relation, compatible with the discussion in § 4.1b .3, is defined by $e^{\lambda_1 x}x^{k_2} \gg e^{\lambda_3 x}x^{k_4}$ iff $\lambda_1 > \lambda_3$ or if $\lambda_1 = \lambda_3$ and $k_2 > k_4$. Consistent with this order relation it is then natural to reorder the expansion(4.34) as follows

$$\sum_{i=1}^{\infty} e^{-\lambda_i x} \sum_{j=1}^{\infty} \frac{c_{ij}}{x^{k_j}} \tag{4.35}$$

Then we can still define the dominance of a structure of the form (4.34).

The question is what is the general *formal* solution of

$$f' + f = x^{-1} \tag{4.36}$$

For this we have to assume we have a space A of formal objects in which all operations involved in (4.36) make sense and have the usual properties. A would be a differential algebra. It is natural to assume that in A $f' = 0$ has the general solution $f = C$ for some constant C. We need A to contain x^{-1} so that the differential equation makes sense, which implies by closure under algebraic operations that A contains all inverse powers of x, including constants (power zero), and A should contain the formal series solution of (4.36)

$$\tilde{y}_0 = \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} \tag{4.37}$$

Exercise 4.38 ** Show that there exist differential fields, containing* $1/x$*,in which (4.36) has no solution.*

Then if $\tilde{y}$ is any solution of (4.36) then $\tilde{f} = \tilde{y} - \tilde{y}_0$ satisfies the homogeneous equation $f' + f = 0$. To proceed, we may include solutions of this homogeneous equation. If we call this solution e^{-x} and the solution of the related equation

$f' - f = 0$ by e^x we see that $(e^{-x}e^x)' = 0$ thus $e^{-x}e^x = C$ for some C and we can normalize our choice of e^x to make $C = 1$. Then the general solution of $y' + y = 0$ is Ce^{-x}. Indeed, we may multiply by e^x and get $(ye^x)' = 0$, i.e. $ye^x = C$ or $y = Ce^{-x}$.

4.1b .5 Exponential power series solutions for (4.29)

To show how transseries arise naturally as solutions of ODEs we take the prototypical nonlinear equation (4.29). The analysis that follows is formal, and assumes the space of formal solutions has all the "expected" properties. A rigorous construction, backing up this analysis will follow in Chapter 3.

To simplify notation, we drop the tildes from formal asymptotic expansions. We have obtained, in Proposition 4.17 a formal series solution (4.29), y_0. We look for possible further solutions. We take $y = y_0 + \delta$. The equation for δ is (4.30) where we search for solutions $\delta \ll 1$, in which assumption the terms on the right side of the equation are subdominant (see footnote 4 on Page 77). We have $\delta' + \delta(1 + o(1)) = 0$ thus $\delta = Ce^{-x+o(x)}$and this suggests the substitution $\delta = e^w$. We get

$$w' + 1 = 3y_0^2 + 3y_0e^w + e^{2w}$$

and since $e^w = \delta \ll 1$ the dominant balance (footnote 4, Page 77) is between the terms on the left side, thus $w = -x + C + w_1$ and we get

$$w_1' = 3y_0^2 + 3y_0e^{-x}e^{w_1} + e^{-2x+2w_1}$$

We have $y_0e^{-x}e^{w_1} = y_0\delta = y_0e^{-x+o(x)}$. Since $-x + o(x) \gg n\ln(x)$ we have $y_0e^{-x}e^{w_1} \ll x^{-n}$ for any n and thus $w_1' = O(x^{-2})$ then $w_1 = O(x^{-1})$. Thus, $e^{w_1} = 1 + w_1 + w_1^2/2 + ...$ and consequently $3y_0e^{-x}e^{w_1} + e^{-2x+2w_1}$ is negligible with respect to y_0^2. Again by dominant balance, to leading order, $w_1' = 3y_0^2$ and thus $w_1 = \int 3y_0^2 + w_2 := \phi_1 + w_2$ (ϕ_1 is a formal power series). It follows that, to leading order, we have

$$w_2' = 3y_0e^{-x}$$

and thus $w_2 = \phi_2e^{-x}$ where ϕ_2 is a power series. Continuing this process of iteration, we can see inductively that w must be of the form

$$w = -x + \sum_{k=0}^{\infty} \phi_k e^{-kx}$$

where ϕ_k are formal power series, which means

$$y = \sum_{k=0}^{\infty} e^{-kx} y_k \tag{4.39}$$

where y_k are also formal power series. Having obtained this information, it is more convenient to plug in(4.39) directly in the equation and solve for the unknown series y_k. We get the system

$$\begin{aligned} y_0' + y_0 &= x^{-1} + y_0^3 \\ y_1' &= 3y_0^2 y_1 \\ &\cdots \\ y_k' - ky_k - 3y_0^2 y_k &= 3y_0 \sum_{k_1+k_2=k;k_i\ge 1} y_{k_1}y_{k_2} + \sum_{k_1+k_2+k_3=k;k_i\ge 1} y_{k_1}y_{k_2}y_{k_3} \\ &\cdots \end{aligned} \tag{4.40}$$

We can easily see by induction that this system of equations does admit a solution where y_k are integer power series. Furthermore, y_1 is defined up to an arbitrary multiplicative constant, and there is no further freedom in y_k, whose equation can be solved by our usual iteration procedure, after placing the subdominant term y_k' on the RHS. We note that all equations for $k \ge 1$ are *linear inhomogeneous*. The fact that high-order equations are linear is a general feature in perturbation theory.

Choosing then y_0 in such a way that $y_1^{[1]} = 1+ax^{-1}+...$ we have $y_1 = Cy_1^{[1]}$. By the special structure of the RHS of the general equation in (4.40) we see that if $y_k^{[1]}$ is the solution with the choice $y_1 = y_1^{[1]}$ we see, by induction, that the solution when $y_1 = Cy_1^{[1]}$ is $C^k y_k^{[1]}$. Thus the general formal solution of (4.29) in our setting should be

$$\sum_{k=0}^{\infty} C^k y_k^{[1]} e^{-kx}$$

where $y_0^{[1]} = y_0$.

Exercise 4.41 *** Complete the details in the previous proof: show that the equation for y_1 in (4.40) has a one parameter family of solutions of the form $y_1 = c(1+s_1)$ where s_1 is a small series, and that this series is unique. Show that for $k > 1$, given $y_0, ..., y_{k-1}$, the equation for y_k in (4.40) has a unique small series solution. Show that there exists exactly a one parameter family of solutions general formal exponential-power series solution of the form (4.39) of (4.29).*

4.2 Preview of general properties of transseries

4.2a Remarks about the form of asymptotic expansions

The asymptotic expansions seen in the previous examples have the common feature that they are written in terms of powers of the variable, exponentials and logs, e.g.

$$\int_x^\infty e^{-s^2}ds \sim e^{-x^2}\left(\frac{1}{2x}-\frac{1}{4x^2}+\frac{5}{8x^3}-...\right) \tag{4.42}$$

$$n! \sim \sqrt{2\pi}e^{n\ln n-n+\frac{1}{2}\ln n}\left(1+\frac{1}{12n}+...\right) \tag{4.43}$$

$$\int_1^x \frac{e^t}{t}dt \sim e^x\left(\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x^3}+...\right) \tag{4.44}$$

Hardy noted that "No function has yet presented itself whose asymptotic expansion cannot be expressed in terms of exponentials, power series and logs". The modern conjecture of Écalle states that functions of natural origin can be isomorphically represented by "transseries" in the same way as an analytic function is locally given by a convergent Taylor series.

4.2b Transseries

Transseries are studied carefully in Chapter 3.

Informally, they are finitely generated asymptotic combinations of powers, exponentials and logs and are defined inductively. In the case of a power series, finite generation means that the series is an integer multiseries in $y_1, ..., y_n$ where $y_j = x^{-\beta_j}, \Re(\beta_j) > 0$. Examples are (4.34), (3.105) and (1.26); a more involved one would be

$$\ln\ln x + \sum_{k=0}^{\infty} e^{-k\exp(\sum_{k=0}^{\infty} k!x^{-k})}$$

A single term in a transseries is a transmonomial.

1. A term of the form $m = x^{-\alpha_1 k_1 - ... - \alpha_n k_n}$ with $\alpha_i > 0$ is a level zero **(trans)monomial**.
2. Real transseries of level zero are simply finitely generated *asymptotic* power series. That is, given $\alpha_1, ..., \alpha_n$ with $\alpha_i > 0$ a level zero transseries is a sum of the form

$$S = \sum_{k_i \geq M_i} c_{k_1,...,k_n} x^{-\alpha_1 k_1 - ... - \alpha_n k_n} \tag{4.45}$$

with $c_{M_1,\ldots M_n} \neq 0$ where $M_1, ..., M_n$ are *integers*, positive or negative; the terms of S are therefore nonincreasing in k_i and bounded *above* by $O(x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n})$.

3. $x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n}$ is the leading order, $c_{M_1,\ldots M_n}$ is the leading constant and $c_{M_1,\ldots M_n} x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n}$ is the dominance of (4.45), $\mathrm{Dom}(S)$.

4. When we will construct transseries more carefully, we will denote $\boldsymbol{\mu}_{\mathbf{k}} =: \mu_1^{k_1} \cdots \mu_n{}^{k_n}$ the monomial $x^{-k_1\alpha_1 - \cdots - k_n\alpha_n}$. We note that $\mathbf{k} \mapsto \boldsymbol{\mu}_{\mathbf{k}}$ defines a morphism between $\mathbb{Z}^n$ and the abelian multiplicative group generated by $\mu_1, ..., \mu_n$.

5. The lower bound for k_i easily implies that there are only finitely many terms with the same monomial. Indeed, the equation $\alpha_1 k_1 + ... + \alpha_n k_n = p$ does not have solutions if $\Re(\alpha_i) k_i > |p| + \sum_{j \neq i} |\alpha_j||M_j|$.

6. A level zero transseries can be decomposed as $L + \mathit{const} + s$ where L, which could be zero, is the purely large part in the sense that it contains only large monomials, and s is small.

   If $S \neq 0$ we can write uniquely

$$S = \mathit{const}\, x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n}(1 + s)$$

   where s is small.

7. Operations are defined on level zero transseries in the natural way. The product of level zero transseries is a level zero transseries where as in (5) above the lower bound for k_i entails that there are only finitely many terms with the same monomial in the product.

8. It is easy to see that the expression $(1 - s)^{-1} := 1 - s + s^2 - ...$ is well defined and this allows definition of division via

$$1/S = \mathit{const}^{-1} x^{\alpha_1 M_1 + \ldots + \alpha_n M_n}(1 - s)^{-1}$$

9. A transmonomial is small $m = o(1)$ and large if $1/m$ is small. m is neither large nor small iff $m = 1$ i.e., $-\alpha_1 k_1 - ... - \alpha_n k_n = 0$; this is a degenerate case and for some purposes it is not considered a monomial.

10. It can be checked that level zero transseries form a differential field. Composition $S(s)$ is also well defined whenever s is a *large* transseries.

In a more abstract language that we will use later, for a given set of monomials $\mu_1, ..., \mu_n$ and the multiplicative group $\mathcal{G}$ generated by them, a transseries of level zero is a function defined on $\mathbb{Z}^n$ with values in $\mathbb{C}$, with the property that for some $\mathbf{k}_0$ we have $F(\mathbf{k}) = \mathbf{0}$ if $\mathbf{k} < \mathbf{k}_0$.

More general transseries are defined inductively; in a first step exponentials of purely large level zero series are level one series.

It is convenient to first construct transseries without logs and then define the general ones by composition to the right with an iterated log.

11. In general, transseries have an exponential level (height) which is the highest order of composition of the exponential, and similarly a logarithmic depth; both of these are finite; $\exp(\exp(x^2)) + \ln x$ has height 2 and depth 1.

12. **Level one**. The exponential e^x has no asymptotic *power* series at infinity (Proposition 4.31) and e^x is taken to be its own expansion. It is a new element.

13. A level one transmonomial is of the form $\mu = me^L$ where m is a level zero transmonomial and L is a purely large level zero transseries. μ is *large* if the leading constant of L is positive and small otherwise. If L is large and positive then e^L is, by definition, much larger than any monomial of level zero. We define naturally $e^{L_1}e^{L_2} = e^{L_1+L_2}$. Note that in our convention bot x and $-x$ are *large* transseries.

14. A level one transseries is of the form

$$S = \sum_{k_i \geq M_i} c_{k_1,\ldots,k_n} \mu_1^{-k_1} \cdots \mu_n^{-k_n} := \sum_{\mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \tag{4.46}$$

    where μ_i are *large* level one transmonomials.

    With the operations defined naturally as above, level one transseries form a differential field.

15. We define, for a small transseries, $e^s = \sum_{k=0}^{\infty} s^k/k!$. If s is of level zero, then e^s is of level zero too.

16. The construction proceeds similarly, by induction and a general exponential-free transseries is one obtained at *some level* of the induction. They form a differential field.

17. It can be shown, by induction, that $S' = 0$ *iff* $S = const$.

18. *Dominance*: If $S \neq 0$ then there is a largest transmonomial $\mu_1^{-k_1} \cdots \mu_n^{-k_n}$ in S, with nonzero coefficient, C. Then $\mathrm{Dom}(S) = C\mu_1^{-k_1} \cdots \mu_n^{-k_n}$. If S is a nonzero transseries, then $S = \mathrm{Dom}(S)(1 + s)$ where s is purely

small, i.e., all the transmonomials in s are small. It can be shown (the construction is given later) that a base of monomials can then be chosen such that all M_i in s are positive.

19. Topology.

    (a) If $\tilde{S}$ is the space of transseries generated by the monomials $\mu_1, ..., \mu_n$ then, by definition, the sequence $S^{[j]}$ converges to S given in (4.46) if for any $\mathbf{k}$ there is a $j_0 = j_0(\mathbf{k})$ such that $c_{\mathbf{k}}^{[j]} = c_{\mathbf{k}}$ for all $j \geq j_0$.

    (b) This topology is metrizable, see the discussion after Definition 22.

    (c) In this topology, addition and multiplication are continuous, but multiplication by scalars is not.

    (d) It is easy to check that any Cauchy sequence is convergent and transseries form a complete linear metric space.

    (e) Contractive mappings: A function (operator) $\mathcal{A} : \tilde{S} \to \tilde{S}$ is contractive if for some $\alpha < 1$ and any $S_1, S_2 \in \tilde{S}$ we have $d(\mathcal{A}(S_1) - \mathcal{A}(S_2)) \leq \alpha\, d(S_1 - S_2)$.

    (f) *Fixed point theorem.* It can be proved in the usual way that if $\mathcal{A}$ is contractive, then the equation $S = S_0 + \mathcal{A}(S)$ has a unique fixed point.

    Examples –This is a convenient way to show the existence of multiplicative inverses. It is enough to invert $1 + s$ with s small. We choose a basis such that all M_i in s are positive. Then the equation $y = 1 - sy$ is contractive.

    –The equation $y = 1/x - y'$ is contractive within level zero transseries; It has a unique solution.

20. If $L_n = \log(\log(... \log(x)))$ n times, and T is an exponential-free transseries then $T(L_n)$ is a general transseries. They form a differential field, closed under integration, composition to the right with large transseries, and many other operations; this closure is proved as part of the general induction.

21. The theory of differential equations in transseries has many similarities with the usual theory. For instance it is easy to show, using an integrating factor and 17 above that the equation $y' = y$ has the general solution Ce^x and that the equation $y" = xy$ has at most two linearly independent solutions. We will find two such solutions in the examples below.

*

The type of exponential growth is related to the factorial power of divergence of the power series. For illustration we take

$$g'' + 2z^{-1}g' - z^{-5}g = 1 \text{ analyzed for } z \downarrow 0 \tag{4.47}$$

To bring it to a canonical form, we would take $z = 1/x$. To get used to various limits, we will rather work with the equation as presented. The dominant balance of the homogeneous equation occurs between g'' and $z^{-5}g$. By WKB we see that the *solution* of this dominant equation is

$$Const. z^{1/4} e^{-2/3\ z^{-3/2}}$$

The presence of a pole of higher order than the equation makes the power series expansion $\sum_k c_k z^k$ of a solution diverge ($c_k \propto (k!)^p,\ p > 0$), since at the level of the recurrence for the c_k it implies that coefficients with larger k are given in terms of earlier ones multiplied by powers of n. In our specific case we get

$$c_{n+3} = n(n+1)c_n$$

with the solution

$$c_{3k} = const. 3^{2k}\Gamma(k + 1/3)\Gamma(k)$$

roughly,

$$c_k \propto (k!)^{2/3} \tag{4.48}$$

Example 2. By a similar method, we can find a formal solution for the Gamma function $a_{n+1} = na_n$. We look directly for transseries of level at least one, $a_n = e^{f_n}$ and thus $f_{n+1} = \ln n + f_n$. It is clear that $f_{n+1} - f_n \ll f_n$; this suggests writing $f_{n+1} = f_n + f'_n + \frac{1}{2}f''_n + ...$ and, taking $f' = h$ we get the equation

$$h_n = \ln n - \frac{1}{2}h'_n - \frac{1}{6}h''_n - ... \tag{4.49}$$

(which is contractive in the space of transseries of zero height as we shall see in Chapter A). We get

$$h = \ln n - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} ...$$

and thus

$$f_n = n\ln n - n - \frac{1}{2}\ln n + \frac{1}{12n} - \frac{1}{360n^3} ... + C$$

4.2c Properties of the Laplace transform

Proposition 4.50 *If $F \in L^1(\mathbb{R}^+)$ then $\mathcal{L}F$ is analytic in the right half plane H and continuous on the imaginary axis ∂H, and $\mathcal{L}\{F\}(x) \to 0$ as $x \to \infty$ in H.*

Proof. Continuity and analyticity are preserved by integration against a finite measure ($F(p)dp$). Equivalently, these properties follow by dominated convergence, as $\epsilon \to 0$, of $\int_0^\infty e^{-isp}(e^{-ip\epsilon} - 1)F(p)dp$ and of $\int_0^\infty e^{-xp}(e^{-p\epsilon} - 1)\epsilon^{-1}F(p)dp$ respectively, the last integral for $\Re(x) > 0$. The stated limit also follows easily from dominated convergence, if $|\arg(x) \pm \pi/2| > \delta$; the general case follows from the case $|\arg(x)| = \pi/2$ which is a consequence of the Riemann-Lebesgue lemma. □

First inversion formula.

Let $\mathcal{H}$ denote the space of analytic functions in H.

Proposition 4.51 *(i) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ and $\|\mathcal{L}\{F\}\|_\infty \le \|F\|_1$.*
(ii) $\mathcal{L} : L^1 \mapsto \mathcal{L}(L^1)$ is invertible, and the inverse is given by

$$F(x) = \hat{\mathcal{F}}^{-1}\{\mathcal{L}\{F\}(it)\}(x) \tag{4.52}$$

for ($x \in \mathbb{R}^+$) where $\hat{\mathcal{F}}$ is the Fourier transform.

Proof. Part (i) is immediate, since $|e^{-xp}| \le 1$. (ii) Extending F on $\mathbb{R}^-$ by zero we have $\mathcal{L}\{F\}(it) = \int_{-\infty}^{\infty} e^{-ipt}F(p)\mathrm{d}p = \hat{\mathcal{F}}F$. □

Second inversion formula.

Laplace transform is not surjective from L^1 to $\mathcal{H}$ but functions in $\mathcal{H}$ with sufficient decay do belong to $\mathcal{L}(L^1)$.

Proposition 4.53 *(i) Assume f is analytic in an open sector $H_\delta := \{x : |\arg(x)| < \pi + \delta\}$, $\delta \ge 0$ and is continuous on ∂H_δ, and that for some $K > 0$ and any $x \in \overline{H_\delta}$ we have*

$$|f(x)| \le K(|x|^2 + 1)^{-1} \tag{4.54}$$

Then $\mathcal{L}^{-1}f$ is well defined by

$$F = \mathcal{L}^{-1}f = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}t\, e^{pt} f(t) \tag{4.55}$$

and

$$\int_0^\infty \mathrm{d}p\, e^{-px}F(p) = \mathcal{L}\mathcal{L}^{-1}f = f(x)$$

and in addition $\|\mathcal{L}^{-1}\{f\}\|_\infty \le K\pi$ *and* $\mathcal{L}^{-1}\{f\} \to 0$ *as* $p \to \infty$.

(ii) If $\delta > 0$ *then* $F = \mathcal{L}^{-1} f$ *is analytic in the sector* $S = \{p \ne 0 : |\arg(p)| < \delta\}$. *In addition,* $\sup_S |F| \le K\pi$ *and* $F(p) \to 0$ *as* $p \to \infty$ *in* S.

Proof. (i) We have

$$\int_0^\infty \mathrm{d}p\, e^{-px} \int_{-\infty}^\infty \mathrm{d}s\, e^{ips} i f(is) = \int_{-\infty}^\infty \mathrm{d}t\, f(is) \int_0^\infty \mathrm{d}p\, e^{-px} e^{ips} \tag{4.56}$$

$$= \int_{-i\infty}^{i\infty} f(z)(x-z)^{-1}\mathrm{d}z = 2\pi i f(x) \tag{4.57}$$

where we applied Fubini's theorem and then pushed the contour of integration past x to infinity. The norm is obtained by majorizing $|fe^{ips}|$ by $K(|x^2|+1)^{-1}$.

(ii) We have for any $\delta' < \delta$, by (4.54),

$$\int_{-i\infty}^{i\infty} \mathrm{d}s\, e^{ps} f(s) = \left(\int_{-i\infty}^0 + \int_0^{i\infty} \right) \mathrm{d}s\, e^{ps} f(s)$$

$$= \left(\int_{-i\infty e^{-i\delta'}}^0 + \int_0^{i\infty e^{i\delta'}} \right) \mathrm{d}s\, e^{ps} f(s) \tag{4.58}$$

and analyticity is clear in (4.58).

For (ii) we note that (i) applies in $\bigcup_{|\delta'|<\delta} e^{i\delta'} H_0$. □

Many cases can be reduced to this one after transformations. For instance if $g = \sum_{j=1}^N a_j x^{-k_j} + f(x)$, with $k_j > 0$ and f satisfying the assumptions above, then g is inverse Laplace transformable since the finite sum in its definition is explicitly transformable.

Proposition 4.59 *Let* F *be analytic in the open sector* $S_p = e^{i\phi}\mathbb{R}^+$ *with* $\phi \in (-\delta, \delta)$ *be such that* $|F(|x|e^{i\phi})| \le g(|x|)$ *for some* $g \in L^1[0, \epsilon)$ *bounded as* $x \to \infty$. *Then* $f = \mathcal{L}F$ *is analytic in the sector* $S_x = \{x : |\arg(x)| < \pi/2 + \delta\}$ *and* $f(x) \to 0$ *as* $|x| \to \infty, \arg(x) = \theta \in (-\pi/2 - \delta, \pi/2 + \delta)$.

Proof. Because of the analyticity of F and the decay conditions for large p, the path of Laplace integration can be rotated by any angle $\phi \in (-\delta, \delta)$ without changing $(\mathcal{L}F)(x)$ (see also the next example). This means Proposition 4.50 applies in $\cup_{|\phi|<\delta} e^{i\phi} H$.

Note that without further assumptions on $\mathcal{L}F$, F is *not* necessarily analytic at $p = 0$.

Corollary 4.60 The kernel of $\mathcal{L}$ is trivial: *if* $F \in L^1(\mathbb{R}^+)$ *and* $\mathcal{L}F = 0$ *then* $F = 0$.

Remark. It is useful to note that by continuity and analyticity, it is enough to have $\mathcal{L}F(x) = 0$ on any set with an accumulation point in the right half plane to ensure $F \equiv 0$.

Proof. An immediate consequence of the first inversion formula. □

4.2c .1 Asymptotic properties Laplace transforms

The asymptotic behavior of Laplace integrals is particularly important given that every analyzable function should be convergently expressed by

4.2d Representability in terms of Laplace transforms

Let us now consider the homogeneous equation associated to (4.47) after the substitution $z = 1/x$. We divide by the exponential and change variable $\frac{2}{3}x^{3/2} = s$ to linearize the exponent and ensure that the transformed function has an asymptotic series with factorial divergence. Such a series can be obtained by Watson's Lemma from a convergent series. Inverse Laplace transform in then likely to regularize the equation.

Taking $f(x) = e^{\frac{2}{3}x^{3/2}}h(\frac{2}{3}x^{3/2})$ we get

$$h'' + \left(2 + \frac{1}{3s}\right)h' + \frac{1}{3s}h = 0 \tag{4.61}$$

and with $H = \mathcal{L}^{-1}(h)$ we get

$$p(p-2)H' = \frac{5}{3}(1-p)H$$

which indeed has a regular singularity at $p = 0$. The solution is

$$H = Cp^{-5/6}(2-p)^{-5/6}$$

and it can be easily checked that any integral of the form

$$h = \int_0^{\infty e^{i\phi}} e^{-ps}H(p)dp$$

for $\phi \neq 0$ is a solution of (4.61) yielding the expression

$$f = e^{\frac{2}{3}x^{3/2}} \int_0^{\infty e^{i\phi}} e^{-\frac{2}{3}x^{3/2}p}p^{-5/6}(2-p)^{-5/6}dp \tag{4.62}$$

for a solution of the Airy equation. A second solution can be obtained in a similar way, replacing $e^{\frac{2}{3}x^{3/2}}$ by $e^{-\frac{2}{3}x^{3/2}}$, or by taking the difference between two integrals of the form (4.62).

For Example 2 above, factorial suggests taking inverse Laplace transform of $g_n = f_n - (n\ln n - n - \frac{1}{2}\ln n)$.

Inverse Laplace transform is given by the *Bromwich integral* along a vertical contour in the right half plane:

$$(\mathcal{L}^{-1}F)(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xp}F(p)dp$$

The recurrence satisfied by g is

$$g_{n+1} - g_n = q_n = 1 - \left(\frac{1}{2} + n\right) \ln\left(1 + \frac{1}{n}\right) = -\frac{1}{12n^2} + \frac{1}{12n^3} + \ldots$$

First note that $\mathcal{L}^{-1}q = p^{-2}\mathcal{L}^{-1}q''$ which can be easily evaluated by residues since

$$q'' = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{(n+1)^2} + \frac{1}{n^2}\right)$$

Thus, with $\mathcal{L}^{-1}g_n := G$ we get

$$(e^{-p} - 1)G(p) = \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2}$$

$$g_n = \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2(e^{-p} - 1)} e^{-np} dp$$

(It is easy to check that the integrand is analytic at zero; its Taylor series is $\frac{1}{12} - \frac{1}{720}p^2 + O(p^3)$.)

The integral is well defined, and it easily follows that

$$f_n = C + n(\ln n - 1) - \frac{1}{2}\ln n + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2(e^{-p} - 1)} e^{-np} dp$$

solves our recurrence. The constant $C = \frac{1}{2}\ln(2\pi)$ is most easily obtained by comparing with Stirling's series (3.54) and we thus get the identity

$$\ln\Gamma(n+1) = n(\ln n - 1) - \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi) + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2(e^{-p} - 1)} e^{-np} dp \tag{4.63}$$

which holds with n replaced by $z \in \mathbb{C}$ as well.

This represents, as we shall soon see, the Borel summed version of Stirling's formula.

Other recurrences can be dealt with in the same way. One can calculate $\sum_{j=1}^{n} j^{-1}$ as a solution of the recurrence

$$s_{n+1} - s_n = \frac{1}{n}$$

Proceeding as in the Gamma function example, we have $f' - \frac{1}{n} = O(n^{-2})$ and the substitution $s_n = \ln n + g_n$ yields

$$g_{n+1} - g_n = \frac{1}{n} + \ln\Big(\frac{n}{n+1}\Big)$$

and in the same way we get

$$f_n = C + \ln n + \int_0^\infty e^{-np}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp$$

where the constant can be obtained from the initial condition, $f_1 = 0$,

$$C = -\int_0^\infty e^{-p}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp$$

which, by comparison with the usual asymptotic expansion of the harmonic sum also gives

$$\gamma = \int_0^\infty e^{-p}\Big(\frac{1}{1-e^{-p}} - \frac{1}{p}\Big)dp$$

Comparison with (4.63) gives

$$\sum_{j=1}^{n-1}\frac{1}{j} - \gamma = \ln n + \int_0^\infty e^{-np}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp = \frac{\Gamma'(n)}{\Gamma(n)} \tag{4.64}$$

Exercise: Zeta function. Use the same strategy to show that

$$(n-1)!\zeta(n) = \int_0^\infty p^{n-1}\frac{e^{-p}}{1-e^{-p}}dp = \int_0^1 \frac{\ln^{n-1} s}{1-s}ds \tag{4.65}$$

4.2d .1 The Euler-Maclaurin Sum formula

We can also generalize the asymptotic evaluation of difference equations in the following setting. Assume $f(n)$ does not increase too rapidly with n and and we want to find the asymptotic behavior of

$$S(n+1) = \sum_{k=k_0}^{n} f(k) \tag{4.66}$$

for large n. We see that $S(k)$ is the solution of the difference equation

$$S(k+1) - S(k) = f(k) \tag{4.67}$$

To be more precise, assume f has a level zero transseries as $n \to \infty$. Then we write $\tilde{S}$ for the transseries of S which we seek at level zero. Then $S(k+1) - S(k) = S'(k) + S''(k)/2 + ... + S^{(n)}(k)/k! + ...$ where the last sum converges in the topology of transseries since differentiation is contractive on level zero transseries, we get

$$\tilde{S}'(k) = f(k) - (\tilde{S}''(k)/2 + ... + \tilde{S}^{(n)}(k)/k! + ...) \tag{4.68}$$

If $Df := f'$, the operator

$$\sum \frac{D^k}{k!} \tag{4.69}$$

is contractive in the space $\mathcal{T}_0$ of transseries of level zero (check)! and we get that $\tilde{S}'$ is uniquely defined as the solution in $\mathcal{T}_0$ of (4.68).

has a unique level zero transseries solution. It is easy to check that there are no other solutions of other levels, and this should be also obvious from the interpretation of S as the solution of (4.67) which is thereby defined, on the positive integers, up to an additive constant. What do we get, in terms of f? We only need to iterate (4.68) At the first step we get

$$\tilde{S}'(k) = f(k) \tag{4.70}$$

At the second step the result is

$$\tilde{S}'(k) = f(k) - \frac{1}{2} f'(k) \tag{4.71}$$

At the third step we obtain

$$\tilde{S}'(k) = f(k) - \frac{1}{2} f'(k) + \frac{1}{12} f''(k)$$

Continuing like this we get

$$\tilde{S}'(k) = f(k) - \frac{1}{2} f'(k) + \frac{1}{12} f''(k) - \frac{1}{720} f^{(4)}(k) + \cdots = \sum_{j=0}^{\infty} C_j f^{(j)}(k) \tag{4.72}$$

What are the coefficients C_j? It is clear from the procedure itself that they should not depend on f. But then it suffices to look at some particular f for which the sum can be calculated explicitly. If $n > 0$ we have

$$\frac{1}{1 - e^{-1/n}} = \sum_{k=0}^{\infty} e^{-k/n} \tag{4.73}$$

while, by one of the definitions of the Bernoulli numbers we have

$$\frac{z}{1 - e^{-z}} = \sum_{k=0}^{\infty} (-1)^j \frac{B_j}{j!} z^j \tag{4.74}$$

By integration we get

$$S(k) \sim \int_{k_0}^{k} f(s) ds + C + \sum_{j=0}^{\infty} \frac{B_{j+1}}{(j+1)!} f^{(j)}(k) \tag{4.75}$$

Exercise 4.76 ** *Complete the details of the calculation involving the identification of coefficients in the Euler-Maclaurin sum formula.*

Exercise 4.77 ** Without using the Euler-Maclaurin sum formula, find rigorously for which values of $a > 0$ the series*

$$\sum_{k=1}^{\infty} \frac{e^{i\sqrt{k}}}{k^a}$$

is convergent.

Exercise 4.78 ** Prove the Euler-Maclaurin sum formula in the case f is C^∞ by first looking at the integral $\int_n^{n+1} f(s)ds$ and expanding f in Taylor by $s = n$. Then correct f to get a better approximation etc.*

That (4.75) gives the correct asymptotic behavior in some generality is proved, for example, in [19]. Eq. (4.75) is called the Euler-Maclaurin sum formula.

We will prove here, under stronger assumptions, a stronger result which implies (4.75). The conditions are often met in applications, after chenges of variables, as our examples showed.

Lemma 4.79 *Assume f is analytic at the origin and $f(z) = O(z^2)$. Then $f(1/n) = \int_0^\infty F(p)e^{-np}dp$, $F(p) = O(p)$ for small p and*

$$\sum_{k=n_0}^{n-1} f(1/n) = \int_0^\infty e^{-np}\frac{F(p)}{e^{-p}-1}dp - \int_0^\infty e^{-n_0 p}\frac{F(p)}{e^{-p}-1}dp \tag{4.80}$$

PROOF The fact that $f(1/n) = \int_0^\infty F(p)e^{-np}dp$ and $F(p) = O(p)$ follows from the general theory of Laplace transforms. We seek a solution of (4.67) in the form $S = C + \int_0^\infty H(p)e^{-kp}dp$, or, in other words we inverse Laplace transform the equation (4.67). We get

$$(e^{-p} - 1)H = F \Rightarrow H(p) = \frac{F(p)}{e^{-p}-1} \tag{4.81}$$

and the conclusion follows by taking the Laplace transform which is well defined since $F(p) = O(p)$, and imposing the initial condition $S(k_0) = 0$. ☐

4.2d .2 Example: The Painlevé equation P1

$$\frac{d^2y}{dz^2} = 6y^2 + z \tag{4.82}$$

We first look for formal solutions. As a transseries of level zero it is easy to see that the only possible balance is $6y^2 + z = 0$ giving

$$y \sim \pm\frac{i}{\sqrt{6}}\sqrt{z}$$

We choose one of the signs, say + and write

$$y = \frac{i}{\sqrt{6}}\sqrt{z - y''} = \frac{i}{\sqrt{6}}\left(\sqrt{z} - \frac{y''}{2\sqrt{z}} - \frac{1}{8z^{3/2}}(y'')^2 \cdots\right) \tag{4.83}$$

Equation (4.83) is contractive in the space of level zero transseries, and it therefore has a unique solution there. A few iterations yield

$$y = \frac{i}{\sqrt{6}}\left(\sqrt{z} + \frac{i\sqrt{6}}{48z^2} + \frac{49i}{768z^{9/2}} \cdots\right) \tag{4.84}$$

Since the procedure placed the highest derivative on the right side, we expect that the series (4.84) is divergent. We could analyze the recurrence relation for the coefficients to determine the rate of divergence, but this is harder than the following process. We first look at the type of possible exponentials besides the power series. If we attempt to represent the solution as a Laplace transform of a nonentire analytic function with respect to a variable s, then choosing two distinct contours would yield in general different solutions. If we have a nearest singularity at, say p_0 in the right half plane, then it is easy to see that the difference of the two Laplace transforms would be roughly of the order $e^{-p_0 s}$. On the other hand, by Cauchy estimates and Watson's Lemma it is easy to see that the formal asymptotic series of the Laplace transform has coefficients increasing roughly like $k!/|p_0|^k$. Thus, the divergence of the series in the adapted variable s is $k!|p_0|^k$. It is easy from here to infer the rate of divergence in the original variable.

We denote by y_0 the series in (4.84) and look for further solutions in the form $y_0 + \delta$, where we assume that δ is exponentially small (indeed, if exponential solutions exist, we should be able to find a direction in $\mathbb{C}$ where the exponential decreases.) We get for $\delta(x)$, discarding the quadratic terms,

$$\delta'' - 12y_0\delta = 0 \tag{4.85}$$

We solve this equation by formal WKB. Substituting $\delta = e^w$ in (4.85) we get

$$w' = \pm\sqrt{2i\sqrt{6}\sqrt{z} - \frac{1}{4z^2} + \cdots - w''} \tag{4.86}$$

which is again contractive in the space of transseries of level zero. We get

$$w = \frac{4}{5}\sqrt{2i}6^{1/4}z^{5/4} + \cdots \tag{4.87}$$

The natural variable is $z^{5/4}$. We further normalize the equation so that the transseries for δ is the simplest. That should bring the equation P1 to its simplest form for the purpose of Borel summation . We let

$$x = \frac{(-24z)^{5/4}}{30};\ y(z) = \sqrt{\frac{-z}{6}}\left(1 - \frac{4}{25x^2} + h(x)\right)$$

P1 becomes

$$h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \tag{4.88}$$

We are now in the adapted variable, the and the divergence of the series is expected to be of order $k!$ which can be compensated by the inverse Laplace transform (by virtue of Watson's Lemma).

If we write $h(x) = \int_0^\infty H(p)e^{-xp}dp$, then the equation for H is

$$(p^2 - 1)H(p) = \frac{196}{1875}p^3 + \int_0^p sH(s)ds + \frac{1}{2}H * H \tag{4.89}$$

where convolution is defined by

$$(F * G)(p) = \int_0^p F(s)G(p-s)ds$$

We will learn how to solve equations of the form (4.89).

4.3 Borel transforms

The Laplace transform is defined on integrable functions of at most exponential growth by

$$\mathcal{L}\{F\}(x) := \int_0^\infty e^{-px}F(p)\mathrm{d}p \quad (\Re(x) > x_0)$$

When dealing with functions defined in the complex domain it is useful to allow for different contours of integration; $\mathcal{L}_\phi$ denotes the Laplace transform in the direction ϕ:

$$\mathcal{L}_\phi\{F\}(x) := \int_0^{\infty e^{i\phi}} e^{-px}F(p)\mathrm{d}p \quad (\Re(xe^{-i\phi}) > x_0)$$

The formal Laplace transform, still denoted $\mathcal{L} : \mathbb{C}[[p]] \mapsto \mathbb{C}[[x^{-1}]]$ is defined by

$$\mathcal{L}\{s\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} c_k p^k\right\} = \sum_{k=0}^{\infty} c_k \mathcal{L}\{p^k\} = \sum_{k=0}^{\infty} c_k k! x^{-k-1} \tag{4.90}$$

(with $\mathcal{L}\{p^{\alpha-1}\} = \Gamma(\alpha)x^{-\alpha}$ the definition extends straightforwardly to noninteger power series).

4.4 The Borel transform $\mathcal{B}$

The **Borel transform**, $\mathcal{B} : \mathbb{C}[[x^{-1}]] \mapsto \mathbb{C}[[p]]$ is the (formal) inverse of the operator $\mathcal{L}$ in (4.90). This is a transform on the space of formal series. By definition, for a monomial we have

$$\mathcal{B}\frac{\Gamma(s+1)}{x^{s+1}} = p^s \tag{4.91}$$

to be compared with the inverse Laplace transform,

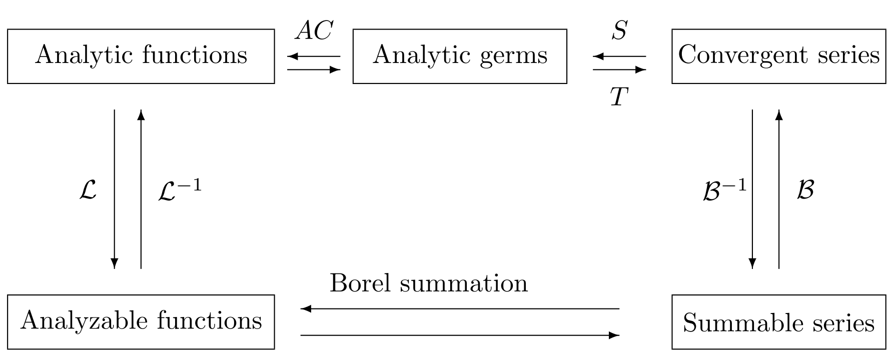
$$\mathcal{L}^{-1}\frac{\Gamma(s+1)}{x^{s+1}} = \begin{cases} p^s & \text{for } \Re p > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.92}$$

For a formal series, the Borel transform is the formal sum of the Borel transforms of its terms.

Because the $k-th$ coefficient of $\mathcal{B}\{\tilde{f}\}$ is smaller by a factor $k!$ than the corresponding coefficient of $\tilde{f}$, $\mathcal{B}\{\tilde{f}\}$ may converge even if $\tilde{f}$ does not. Since factorial divergence is commonplace in analytic problems (for reasons that will become clear in the sequel) this convergence-improving property of $\mathcal{B}$ is very useful.

Also important is that the combination $\mathcal{LB}$ is, formally, **the identity** operator, and must thus have, when properly interpreted, good commutation properties with function operations.

These two facts account for the central role played by $\mathcal{LB}$, the operator of Borel summation in the theory of analyzable functions .



This diagram is crucial to Écalle's theory.

4.5 Definition of Borel summation and basic properties

Series of the form $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-\beta_1 k_1 - \ldots - \beta_m k_m - r}$ with $\Re(\beta_j) > 0$ frequently arise as formal solutions of differential systems. We will first analyze the case $m = 1, r = 1, \beta = 1$ but the theory extends without difficulty to more general series.

Borel summation is relative to a direction, see Remark 4.119. The same formal series $\tilde{f}$ may yield different functions by Borel summation in different directions.

Borel summation along $\mathbb{R}^+$ consists in three operations, assuming they are possible:

1. Borel transform, $\tilde{f} \mapsto \mathcal{B}\{\tilde{f}\}$.

2. Summation of the series $\mathcal{B}\{\tilde{f}\}$ and analytic continuation along $\mathbb{R}^+$; denote this function by F.

3. Laplace transform, $F \mapsto \int_0^\infty F(p)e^{-px}\mathrm{d}p =: \mathcal{L}\mathcal{B}\{\tilde{f}\}$, which requires exponential bounds on F, defined in some half plane $\Re(x) > x_0$.

The *domain* of Borel summation is the subspace $S_{\mathcal{B}}$ of series for which the conditions for the steps 1-3 above are met. For 3 we can require that for some constants C_F, ν_F we have $|F(p)| \le C_F e^{\nu_F p}$. Or we can require that $\|F\|_\nu < \infty$ where, for $\nu > 0$ we define

$$\|F\|_\nu := \int_0^\infty e^{-\nu p}|F(p)|dp \tag{4.93}$$

We note that $L^1_\nu := \{f : \|f\|_\nu < \infty\}$ forms a Banach space, and it is easy to check that

$$L^1_\nu \subset L^1_{\nu'} \text{ if } \nu' > \nu \tag{4.94}$$

and that

$$\|F\|_\nu \to 0 \text{ as } \nu \to \infty \tag{4.95}$$

the latter statement following from dominated convergence.

4.6 General remarks on Borel summation

A few notes are in order, to understand why Borel summation is natural.

1. If a problem has analytic coefficients and is nonsingular, or regularly perturbed, the series expansions are convergent. In differential systems, a problem is singularly perturbed if the highest derivative is formally small, for instance in problems like $f' + f = 1/x$ or, exiting the realm of one variable, $\epsilon f'' + h(x)f = g$.

2. In singularly perturbed problems the highest derivative belongs formally to the right side. One then iterates upon the highest derivative. For generic analytic functions, by Cauchy's formula, $f^{(n)}$ grows roughly like $const^n n!$

3. It is then natural to diagonalize d/dx. Then, by repeated iteration of d/dx yields geometric rather that factorial divergence. This is much easier to resolve.

4. The operator d/dx is diagonalized by the Fourier transform. Since it is often the case that we deal with asymptotic problems, for say a large variable x, we would like to perform it while keeping x large. The Fourier transform on a vertical contour in the complex domain is in fact an inverse Laplace transform, cf. (1.56).

5. $\mathcal{L}^{-1} f' = pf$ thus repeated differentiation means repeated multiplication by p. As noted in 4 above, Factorial growth is replaced by geometric growth, much easier to control.

6. The formal inverse Laplace transform (Borel transform, $\mathcal{B}$) of a small zero level transseries, that is of a small multiseries, is defined, roughly, as the term-by-term inverse Laplace transform of the series. It is still a level zero transseries,

$$\mathcal{B} \sum_{\mathbf{k}>0} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\mathbf{a}} = \sum_{\mathbf{k}>0} c_{\mathbf{k}} p^{\mathbf{k}\cdot\mathbf{a}-1} / (\mathbf{k}\cdot\mathbf{a}-1)! \tag{4.96}$$

   where the factorial is understood in terms of the Gamma function.

   The result of summing a formal series is still a formal series, convergent or not.

7. One difference between $\mathcal{L}^{-1}$ and $\mathcal{B}$ is that $\mathcal{L}^{-1} x^{-b-1} = p^b/b!$ for all small p, not only for $\Re p > 0$.

8. A series is classically Borel summable if (a) the series in p in (4.96) is convergent (as a Puiseux series) for small p, (b) the sum admits analytic continuation along $\mathbb{R}^+$ and (c) the sum f is analytic in a neighborhood of the real line, along which f does not grow faster than exponentially. The norm can be taken the sup norm with weight $e^{-\nu p}$ for some ν, or $L^1(\mathbb{R}^+, e^{-\nu p} dp)$ etc.

9. The Borel sum is then, by definition the Laplace transform of f. As mentioned before the whole process is formally the identity, and it should preserve "all properties".

4.6a Borel summation as analytic continuation

There is another interpretation which clarifies showing Borel summation should commute with all operations. We can take the more general sum

$$\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta k + \beta)}{x^{k+1}} \tag{4.97}$$

which for $\beta = 1$ agrees with (1.50). For $\beta = i$, (4.97) converges if $|x| > 1$, and the sum is, using the integral representation for the Gamma function and dominated convergence,

$$\int_0^{\infty} \frac{e^{-px}}{1 + p^{\beta} x^{\beta - 1}} dp \tag{4.98}$$

Analytic continuation of (4.98) back to $\beta = i$ becomes precisely (1.51).

Exercise 4.99 *Complete the details in the calculations above. Show that continuation to i and to $-i$ give the same result (1.51).*

Thus Borel summation should commute with all operations with which analytic continuation does. This latter commutation is very general, and comes under the umbrella of the vaguely stated "principle of permanence of relations" which can hardly be formulated rigorously without giving up some legitimate "relations".

10. We will extend this summation to transseries. In practice however, rarely does one need to Borel sum several levels of a transseries[1]: once the lowest level has been summed, usually the remaining object is convergent.

11. As we shall see, generic formal solutions which allow for small real valued exponential corrections are not Borel summable, in the classical sense. Even the prototypical example $\sum_{k=0}^{\infty} k! x^{-k-1}$ is not summable since its Borel transform $(1-p)^{-1}$ is not real-analytic. Natural extensions (medianization, multisummation) exist to cover these cases. Higher powers of the factorial can often be easily dealt with by changes of the independent variable. For instance, in $\sum_{k=0}^{\infty} (k!)^2 x^{-k+1}$ we achieve that by taking $x = y^2$ to get $\sum_{k=0}^{\infty} (k!)^2 y^{-2k+2}$. Note that $k!^2$ roughly behaves like $(2k)!$. But at times, mixtures of series may occur and no single change of variable suffices. This is dealt with by *multisummability*.

12. As a rule of thumb, we pass to the variable in which divergence is factorial. It will turn out that this is intimately linked to the form of free

[1]That is, terms appearing in higher iterates of the exponential.

small exponential corrections. If these are of the form e^{-x^q} then divergence is usually like $(n!)^{1/q}$. The variable should then be chosen to be $t = x^q$. This t is the *critical time.*

The reason for this choice can be understood as follows. If we expect the solutions of an equation to be summable with respect to a power of the variable, then the possible freedom in choosing the Laplace contour in the complex domain should be compatible with the type of freedom in the solutions. For instance we can take the Laplace transform of $(1-p)^{-1}$ along any ray other than $\mathbb{R}^+$. An upper half plane transform differs from a lower half plane transform precisely by $2\pi i e^{-x}$. But sub- or super-exponential corrections cannot originate in proper Borel sums. This will be proved shortly.

13. It is crucial to perform Borel summation in the adequate variable. If the divergence is not fully compensated, then obviously we are still left with a divergent series. "Oversummation", the result of overcompensating divergence usually leads to superexponential growth of the transformed function. The presence of singularities in Borel plane is in fact a good sign.

    For equation (4.101), one can check that the divergence is like $\sqrt{n!}$. The equation is oversummed if we inverse Laplace transform it in x; what we get is
$$2H' - pH = 0; \quad H(0) = 1/2 \tag{4.100}$$
    and thus $H = \frac{1}{2}e^{p^2/4}$. There are no singularities anymore but we have superexponential growth; this combination is a sign of oversummation. After oversummation there is no obvious way of taking the Laplace transform close to the real line. In some cases, a simple change of variable, as we have seen, can cure the problem.

14. For instance, to find the antiderivative of e^{x^2}, $g' = e^{x^2}$ we can write $g = ue^{x^2}$ and then
$$2xu' + u = 1 \tag{4.101}$$
    The freedom is that of an additive constant, $g = \mathcal{P}e^{x^2} + C$ and thus the correction is roughly e^{-x^2} times the dominant term. The critical time is $t = x^2$. We then take it is convenient to take $g = h(x^2)e^{x^2}$. The equation for h is
$$h' + h = \frac{1}{2\sqrt{t}} \tag{4.102}$$

15. If the transform of the solution of an equation is summable, then it is expected that the transformed equation should be more regular. In this sense, Borel summation is a regularizing transformation.

In the case of (4.102) it becomes

$$-pH + H = p^{-1/2}\pi^{-1/2} \tag{4.103}$$

an algebraic equation, with algebraic singularities. The irregular singularity has been removed.

16. We see though that H is not Laplace transformable, since it has a pole at $p = 1$. This type of difficulty is not so rare, and it is dealt with by an appropriate Écalle *medianization*. This is a suitable universal linear combination of analytic continuations, chosen in such a way that averaging commutes with the Laplace convolution (1.53) and the Borel sum of a product is the product of Borel transforms. We will return to this. For (4.103) it all amounts to taking the half-sum of the Laplace transforms along contours from 0 to $(1 \pm i\epsilon)\infty$.

Exercise 4.104 *Show, using dominated convergence, Morera's and Fubini's theorems that if $F \in L^1_\nu$ then $\mathcal{L}F$ is analytic in x in the half plane $\Re(x) \geq \nu$.*

Note 4.105 *Equivalently we can say that the series $\tilde{f}$ is Borel summable if it is the asymptotic series as $x \to +\infty$ of $\mathcal{L}F$ with F analytic in a neighborhood $\mathcal{D}_{\mathbb{R}^+}$ of $\mathbb{R}^+$ (in particular, we say such a function is* real-analytic on $[0, +\infty)$*) and exponentially bounded at infinity. The domain $\mathcal{D}_{\mathbb{R}^+}$ as well as the bounds may depend on F. The definition is unambiguous since on the one hand the asymptotic series of a function is unique, and, by Watson's Lemma, if the asymptotic series of $\mathcal{L}F$ is zero, then the Taylor series of F at $p = 0$ is zero as well, and then $F \equiv 0$.*

Definition 4.106 (Inverse Laplace space convolution) *If $f, g \in L^1_{loc}$ then*

$$(f * g)(p) := \int_0^p f(s)g(p-s)ds \tag{4.107}$$

Lemma 4.108 *The space of functions which are in $L^1[0,\epsilon)$ for some $\epsilon > 0$ and real-analytic on $(0,\infty)$ is closed under convolution. If F and G are exponentially bounded then so is $F * G$. If $F, G \in L^1_\nu$ then $F * G \in L^1_\nu$.*

Proof. The statement about L^1 follows easily from Fubini's theorem. Analyticity follows by writing

$$\int_0^p f_1(s)f_2(p-s)ds = p\int_0^1 f_1(pt)f_2(p(1-t))dt \tag{4.109}$$

which is manifestly analytic in p. Clearly, if $|F_1| \leq C_1 e^{\nu_1 p}$ and $|F_2| \leq C_2 e^{\nu_2 p}$, then

$$|F_1 * F_2| \leq C_1 C_2 p\, e^{(\nu_1+\nu_2)p} \leq C_1 C_2\, e^{(\nu_1+\nu_2+1)p}$$

Finally, we note that

$$\int_0^\infty e^{-\nu p}\left|\int_0^p F(s)G(p-s)ds\right| dp \le \int_0^\infty e^{-\nu s}e^{-\nu(p-s)}\int_0^p |F(s)||G(p-s)|dsdp$$
$$= \int_0^\infty\int_0^\infty e^{-\nu s}|F(s)|e^{-\nu\tau}|G(\tau)|d\tau = \|F\|_\nu\|G\|_\nu \quad (4.110)$$

by Fubini.

Remark 4.111 *The results above can be rephrased for more general series of the form* $\sum_{k=0}^\infty c_k x^{-k-r}$ *by noting that for* $\Re(\rho) > -1$ *we have*

$$\mathcal{L}p^\rho = x^{-\rho-1}\Gamma(\rho+1)$$

and thus

$$\mathcal{B}\left(\sum_{k=0}^\infty c_k x^{-k-r}\right) = c_0\frac{p^{r-1}}{\Gamma(r)} + \frac{p^{r-1}}{\Gamma(r)} * \mathcal{B}\left(\sum_{k=1}^\infty c_k x^{-k}\right)$$

Furthermore, Borel summation summation naturally extends to to series of the form

$$\sum_{k=-M}^\infty c_k x^{-k-r}$$

where $M \in \mathbb{N}$ *by defining*

$$\mathcal{LB}\left(\sum_{k=-M}^\infty c_k x^{-k-r}\right) = \sum_{k=-M}^0 c_k x^{-k-r} + \mathcal{LB}\left(\sum_{k=0}^\infty c_k x^{-k-r}\right)$$

and more general powers can be allowed, replacing analyticity in p *with analyticity in* $p^{\beta_1}, ..., p^{\beta_m}$.

Proposition 4.112 *(i)* $S_{\mathcal{B}}$ *is a differential field,*[2] *and* $\mathcal{LB} : S_{\mathcal{B}} \mapsto \mathcal{LB}S_{\mathcal{B}}$ *is a differential algebra isomorphism.*

(ii) If $S_c \subset S_{\mathcal{B}}$ *denotes the differential algebra of convergent power series, and we identify a convergent power series with its sum, then* $\mathcal{LB}$ *is the identity on* S_c.

(iii) In addition, for $\tilde{f} \in S_B$, $\mathcal{LB}\{\tilde{f}\} \sim \tilde{f}$ *as* $|x| \to \infty$, $\Re(x) > 0$.

Proof. (i) Clearly $S_{\mathcal{B}}$ is a linear space; furthermore, $\tilde{f} = 0 \iff \mathcal{B}\tilde{f} = 0 \iff \mathcal{LB}\{\tilde{f}\} = 0$ (the last step follows from the injectivity of $\mathcal{L}$ which, in our case also follows from Watson's Lemma as in Note 4.105 above.)

[2]with respect to formal addition, multiplication, and differentiation of power series.

Exercise 4.113 *Show that if* $\mathcal{B}\tilde{f} = F$ *and* $\mathcal{B}\tilde{g} = G$ *then* $\mathcal{B}\tilde{f}\tilde{g}$ *is the power series in* p *of* $F * G$.

To show multiplicativity, we use Note 4.105. Analyticity and exponential bounds of $|F * G|$ follow from Lemma 4.108. Consequently, $F * G$ is Laplace transformable, and by elementary properties of Laplace transforms (or by performing a simple change of variables in a double integral) we see that

$$\mathcal{L}(F * G) = \mathcal{L}F\,\mathcal{L}G$$

We have to show that if $\tilde{f}$ is a Borel summable series, so is $1/\tilde{f}$. We have $f = Cx^m(1+s)$ for some m where s is a small series.

We want to show that

$$1 - s + s^2 - s^3 + \cdots \tag{4.114}$$

is Borel summable, or that

$$-s + s^2 - s^3 + \cdots \tag{4.115}$$

is Borel summable. Let $\mathcal{B}s = H$. We examine the series

$$S = -H + H * H - H^{*3} + \cdots \tag{4.116}$$

where H^{*n} is the self convolution of H n times. Each term of the series is analytic, by Lemma 4.108. If $\max_{p\in\mathcal{D}} |H(p)| = m$, then it is easy to see that

$$|H^{*n}| \le m^n 1^{*n} = m^n \frac{p^{n-1}}{(n-1)!} \tag{4.117}$$

Thus the function series in (4.116) is absolutely and uniformly convergent in $\mathcal{D}$ and the limit is analytic. Let now ν be large enough so that $\|H\|_\nu < 1$. This is possible by (4.95). Then the series in (4.116) is norm convergent, thus an element of L^1_ν.

Exercise 4.118 *Check that* $(1 + \mathcal{L}H)(1 + \mathcal{L}S) = 1$.

It remains to show that the asymptotic expansion of $\mathcal{L}(F * G)$ is indeed the product of the asymptotic series of $\mathcal{L}F$ and $\mathcal{L}G$, which is a consequence of the more general fact that the asymptotic series of a product is the product of the corresponding asymptotic series.

(ii) Since $\tilde{f}_1 = \tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k-1}$ is convergent, then $|c_k| \le CR^k$ for some C, R and $F(p) = \sum_{k=0}^{\infty} c_k p^k/k!$ is entire, $|F(p)| \le \sum_{k=0}^{\infty} CR^k p^k/k! = Ce^{Rp}$ and thus F is Laplace transformable for $|x| > R$. By dominated convergence we have for $|x| > R$,

$$\mathcal{L}\{\sum_{k=0}^{\infty} c_k p^k/k!\} = \lim_{N\to\infty} \mathcal{L}\{\sum_{k=0}^{N} c_k p^k/k!\} = \sum_{k=0}^{\infty} c_k x^{-k-1} = f(x)$$

(iii) This part follows simply from Watson's Lemma, cf. § 3.1a . $\square$

Remark 4.119 We note that in the last step in Borel summation we may take the integral in p along a different half-line in $\mathbb{C}$, as long as $\Re(xp) > 0$, and the algebraic properties are preserved. But it is also easy to check that the path *matters*, in general. For instance, if $x \in \mathbb{R}^+$ and $\mathcal{B}\tilde{f} = (1-p)^{-1}$, the half line can be any ray in the open right half plane, other than $\mathbb{R}^+$. But

$$\int_0^{\infty e^{i0+}} \frac{e^{-xp}}{1-p} dp - \int_0^{\infty e^{i0-}} \frac{e^{-xp}}{1-p} dp = 2\pi i e^{-x}$$

thus a convention for a choice of ray is needed.

Definition. The Borel sum of a series in the direction ϕ ($\arg x = \phi$), $(\mathcal{LB})_\phi \tilde{f}$ is by convention, the Laplace transform of $\mathcal{B}\tilde{f}$ in the direction that ensures $xp \in \mathbb{R}^+$,

$$(\mathcal{LB})_\phi \tilde{f} = \int_0^{\infty e^{-i\phi}} e^{-px} F(p) dp = \mathcal{L}_{-\phi} F = \mathcal{L}F(\cdot e^{-i\phi}) \tag{4.120}$$

We can also say that Borel summation of $\tilde{f}$ along the ray $\arg(x) = \phi$ is defined as the (real) Borel summation of $\tilde{f}(xe^{i\phi})$.

Control over the analytic properties of $\mathcal{B}\tilde{f}$ near $p = 0$ is essential to Borel summability (classical or generalized). Indeed, by the Borel-Ritt Lemma, §3.77, for any power series $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k}$ and *any* sector S there exist (many) functions f analytic in S and asymptotic in S to $\tilde{f}$ [17]. Now, choosing $\delta > 0$, a sector S of angle larger than $\pi + \delta$, and any f such that $f \sim \tilde{f}$ in S, and denoting $f_1 = f$, then Proposition 4.53 below shows that $f - c_0 - c_1 x^{-1} = \mathcal{L}\{F_1\}$ with F_1 analytic in a sector of angle δ; in addition, by Watson's Lemma (see Lemma 3.37), $\mathcal{L}\{F_1\} \sim \tilde{f}_1$ in S. *Any* series would thus be "summable" (very non-uniquely) in this weak sense. Summable series $\tilde{f}$ are distinguished by the analytic properties of F_1 at $p = 0$.

Since in most cases of interest $\mathcal{B}\tilde{f}$ has singularities in the complex plane, different functions $\mathcal{LB}_\phi \tilde{f}$ are obtained for different ϕ. For example, we have

$$\mathcal{LB}_\phi \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} = \mathcal{L}_{-\phi}\{(1-p)^{-1}\} = \begin{cases} \mathrm{e}^{-x}(\mathrm{Ei}(x) - \pi i) & \text{for } \phi \in (-\pi, 0) \\ \mathrm{e}^{-x}(\mathrm{Ei}(x) + \pi i) & \text{for } \phi \in (0, \pi) \end{cases} \tag{4.121}$$

while the series is *not* classically Borel summable along $\mathbb{R}^+$, because of the pole at $p = 1$.

(iv) On the other hand it can be seen by deforming the contour in $\mathcal{L}$ that if $\mathcal{B}\tilde{f}$ is analytic and has uniform exponential bounds at infinity for $\arg(p) \in (-\delta_1, \delta_2)$, then the function $\mathcal{LB}_\phi \tilde{f}$ is the same for all $\arg(x) \in (-\delta_2, \delta_1)$, in contrast to (4.121).

4.6b Recovering exact solutions from formal series.

If a differential equation has a formal solution $\tilde{f} \in S_{\mathcal{B}}$ then $\mathcal{LB}\tilde{f}$ is an actual solution of the same equation. For example

$$f' - f = x^{-1} \tag{4.122}$$

for $x \to \infty$ has the series solution $\tilde{f} = \sum_{k=0}^{\infty}(-x)^{-k-1}k!$ and $\mathcal{B}\{\tilde{f}\} = \sum_{k=0}^{\infty}(-p)^k$ sums to the Laplace transformable function $(1+p)^{-1}$. Now, for any $\tilde{f} \in S_{\mathcal{B}}$ and $f \in \mathcal{LB}(S_{\mathcal{B}})$ we have

$$\tilde{f}' - \tilde{f} - x^{-1} = 0 \iff \mathcal{LB}\big(\tilde{f}' - \tilde{f} - x^{-1}\big) = 0 \tag{4.123}$$

$$\iff \big(\mathcal{LB}\{\tilde{f}\}\big)' - \mathcal{LB}\{\tilde{f}\} - x^{-1} = 0 \tag{4.124}$$

In particular,

$$\mathcal{LB}\{\tilde{f}\} = \int_0^\infty \frac{e^{-px}\mathrm{d}p}{1+p} = f \tag{4.125}$$

is an actual solution of (4.122). Solving the analytic problem (4.122) in $\mathcal{LB}(S_{\mathcal{B}})$ has reduced thus to an essentially *algebraic* question, that of finding $\tilde{f}$.

4.6c Stokes phenomena: first examples

Rotation of the contour of integration in the complex plane is a convenient way to calculate the change in asymptotic behavior with respect to the sector of analysis. We illustrate this on a simple case:

$$y(x) := \int_0^\infty \frac{e^{-px}}{1+p}dp \tag{4.126}$$

and we would like, say, to find the asymptotic behavior in the complex plane of the analytic continuation of this integral with respect to x after one anticlockwise loop around infinity. A simple estimate of the integral over an arc of radius R shows that for $x \in \mathbb{R}^+$ $y(x)$ also equals

$$y(x) = \int_0^{\infty e^{-i\pi/4}} \frac{e^{-px}}{1+p}dp \tag{4.127}$$

Then the functions given in (4.126) and (4.127) agree in $\mathbb{R}^+$ thus they agree everywhere they are analytic. Furthermore, the expression (4.127) is analytic for $\arg x \in (-\pi/4, 3\pi/4)$ and by the very definition of analytic continuation f admits analytic continuation in a sector $\arg(x) \in (-\pi/2, 3\pi/4)$. Now we take

x with $\arg x = \pi/4$ and note that along this ray, by the same argument as before, the integral equals

$$y(x) = \int_0^{\infty e^{-\pi i/2}} \frac{e^{-px}}{1+p} dp \tag{4.128}$$

we can continue this rotation process until $\arg(x) = \pi - \epsilon$ in which case we have

$$y(x) = \int_0^{\infty e^{-\pi i + i\epsilon}} \frac{e^{-px}}{1+p} dp \tag{4.129}$$

which is now manifestly analytic for $\arg(x) \in (\pi/2 - \epsilon, 3\pi/2 - \epsilon)$. To proceed further, we relate the integral below the pole to the integral above the pole, noting that their difference is simply calculated in terms of the at the pole:

$$\int_0^{\infty e^{-\pi i - i\epsilon}} \frac{e^{-px}}{1+p} dp - \int_0^{\infty e^{-\pi i + i\epsilon}} \frac{e^{-px}}{1+p} dp = 2\pi i e^{x} \tag{4.130}$$

and thus

$$f(x) = \int_0^{\infty e^{-\pi i - i\epsilon}} \frac{e^{-px}}{1+p} dp - 2\pi i e^{x} \tag{4.131}$$

which is manifestly analytic for $\arg(x) \in (\pi/2+\epsilon, 3\pi/2+\epsilon)$. We can now freely proceed with the analytic continuation in similar steps until $\arg(x) = 2\pi$ and get

$$f(xe^{2\pi i}) = f(x) - 2\pi i e^{x} \tag{4.132}$$

The function has *nontrivial monodromy at infinity*. We also note that by Watson's Lemma, as long as f can be written as a pure Laplace-like integral, f has an asymptotic series in a half-plane. The relation (4.131) shows that this ceases to be the case when $\arg(x) = \pi$. This line is called a **Stokes line**. The exponential, "born" there is smaller than the terms of the series until $\arg(x) = 3\pi/2$ when it becomes the dominant term of the expansion. This line is called an **Antistokes line**. The fact that the function itself is not single-valued in a neighborhood of infinity is also seen from the calculation (take first $x \in \mathbb{R}^+$)

$$\begin{aligned} f(x) = e^{-x} \int_1^{\infty} \frac{e^{-xt}}{t} dt = e^{-x} \int_x^{\infty} \frac{e^{-s}}{s} ds = e^{-x} \left(\int_x^1 \frac{e^{-s}}{s} ds + \int_1^{\infty} \frac{e^{-s}}{s} ds \right) \\ = e^{-x} \left(C_1 + \int_x^1 \frac{e^{-s}}{s} ds \right) = e^{-x} \left(C_1 + \int_x^1 \frac{e^{-s} - 1}{s} ds - \ln x \right) \\ = e^{-x} \left(\text{entire} - \ln x \right) \end{aligned} \tag{4.133}$$

The Stokes phenomenon however is *not* due to the multivaluedness of the function but to the divergence of the asymptotic series, as seen from the following simple remark.

Remark 4.134 *Assume f is analytic outside a compact set and is asymptotic to $\tilde{f}$ as $|x| \to \infty$ (in any direction). Then $\tilde{f}$ is convergent.*

PROOF By the change of variable $x = 1/z$ we move the analysis at zero. The existence of an asymptotic series as $z \to 0$ implies in particular that f is bounded at zero. Since it is analytic in $\mathbb{C} \setminus \{0\}$ then zero is a removable siungularity of f, and thus the asymptotic series, which as we know is unique, must coincide with the Taylor series of f at zero, a convergent series. ∎

The exercise below also shows that the Stokes phenomenon is not due to multivaluedness.

Exercise 4.135 ** (1) Show that the function $f(x) = \int_x^\infty e^{-s^2} ds$ is entire.*

(2) Note that

$$\int_x^\infty e^{-s^2} ds = \frac{1}{2}\int_{x^2}^\infty \frac{e^{-t}}{\sqrt{t}} dt = \frac{1}{2x}\int_1^\infty \frac{e^{-x^2 u}}{\sqrt{u}} du = \frac{e^{-x^2}}{2x}\int_0^\infty \frac{e^{-x^2 p}}{\sqrt{1+p}} dp \tag{4.136}$$

Do a similar analysis to the one in the text and identify the Stokes and anti-stokes lines for f. Note that the "natural variable" now is x^2.

4.6d Nonlinear Stokes phenomena and formation of singularities

Let us now look at (4.63). If we take n to be a complex variable, then the Stokes lines are those for which after deformation of contour of the integral

$$\int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right) e^{-p}}{p^2(e^{-p} - 1)} e^{-np} dp \tag{4.137}$$

in (4.63), which is manifestly a Borel sum of a series, will run into singularities of the denominator. This happens when n is purely imaginary. Assume that n is on a ray $\arg n = -\pi/2 + \epsilon$. We let

$$F(p) = \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right) e^{-p}}{p^2(e^{-p} - 1)}$$

We rotate the contour of integration to $\arg p = \pi/2 - \epsilon$, compare with the integral to the left of the imaginary line and get the representation

$$\int_0^{\infty e^{i\pi/2-i\epsilon}} F(p)e^{-np}dp = \int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp + 2\pi i \sum_{j\in\mathbb{N}} \mathrm{Res} F(p)e^{-np}|_{p=2j\pi i}$$
$$= \int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp + \sum_{j\in\mathbb{N}} \frac{1}{je^{2nj\pi i}}$$
$$= \int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp - \ln(1-\exp(-2n\pi i)) \quad (4.138)$$

where the sum is convergent when $\arg n = -\pi/2+\epsilon$, and thus, when $\arg n = -\pi/2+\epsilon$ we can also write

$$\Gamma(n+1) = \frac{1}{1-\exp(-2n\pi i)}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp\right) \quad (4.139)$$

from which it is manifest that Γ is analytic in the complex plane, has Stirling's formula as asymptotic series in the classical sense for $\arg(n) \in (-\pi,\pi)$ and has simple poles at all negative integers.

We see that these poles occur on the antistokes line of Γ, and the first time corrections are visible, they come in the form of singularities. We also note that, after reexpansion of the log, the middle expression in (4.138) is a Borel summed series plus a *transseries* in n (although now we allow n to be complex). Singularities occur precisely where the transseries ceases to be valid formally, on the line where $|e^{2\pi in}| \not\ll 1$. This is typical when there are infinitely many singularities in Borel space, generically the case for nonlinear ODEs that we will study in more detail in the sequel (this is the reason we spoke of nonlinear Stokes phenomena). This is also the case, as we saw, in difference equations. Our calculations also show that we have, for $\arg(n) \neq \pi$

$$\ln\Gamma(n+1) \sim n(\ln n - 1) - \frac{1}{2}\ln(2\pi n) + P.S. \quad (4.140)$$

where $P.S.$ is a power series

$$P.S. = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \cdots$$

(explain the sign pattern and the fact that it is an odd series; this is a special case of Dingle's rule of signs that we shall study later, asserting roughly speaking that small exponentials are born on the rays where all signs are in phase). The power series Borel sums to

$$P.S. \overset{\mathcal{LB}}{=} \begin{cases} \ln\Gamma(n+1) - \ln\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right] & \arg(n \in (-\pi/2,\pi/2) \\ \ln\Gamma(n+1) - \ln\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right] - \ln(1-\exp(-2n\pi i)) & \arg n \in \left(-\pi, -\frac{\pi}{2}\right) \\ \ln\Gamma(n+1) - \ln\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right] - \ln(1-\exp(2n\pi i)) & \arg n \in \left(\frac{\pi}{2}, \pi\right) \end{cases} \quad (4.141)$$

and the lines $\arg n = \pm\pi/2$ are Stokes lines. We note that nothing special happens to the function on these lines, while the asymptotic series has the same shape. The discontinuity lies in the representation of the function as a Borel summed series. Something similar, although the similarity should not be pushed too far, happens with the function $(1+x)^{-1}$ when $x = 1$, in reference to its power series, which diverges at $x = 1$ without $x = 1$ being special in any way for the *function*.

Exercise 4.142 *** Find the asymptotic expansion for large $|x|$ in all directions in $\mathbb{C}$ of a solution of the Airy equation $y'' = xy$ which goes to zero as $x \to +\infty$. It is convenient to use formulas of the type (4.62). (Note that by Remark 4.111, Borel summation allows for the Borel transform to be convergent power series in noninteger powers of p).*

4.6e Overview of Borel summation results obtained so far

- Some series which are not convergent are Borel summable. (After we generalize Borel summation we shall see that this is the case in many problems.)

- Borel summation has many of the good features of usual summation of geometrically convergent series. In particular, it commutes with algebraic operations and differentiation.

- Borel summation gives, via Watson's Lemma, a transparent description of the asymptotic properties in the complex domain of the function obtained.

- The function obtained by nontrivial Borel summation depends on the direction. The asymptotic properties of functions thus obtained also depend on the direction. This is seen as Stokes phenomena or formation of singularities. An example is (4.121).

- If there are finitely many singularities in Borel space (the case, as we shall see, of solutions of linear ODEs), then a solution that decays in some direction increases exponentially in complementary directions. If there are infinitely many singularities (the case of solutions of nonlinear ODEs and difference equations among others), solutions that decrease along some directions typically form singularities in complementary directions. An example is (4.141). These singularities are arranged in almost periodic arrays. The transseries representation of a solution is a very useful tool to find its singularities in the complex domain, as we shall see.

4.6f Some of the difficulties of classical Borel summation

The domain of definition of classical Borel summation is not large enough to be able to use it to sum transseries. First, $\mathcal{LB}$ only applies to power series, while for instance the general solution of (4.122) is $\mathcal{LB}\{\tilde{f}\} + Ce^x$. This is not a serious problem however; the definition $\mathcal{LB}\exp(ax) = \exp(ax)$ solves it.

A more substantial difficulty is encountered within simple power series. The change of variable $x \mapsto (-x)$ in (4.122) leads to the equation $f' + f = 1/x$, with formal series solution $\tilde{f} = \sum_{k=0}^{\infty} k! x^{-k-1}$. We now get $\sum \mathcal{B}\tilde{f} = (1-p)^{-1}$ which is *not* Laplace transformable, because of the nonintegrable singularity at $p = 1$. Although one can avoid the singularity by shifting the contour of $\mathcal{L}$ in the complex plane, there is **no systematic way** to define the shift to allow for arbitrary location of the singularities of a general $\mathcal{B}\tilde{f}$, and if the contour of integration has to depend on $\tilde{f}$, then the linearity of $\mathcal{LB}$ is lost. (Commutation with complex conjugation is also lost if the contour of integration is not fixed.) Restricting however the *location* of singularities would make Borel summability incompatible with the trivial change of variable $x \mapsto \text{Const}.x$.

Finally, $\mathcal{LB}$ cannot be usefully restricted to those $\tilde{f} \in S_{\mathcal{B}}$ for which $F_1 = \mathcal{B}\{\tilde{f}\}$ is entire and $|F_1(p)| \le C_1 e^{C_2|p|}$ in $\mathbb{C}$, because this simply entails the convergence of $\tilde{f}$. Indeed, by shifting the contour of integration in $\int_0^{\infty} e^{-px} F_1(p) dp$ and rotating x simultaneously to keep xp real and positive, we see that $(\mathcal{L}F_1)(x)$ is single-valued near infinity. By Proposition 4.112 $\mathcal{L}F_1 \sim \tilde{f}_1$ as $|x| \to \infty$, $x \in \mathbb{C}$, therefore ∞ is a removable singularity of $\mathcal{L}F_1$ and the series $\tilde{f}_1$ converges.

Furthermore, as a manifestation of Stokes' phenomenon, as we saw in the previous section, a single-valued function f cannot be asymptotic to the same divergent expansion in every direction in the complex plane.

Since the restrictions needed for classical Borel summation to apply do not allow it to define a sufficiently general isomorphism, one looks instead at extensions of $\mathcal{LB}$, as an **operator**.

4.7 Gevrey classes, least term truncation, and Borel summation

In the simple example of $\text{Ei}(x)$, factorial divergence is associated with the possible presence of exponentially small terms, terms beyond all orders. This and the power-of-factorial-like divergence of formal asymptotic series of solutions of differential equations are quite general phenomena, as will become clear in the following chapters.

Let now $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k}$ be a formal power series and f a function asymptotic to it. The definition (1.12) provides estimates of the value of $f(x)$ for large x , within $o(x^{-N})$, $N \in \mathbb{N}$, which are, as we have seen, insufficient to pin down a unique f associated to $\tilde{f}$. Simply widening the sector in which (1.12) is required cannot change this situation since, for instance, $\exp(-x^{1/m})$ is beyond all orders of $\tilde{f}$ in a sector of angle almost $m\pi$.

It seems then reasonable to attempt to (a) lower the errors in the approximation of f by the truncates of $\tilde{f}$ to less than $O(e^{-\text{Const.}|x|})$, to roughly match the "natural" indeterminacy of f, and then (b) look for estimates in a wide enough sector in the hope of ruling out any possible terms beyond all orders, in this way restoring uniqueness of the association between f and $\tilde{f}$. In some cases this strategy is successful. One important approach in this class is due to *Gevrey* (see e.g. [10]).

The formal series

$$\tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k}, \quad x \to \infty$$

is by definition Gevrey of order $1/m$, or Gevrey-$(1/m)$ if $|c_k| \le C_1 C_2^k (k!)^m$ for some C_1, C_2. Taking $x = y^m$ and $\tilde{g}(y) = \tilde{f}(x)$, then $\tilde{g}$ is Gevrey-1 (albeit not necessarily an integer power series, the generalization to noninteger power series is immediate) and we will focus on this case. Also, the corresponding classification for series in z, $z \to 0$ is obtained by taking $z = 1/x$.

Remarks 4.143 (a) The Gevrey order of the series $\sum_k (k!)^r x^{-k}$ $r > 0$, is the same as that of $\sum_k (rk)! x^{-k}$. Indeed, if $\epsilon > 0$ we have, by Stirling's formula,

$$Const\ (1+\epsilon)^{-k} \le (rk)!/(k!)^r \sim Const\ k^{\frac{1}{2}-r} \le Const\ (1+\epsilon)^k$$

(b) There is a simple connection between the Gevrey order of formal power series solutions of a differential equation at an irregular singular point and the type of exponentials of the associated homogeneous equation. For illustration consider the example of the equation $z^{q+1}y' - ay = 1$ in a neighborhood of zero, with $q \in \mathbb{N}$. The coefficients c_k of a formal power series solution $\tilde{y} = \sum_{k \ge 0} c_k z^k$ satisfy the recurrence $a_0 = 0$ and $(k-q)c_{k-q} + ac_k = 0$ if $k - q > 0$. If $q \ge 1$ we get $c_{jq+q} = a^j j!$, the series diverges and $x = 0$ is an irregular singularity. Using part (a) above we see that the series is Gevrey-q. On the other hand, the solution of the homogeneous equation $z^{q+1}y' - ay = 0$ is $C \exp\left(-\frac{a}{q} z^{-q}\right)$. Precise asymptotic control of the coefficients of formal power series solutions can be obtained for quite general differential systems, see e.g. [23].

Exercise. Formulate and prove a more general result in the spirit of Remark 4.143 (b) for n-th order linear differential equations.

*

Let $\tilde{f}$ be Gevrey-1 . A function f is *Gevrey-1 asymptotic* to $\tilde{f}$ as $x \to \infty$ in a sector S if for some C_3, C_4, C_5, and all $x \in S$ with $|x| > C_5$ and all N we have

$$|f(x) - \tilde{f}^{[N]}| \le C_1 C_2^{N+1} |x|^{-N-1} (N+1)! \tag{4.144}$$

i.e. the error $f - \tilde{f}^{[N]}$ is of the same form as the first omitted term in $\tilde{f}$.

Remark 4.145 *If $\tilde{f}$ is Gevrey-1 and f is Gevrey-1 asymptotic to $\tilde{f}$ then f can be approximated by $\tilde{f}$ with exponential precision in the following sense. Let $N = \lfloor |x/C_2| \rfloor$ ($\lfloor \cdot \rfloor$ is the integer part) then for any $C > C_2$ we have*

$$f(x) - \tilde{f}^{[N]}(x) = O(|x|^{-1/2} e^{-|x|/C}) \ |x| \text{ large}) \tag{4.146}$$

Indeed, letting $|x| = NC_2 + \epsilon$ with $\epsilon \in ([0, 1)$ and applying Stirling's formula we have

$$N!(N+1)C_2^N |NC_2 + \epsilon|^{-N-1} = O(|x|^{1/2} e^{-|x|/C_2})$$

□

Notes. (a) A heuristic discussion about the strategy may be helpful now; rigorous statements will follow.

Usually the imprecision implied by (4.146) is larger than the potential terms beyond a Gevrey-1 series $\tilde{f}$, at least in *some* directions.

However, if the estimate (4.146) holds for f in a sector $S_{\pi+}$ of opening more than π, then it is easy to see that (4.146) cannot hold at the same time for f and for $f + C' e^{-C'' x^p} x^{-m}$, no matter what $C'', m, p > 1$ are, unless $C' = 0$. Since terms beyond all orders, if present, are expected to be some combinations of powers, exponentials and logs, these and similar attempts suggest that if f satisfies (4.146) in $S_{\pi+}$, then f is unique. Theorem 4.147 below shows that this is true.

(b) It is also interesting that when there is a unique f in $S_{\pi+}$ with the property (4.146), then $\tilde{f}$ is Borel summable, and f is *precisely the Borel sum of $\tilde{f}$* (Theorem 4.147 below).[3]

(c) However the same theorem suggests that unless the series $\tilde{f}$ is trivial, there must exist *some* $S_{\pi+}$ in which *no* f is Gevrey−1-asymptotic to $\tilde{f}$ and where this method of associating an f to $\tilde{f}$ fails. In addition we note that there is no entire function of exponential order one at infinity (i.e., $f(x) \le C_1 \exp(C_2|x|)$) which is Gevrey−1 asymptotic to a divergent series in more than a half plane. Indeed if there was such a function f then the Phragmén-Lindelöf principle applied in $\mathbb{C} \setminus S_{\pi+}$ would imply that f is bounded at infinity, thus f and $\tilde{f}$ would be constant.

(d) *Summation to the least term* as will be detailed in the Chapter 4, is in a sense a refined version of Gevrey asymptotics. It requires *optimal constants* in addition to an improved form of Rel. (4.144). In this way the imprecision

[3]Borel summability is clearly not ensured by the Gevrey character of $\tilde{f}$ alone, since such estimates give no information about $\sum \mathcal{B}\tilde{f}$ beyond the implied disk of convergence.

of approximation of f by $\tilde{f}$ turns out to be smaller than the largest exponentially small "possible" term beyond all orders, and thus the cases in which uniqueness is ensured are more numerous.

4.7a Connection between Gevrey asymptotics and Borel summation

The following theorem goes back to Watson [12].

Theorem 4.147 *Let* $\tilde{f} = \sum_{k=2}^{\infty} c_k x^{-k}$ *be a Gevrey-1 series and assume the function* f *is analytic for large* x *in* $S_{\pi+} = \{x : |\arg(x)| < \pi/2 + \delta\}$ *for some* $\delta > 0$ *and Gevrey-1 asymptotic to* $\tilde{f}$ *in* $S_{\pi+}$*. Then*

(i) f *is unique.*

(ii) $\tilde{f}$ *is Borel summable in any direction* $e^{i\theta}\mathbb{R}^+$ *with* $|\theta| < \delta$ *and* $f = \mathcal{L}\mathcal{B}_\theta \tilde{f}$.

(iii) $\mathcal{B}(\tilde{f})$ *is analytic (at* $p = 0$ *and) in the sector* $S_\delta = \{p : \arg(p) \in (-\delta, \delta)\}$*, and Laplace transformable in any closed subsector.*

(iv) Conversely, if $\tilde{f}$ *is Borel summable along any ray in the sector* S_δ *given by* $|\arg(x)| < \delta$*, and if* $\mathcal{B}\tilde{f}$ *is uniformly bounded by* $e^{\nu|p|}$ *in any closed subsector of* S_δ*, then* f *is Gevrey-1 with respect to its asymptotic series* $\tilde{f}$ *in the sector* $|\arg(x)| \le \pi/2 + \delta$.

Notes. (i) In particular, when the assumptions of the theorem are met, Borel summability follows using only *asymptotic estimates*.

(ii) We also see that the cases described in Theorem 4.147 in which Gevrey estimates ensure uniqueness of the association between $\tilde{f}$ and f are weaker than those in which $\tilde{f}$ is Borel summable, since Borel summability requires analyticity in some neighborhood of $\mathbb{R}^+$ and not in a sector.

Proof of Theorem 4.147. Let us note first a possible pitfall. Inverse Laplace transformability of f follows immediately from the assumptions. What doesn't follow is analyticity of the transform at zero. On the other hand, the formal inverse Laplace of $\tilde{f}$ trivially converges to an analytic function. But there is no guarantee that this analytic function has anything to do with the inverse Laplace transform of f! This is where Gevrey estimates enter.

(i) If f_1 and f_2 satisfy the assumption of the theorem, then by Proposition 4.145, for some constants C_1, C_2 (same for f_1 and f_2) we have

$$|f_1(x) - f_2(x)| < C_1\sqrt{|x|}e^{-C_2|x|} \tag{4.148}$$

in a sector of opening more then π. Note that by Proposition 4.53 $\mathcal{L}^{-1}\{f_1 - f_2\}$ exists and is analytic for $\arg(p) \in (-\delta, \delta)$ and that, by (4.148), for $|p| < C_2$ the contour of integration in (4.55) can be pushed to infinity implying that $\mathcal{L}^{-1}\{f_1 - f_2\} = 0$ on the interval $(0, C_2)$. By analyticity $\mathcal{L}^{-1}\{f_1 - f_2\} \equiv 0$ and the inversion formula gives $f_1 - f_2 = 0$.

(ii) By a simple change of variables we arrange $C_1 = C_2 = 1$. The series $\tilde{F}_1 = \mathcal{B}\tilde{f}$ is convergent for $|p| < 1$ and defines an analytic function, F_1. By

Proposition 4.53, the function $F = \mathcal{L}^{-1}f$ is analytic for $|p| > 0, |\arg(p)| < \delta$, and $F(p)$ is analytic and uniformly bounded if $|\arg(p)| < \delta_1 < \delta$. We now show that F is analytic for $|p| < 1$. Taking p real, $p \in [0,1)$ we obtain in view of (4.144) that

$$|F(p) - \tilde{F}^{[N-1]}(p)| \leq \int_{-i\infty+N}^{i\infty+N} d|s| \left| f(s) - \tilde{f}^{[N-1]}(s) \right| e^{\Re(ps)}$$
$$\leq N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{|x+iN|^N} = N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+N^2)^{N/2}} \quad (4.149)$$

where we take $x = N \tan t$ and get the estimate for the last term

$$\frac{N! e^{pN}}{N^{N-1}} \int_{-\pi/2}^{\pi/2} \cos^{N-2}(t)\mathrm{d}t \leq const N^{3/2} e^{(p-1)N} \quad (N \to \infty) \qquad (4.150)$$

Since the RHS in (4.150) vanishes in the limit $N \to \infty$ for $p \in [0,1)$, this implies $F = F_1$ for $p \in [0,1)$, thus $F = F_1$ for any p with $|p| < 1$ and also for any p with $|\arg(p)| < \delta$.

Since $\sum \mathcal{B}\tilde{f} = \mathcal{L}^{-1}f$, (iii) follows now from Proposition 4.53.

(iv) Let $|\phi| < \delta$. We have, by integration by parts,

$$f(x) - \tilde{f}^{[N-1]}(x) = x^{-N} \mathcal{L} \frac{d^N}{dp^N} F \qquad (4.151)$$

On the other hand, F is analytic in S_a, some $a = a(\phi)-$neighborhood of the sector $\{p : |\arg(p)| < |\phi|\}$. Estimating Cauchy's formula on an $a-$circle around the point p with $|\arg(p)| < |\phi|$ we get

$$|F^{(n)}(p)| \leq N! a(\phi)^{-N} \|F(p)\|_{\infty;S_a}$$

Thus, by (4.151), with $|\theta| \leq |\phi|$ chosen so that $\gamma = \cos(\theta - \arg(x))$ is maximal we have

$$\left| f(x) - \tilde{f}^{[N]} \right| = \left| x^{-N} \int_0^{\infty \exp(-i\theta)} F^{(N)}(p) e^{-px} dp \right|$$
$$\leq N! a^{-N} |x|^{-N} \|F e^{\nu|p|}\|_{\infty;S_a} \int_0^{\infty} e^{-px+\nu|p|\gamma} dp = const. N! a^{-N} \gamma^{-1} |x|^{-N-1} \|F\|_{\infty;S_a} \quad (4.152)$$

for large enough x. $\square$

4.7b Stokes lines

Theorem 4.147 and the discussion in §4.6f show that for a non-convergent Gevrey-1 series $\tilde{f}$ there must exist sectors of opening more than π where no f is Gevrey−1 asymptotic to $\tilde{f}$. These "singular" directions reflect the presence of the local Stokes phenomenon .

DEFINITION. *Let $\tilde{f}$ be Gevrey-1.*

We say that $\tilde{f}$ is **Gevrey-1 asymptotic in** $S(\phi;\epsilon;R)$ *where*

$$S(\phi;\epsilon;R)=\{x:|x|>R,|\arg(x)-\phi|<\pi/2+\epsilon\}$$

if there exists f analytic $S(\phi;\epsilon;R)$ such that $f\overset{G_1}{\sim}\tilde{f}$ in $S(\phi;\epsilon;R)$ (then this f is unique, by Theorem 4.147).

If ϕ is such that $\tilde{f}$ is not Gevrey-1 asymptotic in $S(\phi;\epsilon;R)$, we say that $d_\phi=\{x:\arg(x)=\phi\}$ is a **Stokes ray** *for $\tilde{f}$.*

Proposition 4.153 *Let $\tilde{f}$ be Gevrey-1. Then $\tilde{f}$ is divergent* **iff** *it has at least a Stokes ray .*

PROOF This property of $\tilde{f}$ is clearly independent of any finite number of terms in $\tilde{f}$ so we may assume $\tilde{f}=\sum_{k=2}^{\infty}f_kx^{-k}$. If $\tilde{f}$ is convergent then clearly it has no Stokes line s. For the converse, we assume that $\tilde{f}$ has no Stokes line s and for $\phi\in[0,2\pi+\delta]$ we let $\epsilon_\phi>0,R_\phi,f_\phi$ be such that $f_\phi\overset{G_1}{\sim}\tilde{f}$ in $S(\phi;\epsilon_\phi;R_\phi)$. If $E(\phi)$ is the sup of ϵ_ϕ such that $\tilde{f}$ is Gevrey-1 asymptotic in $S(\phi;\epsilon_\phi;R_\phi)$ for some R_ϕ then it is easy to check that $E(\phi)$ is continuous in ϕ and then, for some $N\in\mathbb{N}$ we have $\inf_{\phi\in[0,2\pi+\delta]}E(\phi)>(2/N)>0$. In all sectors $S_j=S(j/N;\epsilon_{j/N};R_{j/N})$ with $0\le j/N<2\pi+\delta$ the series $\tilde{f}$ is Gevrey-1 asymptotic, and since $S_j\cap S_{j+1}$ is wider than π we have by Theorem 4.147 that $f_{(j+1)/N}=f_{j/N}$ if $0\le j/N<2\pi+\delta$. Thus $f_{j/N}=f$ is independent of j and in particular f is single-valued at infinity. Thus, by Liouville's theorem f is analytic at infinity and $\tilde{f}$ is convergent. ∎

4.7c Strategies of Borel summation of formal power series solutions: an introduction

Assume we intend to solve using Borel summability techniques an ODE, say

$$y'+y=x^{-2}+y^3 \tag{4.154}$$

To find a formal power series solution we proceed as usual, separating out the dominant terms, in this case y and x^{-2}. We get the iterations scheme

$$y_{[n]}(x)-x^{-2}=y_{[n-1]}^3-y_{[n-1]}' \tag{4.155}$$

with $y_{[0]} = 0$. After a few iterations we get

$$\tilde{y}(x) = x^{-2}+2\,x^{-3}+6\,x^{-4}+24\,x^{-5}+121\,x^{-6}+732\,x^{-7}+5154\,x^{-8}+\cdots \quad (4.156)$$

For differential equations of this kind there exist results in great generality as to the Borel summability of formal transseries solutions, and we shall see a few of these in the sequel. The purpose now is to illustrate a strategy of proof that is convenient and which applies to a reasonably large class of settings.

It would be technically awkward to prove, solely based on (4.156) that the Borel transform of the series is convergent, extends analytically along the real line and that is has the required exponential bounds towards infinity.

A better approach is to control the Borel transform of $\tilde{y}$ via the equation it satisfies. This equation is the formal inverse Laplace transform of (4.155), namely, setting $Y = \mathcal{B}\tilde{y}$

$$-pY + Y = p + Y * Y * Y := p + Y^{*3} \quad (4.157)$$

We then show that the equation (4.157) has a (unique) solution which is analytic in a neighborhood of the origin together with a sector centered on $\mathbb{R}^+$ in which this solution has exponential bounds. Thus Y is Laplace transformable, and immediate verification shows that $y = \mathcal{L}Y$ satisfies (4.154). Furthermore, since the Maclaurin series $S(Y)$ formally satisfies (4.157) then the formal Laplace (inverse Borel) transform $\mathcal{B}^{-1}SY$ is a *formal* solution of (4.154), and thus equals $\tilde{y}$ since this solution, as we proved in many similar settings is unique. But since then $SY = \mathcal{B}\tilde{y}$ it follows that $\tilde{y}$ is Borel summable, and the Borel sum solves (4.154).

The transformed equations are expected to have analytic solutions, therefore to be more regular than the original ones.

4.7c .1 Regularizing the heat equation

$$f_{xx} - f_t = 0 \quad (4.158)$$

Since (4.158) is parabolic, power series solutions

$$f = \sum_{k=0}^{\infty} t^k F_k(x) = \sum_{k=0}^{\infty} \frac{F_0^{(2k)}}{k!} t^k \quad (4.159)$$

are divergent even if F_0 is analytic (but not entire). Nevertheless, under suitable assumptions, Borel summability results of such formal solutions have been shown by Lutz, Miyake, and Schäfke [15] and more general results of multisummability of linear PDEs have been obtained by Balser [10].

The heat equation can be regularized by a suitable Borel summation . The divergence implied, under analyticity assumptions, by (4.159) is $F_k = O(k!)$ which indicates Borel summation with respect to t^{-1}. Indeed, the substitution

$$t = 1/\tau; \quad f(t,x) = t^{-1/2} g(\tau, x) \tag{4.160}$$

yields

$$g_{xx} + \tau^2 g_\tau + \frac{1}{2}\tau g = 0$$

which becomes after formal inverse Laplace transform (Borel transform) in τ,

$$p\hat{g}_{pp} + \frac{3}{2}\hat{g}_p + \hat{g}_{xx} = 0 \tag{4.161}$$

which is brought, by the substitution $\hat{g}(p,x) = p^{-\frac{1}{2}} u(x, 2p^{\frac{1}{2}});\ y = 2p^{\frac{1}{2}}$, to the wave equation, which is hyperbolic, thus *regular*

$$u_{xx} - u_{yy} = 0. \tag{4.162}$$

Existence and uniqueness of solutions to regular equations is guaranteed by Cauchy-Kowalevsky theory. For this simple equation the general solution is certainly available in explicit form: $u = f_1(x-y) + f_2(x+y)$ with f_1, f_2 arbitrary twice differentiable functions. Since the solution of (4.162) is related to a solution of (4.158) through (4.160), to ensure that we do get a solution it is easy to check that we need to choose $f_1 = f_2 =: u$ (up to an irrelevant additive constant which can be absorbed into u) which yields,

$$f(t,x) = t^{-\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}} \left[u\left(x + 2\, y^{\frac{1}{2}}\right) + u\left(x - 2\, y^{\frac{1}{2}}\right)\right] \exp\left(-\frac{y}{t}\right) dy \tag{4.163}$$

which, after splitting the integral and making the substitutions $x \pm 2\, y^{\frac{1}{2}} = s$ is transformed into the usual Heat kernel solution,

$$f(t,x) = t^{-\frac{1}{2}} \int_{-\infty}^{\infty} u(s) \exp\left(-\frac{(x-s)^2}{4t}\right) ds \tag{4.164}$$

*

4.7d Convolutions: elementary properties

The transformed equation (4.157) is a convolution equation and it is useful to list first some elementary properties of convolutions. Some spaces are well suited for the study of convolution algebras.

(1) Let $\nu \in \mathbb{R}^+$ and define $L^1_\nu := \{f : \mathbb{R}^+ : f(p)e^{-\nu p} \in L^1(\mathbb{R}^+)\}$; then the norm $\|f\|_\nu$ is defined as $\|f(p)e^{-\nu p}\|_1$ where $\|\cdot\|_1$ denotes the L^1 norm.

Proposition 4.165 L^1_ν *is a Banach algebra with respect to convolution.*

PROOF Note first that if $f \in L^1_\nu$ then the Laplace transform of f exists for $\Re(x) \ge \nu$ and $f, g \in L^1_\nu$ implies

$$\begin{aligned}\|f * g\|_\nu = \int_0^\infty e^{-\nu p}\left|\int_0^p f(s)g(p-s)ds\right| dp \\ = \int_0^\infty \left|\int_0^p f(s)e^{-\nu s}g(p-s)e^{-\nu(p-s)}ds\right| dp \\ \le \int_0^\infty \int_0^p \left|f(s)e^{-\nu s}\right| \left|g(p-s)e^{-\nu(p-s)}\right| dsdp \\ = \int_0^\infty \left|f(s)e^{-\nu s}\right| \int_0^\infty ds \left|g(s)e^{-\nu s}\right| ds = \|f\|_\nu \|g\|_\nu \end{aligned} \tag{4.166}$$

In particular, convolution is well defined in L^1_ν and we have, by a very similar calculation,

$$\mathcal{L}[f * g] = (\mathcal{L}f)(\mathcal{L}g) \tag{4.167}$$

Furthermore,

$$\mathcal{L}[f * (g * h)] = \mathcal{L}[f]\mathcal{L}[g * h] = \mathcal{L}[f]\mathcal{L}[g]\mathcal{L}[h] = \mathcal{L}[(f * g) * h] \tag{4.168}$$

and since the Laplace transform is injective, we get

$$f * (g * h) = (f * g) * h \tag{4.169}$$

and convolution is associative. Similarly, it is easy to see that

$$f * g = g * f, \qquad f * (g + h) = f * g + f * h \tag{4.170}$$

∎

(2) A simple generalization is to allow p to be complex. We say that $f \in L^1_\nu(\mathbb{R}^+ e^{i\phi})$ (along the ray $\{te^{i\phi} : t \in \mathbb{R}^+\}$) if $f_\phi := f(te^{i\phi}) \in L^1_\nu$. Convolution is defined as

$$\begin{aligned}(f * g)(p) = \int_0^p f(s)g(p-s)ds = \\ e^{i\phi}\int_0^{|p|e^{i\phi}} f(te^{i\phi}))g(e^{i\phi}(|p|-t)dt = e^{i\phi}(f_\phi * g_\phi)(|p|e^{i\phi})\end{aligned} \tag{4.171}$$

and it is clear that $L^1_\nu(\mathbb{R}^+ e^{i\phi})$ is also a Banach algebra with respect to convolution.

Similarly, we say that $f \in L^1_\nu(S)$ where $S = \{te^{i\phi} : t \in \mathbb{R}^+, \phi \in (a,b)\}$ if $f \in L^1_\nu(\mathbb{R}^+ e^{i\phi})$ for all $\phi \in (a,b)$. We define $\|f\|_{\nu,S} = \sup_{\phi \in (a,b)} \|f\|_{L^1_\nu(\mathbb{R}e^{i\phi})}$. $L^1_\nu(S)$ is also a Banach algebra.

(3) The L^1_ν spaces can be restricted to an initial interval along a ray, or a compact subset of S, with the norm restricted to the appropriate subset. For instance,

$$L^1_\nu([0,1]) = \{f : \int_0^1 e^{-\nu s}|f(s)|ds < \infty\} \tag{4.172}$$

These spaces are Banach algebras as well. Obviously, if $A \subset B$, $L^1_\nu(B)$ is naturally imbedded in $L^1_\nu(A)$.

(4) Another important space is $\mathcal{A}_{K;\nu}(\mathcal{N})$, the space of analytic functions analytic in a star-shaped neighborhood $\mathcal{N} \in \mathbb{C}$ of the interval $[0,K]$ in the norm ($\nu \in \mathbb{R}^+$)

$$\|f\| = K \sup_{p\in\mathcal{N}} \left| e^{-\nu|p|} f(p) \right|$$

Note This norm is topologically equivalent with the sup norm, but this form is better suited for controlling exponential growth.

Proposition 4.173 *The space $\mathcal{A}_{K;\nu}$ is a Banach algebra with respect to convolution.*

PROOF Analyticity of convolution is proved in the same way as Lemma 4.108. Associativity and commutativity of convolution are shown either by a strategy similar to the one in the previous proposition, or by direct verification.

To show continuity of convolution we let $|p| = P$, $p = Pe^{i\phi}$ and note that

$$\begin{aligned}
\left| Ke^{-\nu P} \int_0^p f(s)g(p-s)ds \right| &= \left| Ke^{-\nu P} \int_0^P f(te^{i\phi})g((P-t)e^{i\phi})dt \right| \\
&= \left| K^{-1} \int_0^P Kf(te^{i\phi})e^{-\nu t} Kg((P-t)e^{i\phi})e^{-\nu(P-t)}dt \right| \\
&\le K^{-1}\|f\|\|g\| \int_0^P d|t| = \|f\|\|g\|
\end{aligned} \tag{4.174}$$

∎

Note that $\mathcal{A}_{K;\nu} \subset L^1_\nu(\mathcal{N})$.

(5) Finally, we note that the space $\mathcal{A}_{K,\nu;0}(\mathcal{N}) = \{f \in \mathcal{A}_{K,\nu}(\mathcal{N}) : f(0) = 0\}$ is a closed subalgebra of $\mathcal{A}_{K,\nu}$.

Remark 4.175 *In the spaces L^1_ν, $\mathcal{A}_{K;\nu}$, $\mathcal{A}_{K,\nu;0}$ etc. we have, for a bounded function f,*

$$\|fg\| \le \|g\| \max|f|$$

4.7e Spaces of sequences of functions

In Borel summing not simply series but transseries it is convenient to look at sequences of vector-valued functions belonging to one ore more of the spaces introduced before. We let

$$\mathbf{y} = \{\mathbf{y_k}\}_{\mathbf{k}\geq\mathbf{0}}\,; \quad \mathbf{k} \in \mathbb{Z}^m,\ \mathbf{y_k} \in \mathbb{C}^n \tag{4.176}$$

We let

$$\mathbf{Y} = \{\mathbf{Y}\} \tag{4.177}$$

If $\mathbf{F}$ and $\mathbf{G}$ are functions with values in $\mathbb{C}^n$, we write
We let

$$L^1_{\nu,\mu} = \{\overset{\mathbb{N}}{\mathbf{Y}} \in (L^1_\nu)^{\mathbb{N}} : \sum_{k=1}^{\infty} \mu^{-k} \|\mathbf{Y}_k\|_\nu < \infty\} \tag{4.178}$$

$$\overset{\mathbb{N}}{\mathbf{Y}}$$

We introduce the following convolution on $L^1_{\nu,\mu}$

$$\left(\overset{\mathbb{N}}{\mathbf{F}} \overset{*}{*} \overset{\mathbb{N}}{\mathbf{G}}\right)_k = \sum_{j=1}^{n-1} \mathbf{F}_j * \mathbf{G}_{k-j} \tag{4.179}$$

Exercise 4.180 ** Show that*

$$\| \overset{\mathbb{N}}{\mathbf{F}} \overset{*}{*} \overset{\mathbb{N}}{\mathbf{G}} \|_{\nu,\mu} \leq \| \overset{\mathbb{N}}{\mathbf{F}} \|_{\nu,\mu} \| \overset{\mathbb{N}}{\mathbf{G}} \|_{\nu,\mu} \tag{4.181}$$

$(L^1_{\nu,\mu}, +, \overset{*}{*}, \| \|_{\nu,\mu})$ *where* $\| \|_{\nu,\mu}$ *is the norm introduced in (4.178) is a Banach algebra.*

4.7f Focusing spaces and algebras

An important property of the norms introduced, on the spaces L^1_ν and $\mathcal{A}_{K,\nu;0}$ is that for any f in these spaces $\|f\| \to 0$ as $\nu \to \infty$. In the case L^1_ν this is an immediate consequence of dominated convergence.

More generally, we say that a family of norms $\|\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **focusing** if for any f with $\|f\|_{\nu_0} < \infty$

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{4.182}$$

Let $\mathcal{E}$ be a linear space and $\{\|\|\|_\nu\}$ a family of norms satisfying (4.182). For each ν we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{E} : \|f\|_\nu < \infty\}$. Enlarging $\mathcal{E}$ if needed, we may assume that $\mathcal{B}_\nu \subset \mathcal{E}$. For $\alpha < \beta$, (4.182) shows that the identity is an embedding of $\mathcal{B}_\alpha$ in $\mathcal{B}_\beta$. Let $\mathcal{F} \subset \mathcal{E}$ be the projective limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \bigcup_{\nu>0} \mathcal{B}_\nu \tag{4.183}$$

is endowed with the topology in which a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focusing space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ ("convolution") is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focusing algebra**.

Examples. The spaces $\bigcup_{\nu>0} L^1_\nu$ and $\bigcup_{\nu>0} \mathcal{A}_{K;\nu;0}$ and $L^1_{\nu,\mu}$ are focusing algebras. The last space is focusing as $\nu \to \infty$ and/or $\mu \to \infty$.

Remark 4.184 *The following result is immediate. Let A, B be any sets and assume that the equation $f(x) = 0$ is well defined and has a unique solution x_1 in A, a unique solution x_2 in B and a unique solution x_3 in $A \cap B$. Then $x_1 = x_2 = x_3 = x$. In particular, if $A \subset B$ then $x \in A \cap B$.*

4.8 Borel summability of solutions of nonlinear equations: an introduction

4.8a An example. Borel summability of the main series

Note Since we have a Banach algebra structure in Borel plane, differential equations become effectively algebraic equations, much easier to deal with.

The concepts an methods used for the simple equation

$$y' + y = x^{-2} + y^3 \tag{4.185}$$

capture a good part of the ones needed in a general setting.
Formal inverse Laplace transform of (4.185) yields, with the notation $\mathcal{L}^{-1}y = Y$ and $Y^{*3} = Y * Y * Y$,

$$-pY + Y = p + Y^{*3}; \Leftrightarrow Y = \frac{p}{1-p} + \frac{1}{1-p} Y^{*3} := \mathcal{N}(Y) \tag{4.186}$$

Let $[a, b] \in (0, 2\pi)$, and $S = \{p : \arg(p) \in (a, b)$, $S_K = \{p \in S : |p| < K\}$, $B = \{p : |p| < a < 1\}$.

Proposition 4.187 *(i) For large enough ν, Eq. (4.186) has a unique solution in the following spaces: $L^1_\nu(S), L^1_\nu(S_K), \mathcal{A}_{\nu,0}(S_K \cup B)$. (ii) There is a solution of Y which is analytic in $S \cup B$ and is Laplace transformable along any direction in S. The Laplace transform is a solution of (4.185).*

PROOF The proof is the same for all these types of spaces, call them generically $\mathcal{S}_\nu$, since for each type their inductive limit is a focusing algebra. Choose ϵ small enough. Then for large enough ν we have

$$\left\|\frac{p}{1-p}\right\|_\nu < \epsilon/2 \tag{4.188}$$

Let $\mathfrak{B}$ be the ball of radius ϵ in the norm ν. Then if $F \in \mathfrak{B}$ we have

$$\|\mathcal{N}(F)\|_\nu \leq \left\|\frac{p}{1-p}\right\|_\nu < \epsilon/2 + \max\left|\frac{1}{p-1}\right| \|Y\|_\nu^3 = \epsilon/2 + c\epsilon^3 \leq \epsilon \tag{4.189}$$

if ϵ is small enough (that is, if ν is large). Furthermore, for large ν, $\mathcal{N}$ is contractive in $\mathfrak{B}$ for we have, for small ϵ,

$$\|\mathcal{N}(F_1)-\mathcal{N}(F_2)\|_\nu \leq c\|F_1^{*3} - F_2^{*3}\|_\nu = c\|(F_1-F_2)*(F_1^{*2}+F_1*F_2+F_2^{*2})\|_\nu$$
$$\leq c\|(F_1-F_2)\|_\nu(3\epsilon^2) < \epsilon \tag{4.190}$$

(ii) We have the following embeddings: $L^1_\nu(S) \subset L^1_\nu(S_K), \mathcal{A}_{\nu,0}(S_K \cup B) \subset L^1_\nu(S_K)$. Thus, by Remark 4.184, there exists a unique solution Y of (4.186) which belongs to all these spaces.

Thus Y is analytic in S and in $L^1_\nu(S)$, in particular Laplace transformable. The Laplace transform is a solution of (4.185) as it is easy to check.

It also follows from the argument that the formal power series solution $\tilde{y}$ of (4.185) is Borel summable in any sector not containing $\mathbb{R}^+$, which is a Stokes line. We have, indeed, $\mathcal{B}\tilde{y} = Y$ (check!). ∎

4.8a .1 Convergent series composed with Borel summable series

Proposition 4.191 *Assume A is an analytic function in the disk of radius ρ centered at the origin, $a_k = A^{(k)}(0)/k!$, and $\tilde{s} = \sum s_k x^{-k}$ is a small series which is Borel summable along $\mathbb{R}^+$. Then the formal power series obtained by reexpanding*

$$\sum a_k s^k$$

in powers of x is Borel summable along $\mathbb{R}^+$.

PROOF Let $S = \mathcal{B}s$ and choose ν be large enough so that $\|S\|_\nu < \rho^{-1}$ in L^1_ν. Then

$$\|F\|_\nu := \|A(*S)\|_\nu := \|\sum_{k=0}^\infty a_k S^{*k}\|_\nu \leq \sum_{k=0}^\infty a_k \|S\|_\nu^k \leq \sum_{k=0}^\infty a_k \rho^k < \infty \tag{4.192}$$

thus $A(*S) \in L^1_\nu$. Similarly, $A(*S)$ is in $L^1_\nu([0,a))$, in $\mathcal{A}_{K,\nu}\nu([0,a))$ for any a.
∎

4.8b Borel summation of the transseries solution

If we substitute

$$y = \tilde{y}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{y}_k \tag{4.193}$$

in (4.185) and equate the coefficients of e^{-kx} we get the system of equations

$$\tilde{y}_k' + (1 - k - 3\tilde{y}_0^2)\tilde{y}_k = 3\tilde{y}_0^2 \sum_{j=1}^{k-1} \tilde{y}_j \tilde{y}_{k-j} + \sum_{j_1+j_2+j_3=k; j_i \geq 1} \tilde{y}_{j_1} \tilde{y}_{j_2} \tilde{y}_{j_3} \tag{4.194}$$

The equation for y_1 is special, being linear homogeneous.

$$y_1' = 3y_0^2 y_1 \tag{4.195}$$

and thus

$$y_1 = Ce^s; \quad s := \int_{\infty}^{x} 3y_0^2(s)ds \tag{4.196}$$

Since $\tilde{s} = O(x^{-3})$ and it is the product of series which are Borel summable in $\mathbb{C} \setminus \mathbb{R}^+$, $\tilde{s}$ and then, by Proposition 4.191 $e^{\tilde{s}}$ is Borel summable in $\mathbb{C} \setminus \mathbb{R}^+$. We note that $y_1 = 1 + o(1)$ and we cannot take the inverse Laplace transform of y_1 directly. It is convenient to make the substitution $y_k = x^k \varphi_k$ and we get

$$\tilde{\varphi}_k' + (1 - k - 3\tilde{\varphi}_0^2 + kx^{-1})\tilde{\varphi}_k = 3\tilde{\varphi}_0^2 \sum_{j=1}^{k-1} \tilde{\varphi}_j \tilde{\varphi}_{k-j} + \sum_{j_1+j_2+j_3=k; j_i \geq 1} \tilde{\varphi}_{j_1} \tilde{\varphi}_{j_2} \tilde{\varphi}_{j_3} \tag{4.197}$$

or after Borel transform

$$-p\mathbf{\Phi} + (1 - \hat{k})\mathbf{\Phi} = -\hat{k} * \mathbf{\Phi} + 3Y_0^{*2} * \mathbf{\Phi} + 3Y_0 * \mathbf{\Phi} \mathbin{\overset{*}{*}} \mathbf{\Phi} + \mathbf{\Phi} \mathbin{\overset{*}{*}} \mathbf{\Phi} \mathbin{\overset{*}{*}} \mathbf{\Phi} \tag{4.198}$$

where $\mathbf{\Phi} = \{\Phi_j\}_{j \in \mathbb{N}}$, $(\hat{k}\mathbf{\Phi})_k = \Phi_k$ and $(F * \mathbf{G})_k := F * G_k$.

Proposition 4.199 *Given $\Phi_1 \in L^1_\nu$, (4.198) is contractive in the space of sequences $\{\Phi_j\}_{j \geq 2}$ in the norm of $L^1_{\nu,\mu}$ for any μ if ν is large enough. Thus (4.198) has a unique solution in this space. Similarly, it has a unique solution in $L^1_{\nu,\mu}(S), L^1_{\nu,\mu}(S), \mathcal{A}_{\nu,\mu}(S_K)$ for any S and S_K as in Proposition 4.187. Thus there is a ν large enough such that for all k*

$$\varphi_k(x) = \int_0^{\infty e^{-i \arg(x)}} e^{-xp} \Phi_k(p) dp \tag{4.200}$$

exist for $|x| > \nu$. The functions $\varphi_k(x) = \varphi_k(x)^+$ are analytic in x and independent of $\arg x$ for $\arg(x) \in (-\pi/2, 2\pi + \pi/2)$. Similarly, $\varphi_k(x) = \varphi_k(x)^-$ are analytic in x and independent of $\arg x$ for $\arg(x) \in (-2\pi - \pi/2, \pi/2)$.

(ii) The function series

$$\sum_{k=0}^{\infty} C_+^k e^{-kx} x^k \varphi_k^+(x) \tag{4.201}$$

or

$$\sum_{k=0}^{\infty} C_-^k e^{-kx} x^k \varphi_k^-(x) \tag{4.202}$$

solve (4.185) for $\Re(x) > \nu(C_\pm)$, $\arg(x) \in (-\pi/2, \pi/2)$.

Note. The solution cannot be written in the form (4.201) or (4.202) in a sector of opening more than π centered on $\mathbb{R}^+$ because the exponentials would become large and convergence is not ensured anymore. We shall see that, generically, there is in fact blow-up of the actual solutions.

Exercise 4.203 ** Show that distinct values of* C_+ *in (4.201) give rise to distinct solutions. We will see shortly that all* $o(1)$ *slutions are of this form.*

Exercise 4.204 *** Prove Proposition 4.199.*

Proposition 4.205 Any *Solution of (4.185) which is* $o(1)$ *as* $x \to +\infty$ *can be written in the form (4.201) or, equally well, in the form (4.201).*

PROOF Let y^+ be the solution of (4.185) of the form (4.201) with $C = 0$. Let y be another solution which is $o(1)$ as $x \to +\infty$ and let $\delta = y - y^+$. We have

$$\delta' = -\delta + 3y_0^2\delta + 3y_0\delta^2 + \delta^3 \tag{4.206}$$

or

$$\frac{\delta'}{\delta} = -1 + 3y_0^2 + 3y_0\delta + \delta^2 = -1 + o(1) \tag{4.207}$$

Thus (since we can integrate asymptotic series),

$$\ln\delta = -x + C_1 + o(1) \tag{4.208}$$

or

$$\ln\delta = Ce^{-x}(1 + o(1)) \tag{4.209}$$

We then take $\delta = e^{-x+s}$ and obtain

$$s' = 3y_0^2 + 3y_0 e^{-w+s} + e^{-2w+2s} \tag{4.210}$$

or

$$s = C_1 + \int_\infty^x \left(3y_0^2(t) + 3y_0 e^{-x+s(t)} + e^{-2t+2s(t)}\right) dt \tag{4.211}$$

For fixed C_1, (4.211) is contractive in the space of functions $s : [\nu, \infty) \mapsto \mathbb{C}$ in the sup norm. The solution of this equation is then unique. But $s = \ln(y^+ - y_C) + x$ where y_C is the solution of the form (4.201) with $C = \ln C_1$ is already a solution of (4.211). ∎

4.8c Analytic structure along $\mathbb{R}^+$

By Proposition 4.187, $Y = Y_0$ is analytic in any region of the form $B \cup S_K$. We now aim to show that Y_0 has analytic continuation along curves that do not pass through the integers.

For this purpose we shall exploit (4.202) and (4.201). From them we will derive the behavior of Y. It is a way of exploiting what Écalle has discovered in more generality, *bridge equations*.

We start with a relatively trivial, and in fact nongeneric case.

Assume $y_+ = y_- =: y_0$. Since

$$y_\pm = \int_0^{\infty e^{\pm i\epsilon}} Y(p)e^{-px}dp = y_0 \tag{4.212}$$

and Y is analytic in $\mathbb{C} \setminus \mathbb{R}^+$, we have, by the usual contour deformations and Watson's Lemma that $y_0 \sim \tilde{y_0}$ in the sector $S_x\{x : \arg(x) \in (-2\pi - \pi/2, 2\pi + \pi/2)\}$.

Exercise 4.213 ** If we take the sector $S = \{p : |\arg(p)| > \epsilon\}$ we know that for some ν, $Y \in L^1_\nu(S)$. We also know that $Y(0) = 0$ and is analytic near zero. Show that this implies that there is a constant C and a ν so that for $|x| > \nu$ y_0 is analytic in S_x and*

$$|y_0(x)| < C|x|^{-2} \quad \forall x \in S_x, \quad |x| > \nu \tag{4.214}$$

Thus, y_0 is inverse Laplace transformable along any line of the form $e^{i\phi}(c + it), t \in \mathbb{R}, c > \nu$. We let this inverse Laplace transform to be Y_1; it is bounded by $Ke^{\nu|p|}$, for some K independent of $\arg p$ (by immediate estimates). In particular it belongs to $L^1_\nu(\mathbb{R}e^{i\phi})$ for all $\phi \in (-2\pi, 2\pi)$. But it must coincide with Y for any $\phi \neq 0$ by uniqueness of the solution in $L^1_{\nu'}$ for ν' large. We thus get the estimate inherited from Y_1

$$|Y(p)| \le Ke^{\nu|p|}, \quad \forall p, \arg(p) \neq 0 \tag{4.215}$$

Thus Y is uniformly bounded in $\mathbb{C} \setminus \mathbb{R}^+$. Consequently, $J(p) = \int_0^p Y(s)ds$ defined on $\mathbb{C} \setminus \mathbb{R}^+$ has continuous limits as $J_\pm(p)$ as $\mathbb{R}^+$ is approached from above or from below and $|J_\pm(p)| \le |p|e^{\nu|p|}$.

On the other hand, we have for $|x| > \nu$, by dominated convergence,

$$y_0 = \int_0^{\infty e^{\pm i\epsilon}} e^{-px}Y(p)dp = \int_0^{\infty e^{\pm i\epsilon}} e^{-px}\frac{dJ(p)}{dp}dp = x\int_0^\infty J_\pm(p)dp \tag{4.216}$$

Since the Laplace transforms of $J_\pm(p)$ coincide and the kernel of the Laplace transform is $\{0\}$, then $J_+ = J^-$, thus J is continuous, and then by Morera's theorem it in analytic in $\mathbb{C}$. So Y is entire. In this case, y_0 is analytic at infinity and the series $\tilde{y}$ converges.

We now consider the (generic) case when $y_+ \neq y^-$. Then there exists S such that

$$y^+ = \int_0^{\infty e^{i\epsilon}} e^{-px} Y(p) dp = y^- + \sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^-(x) \tag{4.217}$$

Thus

$$\int_{\infty e^{-i\epsilon}}^{\infty e^{+i\epsilon}} e^{-px} Y(p) dp = -\sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^-(x) \tag{4.218}$$

In particular, we have

$$\frac{1}{x} \int_{\infty e^{-i\epsilon}}^{\infty e^{+i\epsilon}} e^{-px} Y(p) dp = -S \int_0^{\infty e^{i\epsilon}} Y_1(p) dp + O(x^2 e^{-2x}) \tag{4.219}$$

4.8d General setting

We consider the differential system

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n \tag{4.220}$$

under the following *assumptions:*

(a1) The function $\mathbf{f}$ is analytic at $(\infty, 0)$.

(a2) A condition slightly weaker than nonresonnance: for any half plane H in $\mathbb{C}$, the eigenvalues λ_i of the linearization

$$\hat{\Lambda} := -\left(\frac{\partial f_i}{\partial y_j}(\infty, 0) \right)_{i,j=1,2,\ldots n} \tag{4.221}$$

lying in H are linearly independent over $\mathbb{Z}$. In particular, all eigenvalues are distinct and none of them is zero.

Pulling out the inhomogeneous and the linear terms (relevant to leading order asymptotics) we get

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x, \mathbf{y}) \tag{4.222}$$

Then matrix $\hat{\Lambda}$ can be diagonalized by a linear change of the dependent variable $\mathbf{y}$. It can be checked that by a further substitution of the form $\mathbf{y}_1 = (I + x^{-1}\hat{V})\mathbf{y}$, the new matrix $\hat{A}$ can be arranged to be diagonal. No assumptions on $\hat{A}$ are needed in this second step. See also [17], [28]. Thus, without loss of generality we can suppose that the system is already presented in *prepared* form, meaning:

(n1) $\hat{\Lambda} = \text{diag}(\lambda_i)$ and (n2) $\hat{B} = \text{diag}(\beta_i)$

For convenience, we rescale x and reorder the components of $\mathbf{y}$ so that

(n3) $\lambda_1 = 1$, and, with $\phi_i = \arg(\lambda_i)$, we have $\phi_i \le \phi_j$ if $i < j$. To simplify notations, we formulate some of our results relative to λ_1; they can be easily adapted to any other eigenvalue.

To unify the treatment we make, by taking $\mathbf{y} = \mathbf{y}_1 x^{-N}$ for some $N > 0$,

(n4) $\Re(\beta_j) < 0,\ j = 1, 2, \ldots, n.$

(there is an asymmetry at this point: the opposite inequality cannot be achieved, in general, as simply and without violating analyticity at infinity). Finally, through a transformation of the form $\mathbf{y} \leftrightarrow \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ we arrange that

(n5) $\mathbf{f}_0 = O(x^{-M-1})$ and $\mathbf{g}(x, \mathbf{y}) = O(\mathbf{y}^2, x^{-M-1}\mathbf{y})$. We choose $M > 1 + \max_i \Re(-\beta_i)$.

Formal solutions. In prepared form, given (a1) and (a2), (4.222) admits an n–parameter family of generalized transseries solutions (check!)

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k} \ge 0; |\mathbf{k}| > 0} C_1^{k_1} \cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k} \cdot \boldsymbol{\lambda}) x} x^{\mathbf{k} \cdot \mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}} \tag{4.223}$$

We note that we allow for complex valued transseries, though we restrict ourselves to the case when the real part is negative. In this sense we speak of them as generalized transseries. In our context we will not need to combine them in a way in which we end up with purely imaginary exponentts, and so the ordering with respect to $\ll$ is still preserved.

These formal solutions have been known, under the name of formal exponential-series solutions, for almost a century now) see [17], [26],[30], and also § Ac below) where $m_i = 1 - \lfloor \beta_i \rfloor$, ($\lfloor \cdot \rfloor$ = integer part), $\mathbf{C} \in \mathbb{C}^n$ is an arbitrary vector of constants, and $\tilde{\mathbf{y}}_{\mathbf{k}} = x^{-\mathbf{k}(\boldsymbol{\beta}+\mathbf{m})} \sum_{l=0}^{\infty} \mathbf{a}_{\mathbf{k};l} x^{-l}$ are formal power series.

When x is large in some direction d in $\mathbb{C}$, an important role is played by the subset of transseries which are at the same time *asymptotic* expressions[4]: When there are infinitely many exponentials in (4.223) we ask that for all i with $C_i \ne 0$ we have $|\mathrm{e}^{-\lambda_i x}| \ll 1$ for large x in the given direction d in $\mathbb{C}$. Formally, agreeing to omit the terms with $C_i = 0$, with x in d, any *ascending* chain $\Re(-\mathbf{k}_1 \cdot \boldsymbol{\lambda} x) \le \Re(-\mathbf{k}_2 \cdot \boldsymbol{\lambda} x) \le \ldots$, $\mathbf{k}_i \ne \mathbf{k}_j$, in (4.223) must be *finite* (the terms of an asymptotic transseries are well-ordered with respect to $'' \ll''$). Thus for x in some direction d we only consider those transseries that satisfy the condition:

(c1) $\xi + \phi_i := \arg(x) + \phi_i \in (-\pi/2, \pi/2)$ for all i such that $C_i \ne 0$. In other words, $C_i \ne 0$ implies that λ_i lies in a half-plane centered on $\overline{d}$, the complex conjugate direction to d.

[4]An asymptotic expansion of a function carries immediate information about behavior of the function near the expansion point (in contrast to antiasymptotic expansions, e.g. a convergent doubly infinite Laurent series)

From now on, $\boldsymbol{\lambda} = (\lambda_{i_1}, \ldots, \lambda_{i_{n_1}})$, $\boldsymbol{\beta} = (\beta_{i_1}, \ldots, \beta_{i_{n_1}})$, $\mathbf{m} = (m_{i_1}, \ldots, m_{i_{n_1}})$ and $\boldsymbol{\beta}' = \boldsymbol{\beta} + \mathbf{m}$ where the indices $i_1, \ldots, i_{n_1}$ satisfy (c1).

We will henceforth consider that (4.222) is presented in prepared form, and use the designation transseries only for those formal solutions satisfying (c1).

The series $\tilde{\mathbf{y}}_0$ is a formal solution of (4.222) while, for $\mathbf{k} \neq 0$, $\tilde{\mathbf{y}}_{\mathbf{k}}$ satisfy a hierarchy of linear differential equations [17] (see also § Ac for a brief exposition and notations).

$$\mathbf{y} \rightleftharpoons \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0; |\mathbf{k}|>0} C_1^{k_1} \cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}} \quad (x \to \infty, \arg(x) = \xi) \tag{4.224}$$

Given $\mathbf{y}$, the value of C_i can change only when $\xi + \arg(\lambda_i - \mathbf{k}\cdot\boldsymbol{\lambda}) = 0$, $k_i \in \mathbb{N} \cup \{0\}$, i.e. when crossing one of the (finitely many by (c1)) Stokes lines. The correspondence (4.224) defines a summation method, in the sense that it is an extension of convergent summation which preserves its basic properties: linearity, multiplicativity, commutation with differentiation and with complex conjugation. These properties are essential for obtaining true solutions out of transseries for nonlinear differential equations. Our procedure is similar to the medianization proposed by Écalle, but (due to the structure of (4.222)) requires substantially fewer analytic continuation paths. In addition we classify in the context of (4.222) all admissible summation methods (there is a one-parameter family of them, preserving the properties of usual summation). Summation recovers from transseries actual solutions of (4.222) without resorting to (4.222) in the process. In addition, the analysis reveals a rich analytic structure and formulas linking the various $\tilde{\mathbf{y}}_{\mathbf{k}}$ among themselves (resurgence relations). In [21] we studied this problem under further restrictions on the transseries (decay of the exponentials in a full half-plane) and on the differential equation. Removing those restrictions creates difficulties that required a new approach. New resurgence relations are found and in addition we provide a complete description, needed in applications, of the singularity structure of the Borel transforms of $\tilde{\mathbf{y}}_{\mathbf{k}}$.

The results proven for this type of equations may be, informally, summarized as follows. The proofs are given in §5.

i) All $\tilde{\mathbf{y}}_{\mathbf{k}}$ are generalized Borel summable at the same time.

ii) The Borel summed series $\mathbf{y}_{\mathbf{k}} = \mathfrak{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ exist in a half plane $H = \{x : \Re(x) > x_0\}$ for some x_0 independent of $\mathbf{k}$ and are analytic there.

iii) There exists a constant $\mathbf{c}$ independent of $\mathbf{k}$ so that $\sup_{x\in H} |\mathbf{y}_{\mathbf{k}}| \leq \mathbf{c}^{\mathbf{k}}$. Thus, the new series,

$$\mathbf{y} = \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x) \tag{4.225}$$

is convergent for any $\mathbf{C}$ *for which the corresponding expansion (4.223) is a transseries, in a region given by the condition* $|C_i e^{-\lambda_i x} x^{\alpha_i}| < c_i^{-1}$ *(remember that* C_i *is zero if* $|e^{-\lambda_i x}|$ *is not small).*

iv) The function $\mathbf{y}$ *obtained in this way is a solution of the differential equation (4.222).*

v) Any solution of the differential equation (4.222) which tends to zero in some direction d *can be written in the form (4.225) for a unique* $\mathbf{C}$*, this constant depending usually on the sector where* d *is. This dependence is a manifestation of the Stokes phenomenon.*

vi) The Borel summation operator $\mathfrak{B}$ *is the usual Borel summation in any direction* d *of* x *which is not a Stokes line . However* $\mathfrak{B}$ *is still an isomorphism, whether* d *is a Stokes direction or not.*

4.8e Some remarks about structure of singularities in Borel space and resurgence phenomena

Let us look at a very simple prototypical example

$$y'' + (2 + x^{-1})y' - (3 + x^{-1})y = x^{-1}y^2$$

We take $y_1 = y, y_2 = y'$ and get a system of equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}$$

Diagonalization of the two 2-by-2 matrices on the right hand side is achieved easily my making a transformation of the dependent variable of the form $\mathbf{y} \mapsto (\hat{M}_1 + x^{-1}\hat{M})\mathbf{y}$ for suitably chosen $\hat{M}_i$ and the system that results is of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1(x^{-1}, y_1, y_2) \\ g_2(x^{-1}, y_1, y_2) \end{pmatrix}$$

satisfying our assumptions. In this particular example, the eigenvalues, though not linearly independent over $\mathbb{Z}$ still satisfy the weaker conditions in [20] and the general theory applies. If the direction of interest for the variable x is $\mathbb{R}^+$, then the only admissible exponential is e^{-x} as e^{2x} tends to infinity instead of being small. Thus there is in the direction of $\mathbb{R}^+$ a one-parameter only family of transseries, in the form

$$\sum_{k=0}^{n} C^k e^{-kx} x^{k\alpha} \tilde{\mathbf{y}}_k(x)$$

4.8e .1 Analytic continuations

The series $\tilde{\mathbf{y}}_k$ will be classically Borel summable in any direction other than $\mathbb{R}^+$ and $\mathbb{R}^-$. It turns out that along any Stokes line , here $\mathbb{R}^+$ and $\mathbb{R}^-$, the Borel transforms $Y_k = \mathcal{B}\tilde{\mathbf{y}}_k$ develop arrays of singularities. These singularities are located at positive multiple integers of 1, and -2. It is proved that the functions $\mathbf{Y}_k$ can be continued analytically along any paths in the complex plane that go towards infinity (the modulus of p increases along the path) and cross between the singular points in the arrays at most once. Borel summability along the special directions of the singularities is ensured both in a sense of distributions, in which generalized Laplace transform is taken through the singular points, or, equivalently, as a specific average of analytic continuations along the paths mentioned above. The averaging formula is the same, irrespective of the differential equation.

4.8e .2 Resurgence

This is another very important phenomenon that occurs in differential systems, in which the higher index series $\tilde{\mathbf{y}}_\mathbf{k}$ are related to $\tilde{\mathbf{y}}_\mathbf{0}$ in a way that does not depend on the differential equation and permits reconstruction of the $\tilde{\mathbf{y}}_\mathbf{k}$, thus of the general formal solution and ultimately of the whole differential equation from the mere knowledge of $\tilde{\mathbf{y}}_\mathbf{0}$. For instance under proper normalization, the $\mathbf{Y}_\mathbf{k}$ are related to differences in the analytic continuations of $\mathbf{Y}_0$ along the various paths between singularities.

4.9 Normalization procedures

Many equations which are not presented in the form (4.222) can be brought to this form by changes of variables. The key idea for so doing in a systematic way is to calculate the transseries solutions of the equation, find the transformations which bring them to the normal form (4.223), and then apply these transformations to the original variables in the differential equation. The first part of the analysis need not be rigorous, as the conclusions are made rigorous in the steps that follow it.

We illustrate this on a simple equation, as $t \to \infty$:

$$u' = u^3 - t \tag{4.226}$$

This is not of the form (4.222) because $g(u,t) = u^3 - t$ is not analytic in t *at* $t = \infty$. This can be however remedied in the way described.

As we have already seen before, dominant balance for large t requires writing the equation (4.226) in the form

$$u = (t + u')^{1/3} \tag{4.227}$$

and we have $u' \ll t$. Three branches of the cubic root are possible and are investigated similarly, but we aim here merely at illustration and choose the simplest. Iterating (4.227) in the usual way, we are lead to a formal series solution in the form

$$\tilde{u} = t^{1/3} + \frac{1}{9}t^{-4/3} + \ldots = t^{1/3}\sum_{k=0}^{\infty}\frac{\tilde{u}_k}{t^{5k/3}} \tag{4.228}$$

To find the full transseries we now substitute $u = \tilde{u} + \delta$ in (4.226) and keep the dominant terms. We get

$$\frac{\delta'}{\delta} = \frac{9}{5}t^{2/3} + \frac{2}{3}\ln t$$

from which it follows that

$$\delta = Ct^{2/3}e^{\frac{9}{5}t^{5/3}} \tag{4.229}$$

Since the normalized transseries must have exponentials of the form e^{-x}, the adequate independent variable must then be $x = -\frac{9}{5}t^{5/3}$. In this variable, the formal power series (4.228) takes the form

$$\tilde{u} = x^{1/5}\sum_{k=0}^{\infty}\frac{\tilde{u}_k}{x^k} \tag{4.230}$$

But the desired form is $\sum_{k=0}^{\infty}\frac{b_k}{x^k}$. Thus the appropriate dependent variable is $h = x^{1/5}u$. In this variable, we are led to the equation

$$h' + \frac{1}{5x}h + 3h^3 - \frac{1}{9} = 0 \tag{4.231}$$

where analyticity at infinity is now ensured! The only remaining transformation is to pull out a few terms out of h, to make the nonlinearity of the order $g = O(x^{-2}, h^2)$. This is done by calculating, again by dominant balance, the first two terms in the $1/x$ power expansion of h, namely $1/3 - x^{-1}/15$ and subtracting them out of h, i.e., changing to the new dependent variable $y = h - 1/3 + x^{-1}/15$. This yields

$$y' = -y + \frac{1}{5x}y + g(y, x^{-1}) \tag{4.232}$$

where

$$g(y, x^{-1}) = -3(y^2 + y^3) + \frac{3y^2}{5x} - \frac{1}{15x^2} - \frac{y}{25x^2} + \frac{1}{3^2 5^3 x^3} \tag{4.233}$$

We see that

$$\lambda = 1, \quad \alpha = 1/5 \tag{4.234}$$

Chapter 5

Ordinary differential equations. Summability of formal solutions

5.0a Nonresonance

(1) λ_i, $i = 1, ..., n_1$ are assumed $\mathbb{Z}$-linearly independent for any d. (2) Let $\theta \in [0, 2\pi)$ and $\tilde{\boldsymbol{\lambda}} = (\lambda_{i_1}, ..., \lambda_{i_p})$ where $\left|\arg \lambda_{i_j} - \theta\right| \in (-\pi/2, \pi/2)$ (those eigenvalues contained in the open half-plane H_θ centered along $e^{i\theta}$). We require that for any θ the complex numbers in the set $\{\tilde{\lambda}_i - \mathbf{k} \cdot \tilde{\boldsymbol{\lambda}} \in H_\theta : \mathbf{k} \in \mathbb{N}^p, \, i = 1, ..., p\}$ (note: the set is *finite*) have *distinct* directions. These are the Stokes lines $d_{i;\mathbf{k}}$.

That the set of $\boldsymbol{\lambda}$ which satisfy (1) and (2) has full measure follows from the fact that (1) and (2) follow from the condition:

$$\Big(\mathbf{m}, \mathbf{m}' \in \mathbb{Z}^n, \;\alpha \in \mathbb{R} \text{ and } (\mathbf{m} - \alpha \mathbf{m}') \cdot \boldsymbol{\lambda} = 0\Big) \;\Rightarrow \Big(\mathbf{m} = \alpha \mathbf{m}'\Big) \tag{5.1}$$

Indeed, if (5.1) fails, one of $\Re\lambda_j, \Im\lambda_j$ is a rational function with rational coefficients of the other $\Re\lambda_j$ and $\Im\lambda_j$, corresponding to a zero measure set in $\mathbb{R}^{2n}$.

5.0b Further notations and conventions

If y_1 and y_2 are inverse Laplace transformable functions, then in a neighborhood of the origin $\mathcal{L}^{-1}(y_1 y_2) = (\mathcal{L}^{-1} y_1) * (\mathcal{L}^{-1} y_2)$, where for $f, g \in L^1$ convolution is given by

$$f * g := p \mapsto \int_0^p f(s) g(p - s) \mathrm{d}s \tag{5.2}$$

We use the convention $\mathbb{N} \ni 0$. Let

$$\mathcal{W} = \{p \in \mathbb{C} : p \neq k\lambda_i \,, \forall k \in \mathbb{N}, i = 1, 2, \ldots, n\} \tag{5.3}$$

The directions $d_j = \{p : \arg(p) = \phi_j\}, j = 1, 2, \ldots, n$ (cf. (a2)) are the *Stokes lines* of $\tilde{\mathbf{y}}_0$ (note: sometimes known as *anti*-Stokes lines!). We construct over $\mathcal{W}$ a surface $\mathcal{R}$, consisting of homotopy classes of smooth curves in $\mathcal{W}$

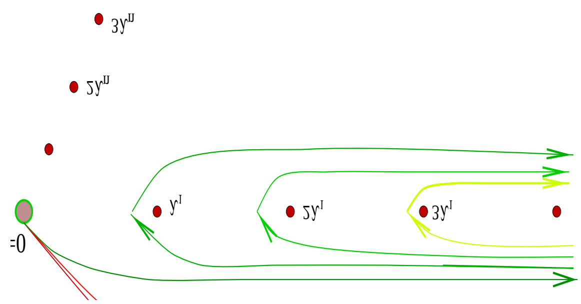


FIGURE 5.1: Constant level lines for the imaginary part of $i\cos(x+iy)$

starting at the origin, moving away from it, and crossing at most one Stokes line, at most once (see Fig. 1):

$$\mathcal{R} := \left\{ \gamma : (0,1) \mapsto \mathcal{W} : \ \gamma(0_+) = 0; \ \frac{\mathrm{d}}{\mathrm{d}t}|\gamma(t)| > 0; \ \arg(\gamma(t)) \text{ monotonic} \right\} \text{modulo homotopies} \tag{5.4}$$

Define $\mathcal{R}_1 \subset \mathcal{R}$ by (5.4) with the supplementary restriction $\arg(\gamma) \in (\psi_n - 2\pi, \psi_2)$ where $\psi_n = \max\{-\pi/2, \phi_n - 2\pi\}$ and $\psi_2 = \min\{\pi/2, \phi_2\}$. $\mathcal{R}_1$ may be viewed as the part of the covering $\mathcal{R}$, above a sector containing the real axis. Similarly we let $\mathcal{R}'_1 \subset \mathcal{R}_1$ with the restriction that the curves γ do not cross the Stokes line s $d_{i,\mathbf{k}}$ (cf. §5.0a), other than $\mathbb{R}^+$, and we let $\psi_\pm = \pm\max(\pm\arg\gamma)$ with $\gamma \in \mathcal{R}'_1$.

Fig 1. *The paths near λ_2 belong to $\mathcal{R}$.*
The paths near λ_1 relate to the balanced average

By $AC_\gamma(f)$ we denote the analytic continuation of f along a curve γ. For the analytic continuations near a Stokes line $d_{i;\mathbf{k}}$ we use symbols similar to Écalle's: f^- is the branch of f along a path γ with $\arg(\gamma) < \phi_i$, while f^{-j+} denotes the branch along a path that crosses the Stokes line between $j\lambda_i$ and $(j+1)\lambda_i$ (see also [21]).

We use the notations $\mathcal{P}f$ for $\int_0^p f(s)\mathrm{d}s$ and $\mathcal{P}_\gamma f$ if integration is along the curve γ.

We write $\mathbf{k} \succeq \mathbf{k}'$ if $k_i \geq k_i'$ for all i and $\mathbf{k} \succ \mathbf{k}'$ if $\mathbf{k} \succeq \mathbf{k}'$ and $\mathbf{k} \neq \mathbf{k}'$. The relation $\succ$ is a well ordering on $\mathbb{N}^{n_1}$. We let $\mathbf{e}_j$ be the unit vector in the j^{th} direction in $\mathbb{N}^{n_1}$.

By symmetry (renumbering the directions) it suffices to analyze the singularity structure of $\mathbf{Y}_0$ in $\mathcal{R}_1$ only. However, (c1) breaks this symmetry for $\mathbf{k} \neq 0$ and the properties of these $\mathbf{Y}_\mathbf{k}$ will be analyzed along some other directions as well.

χ_A will denote the characteristic function of the set A. We write $|\mathbf{f}| := \max_i\{|f_i|\}$. We have

$$\mathbf{g}(x,\mathbf{y}) = \sum_{|\mathbf{l}|\geq 1} \mathbf{g}_\mathbf{l}(x)\mathbf{y}^\mathbf{l} = \sum_{s\geq 0;|\mathbf{l}|\geq 1} \mathbf{g}_{s,\mathbf{l}} x^{-s}\mathbf{y}^\mathbf{l} \quad (|x| > x_0, |\mathbf{y}| < y_0) \tag{5.5}$$

where $\mathbf{y}^\mathbf{l} = y_1^{l_1}\cdots y_n^{l_n}$ and $|\mathbf{l}| = l_1 + \cdots + l_n$. By construction $\mathbf{g}_{s,\mathbf{l}} = 0$ if $|\mathbf{l}| = 1$ and $s \leq M$.

The formal inverse Laplace transform of $\mathbf{g}(x, \mathbf{y}(x))$ (formal since $\mathbf{y}$ is still unrestricted) is given by:

$$\mathcal{L}^{-1}\left(\sum_{|\mathbf{l}|\geq \mathbf{1}} \mathbf{y}(x)^\mathbf{l} \sum_{s\geq 0} \mathbf{g}_{s,\mathbf{l}} x^{-s}\right) = \sum_{|\mathbf{l}|\geq \mathbf{1}} \mathbf{G}_\mathbf{l} * \mathbf{Y}^{*\mathbf{l}} + \sum_{|\mathbf{l}|\geq \mathbf{2}} \mathbf{g}_{0,\mathbf{l}} \mathbf{Y}^{*\mathbf{l}} =: \mathcal{N}(\mathbf{Y}) \tag{5.6}$$

with $\mathbf{G}_\mathbf{l}(p) = \sum_{s=1}^\infty \mathbf{g}_{s,\mathbf{l}} p^{s-1}/s!$ and $(\mathbf{G}_\mathbf{l} * \mathbf{Y}^{*\mathbf{l}})_j := (\mathbf{G}_\mathbf{l})_j * Y_1^{*l_1} * .. * Y_n^{*l_n}$. By (n5), $\mathbf{G}_{1,\mathbf{l}}^{(l)}(0) = 0$ if $|\mathbf{l}| = 1$ and $l \leq M$. The inverse Laplace transform of (4.222) is the convolution equation:

$$-p\mathbf{Y} = \mathbf{F}_0 - \hat{\Lambda}\mathbf{Y} - \hat{B}\mathcal{P}\mathbf{Y} + \mathcal{N}(\mathbf{Y}) \tag{5.7}$$

Let $\mathbf{d}_\mathbf{j}(x) := \sum_{\mathbf{l}\geq\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \mathbf{g}_\mathbf{l}(x)\tilde{\mathbf{y}}_0^{\mathbf{l}-\mathbf{j}}$. Straightforward calculation (see Appendix § Ac ; cf. also [21]) shows that the components $\tilde{\mathbf{y}}_\mathbf{k}$ of the transseries satisfy the hierarchy of differential equations

$$\mathbf{y}_\mathbf{k}' + \left(\hat{\Lambda} + \frac{1}{x}\left(\hat{B} + \mathbf{k}\cdot\mathbf{m}\right) - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{y}_k + \sum_{|\mathbf{j}|=1} \mathbf{d}_\mathbf{j}(x)(\mathbf{y}_\mathbf{k})^\mathbf{j} = \mathbf{t}_\mathbf{k} \tag{5.8}$$

where $\mathbf{t}_\mathbf{k} = \mathbf{t}_\mathbf{k}\left(\mathbf{y}_0, \{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right)$ is a *polynomial* in $\{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}$ and in $\{\mathbf{d}_\mathbf{j}\}_{\mathbf{j}\leq\mathbf{k}}$ (see (5.225)), with $\mathbf{t}(\mathbf{y}_0, \emptyset) = 0$; $\mathbf{t}_\mathbf{k}$ satisfies the homogeneity relation

$$\mathbf{t_k}\left(\mathbf{y}_0,\left\{C^{\mathbf{k}'}\mathbf{y}_{\mathbf{k}'}\right\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right)=C^{\mathbf{k}}\mathbf{t_k}\left(\mathbf{y}_0,\{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) \tag{5.9}$$

Taking $\mathcal{L}^{-1}$ in (5.8) we get, with $\mathbf{D_j}=\sum_{\mathbf{l}\ge\mathbf{j}}\binom{\mathbf{l}}{\mathbf{j}}\left[\mathbf{G_l}*\mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})}+\mathbf{g}_{0,\mathbf{l}}*\mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})}\right]$,

$$\left(-p+\hat{\Lambda}-\mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y_k}+\left(\hat{B}+\mathbf{k}\cdot\mathbf{m}\right)\mathcal{P}\mathbf{Y_k}+\sum_{|\mathbf{j}|=1}\mathbf{D_j}*\mathbf{Y}_{\mathbf{k}}^{*\mathbf{j}}=\mathbf{T_k} \tag{5.10}$$

where $\mathbf{T_k}$ is now a *convolution* polynomial, cf. (5.144).

5.0c Main results

(a) *Analytic structure.*

Theorem 5.11 *(i)* $\mathbf{Y}_0=\mathcal{B}\tilde{\mathbf{y}}_0$ *is analytic in* $\mathcal{R}\cup\{0\}$.

The singularities of $\mathbf{Y}_0$ *(which are contained in the set* $\{l\lambda_j: l\in\mathbb{N}^+, j=1,2,\dots,n\}$*) are described as follows. For* $l\in\mathbb{N}^+$ *and small* z*, using the notations explained in §5.0b ,*

$$\mathbf{Y}_0^{\pm}(z+l\lambda_j)=\pm\Big[(\pm S_j)^l(\ln z)^{0,1}\mathbf{Y}_{l\mathbf{e}_j}(z)\Big]^{(lm_j)}+\mathbf{B}_{lj}(z)=$$
$$\Big[z^{l\beta_j'-1}(\ln z)^{0,1}\,\mathbf{A}_{lj}(z)\Big]^{(lm_j)}+\mathbf{B}_{lj}(z)\ \ (l=1,2,\dots) \tag{5.12}$$

where the power of $\ln z$ *is one iff* $l\beta_j\in\mathbb{Z}$*, and* $\mathbf{A}_{lj},\mathbf{B}_{lj}$ *are analytic for small* z*. The functions* $\mathbf{Y_k}$ *are, exceptionally, analytic at* $p=l\lambda_j$*,* $l\in\mathbb{N}^+$*, iff,*

$$S_j=r_j\Gamma(\beta_j')\left(\mathbf{A}_{1,j}\right)_j(0)=0 \tag{5.13}$$

where $r_j=1-\mathrm{e}^{2\pi i(\beta_j'-1)}$ *if* $l\beta_j\notin\mathbb{Z}$ *and* $r_j=-2\pi i$ *otherwise. The* S_j *are Stokes constants, see Theorem 5.26.*

(ii) $\mathbf{Y_k}=\mathcal{B}\tilde{\mathbf{y}}_\mathbf{k}$*,* $|\mathbf{k}|>1$*, are analytic in* $\mathcal{R}\backslash\{-\mathbf{k}'\cdot\boldsymbol{\lambda}+\lambda_i:\mathbf{k}'\le\mathbf{k}, 1\le i\le n\}$*. For* $l\in\mathbb{N}$ *and* p *near* $l\lambda_j$*,* $j=1,2,\dots,n$ *there exist* $\mathbf{A}=\mathbf{A}_{\mathbf{k}jl}$ *and* $\mathbf{B}=\mathbf{B}_{\mathbf{k}jl}$ *analytic at zero so that (*z *is as above)*

$$\mathbf{Y}_\mathbf{k}^{\pm}(z+l\lambda_j)=\pm\Big[(\pm S_j)^l\binom{k_j+l}{l}(\ln z)^{0,1}\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(z)\Big]^{(lm_j)}+\mathbf{B}_{\mathbf{k}lj}(z)=$$
$$\Big[z^{\mathbf{k}\cdot\boldsymbol{\beta}'+l\beta_j'-1}(\ln z)^{0,1}\,\mathbf{A}_{\mathbf{k}lj}(z)\Big]^{(lm_j)}+\mathbf{B}_{\mathbf{k}lj}(z)\ \ (l=0,1,2,\dots) \tag{5.14}$$

where the power of $\ln z$ *is* 0 *iff* $l=0$ *or* $\mathbf{k}\cdot\boldsymbol{\beta}+l\beta_j-1\notin\mathbb{Z}$ *and* $\mathbf{A}_{\mathbf{k}0j}=\mathbf{e}_j/\Gamma(\beta_j')$*. Near* $p\in\{\lambda_i-\mathbf{k}'\cdot\boldsymbol{\lambda}:0\prec\mathbf{k}'\le\mathbf{k}\}$*, (where* $\mathbf{Y}_0$ *is analytic)* $\mathbf{Y_k}$*,* $\mathbf{k}\neq0$ *have convergent Puiseux series.*

REMARK: The fact that the singular part of $\mathbf{Y_k}(p + l\lambda_j)$ in (5.12) and (5.14) is a *multiple* of $\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(p)$ is the effect of *resurgence* and provides a way of determining the $\mathbf{Y_k}$ given $\mathbf{Y}_0$ provided the S_j are nonzero. Since, generically, the S_j are nonzero this is a surprising upshot: given one formal solution, (generically) an n parameter family of solutions can be constructed out of it, without using (4.222) in the process; the differential equation itself is then recoverable ([25]).

By Theorem 5.11 the Borel transforms $\mathbf{Y_k} = \mathcal{B}\tilde{\mathbf{y}}_\mathbf{k}$ define germs of ramified analytic functions and are continuable on the surface $\mathcal{R}$. In order to be able to take Laplace transforms we need to define $\mathcal{B}\tilde{\mathbf{y}}_\mathbf{k}$ along any direction d in $\mathcal{S}$. If $d \neq d_{j,\mathbf{k}}$ then $\mathbf{Y_k}$ are analytically continuable along d and the continuations turn out to have all the properties that we need. But along Stokes lines $d_{j,\mathbf{k}}$ analytic continuation is impossible: in general the functions $\mathbf{Y_k}$ have an infinite array of branch points (5.14). In addition, while both $\mathbf{Y}_\mathbf{k}^+$ and $\mathbf{Y}_\mathbf{k}^-$ turn out to be Laplace transformable (in distributions) along $d_{j,\mathbf{k}}$, $\mathcal{L}\mathbf{Y}_\mathbf{k}^+$ and $\mathcal{L}\mathbf{Y}_\mathbf{k}^-$ are generically *different*. Neither the upper nor the lower continuation would give rise to a definition of Borel summation which commutes with complex-conjugation, as discussed in the introduction. ther analytic continuations along paths γ that *cross* $d_{j,\mathbf{k}}$ have even worse problems, namely that $AC_\gamma(\mathbf{Y_k} * \mathbf{Y_k}) \neq AC_\gamma(\mathbf{Y_k}) *_\gamma AC_\gamma(\mathbf{Y_k})$, (see [21]). As $\mathcal{B}$ transforms differential equations into convolution equations, the implication is that with such a γ, $\mathcal{L}AC_\gamma(\mathbf{Y_k})$ would *not* be, in general, solutions of their differential equations. Individual analytic continuations are thus inadequate for solving (4.222), but some *averages* of analytic continuations do satisfy all the requirements. Let $\alpha = 1/2 + \mathrm{i}\sigma$ with $\sigma \in \mathbb{R}$ and $\mathcal{B}\tilde{\mathbf{y}}_\mathbf{k}$ be extended along $d_{j,\mathbf{k}}$ by the weighted average of analytic continuations

$$\mathcal{B}_\alpha\tilde{\mathbf{y}}_\mathbf{k} = \mathbf{Y}_\mathbf{k}^\alpha = \mathbf{Y}_\mathbf{k}^+ + \sum_{j=1}^{\infty} \alpha^j \left(\mathbf{Y}_\mathbf{k}^- - \mathbf{Y}_\mathbf{k}^{-(j-1)+} \right) \tag{5.15}$$

Remark 5.16 *Relation (5.15) gives the most general reality preserving, linear operator mapping formal power series solutions of (4.222) to solutions of (5.7) in distributions (more precisely in $\mathcal{D}'_{m,\nu}$; see [20]).*

This remark follows easily from Proposition 23 in [20] and Theorem 5.22 below.

The choice $\alpha = 1/2$ has special properties; we call $\mathcal{B}_{\frac{1}{2}}\tilde{\mathbf{y}}_\mathbf{k} = \mathbf{Y}_\mathbf{k}^{ba}$ the balanced average of $\mathbf{Y_k}$. For this choice the expression (5.15) coincides with the one in which $+$ and $-$ are interchanged (Proposition 34 in [20]), accounting for the reality-preserving property. Clearly, if $\mathbf{Y_k}$ is analytic along $d_{j,\mathbf{k}}$, then the terms in the infinite sum vanish and $\mathbf{Y}_\mathbf{k}^\alpha = \mathbf{Y_k}$; we also let $\mathbf{Y}_\mathbf{k}^\alpha = \mathbf{Y_k}$ if $d \neq d_{j,\mathbf{k}}$, where again $\mathbf{Y_k}$ is analytic. It follows from (5.15) and Theorem 5.17 below that the Laplace integral of $\mathbf{Y}_\mathbf{k}^\alpha$ along $\mathbb{R}^+$ can be deformed into contours as those depicted in Fig. 1, with weight $-(-\alpha)^k$ for a contour turning around $k\lambda_1$.

In addition to symmetry (the balanced average equals the half sum of the upper and lower continuations on $(0, 2\lambda_j)$, [25]), an asymptotic property uniquely picks $C = 1/2$. Namely, for $C = 1/2$ alone are the $\mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}$ always *summable to the least term*, see [20].

(b) *Connection with (4.222) and (5.9).* Generalized Borel summation coincides with the usual Borel summation when the transseries consists of only one term, the first series, when that series is classically Borel summable. This is clear from theorem 5.17 (ii) below. Furthermore, generalized summation is a map from a class of formal series to functions which is linear, multiplicative, commutes with differentiation and complex conjugation [20], so it is a summation procedure, which furthermore, establishes along every direction a one to one correspondence between transseries and decaying actual solutions of (4.222).

For clarity we state the results for $x \in S_x$, a sector in the right half plane containing $\lambda_1 = 1$ in which (c1) holds and for p in the associated domain $\mathcal{R}'_1$, but λ_1 plays no special role as discussed in the introduction.

Theorem 5.17 *(i) The branches of $(\mathbf{Y}_{\mathbf{k}})_\gamma$ in $\mathcal{R}'_1$ ($\mathcal{R}_1$ if $\mathbf{k} = 0$) have limits in a C^*-algebra of distributions, $\mathcal{D}'_{m,\nu}(\mathbb{R}^+) \subset \mathcal{D}'$. Their Laplace transforms in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ $\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_\gamma$ exist simultaneously and with $x \in \mathcal{S}_x$ and for any $\delta > 0$ there is a constant K and an x_1 large enough, so that for $\Re(x) > x_1$ we have $|\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_\gamma(x)| \le K\delta^{|\mathbf{k}|}$.*

In addition, $\mathbf{Y}_{\mathbf{k}}(p\mathrm{e}^{\mathrm{i}\phi})$ are continuous in ϕ with respect to the $\mathcal{D}'_{m,\nu}$ topology, (separately) on $(\psi_-, 0]$ and $[0, \psi_+)$.

If $m > \max_i(m_i)$ and $l < \min_i |\lambda_i|$ then $\mathbf{Y}_0(p\mathrm{e}^{\mathrm{i}\phi})$ is continuous in $\phi \in [0, 2\pi]\backslash\{\phi_i : i \le n\}$ in the $\mathcal{D}'_{m,\nu}(\mathbb{R}^+, l)$ topology and has (at most) jump discontinuities for $\phi = \phi_i$. For each $\mathbf{k}$, $|\mathbf{k}| \ge 1$ and any K there is an $l > 0$ and an m such that $\mathbf{Y}_{\mathbf{k}}(p\mathrm{e}^{\mathrm{i}\phi})$ are continuous in $\phi \in [0, 2\pi]\backslash\{\phi_i; -\mathbf{k}' \cdot \boldsymbol{\lambda} + \lambda_i : i \le n, \mathbf{k}' \le \mathbf{k}\}$ in the $\mathcal{D}'_{m,\nu}((0, K), l)$ topology and have (at most) jump discontinuities on the boundary.

(ii) The sum (5.15) converges in $\mathcal{D}'_{m,\nu}$ (and coincides with the analytic continuation of $\mathbf{Y}_{\mathbf{k}}$ when $\mathbf{Y}_{\mathbf{k}}$ is analytic along $\mathbb{R}^+$). For any δ there is a large enough x_1 independent of $\mathbf{k}$ *so that $\mathbf{Y}^{ba}_{\mathbf{k}}(p)$ with $p \in \mathcal{R}'_1$ are Laplace transformable in $\mathcal{D}'_{m,\nu}$ for $\Re(xp) > x_1$ and furthermore $|(\mathcal{L}\mathbf{Y}^{ba}_{\mathbf{k}})(x)| \le \delta^{|\mathbf{k}|}$. In addition, if $d \ne \mathbb{R}^+$, then for large ν, $\mathbf{Y}_{\mathbf{k}} \in L^1_\nu(d)$.*

The functions $\mathcal{L}\mathbf{Y}^{ba}_{\mathbf{k}}$ are analytic for $\Re(xp) > x_1$. For any $\mathbf{C} \in \mathbb{C}^{n_1}$ there is an $x_1(\mathbf{C})$ large enough so that the sum

$$\mathbf{y} = \mathcal{L}\mathbf{Y}^{ba}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}^{ba}_{\mathbf{k}} \tag{5.18}$$

converges uniformly for $\Re(xp) > x_1(\mathbf{C})$, and $\mathbf{y}$ is a solution of (4.222). When *the direction of p is not the real axis then, by definition, $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$, $\mathcal{L}$ is the usual Laplace transform and (5.18) becomes*

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}} \tag{5.19}$$

In addition, $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba} \sim \tilde{\mathbf{y}}_{\mathbf{k}}$ for large x in the half plane $\Re(xp) > x_1$, for all $\mathbf{k}$, uniformly.

iii) More generally, for any α and any solution $\mathbf{y}$ of (4.222) such that $\mathbf{y} \sim \tilde{\mathbf{y}}_0$ for large x along a ray in S_x there exists a constant vector $\mathbf{C} = \mathbf{C}_{\alpha;\mathbf{y}}$ so that

$$\mathbf{y} = \mathcal{L}\mathcal{B}_\alpha \tilde{\mathbf{y}}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathcal{B}_\alpha \tilde{\mathbf{y}}_{\mathbf{k}} \tag{5.20}$$

Given α the representation (5.20) of $\mathbf{y}$ is unique, with the usual convention of directional Laplace transforms.

Of special interest are the cases $\alpha = 1/2$, discussed above, and also $\alpha = 0, 1$ which give:

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{\pm} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\pm} \tag{5.21}$$

(c) Resurgence properties; local Stokes phenomenon .

It turns out that the formal series $\tilde{\mathbf{y}}_{\mathbf{k}}$ are connected among each-other via their Borel transforms. Resurgence formulas link $\mathbf{Y}_{\mathbf{k}}$ to analytic continuations of $\mathbf{Y}_{\mathbf{k}'}$ with $\mathbf{k}' \prec \mathbf{k}$, in a way that, generically, $\mathbf{Y}_0$ contains enough information to compute all $\mathbf{Y}_{\mathbf{k}}$.

Various resurgence properties have been observed in different contexts, and the term resurgence has been used with slightly different interpretations. In the hyperasymptotic theory of M. Berry, it was discovered that the first asymptotic series reappears in various shapes in the process of computing higher terms of the expansions. J. Écalle, in his comprehensive theory of analyzable functions, has obtained a general resurgence principle, the *bridge equation* [5]. The common denominator of resurgence is the reappearance of "earlier" terms in the formulas of "later" ones. It turns out that, for our problem, resurgence is fundamentally linked to the *Stokes phenomenon* . In the following formulas we make the convention $\mathbf{Y}_{\mathbf{k}}(p-j) = 0$ for $p < j$ as an element of $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$. We again state the results is stated for $p \in \mathcal{R}'_1$ and $x \in S_x$ but hold in any sector where (c1) is valid.

Theorem 5.22 *i) For all $\mathbf{k}$ and $\Re(p) > j, \Im(p) > 0$ as well as in $\mathcal{D}'_{m,\nu}$ we have*

$$\mathbf{Y}_{\mathbf{k}}^{\pm j\mp}(p) - \mathbf{Y}_{\mathbf{k}}^{\pm(j-1)\mp}(p) = (\pm S_1)^j \binom{k_1+j}{j} \left(\mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^{\mp}(p-j)\right)^{(mj)} \tag{5.23}$$

and also,

$$\mathbf{Y}_{\mathbf{k}}^{\pm} = \mathbf{Y}_{\mathbf{k}}^{\mp} + \sum_{j\ge 1} \binom{j+k_1}{k_1} (\pm S_1)^j (\mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^{\mp}(p-j))^{(mj)} \tag{5.24}$$

ii) Local Stokes transition.
Consider the expression of a fixed solution $\mathbf{y}$ *of (4.222) as a Borel summed transseries (5.18). As* $\arg(x)$ *varies, (5.18) changes only through* $\mathbf{C}$*, and that change occurs when Stokes lines are crossed (cf. §5.0a ; the Stokes lines of* $\mathbf{Y}_0$ *are the directions of* λ_i*). We have, in the neighborhood of* $\mathbb{R}^+$*, with* S_1 *defined in (5.13):*

$$\mathbf{C}(\xi) = \begin{cases} \mathbf{C}^- = \mathbf{C}(-0) & \text{for } \xi < 0 \\ \mathbf{C}^0 = \mathbf{C}(-0) + \frac{1}{2}S_1\mathbf{e}_1 & \text{for } \xi = 0 \\ \mathbf{C}^+ = \mathbf{C}(-0) + S_1\mathbf{e}_1 & \text{for } \xi > 0 \end{cases} \tag{5.25}$$

(d) Classical Stokes phenomena and local Stokes transitions. Again we formulate the result below for λ_1 but with straightforward adjustments it holds relative to any other eigenvalue. Let $\mathbf{C}$ be of the form $C_1\mathbf{e}_1$. Along the imaginary axis, condition (c1) fails. The positive and negative imaginary are the *antistokes* lines corresponding to $\lambda_1 = 1$ (note: sometimes called Stokes lines!). If we choose paths in the right half plane approaching the positive/negative imaginary axis in such a way that $|x^{-\beta_1 - l}\mathrm{e}^{-x}| \to K \neq 0$ along them, where $l + \beta \in (0, M)$, then $\mathbf{y} \sim C^{\pm}x^{-l-\beta_1}\mathrm{e}^{-x} + \tilde{\mathbf{y}}_0$ for large x and the term multiplied by K is now the *leading* behavior of $\mathbf{y}$. The particular choice of K and l within this range is rather arbitrary, the main point being that along such special curves, the constant $\mathbf{C}$ is definable in terms of *classical* asymptotics. Within the right half plane, it is only near the imaginary axis that this happens, since otherwise the exponential term is smaller than all terms of $\tilde{\mathbf{y}}_0$. On the other hand Borel summation makes possible the definition of $\mathbf{C}$ throughout the right half plane, and we now address the issue of the relation between classical asymptotics and exponential asymptotics.

Theorem 5.26 *Let* $\gamma^{\pm}$ *be two paths in the right half plane, near the positive/negative imaginary axis such that* $|x^{-\beta_1+1}e^{-x\lambda_1}| \to 1$ *as* $x \to \infty$ *along* $\gamma^{\pm}$*. Consider the solution* $\mathbf{y}$ *of (4.222) given in (5.18) with* $\mathbf{C} = C\mathbf{e}_1$ *and where the path of integration is* $p \in \mathbb{R}^+$*. Then*

$$\mathbf{y} = (C \pm \frac{1}{2}S_1)\mathbf{e}_1 x^{-\beta_1+1}e^{-x\lambda_1}(1 + o(1)) \tag{5.27}$$

for large x along $\gamma^{\pm}$, where S_1 is the same as in (5.13), (5.25).

Classical asymptotics loses track of the value of $\mathbf{C}$ along any ray other than the imaginary directions, as the terms multiplied by $\mathbf{C}$ will be hidden "beyond all orders" of the classically divergent series $\tilde{\mathbf{y}}_0$. In contrast to the classical picture, we see that through generalized Borel summation the constant $\mathbf{C}$ is precisely defined throughout the positive half-plane and the question of where the change in $\mathbf{C}$ occurs is well defined.

Formula (5.25) is the exponential asymptotic expression of the Stokes phenomenon . It shows that the constant jumps as the Stokes line is crossed, (5.25), as originally predicted by Stokes himself [16]. Subsequently, the original ideas of Stokes, based on optimal truncation of series were greatly refined by M. Berry, leading to his theory of hyperasymptotic expansions and a description of Stokes transitions for saddle integrals [31].

If more than one component of $\mathbf{C}$ is nonzero, then in general there is no direction along which $\mathbf{C}$ can be defined through *classical* asymptotics. Part of the difficulty of studying nonlinear Stokes phenomena using classical tools stems from this fact.

Relation (5.25) expresses the evolution of $\mathbf{C}$ and the presence of a Stokes phenomenon beyond all orders of Poincaré asymptotics.

5.1 Comments on the proof

Many complications in this general setting stem from higher dimensionality, which complicates the algebra. Apart from that, a good part of the analysis of the properties of the Borel transformed equation along nonsingular directions is not that different from the analysis in §4.8a , which should be studied first.

Then we look more carefully at the behavior along singular directions. It turns out that the study of the convolution equation is not very difficult even near a singular point. There, by dominant balance we see that the leading behavior is governed by a linear, regularly perturbed, ODE, which is used to rewrite the equation in a contractive form. In nonlinear equations, one singularity is replicated periodically along its complex direction, via the autoconvolution.

The next task is to find a Borel summation valid along the singular directions while preserving all properties of usual Borel summation. The approach is specific to ODEs, and it is not meant to substitute for the very general one of Écalle. It has the advantage of simplicity, and also in finding all possible summation processes, in the context of ODEs. Écalle's approach shows that summation processes such as medianization are universal, *i.e.* would have the expected properties regardless of the problem of origin. They are described in [4].

We work in the restricted setting of [21], where to simplify the analysis we assume further that $\Re(\beta) > 0$ where $\beta = \hat{B}_{1,1}$. Through normalization we make

$$\Re(\beta) \in (0, 1] \tag{5.28}$$

We are interested in the study of the solutions of (4.222) that are decaying for large x, in one of the half-planes $\Re(xe^{-i\phi}) > 0$ with $\phi \in (\arg\lambda_n - 2\pi, \arg\lambda_2)$. These solutions have the same asymptotic behavior at large x, described by a (typically divergent) power series

$$\mathbf{y}(x) \sim \tilde{\mathbf{y}}_0 = \sum_{k=2}^{\infty} \frac{\tilde{\mathbf{y}}_{0,k}}{x^k} \quad (|x| \to \infty;\ \Re\left(xe^{-i\phi}\right) > const > 0) \tag{5.29}$$

For instance, all the solutions of the equation $y' + y = x^{-1}$ have the property $y(x) \sim \sum_{k=0}^{\infty} k! x^{-k-1}$ as $x \to \infty$. If $\phi \neq 0$ there is only one solution of (4.222) satisfying (5.29). A much more interesting case is when we take $\phi = 0$. Then, as it is known (and will also follow from the present paper) there is a one dimensional manifold M^+ of solutions of (4.222) such that (5.29) holds. The manifold $\tilde{M}^+$ of all *formal* solutions which decay in the half plane $\Re x > 0$

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{\mathbf{y}}_k \tag{5.30}$$

also has one free parameter, $C \in \mathbb{C}$. In (5.30), $\tilde{\mathbf{y}}_k$, $k \geq 0$, are formal power series and $\tilde{\mathbf{y}}$ is an instance of a trans-series. In our example $y' + y = x^{-1}$, $\tilde{\mathbf{y}} = \sum_{k=0}^{\infty} k! x^{-k-1} + Ce^{-x}$. See Section 5.2f a heuristic construction leading to trans-series solutions and for references.

The series $\tilde{\mathbf{y}}_k$ satisfy the system of differential equations

$$\mathbf{y}_0' + \left(\hat{\Lambda} + \frac{1}{x}\hat{B}\right)\mathbf{y}_0 = \mathbf{f}_0(x) + \mathbf{g}(x, \mathbf{y}_0)$$

$$\mathbf{y}_k' + \left(\hat{\Lambda} + \frac{1}{x}\hat{B} - k - \partial\mathbf{g}(x, \mathbf{y}_0)\right)\mathbf{y}_k = \sum_{|\mathbf{l}|>1} \frac{\mathbf{g}^{(\mathbf{l})}(x, \mathbf{y}_0)}{\mathbf{l}!} \sum_{\Sigma m = k} \prod_{i=1}^{n} \prod_{j=1}^{l_i} (\mathbf{y}_{m_{i,j}})_i \tag{5.31}$$

where $\mathbf{g}^{(\mathbf{l})} := \partial^{(\mathbf{l})}\mathbf{g}/\partial\mathbf{y}^{\mathbf{l}}$, $(\partial\mathbf{g})\mathbf{y}_k := \sum_{i=1}^{n} (\mathbf{y}_k)_i (\partial\mathbf{g}/\partial y_i)$, and $\sum_{\Sigma m = k}$ stands for the sum over all integers $m_{i,j} \geq 1$ with $1 \leq i \leq n, 1 \leq j \leq l_i$ such that $\sum_{i=1}^{n} \sum_{j=1}^{l_i} m_{i,j} = k$. Because $m_{i,j} \geq 1$, $\sum m_{i,j} = k$ (fixed) and $\mathrm{card}\{m_{i,j}\} = |\mathbf{l}|$, the sums in (5.31) contain only a *finite* number of terms. We use the convention $\prod_{i \in \emptyset} \equiv 0$. The system (5.31) is derived in Section 5.2f .

Starting with $k = 1$ the equations (5.31) are linear. Note that the inhomogeneous term in these linear equations is zero for $k = 1$, and for $k > 1$ it involves only $\mathbf{y}_n$ with $n < k$.

We show that the general solution of (4.222), (5.29) is obtained by replacing each formal series in (5.30) by its Borel sum which gives a one-to-one correspondence between the formal solutions (trans-series) and the actual solutions of (4.222), (5.29):

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{\mathbf{y}}_k \;\longleftrightarrow\; \mathcal{L}_\phi \mathcal{B}_\phi \tilde{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \mathcal{L}_\phi \mathcal{B}_\phi \tilde{\mathbf{y}}_k = \mathbf{y} \qquad (5.32)$$

The Borel summation operator, $\mathcal{LB}$ will be defined precisely. The function $\mathbf{y} \in M^+$ is convergently defined by (5.32) for large x. The left arrow in (5.32) means that $\mathcal{L}_\phi \mathcal{B}_\phi \tilde{\mathbf{y}}_k(x) \sim \tilde{\mathbf{y}}_k(x)$ for $x \to \infty$. The exact statement corresponding to (5.32) is given in Theorem 5.45.

We study in detail the features of the representation (5.32) and the properties of the objects involved. The technique that we use differs from that of [4], [5], [6] and leads to new results. In particular we obtain for the Borel transform of the formal series solutions of differential systems an averaging formula, having, as the medianization of Ecalle the quality of preserving reality and of commuting with convolution, but involving a smaller number of analytic continuations and in addition satisfying the condition of at most exponential growth at infinity.

For $m > 1$, the inverse Laplace transform of x^{-m} is

$$\mathcal{L}^{-1} x^{-m} = p^{m-1}/\Gamma(m-1) = \mathcal{B}\, x^{-m}$$

The *Borel transform* $\mathcal{B}$ of a formal series

$$\tilde{\mathbf{y}} = x^r \sum_{k=1}^{\infty} \tilde{\mathbf{y}}_k x^{-k},\; r \in (0,1) \qquad (5.33)$$

is by definition the formal series gotten by taking $\mathcal{L}^{-1}$ term by term:

$$\mathcal{B}\,\tilde{\mathbf{y}} = \mathbf{Y} := p^{-r} \sum_{k=0}^{\infty} \frac{\tilde{\mathbf{y}}_{k+1}}{\Gamma(k-r)} p^k \qquad (5.34)$$

Of all the formal solutions (5.30), only the one with $C = 0$ (formally) decays in a half-plane, if the half-plane *is not* centered on the real axis. On the other hand, $\mathcal{L}_\phi \mathcal{B} \tilde{\mathbf{y}}_0$ turns out to be the only solution of (4.222), (5.29) which decays in the same half-plane centered on Φ. Borel summation associates uniquely a true solution to $\mathbf{Y}_0$.

The situation is more complicated and more interesting along Stokes ray s (we focus on one of them, $\Phi = \mathbb{R}^+$). Condition 2) above is violated and, generically, the functions $\mathbf{Y}_k$ have an array of branch points along $\mathbb{R}^+$. If we reinterpret 2) and consider paths that avoid the singularities then first of all, analytic continuation is (a priori) ambiguous. What is worse, the Laplace transform of such analytic continuations of $\mathbf{Y}_0$ are, typically, *not* solutions of

(4.222) (see Section Ab). However, Laplace transforms of (a one-parameter family) of suitable weighted combinations of analytic continuations of $\mathbf{Y}_0$ *are*, as we will prove, solutions of (4.222). If we require in addition that real series are Borel-summed to real-valued functions then one of weighted average of analytic continuations appears as more natural (see also Theorem 5.61 below).

*

To define the Borel transform along the Stokes line $\mathbb{R}^+$ we construct a suitable space of analytic functions. Let $\phi_+ = \arg\lambda_2$, $\phi_- = 2\pi - \arg\lambda_n$, and

$$\mathcal{W}_1 := \{p : p \notin \mathbb{N} \cup \{0\} \text{ and } \arg p \in (-\phi_-, \phi_+)\} \tag{5.35}$$

(Fig. 1), a sector containing only the eigenvalue $\lambda_1 = 1$ and punctured at all the integers (where the functions $\mathcal{B}\tilde{\mathbf{y}}_k$ are typically singular; if $n = 1$ the condition on the argument is dropped). We construct over $\mathcal{W}_1$ a surface $\mathcal{R}_1$, consisting of homotopy classes of curves starting at the origin, going only forward and crossing the real axis at most once:

$$\mathcal{R}_1 := \Big\{\gamma : (0,1) \mapsto \mathcal{W}_1 \text{ s.t. } \gamma(0_+) = 0;\ \Re(\gamma(t)) \text{ increases in } t \text{ and}$$
$$0 = \Im(\gamma(t_1)) = \Im(\gamma(t_2)) \Rightarrow t_1 = t_2\Big\} \tag{5.36}$$

modulo homotopies. Let also

$$\mathcal{D} := \mathbb{C} \setminus \cup_{i=1}^{n} \{\alpha\lambda_i : \alpha \geq 1\} \tag{5.37}$$

be the complex plane without the rays originating at the eigenvalues λ_i of $\hat{\Lambda}$.

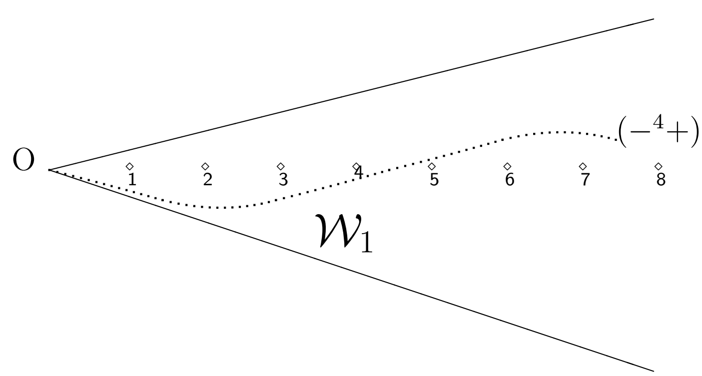


Fig 1. *The region $\mathcal{W}_1$. The dotted line is one of the paths that generate $\mathcal{R}_1$.*

Using notations similar to those of Ecalle, we symbolize the paths in $\mathcal{R}_1$ by a sequence of signs $\epsilon_1, .., \epsilon_j, .., \epsilon_n,\ \epsilon_j = +$ or $-$. For example, $----+ = -^4+$ will symbolize a path in $\mathcal{R}_1$ that crosses the real line from below through the interval $(4,5)$, and then goes only through the upper half-plane (Fig.1); $''+''$ is a path confined to the upper half plane, etc. The analytic continuation of a function $\mathbf{Y}$ along the path $-^4+$ will be denoted $\mathbf{Y}^{-^4+}$.

The result below gives a first characterization of the analytic properties of $\mathcal{B}\tilde{\mathbf{y}}_k$. (In the following, we choose the determination of the logarithm which is real for positive argument.)

Proposition 5.38 *i) The function $\mathbf{Y}_0 := \mathcal{B}\tilde{\mathbf{y}}_0$ is analytic in $\mathcal{D}$ and Laplace transformable along any direction in $\mathcal{D}$. In a neighborhood of $p = 1$*

$$\mathbf{Y}_0(p) = \begin{cases} S_\beta(1-p)^{\beta-1}\mathbf{A}(p) + \mathbf{B}(p) \ \textit{for}\ \beta \neq 1 \\ S_\beta \ln(1-p)\mathbf{A}(p) + \mathbf{B}(p)\ \textit{for}\ \beta = 1 \end{cases} \tag{5.39}$$

(see (5.28)), where $\mathbf{A}$, $\mathbf{B}$ are ($\mathbb{C}^n$-valued) analytic functions in a neighborhood of $p = 1$.

ii) The functions $\mathbf{Y}_k := \mathcal{B}\tilde{\mathbf{y}}_k,\ k = 0, 1, 2, ..$ are analytic in $\mathcal{R}_1$.
iii) For small p,

$$\mathbf{Y}_0(p) = p\mathbf{A}_0(p);\quad \mathbf{Y}_k(p) = p^{k\beta-1}\mathbf{A}_k(p),\ k = 1, 2, .. \tag{5.40}$$

where $\mathbf{A}_k$, $k \geq 0$, are analytic functions in a neighborhood of $p = 0$ in $\mathbb{C}$.

iv) If $S_\beta = 0$ then $\mathbf{Y}_k$, $k \geq 0$, are analytic in $\mathcal{W}_1 \cup \mathbb{N}$.

v) The analytic continuations of $\mathbf{Y}_k$ along paths in $\mathcal{R}_1$ are in $L^1_{loc}(\mathbb{R}^+)$ (their singularities along $\mathbb{R}^+$ are integrable). The analytic continuations of the $\mathbf{Y}_k$ in $\mathcal{R}_1$ can be expressed in terms of each other through "resurgence" relations of the type:

$$S_\beta^k \mathbf{Y}_k = \left(\mathbf{Y}_0^- - \mathbf{Y}_0^{-^{k-1}+}\right) \circ \tau_k, \qquad \textit{on}\ (0,1); \quad (\tau_a := p \mapsto p - a) \tag{5.41}$$

relating the higher order series in the trans-series to the first series and

$$\mathbf{Y}_k^{-^m+} = \mathbf{Y}_k^+ + \sum_{j=1}^{m} \binom{k+j}{k} S_\beta^j \mathbf{Y}_{k+j}^+ \circ \tau_j \tag{5.42}$$

S_β is related to the Stokes constant S by

$$S_\beta = \begin{cases} \dfrac{iS}{2\sin(\pi(1-\beta))} \ \text{for}\ \beta \neq 1 \\ \dfrac{iS}{2\pi}\ \text{for}\ \beta = 1 \end{cases}$$

The Borel transformability of the principal series $\tilde{\mathbf{y}}_0$ has been considered for general systems of differential equations, allowing for resonances (see [8],[9]).

Let $\mathbf{Y}$ be one of the functions $\mathbf{Y}_k$ and define, on $\mathbb{R}^+ \cap \mathcal{R}_1$ the "balanced average" of $\mathbf{Y}$:

$$\mathbf{Y}^{ba} = \mathbf{Y}^+ + \sum_{k=1}^{\infty} 2^{-k} \left(\mathbf{Y}^- - \mathbf{Y}^{-^{k-1}+} \right) \mathcal{H} \circ \tau_k \tag{5.43}$$

($\mathcal{H}$ is Heaviside's function). For any value of the argument, only finitely many terms (5.43) are nonzero. Moreover, the balanced average preserves reality in the sense that if (4.222) is real and $\tilde{\mathbf{y}}_0$ is real then $\mathbf{Y}^{ba}$ is real on $\mathbb{R}^+ - \mathbb{N}$ (and in this case the formula can be symmetrized by taking $1/2$ of the expression above plus $1/2$ of the same expression with $+$ and $-$ interchanged). Equation (5.43) has the main features of medianization (cf. [5]), in particular (unlike individual analytic continuations, see Appendix Ab) commutes with convolution (cf. Theorem 5.61). As it will become clear, the advantage of the definition (5.43) is that $\mathbf{Y}^{ba}$ is exponentially bounded at infinity for the functions we are dealing with.

Let again $\tilde{\mathbf{y}}$ be one of $\tilde{\mathbf{y}}_k$ and $\mathbf{Y} = \mathcal{B}\tilde{\mathbf{y}}$. We define:

$$\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}} := \mathcal{L}_\phi \mathbf{Y} = x \mapsto \int_0^{\infty e^{i\phi}} \mathbf{Y}(p) e^{-px} dp \quad \text{if } \Phi \neq \mathbb{R}^+$$

$$\mathcal{L}_0 \mathcal{B}\tilde{\mathbf{y}} := \mathcal{L}_0 \mathbf{Y} = x \mapsto \int_0^{\infty} \mathbf{Y}^{ba}(p) e^{-px} dp \quad \text{if } \Phi = \mathbb{R}^+ \tag{5.44}$$

The connection between true and formal solutions of the differential equation is given in the following theorem:

Theorem 5.45 *i) There is a large enough b such that, for $\Re(x) > b$ the Laplace transforms $\mathcal{L}_\phi \mathbf{Y}_k$ exist for all $k \geq 0$ and $\phi \in (-\phi_-, \phi_+)$, cf. (5.35).*

For $\phi \in (-\phi_-, \phi_+)$ and any C the series

$$\mathbf{y}(x) = (\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_0)(x) + \sum_{k=1}^{\infty} C^k e^{-kx} (\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_k)(x) \tag{5.46}$$

is convergent for large enough x in the right half plane.

The function $\mathbf{y}$ in (5.46) is a solution of the differential equation (4.222).

Furthermore, for any $k \geq 0$ we have $\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_k \sim \tilde{\mathbf{y}}_k$ in the right half plane and $\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_k$ is a solution of the corresponding equation in (5.31).

ii) Conversely, given ϕ, any *solution of (4.222) having $\tilde{\mathbf{y}}_0$ as an asymptotic series in the right half plane can be written in the form (5.46), for a unique C.*

iii) The constant C, associated in ii) with a given solution $\mathbf{y}$ of (4.222), depends on the angle ϕ:

$$C(\phi) = \begin{cases} C(0_+) & \textit{for } \phi > 0 \\ C(0_+) - \frac{1}{2}S_\beta & \textit{for } \phi = 0 \\ C(0_+) - S_\beta & \textit{for } \phi < 0 \end{cases} \tag{5.47}$$

(see also (5.39)).

Note that by (5.47) the change in the correspondence (5.32) occurs when the Stokes line $\arg x = 0$ is crossed. This is a *local* manifestation of the Stokes phenomenon ([16], [17], [18]).

**

Next, we study the correspondence between the solutions of the differential equations (4.222), (5.55), their formal solutions and the solutions of the inverse Laplace transform of these equations, which, in the transformed space, are convolution equations.

With the convolution defined as

$$f * g := p \mapsto \int_0^p f(s)g(p-s)ds \tag{5.48}$$

we have, as is well known, $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$, $\mathcal{L}(-pf(p)) = \mathcal{L}(f(p))'$. (See Section C for a few more useful formulas.) In (4.222) we write

$$\mathbf{g}(\xi^{-1}, \mathbf{y}) = \sum_{|\mathbf{l}|\geq 1} \mathbf{g}_\mathbf{l}(\xi)\mathbf{y}^\mathbf{l} = \sum_{m\geq 0; |\mathbf{l}|\geq 1} \mathbf{g}_{m,\mathbf{l}}\xi^m \mathbf{y}^\mathbf{l} \quad (|\xi| < \xi_0, |\mathbf{y}| < y_0) \tag{5.49}$$

where by construction $\mathbf{g}_{0,\mathbf{l}} = \mathbf{g}_{1,\mathbf{l}} = 0$ if $|\mathbf{l}| = 1$ and the notation $\mathbf{z}^\mathbf{l}$ means $z_1^{l_1} \cdot z_n^{l_n}$ and $|\mathbf{l}| = l_1 + .. + l_n$. The formal inverse Laplace transform of $\mathbf{g}(x, \mathbf{y}(x))$ is given by:

$$\mathcal{L}^{-1} \sum_{|\mathbf{l}|\geq \mathbf{1}} \mathbf{y}(x)^\mathbf{l} \left(\sum_{m\geq 0} \mathbf{g}_{m,\mathbf{l}} x^{-m} \right) = \sum_{|\mathbf{l}|\geq \mathbf{1}} \mathbf{G}_\mathbf{l} * \mathbf{Y}^{*\mathbf{l}} + \sum_{|\mathbf{l}|\geq \mathbf{2}} \mathbf{g}_{0,\mathbf{l}} \mathbf{Y}^{*\mathbf{l}} =: \mathcal{N}(\mathbf{Y}) \tag{5.50}$$

where

$$\mathbf{G}_\mathbf{l}(p) = \sum_{m=1}^{\infty} \mathbf{g}_{m,\mathbf{l}} \frac{p^{m-1}}{m!} \quad (\mathbf{G}_{1,\mathbf{l}}(0) = 0 \ \text{ if } |\mathbf{l}| = 1) \tag{5.51}$$

$$\mathbf{G}_\mathbf{l} * \mathbf{Y}^{*\mathbf{l}} \in \mathbb{C}^n; \quad (\mathbf{G}_\mathbf{l} * \mathbf{Y}^{*\mathbf{l}})_j := (\mathbf{G}_\mathbf{l})_j * Y_1^{*l_1} * .. * Y_n^{*l_n} \tag{5.52}$$

The inverse Laplace transform of (4.222) is the convolution equation:

$$-p\mathbf{Y}(p) = \mathbf{F}_0(p) - \hat{\Lambda}\mathbf{Y}(p) - \hat{B}\int_0^p \mathbf{Y}(s)ds + \mathcal{N}(\mathbf{Y})(p) \tag{5.53}$$

(see (5.50)) where, since $\mathbf{f}_0(x) = O(x^{-2})$,

$$\mathbf{F}_0(0) = 0 \tag{5.54}$$

By transforming (5.31) we get, similarly:

$$(\hat{\Lambda} - p - k)\mathbf{Y}_k(p) + \hat{B}\int_0^p \mathbf{Y}_k(s)ds - \sum_{j=1}^{n}\int_0^p (\mathbf{Y}_k)_j(s)\mathbf{D}_j(p-s)ds =$$

$$\sum_{|\mathbf{l}|>1} \mathbf{d}_{\mathbf{l}} * \sum_{\Sigma m = k} * \prod_{i=1}^{n} * \prod_{j=1}^{l_i} (\mathbf{Y}_{m_{i,j}})_i =: \mathbf{R}_k(p) \quad (k = 1, 2, ..) \tag{5.55}$$

with $\mathbf{d}_{\mathbf{m}} := \mathcal{L}^{-1}(\mathbf{g}^{(\mathbf{m})}(x, \mathbf{y}_0)/\mathbf{m}!)$, $\mathbf{D}_j := \mathcal{L}^{-1}(\partial \mathbf{g}(\mathbf{x}, \mathbf{y_0})/\partial \mathbf{y}_{\rm j})$ and $*\prod$ standing for the convolution product.

For a given ray Φ we consider the equations (5.53) and (5.55) in $L^1_{loc}(\Phi)$. When Φ is not a Stokes line, the description of the solutions is quite simple:

Proposition 5.56 *i) If Φ is a ray in $\mathcal{D}$, then the equation (5.53) has a unique solution in $L^1_{loc}(\Phi)$, namely $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$.*

ii) For any ray in $\mathcal{W}_1$, the system (5.53), (5.55) has the general solution solution $C^k\mathbf{Y}_k = C^k\mathcal{B}\tilde{\mathbf{y}}_k$, $k \geq 0$.

The more interesting case $\Phi = \mathbb{R}^+$ is dealt with in the following theorem:

Theorem 5.57 *i) The general solution in $L^1_{loc}(\mathbb{R}^+)$ of the equation (5.53) can be written in the form:*

$$\mathbf{Y}_C(p) = \sum_{k=0}^{\infty} C^k \mathbf{Y}_k^{ba}(p-k)\mathcal{H}(p-k) \tag{5.58}$$

with $C \in \mathbb{C}$ arbitrary.

ii) Near $p = 1$, $\mathbf{Y}_C$ is given by:

$$\mathbf{Y}_C(p) = \begin{cases} S_\beta(1-p)^{\beta-1}\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p < 1 \\ C(1-p)^{\beta-1}\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p > 1 \end{cases} \quad (\beta \neq 1) \tag{5.59}$$

$$\mathbf{Y}_C(p) = \begin{cases} S_\beta \ln(1-p)\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p < 1 \\ (S_\beta \ln(1-p) + C)\mathbf{A}(p) + \mathbf{B}(p) \text{ for } p > 1 \end{cases} \quad (\beta = 1)$$

where $\mathbf{A}$ and $\mathbf{B}$ extend to analytic functions in a neighborhood of $p = 1$.

iii) With the choice $\mathbf{Y}_0 = \mathbf{Y}_0^{ba}$, the general solution of (5.55) in $L^1_{loc}(\mathbb{R}^+)$ is $C^k\mathbf{Y}_k^{ba}, k \in \mathbb{N}$.

Comparing (5.59) with (5.39) we see that if $S \neq 0$ (which is the generic case) the general solution of (5.53) can be written on the interval $(0,2)$ as a linear combination of the upper and lower analytic continuations of $\mathcal{B}\tilde{\mathbf{y}}_0$:

$$\mathbf{Y}_C = \lambda_C \mathbf{Y}_0^+ + (1-\lambda_C)\mathbf{Y}_0^- \tag{5.60}$$

Finally we mention the following result, which shows that the balanced average, like medianization [5], commutes with convolution.

Theorem 5.61 *If f and g are analytic in $\mathcal{R}_1$ then $f * g$ extends analytically in $\mathcal{R}_1$ and furthermore,*

$$(f * g)^{ba} = f^{ba} * g^{ba} \tag{5.62}$$

As a consequence of the linearity of the balanced averaging and its commutation with convolution, if $\tilde{\mathbf{t}}_{1,2}$ are the trans-series of the solutions $\mathbf{f}_{1,2}$ of differential equations of the type considered in the present paper (cf. (5.32)), and if $\mathcal{LB}\tilde{\mathbf{t}}_{1,2} = \mathbf{f}_{1,2}$ then

$$\mathcal{LB}\left(a\tilde{\mathbf{t}}_1 + b\tilde{\mathbf{t}}_2\right) = a\mathbf{f}_1 + b\mathbf{f}_2 \tag{5.63}$$

Moreover, what is less obvious, we have for the component-wise product

$$\mathcal{LB}(\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_2) = \mathbf{f}_1\mathbf{f}_2 \tag{5.64}$$

Borel summation is in fact an isomorphism between a sub-algebra of trans-series and a function algebra.

5.2 Proofs and further results

5.2a Outline of the proofs of the main results

To show the results stated in the previous section, we first obtain the general solution in L^1_{loc} of the convolution system (5.55) in $\mathcal{D}$ and then, separately, on the Stokes line $\mathbb{R}^+$. We show that along a ray in $\mathcal{D}$, the solution is unique whereas along the ray $\mathbb{R}^+$ there is a one-parameter family of solutions of the system, branching off at $p = 1$. We show that any L^1_{loc} solution of the system is (uniformly in k) exponentially bounded at infinity therefore Laplace transformable and (by the usual properties of the Laplace transform) these transforms solve (4.222). Conversely, any solution of (4.222) with the required asymptotic properties is inverse Laplace transformable, therefore it has to be one of the previously obtained solutions of the equation corresponding to $k = 0$. We then study the regularity properties of the solutions of the convolution equation by local analysis.

Having the complete description of the family of L^1_{loc} solutions we compare different ways that lead to the same solution and obtain interesting identities; the identities, together with the local properties of the solutions are instrumental in finding the analytic properties of $\mathbf{Y}_k$ in $\mathcal{R}_1$.

Key to the main proofs. The complete connection with Equation (5.43) is established in Section 5.2g . For the remaining parts: *Proposition 5.38*: i) follows from Proposition 5.65 and Lemma 5.108; ii) and iii) follow from Proposition 5.187. The proof of (5.40) is given in Remark 5.176 and iv) is shown in Remark 5.194. Part v) follows from Proposition 5.181 and Proposition 5.187. *Theorem 5.45*: i) and ii) follow from Lemma 5.179 and Proposition 5.169; iii) is Equation (5.185). *Proposition 5.56* follows from Proposition 5.65 and Lemma 5.179. *Theorem 5.57*: follows from Proposition 5.145, Lemma 5.125, Proposition 5.166. The proof of *Theorem 5.61* starts with Proposition 5.198 and is continued after it.

5.2b The convolution equation away from Stokes rays

For any star-shaped set $\mathcal{E}$ in the complex plane containing the origin (i.e., a region such that the origin can be connected with any other point in $\mathcal{E}$ by a straight line segment contained in $\mathcal{E}$) we denote by $L_{ray}(\mathcal{E})$ the set of functions which are locally integrable along each ray in $\mathcal{E}$.

Proposition 5.65 *There is a unique solution of (5.53) in $L_{ray}(\mathcal{D})$ (cf. (5.37)) namely $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$.*

This solution is analytic in $\mathcal{D}$, Laplace transformable along any ray Φ contained in $\mathcal{D}$ and $\mathcal{L}_\phi \mathbf{Y}_0$ is a solution of (4.222).

For the proof we need a few more results.

Remark 5.66 *There is a constant $K > 0$ (independent of p and $\mathbf{l}$) such that for all $p \in \mathbb{C}$ and all $\mathbf{l} \geq \mathbf{0}$*

$$|\mathbf{G}_\mathbf{l}(p)|_\wedge < K\mu^{|\mathbf{l}|} e^{\mu|p|} \tag{5.67}$$

for $\mu > \max\{\xi_0^{-1}, y_0^{-1}\}$ (cf. (5.49)) ($|\mathbf{f}|_\wedge := \max_{1..n}\{|f_1|, .., |f_n|\}$ is an Euclidean norm; for the definition of $\mathbf{G}$ see (5.51), (5.49) and (4.222)).

Proof.
From the analyticity assumption it follows that

$$|\mathbf{g}_{m,\mathbf{l}}|_\wedge < \text{Const } \mu^{m+|\mathbf{l}|} \tag{5.68}$$

where the constant is independent on m and $\mathbf{l}$.
Then, by (5.51),

$$|\mathbf{G}_\mathbf{l}(p)|_\wedge < \text{Const } \mu^{|\mathbf{l}|+1} \frac{e^{\mu|p|} - 1}{\mu|p|} < \text{Const } \mu^{|\mathbf{l}|+1} e^{\mu|p|}$$

□

Consider the ray segments

$$\Phi_D = \{\alpha e^{i\phi} : 0 \le \alpha < D\} \tag{5.69}$$

and along Φ_D the L^1 norm with exponential weight

$$\|f\|_{b,\Phi} = \|f\|_b := \int_\Phi e^{-b|p|}|f(p)||dp| = \int_0^D e^{-bt}|f(te^{i\phi})|dt \tag{5.70}$$

and the space

$$L^1_b(\Phi_D) := \{f : \|f\|_b < \infty\}$$

(if $D < \infty$, $L^1_b(\Phi_D) = L^1_{loc}(\Phi_D)$). We mention the following elementary property:

Remark 5.71 *The Laplace transform $\mathcal{L}$ is a continuous operator from $L^1_b(\Phi_D)$ to the space of analytic functions in the half plane $\Re(x) > b$ with the uniform norm.*

□

Let $\mathcal{K} \in \mathbb{C}$ be a bounded domain, diam $(\mathcal{K}) = D < \infty$. On the space of continuous functions on $\mathcal{K}$ we take the uniform norm with exponential weight:

$$\|f\|_u := D \sup_{p\in\mathcal{K}}\{|f(p)|e^{-b|p|}\} \tag{5.72}$$

(which is equivalent to the usual uniform norm).

Let $\mathcal{O} \subset \mathcal{D}$, $\mathcal{O} \ni 0$ be a *star-shaped, open set*, diam$(\mathcal{O}) = D$ containing a ray segment Φ. Let $\mathcal{A}$ be the space of analytic functions f in $\mathcal{O}$ such that $f(0) = 0$, endowed with the norm (5.72).

Proposition 5.73 *The spaces $L^1_b(\Phi_D)$ and $\mathcal{A}$ are Banach algebras with respect to the usual addition of functions and the convolution (5.48). Furthermore*

$$\begin{gathered}\|f * g\|_b \le \|f\|_b\|g\|_b \quad (f, g \in L^1_b(\Phi_D))\\ \|f * g\|_u \le \|f\|_u\|g\|_u \quad (f, g \in \mathcal{A})\\ \|f * g\|_u \le \|f\|_u\|g\|_b \quad (f \in C(\Phi_D), g \in L^1_b(\Phi_D)\end{gathered} \tag{5.74}$$

($D = \infty$ is allowed in the first inequality).

With $F(s) := f(se^{i\phi})$ and $G(s) := g(se^{i\phi})$ we have:

$$\int_0^D dte^{-bt}\left|\int_0^t dsF(s)G(t-s)\right| \le \int_0^D dte^{-bt}\int_0^t ds|F(s)G(t-s)| =$$
$$\int_0^D\int_0^{D-v} e^{-b(u+v)}|F(v)||G(u)|dudv \le$$
$$\int_0^D\int_0^D e^{-b(u+v)}|F(v)||G(u)|dudv = \|f\|_b\|g\|_b \tag{5.75}$$

On the other hand, for $f, g \in \mathcal{A}$ we have $f * g \in \mathcal{A}$. Also,

$$\|f*g\|_u = D\sup_{p\in\mathcal{O}} e^{-b|p|}\left|\int_0^p f(s)g(p-s)ds\right| \le$$
$$D\sup_{p\in\mathcal{O}}\int_0^{|p|}\left|f(te^{i\arg p})e^{-bt}g(p-te^{i\arg p})e^{-b(|p|-t)}\right|dt \tag{5.76}$$

which is less than both $\|f\|_u\|g\|_u$ and $\|f\|_u\|g\|_b$.

□

Remark 5.77 *For f in $\mathcal{A}$ or f in $L^1_b(\Phi_D)$,*

$$\|f\|_{u,b} \to 0 \quad as \quad b\to\infty \tag{5.78}$$

where $\|\|_{u,b}$ is either of the $\|\|_u$ or $\|\|_b$ and $D=\infty$ is allowed in the second case.

For $\|\|_b$, Eq. (5.78) is an immediate consequence of the dominated convergence theorem whereas for $\|\|_u$ it follows from the definition of $\mathcal{A}$.

□

Corollary 5.79 *Let f be continuous along Φ_D, $D<\infty$ and $g\in L^1_b(\Phi_D)$. Given $\epsilon>0$ there exists a large enough b and $K=K(\epsilon,\Phi_D)$ such that for all k*

$$\|f*g^{*k}\|_u < K\,\epsilon^k$$

By Remark 5.77 we can choose $b=b(\epsilon,\Phi_D)$ so large that $\|g\|_b<\epsilon$. Then, by Proposition 5.73 and Eq. (5.72) we have:

$$\left|\int_0^{pe^{i\phi}} f(pe^{i\phi}-s)g^{*k}(s)ds\right| \le D^{-1}e^{b|p|}\|f\|_u\int_0^{pe^{i\phi}} e^{-b|s|}|g^{*k}(s)||ds| \le$$

$$D^{-1}e^{b|p|}\|f\|_u\|g\|_b^k < K\,\epsilon^k$$

□

Remark 5.80 *By (5.67), for any $b > \mu$, and $\Phi_D \subset \mathbb{C}$, $D \le \infty$*

$$\|\mathbf{G_l}\|_b \le K\mu^{|\mathbf{l}|}\int_0^\infty |dp|e^{|p|(\mu-b)} = K\frac{\mu^{|\mathbf{l}|}}{b-\mu} \tag{5.81}$$

where, to avoid cumbersome notations, we write

$$\mathbf{f} \in L_b^1(\Phi_D) \text{ iff } \ \||\mathbf{f}|_\wedge\|_b \in L_b^1(\Phi_D) \tag{5.82}$$

(and similarly for other norms of vector functions).

Proof of Proposition 5.65.

We first show existence and uniqueness in $L_{ray}(\mathcal{D})$ which amounts to nothing more then existence and uniqueness along each $\Phi_D \subset \mathcal{D}$.

Then we show that for large enough b there exists a unique solution of (5.53) in $L_b^1(\Phi_\infty)$. Since this solution is also in $L_{loc}^1(\Phi_\infty)$ it follows that our (unique) L_{loc}^1 solution is Laplace transformable. Analyticity is proven by finding the solution as a fixed point of a contraction with respect to the uniform norm in a suitable space of analytic functions.

Proposition 5.83 *i) For $\Phi_D \in \mathcal{D}$ and large enough b, the operator*

$$\mathcal{N}_1 := \mathbf{Y}(p) \mapsto (\hat{\Lambda} - p)^{-1}\left(\mathbf{F}_0(p) - \hat{B}\int_0^p \mathbf{Y}(s)ds + \mathcal{N}(\mathbf{Y})(p)\right) \tag{5.84}$$

is a contraction in a small enough neighborhood of the origin with respect to $\|\|_u$ if $D < \infty$ and with respect to $\|\|_b$ for $D \le \infty$.

ii) For $D \le \infty$ the operator $\mathcal{N}$ given formally in (5.50) is continuous in $L_{loc}^1(\Phi_D)$. The last sum in (5.50) converges uniformly on compact subsets of Φ_D. $\mathcal{N}(L_{loc}^1(\Phi_D)) \subset AC(\Phi_D)$, the absolutely continuous functions on Φ_D. Moreover, if $\mathbf{v}_n \to \mathbf{v}$ in $\|\|_b$ on Φ_D, $D \le \infty$, then for $b' \ge b$ large enough, $\mathcal{N}(\mathbf{v}_n)$ exist and converge in $\|\|_{b'}$ to $\mathbf{v}$.

The last statements amounts to saying that $\mathcal{N}$ is continuous in the topology of the inductive limit of the L_b^1.

Proof.

Since $\hat{\Lambda}$ and $\hat{B}$ are constant matrices,

$$\|\mathcal{N}_1(\mathbf{Y})\|_{u,b} \le \text{Const}(\Phi)\left(\|\mathbf{F}_0\|_{u,b} + \|\mathbf{Y}\|_{u,b}\|1\|_b + \|\mathcal{N}(\mathbf{Y})\|_{u,b}\right) \tag{5.85}$$

As both $\|1\|_b$ and $\|\mathbf{F}_0\|_{u,b}$ are $O(b^{-1})$ for large b, the fact that $\mathcal{N}_1$ maps a small ball into itself follows from the following Remark.

Remark 5.86 *Let $\epsilon > 0$ be small enough. Then, there is a K such that for large b and all $\mathbf{v}$ such that $\|\mathbf{v}\|_{u,b} =: \delta < \epsilon$,*

$$\|\mathcal{N}(\mathbf{v})\|_{u,b} \le K\left(b^{-1} + \|\mathbf{v}\|_{u,b}\right)\|\mathbf{v}\|_{u,b} \tag{5.87}$$

By (5.68) and (5.81), for large b and some positive constants $C_1, .., C_5$,

$$\begin{aligned}
\|\mathcal{N}(\mathbf{v})\|_{u,b} &\le C_1\left(\sum_{|\mathbf{l}|\ge 1}\|\mathbf{G_l}\|_b\|\mathbf{v}\|_{u,b}^{|\mathbf{l}|} + \sum_{|\mathbf{l}|\ge 2}\|\mathbf{g}_{0,\mathbf{l}}\|_b\|\mathbf{v}\|_{u,b}^{|\mathbf{l}|}\right)\\
&\le \frac{C_2}{b}\left(\sum_{|\mathbf{l}|\ge 1}\frac{\mu^{|\mathbf{l}|}}{b-\mu}\delta^{|\mathbf{l}|} + \sum_{|\mathbf{l}|\ge 2}\mu^{|\mathbf{l}|}\delta^{|\mathbf{l}|}\right) \le \left(C_2\sum_{m=1}^{\infty} + \sum_{m=2}^{\infty}\right)\mu^m\delta^m\sum_{|\mathbf{l}|=m}1\\
&\le \left(\frac{C_4}{b}+\mu\delta\right)\sum_{m=1}^{\infty}\mu^m\delta^m(m+4)^n \le \left(\frac{C_4}{b}+\mu\delta\right)C_5\delta
\end{aligned} \tag{5.88}$$

□

To show that $\mathcal{N}_1$ is a contraction we need the following:

Remark 5.89

$$\|\mathbf{h_l}\| := \|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}} - \mathbf{f}^{*\mathbf{l}}\| \le |\mathbf{l}|\left(\|\mathbf{f}\| + \|\mathbf{h}\|\right)^{|\mathbf{l}|-1}\|\mathbf{h}\| \tag{5.90}$$

where $\|\| = \|\|_u$ or $\|\|_b$.

This estimate will be useful to us when $\mathbf{h}$ is a "small perturbation". The proof of (5.90) is a simple induction on $\mathbf{l}$, with respect to the lexicographic ordering. For $|\mathbf{l}| = 1$, (5.90) is trivial; assume (5.90) holds for all $\mathbf{l} < \mathbf{l}_1$ and that $\mathbf{l}_1$ differs from its predecessor $\mathbf{l}_0$ at the position k (we can take $k = 1$), i.e., $(\mathbf{l}_1)_1 = 1 + (\mathbf{l}_0)_1$. We have:

$$\begin{aligned}
&\|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}_1} - \mathbf{f}^{*\mathbf{l}_1}\| = \|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}_0} * (\mathbf{f}_1+\mathbf{h}_1) - \mathbf{f}^{*\mathbf{l}_1}\| =\\
&\quad\|(\mathbf{f}^{*\mathbf{l}_0} + \mathbf{h}_{\mathbf{l}_0}) * (f_1+h_1) - \mathbf{f}^{*\mathbf{l}_1}\| = \|\mathbf{f}^{*\mathbf{l}_0} * h_1 + \mathbf{h}_{\mathbf{l}_0} * f_1 + \mathbf{h}_{\mathbf{l}_0} * h_1\| \le\\
&\quad\|\mathbf{f}\|^{|\mathbf{l}_0|}\|\mathbf{h}\| + \|\mathbf{h}_{\mathbf{l}_0}\|\|\mathbf{f}\| + \|\mathbf{h}_{\mathbf{l}_0}\|\|\mathbf{h}\| \le\\
&\quad\|\mathbf{h}\|\left(\|\mathbf{f}\|^{|\mathbf{l}_0|} + |\mathbf{l}_0|(\|\mathbf{f}\|+\|\mathbf{h}\|)^{|\mathbf{l}_0|}\right) \le\\
&\quad\|\mathbf{h}\|(|\mathbf{l}_0|+1)(\|\mathbf{f}\|+\|\mathbf{h}\|)^{|\mathbf{l}_0|}
\end{aligned} \tag{5.91}$$

Remark 5.92 *For small δ and large enough b, $\mathcal{N}_1$ defined in a ball centered at zero, of radius δ in the norms $\|\|_{u,b}$ is contractive.*

By (5.85) and (5.87) we know that the ball is mapped into itself for large b. Let $\epsilon > 0$ be small and let $\mathbf{f}, \mathbf{h}$ be such that $\|\mathbf{f}\| < \delta - \epsilon$, $\|\mathbf{h}\| < \epsilon$. Using (5.90) and the notations (5.53) (5.85) and $\|\| = \|\|_{u,b}$ we obtain, for some positive constants $C_1, .., C_4$ and large b,

$$\|\mathcal{N}_1(\mathbf{f}+\mathbf{h}) - \mathcal{N}_1(\mathbf{f})\| \le C_1 \| \left(\sum_{|\mathbf{l}|\ge \mathbf{2}} \mathbf{g}_{0,\mathbf{l}} \cdot + \sum_{|\mathbf{l}|\ge \mathbf{1}} \mathbf{G}_{\mathbf{l}} * \right) \left((\mathbf{f}+\mathbf{h})^{*\mathbf{l}} - \mathbf{f}^{*\mathbf{l}} \right) \| \le$$
$$C_2 \|\mathbf{h}\| \left(\sum_{|\mathbf{l}|\ge \mathbf{1}} \frac{\mu^{|\mathbf{l}|}}{b-\mu} |\mathbf{l}| \delta^{|\mathbf{l}|-1} + \sum_{|\mathbf{l}|\ge \mathbf{2}} |\mathbf{l}| \mu^{|\mathbf{l}|} \delta^{|\mathbf{l}|-\mathbf{1}} \right) < (C_3 b^{-1} + C_4 \delta)\|\mathbf{h}\| \tag{5.93}$$

To finish the proof of Proposition 5.83 take $\mathbf{v} \in \mathcal{A}$. Given $\epsilon > 0$ we can choose b large enough (by Remark 5.77) to make $\|\mathbf{v}\|_u < \epsilon$. Then the sum in the formal definition of $\mathcal{N}$ is convergent in $\mathcal{A}$, by (5.88). Now, if $D < \infty$ $L^1_{loc}(\Phi_D) = L^1_b(\Phi_D)$ for any $b > 0$. If $\mathbf{v}_n \to \mathbf{v}$ in $L^1_b(\Phi_D)$, we choose ϵ small enough, then b large so that $\|\mathbf{v}\|_b < \epsilon$, and finally n_0 large so that for $n > n_0$ $\|\mathbf{v}_n - \mathbf{v}\|_b < \epsilon$ (note that $\|\|\|_b$ decreases w.r. to b) thus $\|\mathbf{v}_n\|_b < 2\epsilon$ and continuity (in $L^1_b(\Phi_D)$ as well as in $L^1_{loc}(\Phi_\infty) \equiv \cup_{k\in\Phi_\infty} L^1_b(0,k)$) follows from Remark 5.92. Continuity with respect to the topology of the inductive limit of the L^1_b is proven in the same way. It is straightforward to show that $\mathcal{N}(L^1_{loc}(\Phi)) \subset AC(\Phi)$.

□ $P_{5.83}$

The fact that $\mathcal{L}_\phi \mathbf{Y}_0$ is a solution of (4.222) follows from Proposition 5.83, from Remark 5.71 and the elementary properties of $\mathcal{L}$ (see also the proof of Proposition 5.150).

Since $\mathbf{Y}_0(p)$ is analytic for small p, $(\mathcal{L}\mathbf{Y}_0)(x)$ has an asymptotic series for large x, which has to agree with $\tilde{\mathbf{y}}_0$ since $\mathcal{L}\mathbf{Y}_0$ solves (4.222). This shows that $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$.

□ $P5.65$

Remark 5.94 *For any δ there is a constant $K_2 = K_2(\delta, |p|)$ so that for all* $\mathbf{l}$ *we have*

$$|\mathbf{Y}_0^{*\mathbf{l}}(p)|_\wedge \le K_2 \delta^{|\mathbf{l}|} \tag{5.95}$$

The estimates (5.95) follow immediately from analyticity and from Corollary 5.79.

□

5.2c Behavior of $\mathbf{Y}_0(p)$ near $p = 1$.

Let $\mathbf{Y}_0$ be the unique solution in $L_{ray}(\mathcal{D})$ of (5.53) and let $\epsilon > 0$ be small. Define

$$\mathbf{H}(p) := \begin{cases} \mathbf{Y}_0(p) & \text{for } p \in \mathcal{D}\,, |p| < 1-\epsilon \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{h}(1-p) := \mathbf{Y}_0(p) - \mathbf{H}(p) \tag{5.96}$$

In terms of $\mathbf{h}$, for real $z = 1-p, z < \epsilon$, the equation (5.53) reads:

$$-(1-z)\mathbf{h}(z) = \mathbf{F}_1(z) - \hat{\Lambda}\mathbf{h}(z) + \hat{B}\int_\epsilon^z \mathbf{h}(s)ds + \mathcal{N}(\mathbf{H}+\mathbf{h}) \tag{5.97}$$

where

$$\mathbf{F}_1(1-s) := \mathbf{F}_0(s) - \hat{B}\int_0^{1-\epsilon} \mathbf{H}(s)ds$$

Proposition 5.98 *i) For small ϵ, $\mathbf{H}^{*\mathbf{l}}(1+z)$ extends to an analytic function in the disk $\mathbb{D}_\epsilon := \{z : |z| < \epsilon\}$. Furthermore, for any δ there is an ϵ and a constant $K_1 := K_1(\delta,\epsilon)$ such that for $z \in \mathbb{D}_\epsilon$ the analytic continuation satisfies the estimate*

$$|\mathbf{H}^{*\mathbf{l}}(1+z)|_\wedge < K_1\delta^l \tag{5.99}$$

Proof.

The case $|\mathbf{l}| = 1$ is trivial: $\mathbf{H}$ itself extends as the zero analytic function. We assume by induction on $|\mathbf{l}|$ that Proposition 5.98 is true for all $\mathbf{l}$, $|\mathbf{l}| \le l$ and show that it then holds for (e.g.) $H_1 * \mathbf{H}^{*\mathbf{l}}$, for all $\mathbf{l}$, $|\mathbf{l}| \le l$.

$\mathbf{H}$ is analytic in an ϵ–neighborhood of $[0, 1-2\epsilon]$, and therefore so is $\mathbf{H}^{*\mathbf{l}}$. Taking first $z \in \mathbb{R}^+,\ z < \epsilon$, we have

$$\int_0^{1-z} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds = \int_0^{1-\epsilon} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds = \\ \int_0^{1/2} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds + \int_{1/2}^{1-\epsilon} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds \tag{5.100}$$

The integral on $[1/2, 1-\epsilon]$ is analytic for small z, since the argument of $\mathbf{H}^{*\mathbf{l}}$ varies in an ϵ-neighborhood of $[0, 1/2]$; the integral on $[0, 1/2)$ equals

$$\int_{1/2-z}^{1-z} H_1(1-z-t)\mathbf{H}^{*\mathbf{l}}(t)dt = \left(\int_{1/2-z}^{1/2} + \int_{1/2}^{1-\epsilon} + \int_{1-\epsilon}^{1-z}\right) H_1(1-z-t)\mathbf{H}^{*\mathbf{l}}(t)dt \tag{5.101}$$

In (5.101) the integral on $[1/2-z,1/2]$ is clearly analytic in $\mathbb{D}_\epsilon$, the following one is the integral of an analytic function of the parameter z with respect to the absolutely continuous measure $\mathbf{H}^{*\mathbf{l}}dt$ whereas in the last integral, both $\mathbf{H}^{*\mathbf{l}}$ (by induction) and H_1 extend analytically in $\mathbb{D}_\epsilon$.

To prove now the the induction step for the estimate (5.99), fix δ small and let:

$$\eta<\delta;\ M_1:=\max_{|p|<1/2+\epsilon}|\mathbf{H}(p)|_\wedge;\ M_2(\epsilon):=\max_{0\le x\le 1-\epsilon}|\mathbf{H}(p)|_\wedge;\ \ \epsilon<\frac{\delta}{4M_1} \tag{5.102}$$

Let $K_2:=K_2(\eta;\epsilon)$ be large enough so that (5.95) holds with η in place of δ for *real* $x\in[0,1-\epsilon]$ and *also* in an ϵ neighborhood in $\mathbb{C}$ of the interval $[0,1/2{+}2\epsilon]$. We use (5.95) to estimate the second integral in the decomposition (5.100) and the first two integrals on the r.h.s. of (5.101). For the last integral in (5.101) we use the induction hypothesis. If $K_1>2K_2\,(2M_1+M_2)$, it follows that $|\mathbf{H}^{*\mathbf{l}}*H_1|_\wedge$ is bounded by (the terms are in the order explained above):

$$M_2(\epsilon)K_2\eta^l+M_1K_2\eta^l+M_1K_2\eta^l+(2\epsilon)M_1K_1\delta^l<K_1\delta^{l+1} \tag{5.103}$$

□

Proposition 5.104 *The equation (5.97) can be written as*

$$-(1-z)\mathbf{h}(z)=\mathbf{F}(z)-\hat\Lambda\mathbf{h}(z)+\hat B\int_\epsilon^z\mathbf{h}(s)ds-\sum_{j=1}^n\int_\epsilon^z h_j(s)\mathbf{D}_j(s-z)ds \tag{5.105}$$

where

$$\mathbf{F}(z):=\mathcal{N}(\mathbf{H})(1-z)+\mathbf{F}_1(z) \tag{5.106}$$

$$\mathbf{D}_j=\sum_{|\mathbf{l}|\ge1}l_j\mathbf{G}_\mathbf{l}*\mathbf{H}^{*\bar{\mathbf{l}}^j}+\sum_{|\mathbf{l}|\ge2}l_j\mathbf{g}_{0,\mathbf{l}}\mathbf{H}^{*\bar{\mathbf{l}}^j};\ \bar{\mathbf{l}}^j:=(l_1,l_2,..(l_j-1),..l_n) \tag{5.107}$$

(cf. also (5.52)) extend to analytic functions in $\mathbb{D}_\epsilon$ (cf. Proposition 5.98). Moreover, if $\mathbf{H}$ *is a vector in $L^1_b(\mathbb{R}^+)$ then, for large b, $\mathbf{D}_j\in L^1_b(\mathbb{R}^+)$ and the functions $\mathbf{F}(z)$ and $\mathbf{D}_j$ extend to analytic functions in $\mathbb{D}_\epsilon$.*

Proof.

Noting that $(\mathbf{Y}_0-\mathbf{H})^{*2}(1-z)=0$ for $\epsilon<1/2$ and $z\in\mathbb{D}_\epsilon$ the result is easily obtained by re-expanding $\mathcal{N}(\mathbf{H}+\mathbf{h})$ since Proposition 5.98 guarantees the uniform convergence of the series thus obtained. The proof that $\mathbf{D}_j\in L^1_b$

for large b is very similar to the proof of (5.93). The analyticity properties follow easily from Proposition 5.98, since the series involved in $\mathcal{N}(\mathbf{H})$ and $\mathbf{D}_j$ converge uniformly for $|z| < \epsilon$.

□

Consider again the equation (5.105). Let $\hat{\Gamma} = \hat{\Lambda} - (1-z)\hat{1}$, where $\hat{1}$ is the identity matrix. By construction $\hat{\Gamma}$ and $\hat{B}$ are block-diagonal, their first block is one-dimensional: $\hat{\Gamma}_{11} = z$ and $\hat{B}_{11} = \beta$. We write this as $\hat{\Gamma} = z \oplus \hat{\Gamma}_c(z)$ and similarly, $\hat{B} = \beta \oplus \hat{B}_c$, where $\hat{\Gamma}_c$ and $\hat{B}_c$ are $(n-1)\times(n-1)$ matrices. $\hat{\Gamma}_c(z)$ and $\hat{\Gamma}_c^{-1}(z)$ are analytic in $\mathbb{D}_\epsilon$.

Lemma 5.108 *The function* $\mathbf{Y}_0$ *given in Proposition 5.65 can be written in the form*

$$\mathbf{Y}_0(p) = (1-p)^{\beta-1}\mathbf{a}_1(p) + \mathbf{a}_2(p) \quad (\beta \neq 1)$$
$$\mathbf{Y}_0(p) \;=\; \ln(1-p)\mathbf{a}_1(p) + \mathbf{a}_2(p) \quad (\beta = 1) \qquad (5.109)$$

for p *in the region* $(\mathbb{D}_\epsilon + 1) \cap \mathcal{D}$ *(*$\mathbb{D}_\epsilon + 1 := \{1 + z : z \in \mathbb{D}_\epsilon\}$*) where* $\mathbf{a}_1,\ \mathbf{a}_2$ *are analytic functions in* $\mathbb{D}_\epsilon + 1$ *and* $(\mathbf{a}_1)_j = 0$ *for* $j > 1$.

Proof.
Let $\mathbf{Q}(z) := \int_\epsilon^z \mathbf{h}(s)ds$. By Proposition 5.65, $\mathbf{Q}$ is analytic in $\mathbb{D}_\epsilon \cap (1-\mathcal{D})$. From (5.105) we obtain

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{Q}'(z) - (\beta \oplus \hat{B}_c)\mathbf{Q}(z) = \mathbf{F}(z) - \sum_{j=1}^{n}\int_\epsilon^z \mathbf{D}_j(s-z)Q_j'(s)ds \quad (5.110)$$

or, after integration by parts in the r.h.s. of (5.110), ($\mathbf{D}_j(0) = 0$, cf. (5.107)),

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{Q}'(z) - (\beta \oplus \hat{B}_c)\mathbf{Q}(z) = \mathbf{F}(z) + \sum_{j=1}^{n}\int_\epsilon^z \mathbf{D}_j'(s-z)Q_j(s)ds \quad (5.111)$$

With the notation $(Q_1, \mathbf{Q}_\perp) := (Q_1, Q_2, .., Q_n)$ we write the system in the form

$$(z^{-\beta}Q_1(z))' = z^{-\beta-1}\left(F_1(z) + \sum_{j=1}^{n}\int_\epsilon^z D_{1j}'(s-z)Q_j(s)ds\right)$$
$$(e^{\hat{C}(z)}\mathbf{Q}_\perp)' = e^{\hat{C}(z)}\hat{\Gamma}_c(z)^{-1}\left(\mathbf{F}_\perp + \sum_{j=1}^{n}\int_\epsilon^z \mathbf{D}_\perp'(s-z)Q_j(s)ds\right)$$
$$\hat{C}(z) := -\int_0^z \hat{\Gamma}_c(s)^{-1}\hat{B}_c(s)ds$$
$$\mathbf{Q}(\epsilon) = 0 \quad (5.112)$$

After integration we get:

$$Q_1(z) = R_1(z) + J_1(\mathbf{Q})$$
$$\mathbf{Q}_\perp(z) = \mathbf{R}_\perp(z) + J_\perp(\mathbf{Q}) \quad (5.113)$$

with

$$J_1(\mathbf{Q}) = z^\beta \int_\epsilon^z t^{-\beta-1} \sum_{j=1}^n \int_\epsilon^t Q_j(s) D'_{1j}(t-s)dsdt$$
$$J_\perp(\mathbf{Q})(z) := e^{-\hat{C}(z)} \int_\epsilon^z e^{\hat{C}(t)} \hat{\Gamma}_c(t)^{-1} \left(\sum_{j=1}^n \int_\epsilon^z \mathbf{D}'_\perp(s-z) Q_j(s) ds \right) dt$$
$$\mathbf{R}_\perp(z) := e^{-\hat{C}(z)} \int_\epsilon^z e^{\hat{C}(t)} \hat{\Gamma}_c(t)^{-1} \mathbf{F}_\perp(t) dt$$
$$R_1(z) = z^\beta \int_\epsilon^z t^{-\beta-1} F_1(t) dt \qquad (\beta \neq 1)$$
$$R_1(z) = F_1(0) + F_1'(0) z \ln z + z \int_\epsilon^z \frac{F_1(s) - F_1(0) - sF_1'(0)}{s} ds \quad (\beta = 1)$$
$$(5.114)$$

Consider the following space of functions:

$$\mathcal{T}_\beta = \Big\{ \mathbf{Q} \text{ analytic in } \mathbb{D}_\epsilon \cap (\mathcal{D} - 1) : \mathbf{Q} = z^\beta \mathbf{A}(z) + \mathbf{B}(z) \Big\} \text{ for } \beta \neq 1 \text{ and}$$
$$\mathcal{T}_1 = \Big\{ \mathbf{Q} \text{ analytic in } \mathbb{D}_\epsilon \cap (\mathcal{D} - 1) : \mathbf{Q} = z \ln z \mathbf{A}(z) + \mathbf{B}(z) \Big\} \quad (5.115)$$

where $\mathbf{A}, \mathbf{B}$ are analytic in $\mathbb{D}_\epsilon$. (The decomposition of $\mathbf{Q}$ in (5.115) is unambiguous since z^β and $z \ln z$ are not meromorphic in $\mathbb{D}_\epsilon$.)

The norm

$$\|\mathbf{Q}\| = \sup \{ |\mathbf{A}(z)|_\wedge, |\mathbf{B}(z)|_\wedge : z \in \mathbb{D}_\epsilon \} \quad (5.116)$$

makes $\mathcal{T}_\beta$ a Banach space.

For $A(z)$ analytic in $\mathbb{D}_\epsilon$ the following elementary identities are useful in what follows:

$$\int_\epsilon^z A(s) s^r ds = Const + z^{r+1} \int_0^1 A(zt) t^r dt = Const + z^{r+1} Analytic(z)$$
$$\int_0^z s^r \ln s \, A(s) ds = z^{r+1} \ln z \int_0^1 A(zt) t^r dt + z^{r+1} \int_0^1 A(zt) t^r \ln t dt$$
$$(5.117)$$

where the second equality is obtained by differentiating with respect to r the first equality.

Using (5.117) it is straightforward to check that the r.h.s. of (5.113) extends to a linear inhomogeneous operator on $\mathcal{T}_\beta$ with image in $\mathcal{T}_\beta$ and that the norm of J is $O(\epsilon)$ for small ϵ. For instance, one of the terms in J for $\beta = 1$,

$$z\int_0^z t^{-2}\int_0^t s\ln s\, A(s)D'(t-s)ds = \\ z^2\ln z\int_0^1\int_0^1 \sigma A(z\tau\sigma)D'(z\tau - z\tau\sigma)d\sigma d\tau + \\ z^2\int_0^1 d\tau\int_0^1 d\sigma\sigma(\ln\tau+\ln\sigma)A(z\tau\sigma)D'(z\tau - z\tau\sigma) \tag{5.118}$$

manifestly in $\mathcal{T}_\beta$ if A is analytic in $\mathbb{D}_\epsilon$. Comparing with (5.115), the extra power of z accounts for a norm $O(\epsilon)$ for this term.

Therefore, in (5.112) $(1-J)$ is invertible and the solution $\mathbf{Q}\in\mathcal{T}_\beta\subset\mathcal{L}(\mathcal{D})$. In view of the the uniqueness of $\mathbf{Y}_0$ (cf. Proposition 5.65), the rest of the proof of Lemma 5.108 is immediate.

5.2d The solutions of (5.53) on

$[0, 1+\epsilon]$

Let $\mathbf{Y}_0$ be the solution given by Proposition 5.65, take ϵ small enough and denote by $\mathcal{O}_\epsilon$ a neighborhood in $\mathbb{C}$ of width ϵ of the interval $[0, 1+\epsilon]$.

Remark 5.119 . *$\mathbf{Y}_0\in L^1(\mathcal{O}_\epsilon)$. As $\phi\to\pm 0$, $\mathbf{Y}_0(pe^{i\phi})\to\mathbf{Y}_0^{\pm}(p)$ in the sense of $L^1([0,1+\epsilon])$ and also in the sense of pointwise convergence for $p\neq 1$, where*

$$\mathbf{Y}_0^{\pm} := \begin{cases}\mathbf{Y}_0(p) & p<1 \\ (1-p\pm 0i)^{\beta-1}\mathbf{a}_1(p)+\mathbf{a}_2(p) & p>1\end{cases} \quad (\beta\neq 1)$$

$$\mathbf{Y}_0^{\pm} := \begin{cases}\mathbf{Y}_0(p) & p<1 \\ \ln(1-p\pm 0i)\mathbf{a}_1(p)+\mathbf{a}_2(p) & p>1\end{cases} \quad (\beta = 1) \tag{5.120}$$

Moreover, $\mathbf{Y}_0^{\pm}$ are L^1_{loc} solutions of the convolution equation (5.53) on the interval $[0,1+\epsilon]$.

The proof is immediate from Lemma 5.108 and Proposition 5.83.

□

Proposition 5.121 *For any $\lambda\in\mathbb{C}$ the combination $\mathbf{Y}_\lambda = \lambda\mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$ is a solution of (5.53) on $[0,1+\epsilon]$.*

Proof. For $p \in [0,1) \cup (1, 1+\epsilon]$ let $\mathbf{y}_\lambda(p) := \mathbf{Y}_\lambda - \mathbf{H}(p)$. Since $\mathbf{y}_\lambda^{*2} = 0$ the equation (5.53) is actually linear in $\mathbf{y}_\lambda$ (compare with (5.105)).

□

*

Note: We consider the application $\mathcal{Y} := \mathbf{y}_0 \mapsto \mathbf{Y}_\lambda$ and require that it is compatible with complex conjugation of functions $\mathcal{Y}(\mathbf{y}_0{}^*) = (\mathcal{Y}(\mathbf{y}_0))^*$ where $F^*(z) := \overline{F(\overline{z})}$. We get $\Re\lambda = 1/2$. It is natural to choose $\lambda = 1/2$ to make the linear combination a true average. This choice corresponds, on $[0, 1+\epsilon]$, to the balanced averaging (5.43).

*

Remark 5.122 *For any $\delta > 0$ there is a constant $C(\delta)$ such that for large b*

$$\|(\mathbf{Y}_0^{ba})^{*\mathbf{l}}\|_u < C(\delta)\delta^{|\mathbf{l}|} \quad \forall \mathbf{l} \text{ with } |\mathbf{l}| > 1 \tag{5.123}$$

($\|\|_u$ is taken on the interval $[0, 1+\epsilon]$).

Without loss of generality, assume that $l_1 > 1$. Using the notation (5.107),

$$\left\| \int_0^p (\mathbf{Y}_0)_1^{ba}(s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(p-s)ds \right\|_u \le$$
$$\left\| \int_0^{\frac{p}{2}} (\mathbf{Y}_0^{ba})_1(s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(p-s)ds \right\|_{u_2} + \left\| \int_0^{\frac{p}{2}} (\mathbf{Y}_0)_1(p-s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(s)ds \right\|_{u_2} \tag{5.124}$$

($\|\|_{u_2}$ refers to the interval $p \in [0, 1/2 + \epsilon/2]$.) The first u_2 norm can be estimated directly using Corollary 5.79 whereas we majorize the second one by

$$\|(\mathbf{Y}_0^{ba})_1\|_b \|(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(x)\|_{u_2}$$

and apply Corollary 5.79 to it for $|\mathbf{l}| > 2$ (if $|\mathbf{l}| = 2$ simply observe that $(\mathbf{Y}_0^{ba})^{*\mathbf{l}}$ is analytic on $[0, 1/2 + \epsilon/2]$).

□

Lemma 5.125 *The set of all solutions of (5.53) in $L^1_{loc}([0, 1+\epsilon])$ is parameterized by a complex constant C and is given by*

$$\mathbf{Y}_0(p) = \begin{cases} \mathbf{Y}_0^{ba}(p) & \text{for } p \in [0,1) \\ \mathbf{Y}_0^{ba}(p) + C(p-1)^{\beta-1}\mathbf{A}(p) & \text{for } p \in (1, 1+\epsilon] \end{cases} \tag{5.126}$$

for $\beta \ne 1$ or, for $\beta = 1$,

$$\mathbf{Y}_0(p) = \begin{cases} \mathbf{Y}_0^{ba}(p) & \textit{for } p \in [0,1) \\ \mathbf{Y}_0^{ba}(p) + C(p-1)\mathbf{A}(p) & \textit{for } p \in (1, 1+\epsilon] \end{cases} \tag{5.126$'$}$$

where $\mathbf{A}$ *extend analytically in a neighborhood of* $p = 1$.

Different values of C *correspond to different solutions.*

This result remains true if $\mathbf{Y}_0^{ba}$ *is replaced by any other combination* $\mathbf{Y}_\lambda := \lambda\mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$, $\lambda \in \mathbb{C}$.

Proof.

We look for solutions of (5.53) in the form

$$\mathbf{Y}^{ba}(p) + \mathbf{h}(p-1) \tag{5.127}$$

By Lemma 5.108 , $\mathbf{h}(p-1) = 0$ for $p < 1$. Note that

$$\mathcal{N}(\mathbf{Y}_0^{ba} \circ \tau_{-1} + \mathbf{h})(z) = \mathcal{N}(\mathbf{Y}_0^{ba})(1+z) + \sum_{j=1}^{n} \int_0^z h_j(s)\mathbf{D}_j(z-s)ds \tag{5.128}$$

where the $\mathbf{D}_j$ are given in (5.107), and by Remark 5.123 all the infinite sums involved are uniformly convergent. For $z < \epsilon$ (5.53) translates to (compare with (5.105)):

$$-(1+z)\mathbf{h}(z) = -\hat{\Lambda}\mathbf{h}(z) - \hat{B}\int_0^z \mathbf{h}(s)ds + \sum_{j=1}^{n} \int_0^z h_j(s)\mathbf{D}_j(z-s)ds \tag{5.129}$$

Let

$$\mathbf{Q}(z) := \int_0^z \mathbf{h}(s)ds \tag{5.130}$$

As we are looking for solutions $\mathbf{h} \in L^1$, we have $\mathbf{Q} \in AC[0,\epsilon]$ and $\mathbf{Q}(0) = 0$. Following the same steps as in the proof of Lemma 5.108 we get the system of equations:

$$\begin{gathered}
(z^{-\beta}Q_1(z))' = z^{-\beta-1}\sum_{j=1}^{n}\int_0^z D'_{1j}(z-s)Q_j(s)ds \\
(e^{\hat{C}(z)}\mathbf{Q}_\perp)' = e^{\hat{C}(z)}\hat{\Gamma}_c(z)^{-1}\sum_{j=1}^{n}\int_0^z \mathbf{D}'_\perp(z-s)Q_j(s)ds \\
\hat{C}(z) := -\int_0^z \hat{\Gamma}_c(s)^{-1}\hat{B}_c(s)ds \\
\mathbf{Q}(0) = 0
\end{gathered} \tag{5.131}$$

which by integration gives

$$(\hat{1} + J)\mathbf{Q}(z) = C\mathbf{R}(z) \tag{5.132}$$

where $C \in \mathbb{C}$ and

$$\begin{aligned}
(J(\mathbf{Q}))_1(z) &= z^\beta \int_0^z t^{-\beta-1} \sum_{j=1}^n \int_0^t Q_j(s) D'_{1j}(t-s) ds dt \\
J(\mathbf{Q})_\perp(z) &:= e^{-\hat{C}(z)} \int_0^z e^{\hat{C}(t)} \hat{\Gamma}_c(t)^{-1} \left(\sum_{j=1}^n \int_0^z \mathbf{D}'_\perp(z-s) Q_j(s) ds \right) dt \\
\mathbf{R}_\perp &= 0 \\
R_1(z) &= z^\beta
\end{aligned} \tag{5.133}$$

First we note the presence of an arbitrary constant C in (5.132) (Unlike in Lemma 5.108 when the initial condition, given at $z = \epsilon$ was determining the integration constant, now the initial condition $\mathbf{Q}(0) = 0$ is satisfied for all C).

For small ϵ the norm of the operator J defined on $AC[0, \epsilon]$ is $O(\epsilon)$, as in the proof of Lemma 5.108. Given C the solution of the system (5.131) is unique and can be written as

$$\mathbf{Q} = C\mathbf{Q}_0; \quad \mathbf{Q}_0 := (\hat{1} + J)^{-1}\mathbf{R} \neq 0 \tag{5.134}$$

It remains to find the analytic structure of $\mathbf{Q}_0$. We now introduce the space

$$\mathcal{T} = \left\{ \mathbf{Q} : [0, \epsilon) \mapsto \mathbb{C}^n : \mathbf{Q} = z^\beta \mathbf{A}(z) \right\} \tag{5.135}$$

where $\mathbf{A}(z)$ extends to an analytic function in $\mathbb{D}_\epsilon$. With the norm (5.116) (with $\mathbf{B} \equiv \mathbf{0}$), $\mathcal{T}$ is a Banach space. As in the proof of Lemma 5.108 the operator J extends naturally to $\mathcal{T}$ where it has a norm $O(\epsilon)$ for small ϵ. It follows immediately that

$$\mathbf{Q}_0 \in \mathcal{T} \tag{5.136}$$

The formulas (5.126), (5.126') follow from (5.127) and (5.130).

□

Remark 5.137 *If $S_\beta \neq 0$ (cf. Lemma 5.108) then the* general *solution of (5.53) is given by*

$$\mathbf{Y}_0(p) = (1 - \lambda)\mathbf{Y}_0^+(p) + \lambda \mathbf{Y}_0^-(p) \tag{5.138}$$

with $\lambda \in \mathbb{C}$.

Indeed, if $\mathbf{a}_1 \not\equiv 0$ (cf. Lemma 5.108) we get at least two distinct solutions of (5.132) (i.e., two distinct values of C) by taking different values of λ in (5.138). The remark follows from (5.136) (5.135) and Lemma 5.125..

□

5.2e The solutions of (5.53) on $[0, \infty)$

In this section we show that the leading asymptotic behavior of $\mathbf{Y}_p$ as $p \to 1_+$ determines a unique solution of (5.53) in $L^1_{loc}(\mathbb{R}^+)$. Furthermore, any L^1_{loc} solution of (5.53) is exponentially bounded at infinity and thus Laplace transformable. We also study some properties of these solutions and of their Laplace transforms.

Let $\mathbf{H}$ be a solution of (5.53) on an interval $[0, 1+\epsilon]$, which we extend to $\mathbb{R}^+$ letting $\mathbf{H}(p) = 0$ for $p > 1+\epsilon$. For a large enough b, define

$$\mathcal{S}_{\mathbf{H}} := \{f \in L^1_{loc}([0,\infty)) : f(p) = \mathbf{H}(p) \text{ on } [0, 1+\epsilon]\} \tag{5.139}$$

and

$$\mathcal{S}_0 := \{f \in L^1_{loc}([0,\infty)) : f(p) = 0 \text{ on } [0, 1+\epsilon]\} \tag{5.140}$$

We extend $\mathbf{H}$ to $\mathbb{R}^+$ by putting $\mathbf{H}(p) = 0$ for $p > 1+\epsilon$; for $p \geq 1+\epsilon$ (5.53) reads:

$$-p(\mathbf{H}+\mathbf{h}) = F_0 - \hat{\Lambda}(\mathbf{H}+\mathbf{h}) - \hat{B}\int_0^p (\mathbf{H}+\mathbf{h})(s)ds + \mathcal{N}(\mathbf{H}+\mathbf{h}) \tag{5.141}$$

with $\mathbf{h} \in \mathcal{S}_0$, or

$$\mathbf{h} = -\mathbf{H} + (\hat{\Lambda} - p)^{-1}\left(F_0 - \hat{B}\int_0^p (\mathbf{H}+\mathbf{h})(s)ds + \mathcal{N}(\mathbf{H}+\mathbf{h})\right) := \mathcal{M}(\mathbf{h}) \tag{5.142}$$

For small $\phi_0 > 0$ and $0 \leq \rho_1 < \rho_2 \leq \infty$, consider the truncated sectors

$$S^{\pm}_{(\rho_1,\rho_2)} := \{z : z = \rho e^{\pm i\phi}, \rho_1 < \rho < \rho_2;\ 0 \leq \phi < \phi_0\} \tag{5.143}$$

and the spaces of functions analytic in $S^{\pm}_{(\rho_1,\rho_2)}$ and continuous in its closure:

$$\mathcal{T}^{\pm}_{\rho_1,\rho_2} = \left\{f : f \in C(\overline{S_{(\rho_1,\rho_2)}});\ f \text{ analytic in } S^{\pm}_{(\rho_1,\rho_2)}\right\} \tag{5.144}$$

which are Banach spaces with respect to $\|\|_u$ on compact subsets of $\overline{S_{(\rho_1,\rho_2)}}$.

Proposition 5.145 *i) Given* $\mathbf{H}$*, the equation (5.142) has a unique solution in* $L^1_{loc}[1+\epsilon, \infty)$*. For large* b*, this solution is in* $L^1_b([1+\epsilon,\infty))$ *and thus Laplace transformable.*

ii) Let $\mathbf{Y}_0$ *be the solution defined in Proposition 5.65. Then*

$$\mathbf{Y}_0^{\pm}(p) := \lim_{\phi\to\pm 0} \mathbf{Y}_0(pe^{i\phi}) \in C(\mathbb{R}^+\backslash\{1\}) \cap L^1_{loc}(\mathbb{R}^+) \tag{5.146}$$

(and the limit exists pointwise on $\mathbb{R}^+\backslash\{1\}$ *and in* $L^1_{loc}(\mathbb{R}^+)$*.)*
Furthermore, $\mathbf{Y}_0^{\pm}$ *are particular solutions of (5.53) and*

$$\mathbf{Y}_0^{\pm}(p) = (1-p)^{\beta-1}\mathbf{a}^{\pm}(p) + \mathbf{a}_1^{\pm}(p) \quad (\beta \neq 1)$$
$$\mathbf{Y}_0^{\pm}(p) = \ln(1-p)\,\mathbf{a}^{\pm}(p) + \mathbf{a}_1^{\pm}(p) \quad (\beta = 1) \tag{5.147}$$

where $\mathbf{a}^{\pm}$ *and* $\mathbf{a}_1^{\pm}$ *are in* $\mathcal{T}^{\pm}_{0,\infty}$.

Proof

Note first that by Proposition 5.83, $\mathcal{M}$ (eq. (5.142)) is well defined on $\mathcal{S}_0$, (eq.(5.140)). Moreover, since $\mathbf{H}$ is a solution of (5.53) on $[0, 1+\epsilon)$, we have, for $\mathbf{h}_0 \in \mathcal{S}_0$, $\mathcal{M}(\mathbf{h}) = 0$ a.e. on $[0, 1+\epsilon)$, i.e.,

$$\mathcal{M}(\mathcal{S}_0) \subset \mathcal{S}_0$$

Remark 5.148 *For large b,* $\mathcal{M}$ *is a contraction in a small neighborhood of the origin in* $\|\|_{u,b}$.

Indeed, $\sup\{\|(\hat{\Lambda}-p)^{-1}\|_{\mathbb{C}^n\mapsto\mathbb{C}^n} : p \geq 1+\epsilon\} = O(\epsilon^{-1})$ so that

$$\|\mathcal{M}(\mathbf{h}_1) - \mathcal{M}(\mathbf{h}_2)\|_{u,b} \leq \frac{\text{Const}}{\epsilon}\|\mathcal{N}(\mathbf{f}+\mathbf{h}) - \mathcal{N}(\mathbf{f})\|_{u,b} \tag{5.149}$$

The rest follows from (5.93) —Proposition 5.83 and Remark 5.77 applied to $\mathbf{H}$.

□

The existence of a solution of (5.142) in $\mathcal{S}_0 \cap L^1_b([0,\infty))$ for large enough b is now immediate.

Uniqueness in L^1_{loc} is tantamount to uniqueness in $L^1([1+\epsilon, K]) = L^1_b([1+\epsilon, K]$, for all $K-1-\epsilon \in \mathbb{R}^+$. Now, assuming $\mathcal{M}$ had two fixed points in $L^1_b([1+\epsilon, K])$, by Remark 5.77, we can choose b large enough so that these solutions have arbitrarily small norm, in contradiction with Remark 5.148.

ii). For $p < 1, \mathbf{Y}_0^{\pm}(p) = \mathbf{Y}_0(p)$. For $p \in (1, 1+\epsilon)$ the result follows from Lemma 5.108. Noting that (in view of the estimate (5.88)) $\mathcal{M}(\mathcal{T}^{\pm}{}_{1+\epsilon,\infty}) \subset \mathcal{T}^{\pm}{}_{1+\epsilon,\infty}$, the rest of the proof follows from the Remark 5.148 and Lemma 5.108.

□

Proposition 5.150 *There is a one parameter family of solutions of equation (5.53) in* $L^1_{loc}[0,\infty)$*, branching off at* $p = 1$ *and in a neighborhood of* $p = 1$ *all solutions are of the form (5.126), (5.126'). The general solution of (5.53) is Laplace transformable for large b and the Laplace transform is a solution of the original differential equation in the half-space* $\Re(x) > b$.

Note: As of now, the correspondence (5.126), (5.126') with the balanced average (5.43) is proven only near $p = 1$; the complete correspondence is established in Section 5.2g .

Proof.

Let $\mathbf{Y}$ be any solution of (5.53). By Lemma 5.125 and Proposition 5.145, b large implies that $\mathbf{Y} \in L^1_b([0,\infty))$ (thus $\mathcal{L}\mathbf{Y}$ exists), that $\|\mathbf{Y}\|_b$ is small and, in particular, that the sum defining $\mathcal{N}$ in (5.50) is convergent in $L^1_b(\mathbb{R}^+)$.

By Remark 5.71,

$$\mathcal{L}\left(\sum_{|\mathbf{l}|\ge \mathbf{1}} \mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*\mathbf{l}} + \sum_{|\mathbf{l}|\ge 2} \mathbf{g}_{0,\mathbf{l}} \mathbf{Y}^{*\mathbf{l}}\right)(x) = \sum_{|\mathbf{l}|\ge \mathbf{1}} (\mathcal{L}\mathbf{G}_{\mathbf{l}})(\mathcal{L}\mathbf{Y})^{\mathbf{l}}(x) + \sum_{|\mathbf{l}|\ge 2} \mathbf{g}_{0,\mathbf{l}}\,(\mathcal{L}\mathbf{Y})^{\mathbf{l}} = \sum_{|\mathbf{l}|\ge \mathbf{1}} \mathbf{g}_{\mathbf{l}}(x)\mathbf{y}^{\mathbf{l}}(x) = \mathbf{g}(x,\mathbf{y}(x)) \tag{5.151}$$

(and $\mathbf{g}(x,\mathbf{y}(x))$ is analytic for $\Re(x) > b$). The rest is straightforward.

□

5.2f Correspondence with formal solutions

Finally we consider formal solutions *for large argument* of the differential equation, in the differential algebra generated by formal power series (in decreasing powers of the large variable) and (decreasing) exponentials, i.e. solutions as formal asymptotic expansions. The theory of formal solutions is classical ([29], [26] [27]); see also [5] for a vast and very interesting generalization. We only sketch the facts that are relevant to us.

The simplest formal solution of (4.222) is an asymptotic series $\tilde{\mathbf{y}}_0$.

$$\tilde{\mathbf{y}}_0 = \sum_{m=2}^{\infty} \frac{\mathbf{y}_{0,m}}{x^m}$$

In view of the invertibility of $\hat{\Lambda}$, the coefficients $\{\mathbf{y}_{0,m}\}_{m\in\mathbb{N}} \subset \mathbb{C}^n$ can be determined uniquely by expanding in (4.222) in powers of $1/x$ and equating the coefficients of the $x^{-m}, m \ge 2$. The series $\tilde{\mathbf{y}}_0$ is generically divergent.

Since we expect an $n-$*parameter* family of solutions, we look for further solutions as perturbations of $\tilde{\mathbf{y}}_0$. Because of the uniqueness of $\tilde{\mathbf{y}}_0$ a perturbation must be smaller than all powers of x^{-1} i.e., "beyond all orders" of $\tilde{\mathbf{y}}_0$.

Taking $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \tilde{\mathbf{y}}_1$ we get, to the lowest order of approximation, $\tilde{\mathbf{y}}_1' = -\hat{\Lambda}\tilde{\mathbf{y}}_1$. The solutions to this approximate equation are linear combinations of $e^{-\lambda x}$, $\lambda \in \mathrm{spec}\hat{\Lambda}$. We only consider solutions $\tilde{\mathbf{y}}_1$ that are (formally) small perturbations of $\tilde{\mathbf{y}}_0$ in the half-plane $\Re(x) > 0$; this condition selects out the eigenvalue $\lambda = 1$.

Continuing the perturbative procedure until we reach a formal solution of (4.222), we end up with an exponential series

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{k=1}^{\infty} e^{-kx} \tilde{\mathbf{y}}_k \tag{5.152}$$

where $\tilde{\mathbf{y}}_k$ are formal power series. Substituting (5.152) into (4.222) and using the fact that $\tilde{\mathbf{y}}_0$ is already a formal solution we get for $\tilde{\mathbf{y}}_k, k \geq 1$:

$$\begin{aligned} \sum_{k=1}^{\infty} e^{-kx} \left[\tilde{\mathbf{y}}_k' - \left(k - \hat{\Lambda} - \frac{1}{x}\hat{B} + \partial \mathbf{g}(x, \tilde{\mathbf{y}}_0) \right) \tilde{\mathbf{y}}_k \right] &= \\ \sum_{|\mathbf{l}|>1} \frac{\mathbf{g}^{(\mathbf{l})}(x, \tilde{\mathbf{y}}_0)}{\mathbf{l}!} \left(\sum_{k=1}^{\infty} e^{-kx} \tilde{\mathbf{y}}_k \right)^{\mathbf{l}} &= \\ \sum_{k=2}^{\infty} e^{-kx} \sum_{|\mathbf{l}|>1} \frac{\mathbf{g}^{(\mathbf{l})}(x, \tilde{\mathbf{y}}_0)}{\mathbf{l}!} \sum_{\Sigma m=} \prod_{i=1}^{n} \prod_{j=1}^{l_i} (\tilde{\mathbf{y}}_{m_{i,j}})_i & \end{aligned} \tag{5.153}$$

Equating the coefficients of $e^{-kx}, k \geq 0$ we get the system (5.31).

By assumption, $\hat{\Lambda} - 1$ has a one-dimensional null-space. Thus, by (5.31), $\tilde{\mathbf{y}}_1$ has the freedom of an arbitrary multiplicative constant. We make a definite choice of $\tilde{\mathbf{y}}_1$ by requiring that the first component of the coefficient of the leading power of x is one.

Still by assumption, for $k \neq 1$ $\hat{\Lambda} - k$ is invertible, so that, taking $C = 1$, all $\tilde{\mathbf{y}}_k$, $k \geq 1$, are uniquely determined. Letting C be arbitrary we get instead $C\tilde{\mathbf{y}}_1$ for $k = 1$, $C^2\tilde{\mathbf{y}}_2$ for $k = 2$ (because of the condition $\sum m = 2$), etc, so that the general formal solution of type (5.152) is

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{\mathbf{y}}_k$$

The existence of formal exponential solutions has been considered in [29], [26], [30] and a very comprehensive theory can be found in Ecalle [4], [5], [6].

The following proposition is a classical result and is a specialization of general theorems (see [30]).

Proposition 5.154 *There is exactly a one parameter family of solutions of (4.222) having the asymptotic behavior described by* $\tilde{\mathbf{y}}_0$ *in the half-plane* $\Re(x) > 0$.

Proof. Any solution with the properties stated in Proposition 5.154 is inverse Laplace transformable and its inverse Laplace transform has to be one of the L^1_{loc} solutions of the convolution equation (5.53). The rest of the proof follows from Proposition 5.150.

□

Proposition 5.155 *Let* $\mathbf{Y}$ *be any* $L^1_{loc}(\mathbb{R}^+)$ *solution of (5.53). For large* b *and some* $\nu > 0$ *the coefficients* $\mathbf{d_m}$ *in (5.55) are bounded by*

$$|\mathbf{d_m}(p)|_\wedge \le e^{\mu p}\nu^{|\mathbf{m}|}$$

Note that $\mathcal{L}^{-1}(\mathbf{g}^{(\mathbf{m})}(x,\mathbf{y})/\mathbf{m}!)$ is the coefficient of $\mathbf{Z}^{*\mathbf{m}}$ in the expansion of $\mathcal{N}(\mathbf{Y}+\mathbf{Z})$ in convolution powers of Z (5.50):

$$\left(\left(\sum_{|\mathbf{l}|\ge 2}\mathbf{g}_{0,\mathbf{l}}\cdot + \sum_{|\mathbf{l}|\ge 1}\mathbf{G_l}*\right)(\mathbf{Y}+\mathbf{Z})^{*\mathbf{l}}\right)_{\mathbf{Z}^{*\mathbf{m}}} =$$
$$\left(\left(\sum_{|\mathbf{l}|\ge 2}\mathbf{g}_{0,\mathbf{l}}\cdot + \sum_{|\mathbf{l}|\ge 1}\mathbf{G_l}*\right)\sum_{0\le\mathbf{k}\le\mathbf{l}}\binom{\mathbf{l}}{\mathbf{k}}\mathbf{Z}^{*\mathbf{k}}\mathbf{Y}^{*(\mathbf{l}-\mathbf{k})}\right)_{\mathbf{Z}^{*\mathbf{m}}} =$$
$$\left(\sum_{|\mathbf{l}|\ge 2}\mathbf{g}_{0,\mathbf{l}}\cdot + \sum_{|\mathbf{l}|\ge 1}\mathbf{G_l}*\right)\sum_{\mathbf{l}\ge\mathbf{m}}\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G_l}*\mathbf{Y}^{*(\mathbf{l}-\mathbf{m})} \tag{5.156}$$

($\mathbf{m}$ is fixed) where $\mathbf{l}\ge\mathbf{m}$ means $l_i \ge m_i, i = 1..n$ and $\binom{\mathbf{l}}{\mathbf{k}} := \prod_{i=1}^n \binom{l_i}{k_i}$.

Let ϵ be small and b large so that $\|\mathbf{Y}\|_b < \epsilon$. Then, for some constant K, estimate (cf. (5.67))

$$\left|\left(\sum_{II}\mathbf{g}_{0,\mathbf{l}}\cdot + \sum_{I}\mathbf{G_l}*\right)\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G_l}*\mathbf{Y}^{*(\mathbf{l}-\mathbf{m})}\right|_\wedge \le \sum_I Ke^{\mu|p|}(\mu\epsilon)^{|\mathbf{l}-\mathbf{m}|}\binom{\mathbf{l}}{\mathbf{m}} =$$

$$\epsilon^{-|\mathbf{m}|}Ke^{\mu|p|}\prod_{i=1}^n\sum_{l_i\ge m_i}\binom{l_i}{m_i}(\mu\epsilon)^{l_i} = K\frac{e^{\mu|p|}\mu^{|\mathbf{m}|}}{(1-\epsilon\mu)^{|\mathbf{m}|+n}} < e^{\mu|p|}\nu^{|\mathbf{m}|} \tag{5.157}$$

(where $I(II) \equiv \{|\mathbf{l}| \ge 1(2); \mathbf{l}\ge\mathbf{m}\}$) for large enough ν.

□

For $k = 1$, $\mathbf{R}_1 = 0$ and equation (5.55) is (5.129) (with $p \leftrightarrow z$) but now on the whole line $\mathbb{R}^+$. For small z the solution is given by (5.134) (note that $\mathbf{D}_1 = \mathbf{d}_{(1,0,..,0)}$ and so on) and depends on the free constant C (5.134). We choose a value for C (the values of $\mathbf{Y}_1$ on $[0,\epsilon]$ are then determined) and we write the equation of $\mathbf{Y}_1$ for $p \ge \epsilon$:

$$(\hat{\Lambda} - 1 - p)\mathbf{Y}_1(p) + \hat{B}\int_\epsilon^p \mathbf{Y}_1(s)ds - \sum_{j=1}^n \int_\epsilon^p (\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)ds =$$

$$\mathbf{R}(p) := \int_0^\epsilon \mathbf{Y}_1(s)ds + \sum_{j=1}^n \int_0^\epsilon (\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)ds \quad (5.158)$$

($\mathbf{R}$ only depends on the values of $\mathbf{Y}_1(p)$ on $[0,\epsilon]$). We write

$$(1 + J_1)\mathbf{Y}_1 = \hat{Q}_1^{-1}\mathbf{R} \quad (5.159)$$

with $Q_1 = 1 - \hat{\Lambda} + p$. The operator J_1 is defined by $(J_1\mathbf{Y}_1)(p) := 0$ for $p < \epsilon$, while, for $p > \epsilon$,

$$(J_1\mathbf{Y}_1)(p) := Q_1^{-1}\left(\hat{B}\int_\epsilon^p \mathbf{Y}_1(s)ds - \sum_{j=1}^n \int_\epsilon^p (\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)ds\right)$$

By Proposition 5.104, (5.74) and Remark 5.77, noting that $\sup_{p>\epsilon}\|Q_1^{-1}\| = O(\epsilon^{-1})$, b we find that $(1 + J_1)$ is invertible as an operator in L^1_b since:

$$\|J_1\|_{L^1_b \mapsto L^1_b} < \sup_{p>\epsilon}\|\hat{Q}_1^{-1}\|\left(\|\hat{B}\|\|1\|_b + n\max_{1\le j\le n}\|\mathbf{D_j}\|_b\right) \to 0 \text{ as } b \to \infty \quad (5.160)$$

Given C, $\mathbf{Y}_1$ is therefore uniquely determined from (5.159) as an $L^1_b(\mathbb{R}^+)$ function.

The analytic structure of $\mathbf{Y}_1$ for small z is contained in in (5.126), (5.126'). As a result,

$$\mathcal{L}(\mathbf{Y}_1)(x) \sim C\sum_{k=0}^\infty \frac{\Gamma(k-\beta)}{x^{k-\beta}}\mathbf{a}_k \quad (5.161)$$

where $\sum_{k=0}^\infty \mathbf{a}_k z^k$ is the series of $\mathbf{a}(z)$ near $z = 0$.

Correspondingly, we write (5.55) as

$$(1 + J_k)\mathbf{Y}_k = \hat{Q}_k^{-1}\mathbf{R}_k \quad (5.162)$$

with $\hat{Q}_k := (-\hat{\Lambda} + p + k)$ and

$$(J_k\mathbf{h})(p) := \hat{Q}_k^{-1}\left(\hat{B}\int_0^p \mathbf{h}(s)ds - \sum_{j=1}^n \int_0^p h_j(s)\mathbf{D}_j(p-s)ds\right) \quad (5.163)$$

$$\|J_k\|_{L^1_b \mapsto L^1_b} < \sup_{p\ge 0}\|\hat{Q}_k^{-1}\|\left(\|\hat{B}\|\|1\|_b + n\max_{1\le j\le n}\|\mathbf{D_j}\|_b\right) \quad (5.164)$$

Since $\sup_{p\ge 0}\|\hat{Q}_k^{-1}\|\to 0$ as $k\to\infty$ we have

$$\sup_{k\ge 1}\left\{\|J_k\|_{L_b^1\mapsto L_b^1}\right\}\to 0 \text{ as } b\to\infty \tag{5.165}$$

Thus,

Proposition 5.166 *For large b, $(1+J_k), k\ge 1$ are simultaneously invertible in L_b^1, (cf. 5.165). For specified $\mathbf{Y}_0$ and C, $\mathbf{Y}_k, k\ge 1$ are uniquely determined and moreover, for $k\ge 2$,*

$$\|\mathbf{Y}_k\|_b \le \frac{\sup_{p\ge 0}\|\hat{Q}_k^{-1}\|}{1-\sup_{k\ge 1}\|J_k\|_{L_b^1\mapsto L_b^1}}\|\mathbf{R}_k\|_b := K\|\mathbf{R}_k\|_b \tag{5.167}$$

□

(Note: As we will see later, there only is a *one-parameter* freedom in $\mathbf{Y}_k$: a change in $\mathbf{Y}_0$ can be compensated by a corresponding change in C.)

Because of condition $\sum m = k$ in the definition of $\mathbf{R}_\mathrm{k}$, we get, by an easy induction, the homogeneity relation with respect to the free constant C,

$$\mathbf{Y}_k^{[C]} = C^k\mathbf{Y}_k^{[C=1]} =: C^k\mathbf{Y}_k \tag{5.168}$$

Proposition 5.169 *For any $\delta>0$ there is a large enough b, so that*

$$\|\mathbf{Y}_k\|_b < \delta^k,\ \ k=0,1,.. \tag{5.170}$$

Each $\mathbf{Y}_k$ is Laplace transformable and $\mathbf{y}_k=\mathcal{L}(\mathbf{Y}_k)$ solve (5.31).

Proof

We first show inductively that the $\mathbf{Y}_k$ are bounded. Choose r small enough and b large so that $\|\mathbf{Y}_0\|_b<r$. Note that in the expression of $\mathbf{R}_k$, only $\mathbf{Y}_i$ with $i<k$ appear. We show by induction that $\|\mathbf{Y}_k\|_b<r$ for all k. Using (5.167), (5.55) the explanation to (5.31) and Proposition 5.155 we get

$$\|\mathbf{Y}_k\|_b < K\|\mathbf{R}_k\|_b \le \sum_{|\mathbf{l}|>1}\mu^{|\mathbf{l}|}r^k\sum_{\Sigma m=k}1 \le r^k\left(\sum_{l>1}\binom{l}{k}\mu^l\right)^n \le (r(1+\mu)^n)^k < r \tag{5.171}$$

if r is small which completes this induction step. But now if we look again at (5.171) we see that in fact $\|\mathbf{Y}_k\|_b\le (r(1+\mu)^n)^k$. Choosing r small enough, (and to that end, b large enough) the first part of Proposition 5.169 follows. Laplace transformability as well as the fact that $\mathbf{y}_k$ solve (5.31) follow immediately from (5.170) (observe again that, given k, there are only finitely many terms in the sum in $\mathbf{R}_k$).

□

Therefore,

Remark 5.172 *The series*

$$\sum_{k=0}^{\infty} C^k(\mathbf{Y}_k \cdot \mathcal{H}) \circ \tau_k \tag{5.173}$$

is convergent in L^1_b for large b and thus the sum is Laplace transformable. By Remark 5.71 and Proposition 5.170

$$\mathcal{L}\left(\sum_{k=0}^{\infty} C^k(\mathbf{Y}_k \mathcal{H}) \circ \tau_k\right)(x) = \sum_{k=0}^{\infty} e^{-kx}\mathcal{L}(\mathbf{Y}_k)(x) \tag{5.174}$$

is uniformly convergent for large x (together with its derivatives with respect to x). Thus (by its formal construction) (5.174) is a solution of (4.222).

□

(Alternatively, we could have checked in a straightforward way that the series (5.173), truncated to order N is a solution of the convolution equation (5.53) on the interval $p \in [0, N)$ and in view of the $L^1_b(\mathbb{R}^+)$ (or even L^1_{loc}) convergence it has to be one of the general solutions of the convolution equation and therefore provide a solution to (4.222).)

Proof of Proposition 5.38, ii)

We now show (5.40). This is done from the system (5.55) by induction on k. For $k = 0$ and $k = 1$ the result follows from Proposition 5.65 and Proposition 5.119. For the induction step we consider the operator J_k (5.163) on the space

$$\mathcal{T}_k = \left\{\mathbf{Q} : [0, \epsilon) \mapsto \mathbb{C} : \mathbf{Q}(z) = z^{k\beta-1}\mathbf{A}_k(z)\right\} \tag{5.175}$$

where $\mathbf{A}_k$ extends as an analytic function in a neighborhood $\mathbb{D}_\epsilon$ of $z = 0$. Endowed with the norm

$$\|\mathbf{Q}\|_{\mathcal{T}_\mathrm{k}} := \sup_{z\in\mathbb{D}_\epsilon} |\mathbf{A}_k(z)|_\wedge$$

$\mathcal{T}_k$ is a Banach space.

Remark 5.176 *For $k \in \mathbb{N}$ the operators J_k in (5.163) extend continuously to $\mathcal{T}_k$ and their norm is $O(\epsilon)$. The functions $\mathbf{R}_k$, $k \in \mathbb{N}$ (cf. (5.162), (5.55)), belong to $\mathcal{T}_k$. Thus for $k \in \mathbb{N}$, $\mathbf{Y}_k \in \mathcal{T}_k$.*

If A, B are analytic then for $z < \epsilon$

$$\int_0^z ds\, s^{k\beta-1}A(s)B(z-s) = z^{k\beta}\int_0^1 dt\, t^r A(zt)B(z(1-t)) \tag{5.177}$$

is in $\mathcal{T}_\mathrm{k}$ with norm $O(\epsilon)$ and the assertion about J_k follows easily. Therefore $\mathbf{Y}_k \in \mathcal{T}_\mathrm{k}$ if $\mathbf{R}_k \in \mathcal{T}_\mathrm{k}$. We prove both these properties by induction and (by the homogeneity of $\mathbf{R}_k$ and the fact that $\mathbf{R}_k$ depends only on $\mathbf{Y}_m, m < k$)this amounts to checking that if $\mathbf{Y}_m \in \mathcal{T}_m$ and $\mathbf{Y}_n \in \mathcal{T}_n$ then

$$\mathbf{Y}_m * \mathbf{Y}_n \in \mathcal{T}_{m+n}$$

This follows from the identity

$$\int_0^z ds\, s^r A(s)(z-s)^q B(z-s) = z^{r+q+1} \int_0^1 dt\, t^r (1-t)^q A(zt) B(z-zt)$$

□

It is now easy to see that $\mathcal{L}_\phi \mathcal{B} \tilde{\mathbf{y}}_k \sim \tilde{\mathbf{y}}_k$ (cf. Theorem 5.45). Indeed, note that in view of Remark 5.176 and Proposition 5.169, $\mathcal{L}(\mathbf{Y}_k)$ have asymptotic power series that can be differentiated for large x in the positive half plane. Since $\mathcal{L}(\mathbf{Y}_k)$ are true solutions of the system (5.31) their asymptotic series are formal solutions of (5.31) and by the uniqueness of the formal solution of (5.31) once C is given, the property follows.

In the next subsection, we prove that the general solution of the system (5.31) can be obtained by means of Borel transform of formal series and analytic continuation.

We define $\mathbf{Y}^+$ to be the function defined in Proposition 5.145, extended in $\mathcal{D} \cap \mathbb{C}^+$ by the unique solution of (5.53) $\mathbf{Y}_0$ provided by Proposition 5.65. (We define $\mathbf{Y}^-$ correspondingly.)

By Proposition 5.145, *ii*) $\mathbf{Y}^\pm$ are solutions of (5.53) on $[0, \infty)$ (cf. (5.144)). By Lemma 5.125 any solution on $[0, \infty)$ can be obtained from, say, $\mathbf{Y}^+$ by choosing C and then solving uniquely (5.142) on $[1+\epsilon, \infty)$ (Proposition 5.145). We now show that the solutions of (5.159), (5.162) are continuous boundary values of functions analytic in a region bounded by $\mathbb{R}^+$.

Remark 5.178 *The function* $\mathbf{D}(s)$ *defined in (5.107) by substituting* $\mathbf{H} = \mathbf{Y}^\pm$*, is in* $\mathcal{T}^\pm_{0,\infty}$ *(cf. (5.144)).*

By Proposition 5.145, *ii*) it is easy to check that if $\mathbf{H}$ is any function in $\mathcal{T}^+_{0,A}$ then $\mathbf{Y}^+ * \mathbf{Q} \in \mathcal{T}^+_{0,A}$. Thus, with $\mathbf{H} = \mathbf{Y}^+$, ll the terms in the infinite sum in (5.107) are in $\mathcal{T}^+_{0,A}$. For fixed $A > 0$, taking b large enough, the norm ρ_b of $\mathbf{Y}^+$ in L^1_b can be made arbitrarily small uniformly in all rays in $S^+_{0,A}$ (5.144) (Proposition 5.145). Then by Corollary 5.79 and Proposition 5.145 *ii*), the uniform norm of each term in the series (5.107) can be estimated by $Const\, \rho_b^{|\mathbf{l}-1|} \nu^{|\mathbf{l}|}$ and thus the series converges uniformly in $\mathcal{T}^+{}_{0,\infty}$, for large b.

□

Lemma 5.179 *i) The system (5.55) with* $\mathbf{Y}_0 = \mathbf{Y}^+$ *(or* $\mathbf{Y}^-$*) and given* C *(say* $C = 1$*) has a unique solution in* $L^1_{loc}(\mathbb{R}^+)$*, namely* $\mathbf{Y}^+_k$*, (*$\mathbf{Y}^-_k$*, resp.),* $k \in \mathbb{N}$*. Furthermore, for large* b *and all* k*,* $\mathbf{Y}^+_k \in \mathcal{T}^+_{0,\infty}$ *(*$\mathbf{Y}^-_k \in \mathcal{T}^-_{0,\infty}$*) (cf. (5.144)).*

ii) The general solution of the equation (5.53) in $L^1_{loc}(\mathbb{R}^+)$ can be written in either of the forms:

$$\mathbf{Y}^+ + \sum_{k=1}^{\infty} C^k(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k$$

$$\mathbf{Y}^- + \sum_{k=1}^{\infty} C^k(\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k \quad (5.180)$$

Proof.
i) The first part follows from the same arguments as Proposition 5.166. For the last statement it is easy to see (cf. (5.177)) that $J_k\mathcal{T}^+_{0,\infty} \subset \mathcal{T}^+_{0,\infty}$ and by Proposition 5.74 the inequalities (5.164), (5.165) hold for $\|\|\|_{\mathcal{T}_{0,A}\mapsto\mathcal{T}_{0,A}}$ (A arbitrary) replacing $\|\|\|_{L^1_b\mapsto L^1_b}$.

ii) We already know that $\mathbf{Y}^+$ solves (5.55) for $k = 0$. For $k > 0$ by *i*) $C^k\mathbf{Y}_k \in \mathcal{T}_{0,\infty}$ and so, by continuity, the boundary values of $\mathbf{Y}_k^+$ on $\mathbb{R}^+$ solve the system (5.55) on $\mathbb{R}^+$ in L^1_{loc}. The rest of *ii)* follows from Lemma 5.125, Proposition 5.145 and the arbitrariness of C in (5.180) (cf. also (5.134).

$\square_{L_4}$

5.2g Analytic structure and averaging

Having the general structure of the solutions of (5.53) given in Proposition 5.56 and in Lemma 5.179 we can obtain various analytic identities. The function $\mathbf{Y}_0^{\pm} := \mathbf{Y}^{\pm}$ has been defined in the previous section.

Proposition 5.181 *For $m \geq 0$,*

$$\mathbf{Y}_m^- = \mathbf{Y}_m^+ + \sum_{k=1}^{\infty} \binom{m+k}{m} S_\beta^k(\mathbf{Y}_{m+k}^+ \cdot \mathcal{H}) \circ \tau_k \quad (5.182)$$

Proof.
$\mathbf{Y}_0^-(p)$ is a particular solution of (5.53). It follows from Lemma 5.179 that the following identity holds on $\mathbb{R}^+$:

$$\mathbf{Y}_0^- = \mathbf{Y}_0^+ + \sum_{k=1}^{\infty} S_\beta^k(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k \quad (5.183)$$

since, by (5.59) and (5.39), (5.183) holds for $p \in (0, 2)$.

By Lemma 5.179 for any C_+ there is a C_- such that

$$\mathbf{Y}_0^+ + \sum_{k=1}^{\infty} C_+^k(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = \mathbf{Y}_0^- + \sum_{k=1}^{\infty} C_-^k(\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k \quad (5.184)$$

To find the relation C_+ and C_- we take $p \in (1,2)$; we get, comparing with (5.183):

$$\mathbf{Y}_0^+(p) + C_+\mathbf{Y}_1(p-1) = \mathbf{Y}_0^-(p) + C_-\mathbf{Y}_1(p-1) \Rightarrow C_+ = C_- + S_\beta \quad (5.185)$$

whence, for any $C \in \mathbb{C}$,

$$\mathbf{Y}_0^+ + \sum_{k=1}^{\infty}(C+S_\beta)^k(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = \mathbf{Y}_0^- + \sum_{k=1}^{\infty} C^k(\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k \qquad (5.186)$$

Differentiating m times w.r. to C and taking $C = 0$ we get

$$\sum_{k=m}^{\infty} \frac{k!}{(k-m)!} S_\beta^{k-m}(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = m!(\mathbf{Y}_m^- \cdot \mathcal{H}) \circ \tau_m$$

from which we obtain (5.182) by rearranging the terms and applying τ_{-m}.

□

Proposition 5.187 *The functions $\mathbf{Y}_k$, $k \ge 0$, are analytic in $\mathcal{R}_1$.*

Proof.

Starting with (5.183), if we take $p \in (1,2)$ and obtain:

$$\mathbf{Y}_0^-(p) = \mathbf{Y}_0^+(p) + S_\beta \mathbf{Y}_1(p-1) \qquad (5.188)$$

By Proposition 5.145 and Lemma 5.179 the l.h.s of (5.188) is analytic in a lower half plane neighborhood of $(\varepsilon, 1-\varepsilon)$, $(\forall\varepsilon \in (0,1))$ and continuous in the closure of such a neighborhood. The r.h.s. is analytic in an upper half plane neighborhood of $(\varepsilon, 1-\varepsilon)$, $(\forall\varepsilon \in (0,1))$ and continuous in the closure of such a neighborhood. Thus, $\mathbf{Y}_0^-(p)$ can be analytically continued along a path crossing the interval $(1,2)$ from below, i.e., $\mathbf{Y}_0^{-+}$ exists and is analytic.

Now, in (5.183), let $p \in (2,3)$:

$$\begin{aligned} S_\beta^2\mathbf{Y}_2(p-2) &= \mathbf{Y}_0(p)^- - \mathbf{Y}(p)^+ - S_\beta\mathbf{Y}_1(p-1)^+ = \\ &\mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^+ - \mathbf{Y}_0(p)^{-+} + \mathbf{Y}_0(p)^+ = \mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^{-+} \end{aligned} \qquad (5.189)$$

and, in general, taking $p \in (k, k+1)$ we get

$$S_\beta^k\mathbf{Y}_k(p-k) = \mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^{-^{k-1}+} \qquad (5.190)$$

Using (5.190) inductively, the same arguments that we used for $p \in (0,1)$ show that $\mathbf{Y}_0^{-^k}(p)$ can be continued analytically in the upper half plane. Thus, we have

Remark 5.191 *The function* $\mathbf{Y}_0$ *is analytic in* $\mathcal{R}_1$*. In fact, for* $p \in (j, j+1)$*,* $k \in \mathbb{N}$*,*

$$\mathbf{Y}_0^{-^j+}(p) = \mathbf{Y}_0^+(p) + \sum_{k=1}^{j} S_\beta^k \mathbf{Y}_k^+(p-k)\mathcal{H}(p-k) \tag{5.192}$$

The relation (5.192) follows from (5.190) and (5.183).

$\Box_{R5.191}$

Note: Unlike (5.183), in (5.192) the sum contains a finite number of terms. For instance we have:

$$\mathbf{Y}_0^{-+}(p) = \mathbf{Y}_0^+(p) + \mathcal{H}(p-1)\mathbf{Y}_1^+(p-1). \quad (\forall p \in \mathbb{R}^+) \tag{5.193}$$

The analyticity of $\mathbf{Y}_m$, $m \geq 1$ is shown inductively on m, using (5.182) and following exactly the same course of proof as for $k = 0$.

$\Box$

Remark 5.194 *If* $S_\beta = 0$ *then* $\mathbf{Y}_k$ *are analytic in* $\mathcal{W}_1 \cup \mathbb{N}$*.*

Indeed, this follows from (5.183) (5.182) and Lemma 5.179, i)

$\Box$

On the other hand, if $S_\beta \neq 0$, then all $\mathbf{Y}_k$ are analytic continuations of the Borel transform of $\mathbf{y}_0$ (cf. (5.189)). This is an instance of the so-called resurgence .
Moreover, we can now calculate $\mathbf{Y}_0^{ba}$. By definition, (see the discussion before Remark 5.122) on the interval $(0, 2)$,

$$\mathbf{Y}_0^{ba} = \frac{1}{2}(\mathbf{Y}_0^+ + \mathbf{Y}_0^-) = \mathbf{Y}_0^+ + \frac{1}{2} S_\beta(\mathbf{Y}_1 \mathcal{H}) \circ \tau_1 \tag{5.195}$$

Now we are looking for a solution of (5.53) which satisfies the condition (5.195). By comparing with Lemma 5.179, which gives the general form of the solutions of (5.53), we get, now on the whole positive axis,

$$\mathbf{Y}_0^{ba} = \mathbf{Y}_0^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} S_\beta^k (\mathbf{Y}_k^+ \mathcal{H}) \circ \tau_k \ (\text{on } \mathbb{R}^+) \tag{5.196}$$

which we can rewrite using (5.190):

$$\mathbf{Y}_0^{ba} = \mathbf{Y}_0^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\mathbf{Y}_0^{-^k} - \mathbf{Y}_0^{-^{k-1},+} \right) (\mathcal{H} \circ \tau_k) \tag{5.197}$$

Proposition 5.198 *Let* $y_1(p), y_2(p)$ *be analytic in* $\mathcal{R}_1$*, and such that for any path* $\gamma = t \mapsto t\exp(i\phi(t))$ *in* $\mathcal{R}_1$*,*

$$|y_{1,2}(\gamma(t))| < f_\gamma(t) \in L^1_{loc}(\mathbb{R}^+) \tag{5.199}$$

Assume further that for some large enough b, M and any path γ in $\mathcal{R}_1$:

$$\int_\gamma |y_{1,2}|(s)e^{-b|s|}|ds| < M \tag{5.200}$$

Then the analytic continuation $AC_\gamma(y_1 * y_2)$ along a path γ in $\mathcal{R}_1$, of their convolution product $y_1 * y_2$ (defined for small p by (5.48)) exists, is locally integrable and satisfies (5.199) and, for the same b and some γ-independent $M' > 0$,

$$\int_\gamma |y_1 * y_2|(s)e^{-b|s|}|ds| < M' \tag{5.201}$$

Proof.
Since

$$2y_1 * y_2 = (y_1 + y_2) * (y_1 + y_2) - y_1 * y_1 - y_2 * y_2 \tag{5.202}$$

it is enough to take $y_1 = y_2 = y$. For $p \in \mathbb{R}^+ \backslash \mathbb{N}$ we write:

$$y^- = y^+ + \sum_{k=1}^{\infty} (\mathcal{H} \cdot y_k^+) \circ \tau_k \tag{5.203}$$

The functions y_k are *defined* inductively (the superscripts "+,(-)" mean, as before, the analytic continuations in $\mathcal{R}_1$ going below(above) the real axis). In the same way (5.190) was obtained we get by induction:

$$y_k = (y^- - y^{-^{k-1}+}) \circ \tau_{-k} \tag{5.204}$$

where the equality holds on $\mathbb{R}^+ \backslash \mathbb{N}$ and $+, -$ mean the upper and lower continuations. For any p only finitely many terms in the sum in (5.203) are nonzero. The sum is also convergent in $\| \|_b$ (by dominated convergence; note that, by assumption, the functions $y^{--\cdot\cdot-\pm}$ belong to the same L^1_b).

If $t \mapsto \gamma(t)$ in $\mathcal{R}_1$, is a straight line, other than $\mathbb{R}^+$, then:

$$AC_\gamma((y * y)) = AC_\gamma(y) *_\gamma AC_\gamma(y) \text{ if } \arg(\gamma(t))=\text{const}\neq 0 \tag{5.205}$$

(Since y is analytic along such a line). The notation $*_\gamma$ means (5.48) with $p = \gamma(t)$.

Note though that, suggestive as it might be, (5.205) is *incorrect* if the condition stated there is not satisfied and γ is a path that crosses the real line (see the Appendix, Section Ab)!

We get from (5.205), (5.203) (see also (8.119), in the Appendix):

$$(y*y)^- = y^- * y^- = y^+ * y^+ + \sum_{k=1}^{\infty}\left(\mathcal{H}\sum_{m=0}^{k} y_m^+ * y_{k-m}^+\right)\circ\tau_k =$$
$$(y*y)^+ + \sum_{k=1}^{\infty}\left(\mathcal{H}\sum_{m=0}^{k}\big(y_m * y_{k-m}\big)^+\right)\circ\tau_k \quad (5.206)$$

and now the analyticity of $y*y$ in $\mathcal{R}_1$ follows: on the interval $p\in(m,m+1)$ we have from (5.204)

$$\big(y*y\big)^{-^j}(p) = \big(y*y\big)^-(p) = \big(y^{*2}\big)^+(p) + \sum_{k=1}^{j}\sum_{m=0}^{k}\big(y_m*y_{k-m}\big)^+(p-k) \quad (5.207)$$

Again, formula (5.207) is useful for analytically continuing $\big(y*y\big)^{-^j}$ along a path as the one depicted in Fig.1. By dominated convergence, $(y*y)^{\pm}\in\mathcal{T}^{\pm}_{(0,\infty)}$, (5.144). By (5.204), y_m are analytic in $\mathcal{R}_1^+ := \mathcal{R}_1\cap\{p:\Im(p)>0\}$ and thus by (5.205) the r.h.s. of (5.207) can be continued analytically in $\mathcal{R}_1^+$. The same is then true for $(y*y)^-$. The function $(y*y)$ can be extended analytically along paths that cross the real line from below. Likewise, $(y*y)^+$ can be continued analytically in the lower half plane so that $(y*y)$ is analytic in $\mathcal{R}_1$.

Combining (5.207), (5.205) and (5.202) we get a similar formula for the analytic continuation of the convolution product of two functions, f,g satisfying the assumptions of Proposition 5.198

$$(f*g)^{-^j+} = f^+ * g^+ + \sum_{k=1}^{j}\left(\mathcal{H}\sum_{m=0}^{k} f_m^+ * g_{k-m}^+\right)\circ\tau_k \quad (5.208)$$

Note that (5.208) corresponds to (5.203) and in those notations we have:

$$\big(f*g\big)_k = \sum_{m=0}^{k} f_m * g_{k-m} \quad (5.209)$$

Integrability as well as (5.201) follow from (5.204), (5.207) and Remark 5.73. $\Box_{P5.198}$

By (5.43) and (5.204),

$$y^{ba} = y^+ + \sum_{k=1}^{\infty}\frac{1}{2^k}(y_k^+\mathcal{H})\circ\tau_k$$

so that (see (8.119))

$$y^{ba} * y^{ba} = \left(y^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} (\mathcal{H} \circ \tau_k)(y_k^+ \circ \tau_k) \right)^{*2} =$$
$$y^+ * y^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} \mathcal{H} \circ \tau_k \sum_{m=0}^{k} (y_m^+ \circ \tau_m) * (y_{k-m}^+ \circ \tau_{k-m}) \circ \tau_k =$$
$$y^+ * y^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} \mathcal{H} \circ \tau_k \sum_{m=0}^{k} (y_m * y_{k-m})^+ \circ \tau_k = (y^{*2})^{ba} \tag{5.210}$$

To finish the proof of Theorem 5.45 note that on any finite interval the sum in (5.43) has only a finite number of terms and by (5.210) balanced averaging commutes with any finite sum of the type

$$\sum_{k_1,..,k_n} c_{k_1..k_n} f_{k_1} * .. * f_{k_n} \tag{5.211}$$

and then, by continuity, with any sum of the form (5.211), with a finite or infinite number of terms, provided it converges in L^1_{loc}. Averaging thus commutes with all the operations involved in the equations (5.162). By uniqueness therefore, if $\mathbf{Y}_0 = \mathbf{Y}^{ba}$ then $\mathbf{Y}_k = \mathbf{Y}_k^{ba}$ for all k. Preservation of reality is immediate since (5.53), (5.55) are real if (4.222) is real, therefore $\mathbf{Y}_0^{ba}$ is real-valued on $\mathbb{R}^+ \backslash \mathbb{N}$ (since it is real-valued on $[0, 1) \cup (1, 2)$) and so are, inductively, all $\mathbf{Y}_k$.

A Appendix

Aa Example of non-typical behavior

Consider the equation

$$f' = -f - \frac{1}{2x} f + \frac{1}{x} - \frac{1}{2x^2} \tag{5.212}$$

The general solution of this equation is given by

$$f = \frac{1}{x} + Cx^{-1/2} e^{-x} = \int_0^{\infty} \left(p + \frac{C}{\sqrt{p-1}} \mathcal{H}(1-p) \right) dp \tag{5.213}$$

We see that the asymptotic series of f for $x \to \infty$, $\Re(x) > 0$, $\tilde{\mathbf{y}}_0 = 1/x$. The inverse Laplace transform of f is

$$\mathcal{L}^{-1} f = p + \frac{C}{\sqrt{p-1}} \mathcal{H}(1-p) \tag{5.214}$$

i) The Stokes constant is zero and $\mathbf{Y}_0 = \mathcal{B}(\tilde{\mathbf{y}}_0) = p$ is entire.

ii) All combinations $\lambda \mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$ coincide. Therefore (5.60) does not hold.

Equation (5.212) is exceptional, in the sense that the properties $i), ii)$ above do not withstand a small perturbation. Indeed, for the equation

$$f' = -f - \frac{1}{2x} f + \frac{1+\epsilon}{x} - \frac{1}{2x^2} \tag{5.215}$$

we have $\mathcal{B}(\tilde{\mathbf{y}}_0) = 2\epsilon + p + \epsilon(1-p)^{-1/2}$ and the inverse Laplace transform of the general solution is

$$\mathcal{L}^{-1}(f) = \begin{cases} 2\epsilon + p + \epsilon(1-p)^{-1/2} \text{ for } p < 1 \\ 2\epsilon + p + C(p-1)^{-1/2} \text{ for } p > 1 \end{cases}$$

Ab $AC(f * g)$ versus $AC(f) * AC(g)$

Typically, the analytic continuation along curve in $\mathcal{W}_1$ which is not homotopic to a straight line will not commute with convolution. For example, in equation (5.215), $\mathcal{B}(\tilde{\mathbf{y}}_0)^{-+} * \mathcal{B}(\tilde{\mathbf{y}}_0)^{-+} \neq [\mathcal{B}(\tilde{\mathbf{y}}_0) * \mathcal{B}(\tilde{\mathbf{y}}_0)]^{-+}$, as it can be seen from Remark 5.216 below (or by direct calculation). This situation is generic:

Remark 5.216 *Let y be a function satisfying the conditions stated in Proposition 5.198 and assume that $p = 1$ is a branch point of y. Then,*

$$(y * y)^{-+} \neq y^{-+} * y^{-+} \tag{5.217}$$

Proof

Indeed, by (5.208) and (5.204)

$$\begin{aligned} (y * y)^{-+} &= y^+ * y^+ + 2[(y^+ * y_1^+)\mathcal{H}] \circ \tau_1 \neq y^{-+} * y^{-+} = \\ &[y^+ + (\mathcal{H} y_1^+) \circ \tau_1]^{*2} = y^+ * y^+ + 2[(y^+ * y_1^+)\mathcal{H}] \circ \tau_1 + [\mathcal{H}(y_1^+ * y_1^+)] \circ \tau_2 \end{aligned} \tag{5.218}$$

since in view of (5.204), in our assumptions, $y_1 \not\equiv 0$ and thus $y_1 * y_1 \not\equiv 0$.

□

There is also the following intuitive reasoning leading to the same conclusion. For a generic system of the form (4.222)–(5.29), $p = 1$ is a branch point of $\mathbf{Y}_0$ and so $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$. On the other hand, if AC_{-+} commuted with convolution, then $\mathcal{L}(\mathbf{Y}_0^{-+})$ would provide a solution of (4.222). By Lemma 5.179, $\mathcal{L}(\mathbf{Y}_0^-)$ is a different solution (since $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$). As $\mathbf{Y}_0^-$ and $\mathbf{Y}_0^{-+}$ coincide up to $p = 2$ we have $\mathcal{L}(\mathbf{Y}_0^{-+}) - \mathcal{L}(\mathbf{Y}_0^-) = O(e^{-2x} x^{power})$ for $x \to +\infty$. By Theorem 5.45 however, no two solutions of (4.222)–(5.29) can differ by less than $e^{-x} x^{power}$ without actually being equal (also, heuristically, this can be checked using formal perturbation theory), contradiction.

Ac Derivation of the equations for the transseries.

Consider first the scalar equation

$$y' = f_0(x) - \lambda y - x^{-1}By + g(x,y) = -y + x^{-1}By + \sum_{k=1}^{\infty} g_k(x)y^k \tag{5.219}$$

For $x \to +\infty$ we take

$$y = \sum_{k=0}^{\infty} y_k \mathrm{e}^{-kx} \tag{5.220}$$

where y_k will be either formal series $x^{-s_k}\sum_{n=0}^{\infty} a_{kn}x^{-n}$, with $a_{k,0} \neq 0$ or actual functions with the condition that (5.220) converges uniformly. As a transseries, (5.220) can be also understood as a well ordered double sequence $t_{kn} = x^{p_{kn}}\mathrm{e}^{-kx}$, with $p_{k\,n+1} < p_{kn}$. (The order relation is $x^p\mathrm{e}^{-kx} \gg x^{p'}\mathrm{e}^{-k'x}$ as $x \to +\infty$ iff $k < k'$ or $k = k'$ and $p > p'$; thus a strictly *increasing* sequence of terms of a transseries necessarily terminates.) Power series are a special case of transseries, with $y_1 = y_2 = \ldots = 0$. Two transseries $\sum_{k=0}^{\infty} y_k \mathrm{e}^{-kx}$ coincide iff all corresponding component power series y_k coincide. Transseries of this type are closed under addition, multiplication and infinite sums of the form involved in (5.219) (this last aspect will become clear in the calculation leading to (5.222) below). Note that well-ordering plays an important part in defining multiplication of transseries; in contrast, for the unrestricted formal expansion $S = \sum_{k=-\infty}^{\infty} x^k$, no immediate meaning can be given to S^2. Let y_0 be the first term in (5.220) and $\delta = y - y_0$. We have

$$y^k - y_0^k - ky_0^{k-1}\delta = \sum_{j=2}^{k}\binom{k}{j} y_0^{k-j}\delta^j = \sum_{j=2}^{k}\binom{k}{j} y_0^{k-j} \sum_{i_1,\ldots,i_j=1}^{\infty}\prod_{s=1}^{j}\left(y_{i_s}\mathrm{e}^{-i_s x}\right)$$
$$= \sum_{m=1}^{\infty}\mathrm{e}^{-mx}\sum_{j=2}^{k}\binom{k}{j} y_0^{k-j}\sum_{(i_s)}^{(m;j)}\prod_{s=1}^{j} y_{i_s} \tag{5.221}$$

where $\sum_{(i_s)}^{(m;j)}$ means the sum over all positive integers $i_1, i_2, \ldots, i_j$ with the restriction $i_1 + i_2 + \cdots + i_j = m$. Let $d_1 = \sum_{k\geq 1} kg_k y_0^{k-1}$. Introducing $y = y_0 + \delta$ in (5.219) and equating the coefficients of e^{-lx} we get, by separating the terms containing y_l for $l \geq 1$ and interchanging the j,k orders of summation,

$$y_l' + (\lambda(1-l) + x^{-1}B - d_1(x))y_l = \sum_{j=2}^{\infty}\sum_{(i_s)}^{(l;j)}\prod_{s=1}^{j} y_{i_s}\sum_{k\geq\{2,j\}}\binom{k}{j} g_k y_0^{k-j}$$
$$= \sum_{j=2}^{l}\sum_{(i_s)}^{(l;j)}\prod_{s=1}^{j} y_{i_s}\sum_{k\geq\{2,j\}}\binom{k}{j} g_k y_0^{k-j} =: \sum_{j=2}^{l} d_j(x)\sum_{(i_s)}^{(l;j)}\prod_{s=1}^{j} y_{i_s} \tag{5.222}$$

where for the middle equality we note that the infinite sum terminates because $i_s \geq 1$ and $\sum_{s=1}^{j} i_s = l$. The fact mentioned before that $\sum_{k=1}^{\infty} g_k(x)y_k$ is well defined when y_k are formal series is now visible: collecting the coefficient of $x^p e^{-kx}$, only *finite* sums of coefficients appear.

For a vectorial equation like (4.222) we first write

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} + x^{-1}\hat{B}\mathbf{y} + \sum_{\mathbf{k}\succ 0} \mathbf{g}_{\mathbf{k}}(x)\mathbf{y}^{\mathbf{k}} \tag{5.223}$$

with $\mathbf{y}^{\mathbf{k}} := \prod_{i=1}^{n_1} (\mathbf{y})_i^{k_i}$. The formal operations and ordering extend naturally to the vectorial general transseries (4.223), under the restriction $\Re(\mathbf{k}\cdot\boldsymbol{\lambda}x) > 0$ As with (5.222), we introduce the transseries (4.223) in (5.223) and equate the coefficients of $\exp(-\mathbf{k}\cdot\boldsymbol{\lambda}x)$. Let $\mathbf{v}_{\mathbf{k}} = x^{-\mathbf{k}\cdot\mathbf{m}}\mathbf{y}_{\mathbf{k}}$ and

$$\mathbf{d}_{\mathbf{j}}(x) = \sum_{\mathbf{l}\geq\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \mathbf{g}_{\mathbf{l}}(x)\mathbf{v}_0^{\mathbf{l}-\mathbf{j}} \tag{5.224}$$

Noting that, by assumption, $\mathbf{k}\cdot\boldsymbol{\lambda} = \mathbf{k}'\cdot\boldsymbol{\lambda} \Leftrightarrow \mathbf{k} = \mathbf{k}'$ we obtain, for $\mathbf{k} \in \mathbb{N}^{n_1}$, $\mathbf{k} \succ 0$

$$\mathbf{v}_{\mathbf{k}}' + \left(\hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\hat{I} + x^{-1}\hat{B}\right)\mathbf{v}_{\mathbf{k}} + \sum_{|\mathbf{j}|=1} \mathbf{d}_{\mathbf{j}}(x)(\mathbf{v}_{\mathbf{k}})^{\mathbf{j}}$$
$$= \sum_{\substack{\mathbf{j}\leq\mathbf{k} \\ |\mathbf{j}|\geq 2}} \mathbf{d}_{\mathbf{j}}(x) \sum_{(\mathbf{i}_{mp}:\mathbf{k})} \prod_{m=1}^{n} \prod_{p=1}^{j_m} \left(\mathbf{v}_{\mathbf{i}_{mp}}\right)_m = \mathbf{t}_{\mathbf{k}}(\mathbf{v}) \tag{5.225}$$

where $\binom{\mathbf{l}}{j} = \prod_{j=1}^{n} \binom{l_i}{j_i}$, $(\mathbf{v})_m$ means the component m of $\mathbf{v}$, and $\sum_{(\mathbf{i}_{mp}:\mathbf{k})}$ stands for the sum over all vectors $\mathbf{i}_{mp} \in \mathbb{N}^n$, with $p \leq j_m, m \leq n$, such that $\mathbf{i}_{mp} \succ 0$ and $\sum_{m=1}^{n}\sum_{p=1}^{j_m} \mathbf{i}_{mp} = \mathbf{k}$. We use the convention $\prod_{\emptyset} = 1, \sum_{\emptyset} = 0$. With $m_i = 1 - \lfloor \beta_i \rfloor$ we obtain for $\mathbf{y}_{\mathbf{k}}$

$$\mathbf{y}_{\mathbf{k}}' + \left(\hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\hat{I} + x^{-1}(\hat{B} + \mathbf{k}\cdot\mathbf{m})\right)\mathbf{y}_{\mathbf{k}} + \sum_{|\mathbf{j}|=1} \mathbf{d}_{\mathbf{j}}(x)(\mathbf{y}_{\mathbf{k}})^{\mathbf{j}} = \mathbf{t}_{\mathbf{k}}(\mathbf{y}) \tag{5.226}$$

There are clearly finitely many terms in $\mathbf{t}_{\mathbf{k}}(\mathbf{y})$. To find a (not too unrealistic) upper bound for this number of terms, we compare with $\sum_{(\mathbf{i}_{mp})'}$ which stands for the same as $\sum_{(\mathbf{i}_{mp})}$ except with $\mathbf{i} \geq 0$ instead of $\mathbf{i} \succ 0$. Noting that $\binom{k+s-1}{s-1} = \sum_{a_1+\ldots+a_s=k} 1$ is the number of ways k can be written as a sum of s integers, we have

$$\sum_{(\mathbf{i}_{mp})} 1 \leq \sum_{(\mathbf{i}_{mp})'} 1 = \prod_{l=1}^{n_1} \sum_{(\mathbf{i}_{mp})_l} 1 = \prod_{l=1}^{n_1} \binom{k_l + |\mathbf{j}| - 1}{|\mathbf{j}| - 1} \leq \binom{|\mathbf{k}| + |\mathbf{j}| - 1}{|\mathbf{j}| - 1}^{n_1} \tag{5.227}$$

Remark 5.228 *Equation (5.225) can be written in the form (5.9)*

Proof. The fact that only predecessors of $\mathbf{k}$ are involved in $\mathbf{t}(\mathbf{y}_0,\cdot)$ and the homogeneity property of $\mathbf{t}(\mathbf{y}_0,\cdot)$ follow immediately by combining the conditions $\sum \mathbf{i}_{mp} = \mathbf{k}$ and $\mathbf{i}_{mp} \succ 0$. □

The formal inverse Laplace transform of (5.226) is then

$$\left(-p+\hat{\Lambda}-\mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y}_{\mathbf{k}}+\left(\hat{B}+\mathbf{k}\cdot\mathbf{m}\right)\mathcal{P}\mathbf{Y}_{\mathbf{k}}+\sum_{|\mathbf{j}|=1}\mathbf{D}_{\mathbf{j}}*(\mathbf{Y}_{\mathbf{k}})^{\mathbf{j}}=\mathbf{T}_{\mathbf{k}}(\mathbf{Y}) \tag{5.229}$$

with

$$\mathbf{T}_{\mathbf{k}}(\mathbf{Y})=\mathbf{T}\left(\mathbf{Y}_0,\{\mathbf{Y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right)=\sum_{\mathbf{j}\le k;\ |\mathbf{j}|>1}\mathbf{D}_{\mathbf{j}}(p)*\sum_{(\mathbf{i}_{mp};\mathbf{k})}{}^{*}\prod_{m=1}^{n_1}{}^{*}\prod_{p=1}^{j_m}\left(\mathbf{Y}_{\mathbf{i}_{mp}}\right)_m \tag{5.230}$$

and

$$\mathbf{D}_{\mathbf{j}}=\sum_{\mathbf{l}\ge\mathbf{m}}\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G}_{\mathbf{l}}*\mathbf{Y}_0^{*(\mathbf{l}-\mathbf{m})}+\sum_{\mathbf{l}\ge\mathbf{m};|\mathbf{l}|\ge 2}\binom{\mathbf{l}}{\mathbf{m}}\mathbf{g}_{0,\mathbf{l}}\mathbf{Y}_0^{*(\mathbf{l}-\mathbf{m})} \tag{5.231}$$

Chapter 6

Difference equations; formal solutions and summability

d Setting

Let us now look at difference systems of equations which can be brought to the form

$$\mathbf{x}(n+1) = \hat{\Lambda}\left(I + \frac{1}{n}\hat{A}\right)\mathbf{x}(n) + \mathbf{g}(n, \mathbf{x}(n)) \tag{6.1}$$

where $\hat{\Lambda}$ and $\hat{A}$ are constant coefficient matrices, $\mathbf{g}$ is convergently given for small $\mathbf{x}$ by

$$\mathbf{g}(n, \mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{g}_{\mathbf{k}}(n)\mathbf{x}^{\mathbf{k}} \tag{6.2}$$

with $\mathbf{g}_{\mathbf{k}}(n)$ analytic in n at infinity and

$$\mathbf{g}_{\mathbf{k}}(n) = O(n^{-2}) \text{ as } n \to \infty, \text{ if } \sum_{j=1}^{m} k_j \le 1 \tag{6.3}$$

under nonresonance conditions: Let $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ and $\mathbf{a} = (a_1, ..., a_n)$ where $e^{-\mu_k}$ are the eigenvalues of $\hat{\Lambda}$ and the a_k are the eigenvalues of $\hat{A}$. Then the nonresonance condition is

$$(\mathbf{k}\cdot\boldsymbol{\mu} = 0 \mod 2\pi i \text{ with } \mathbf{k} \in \mathbb{Z}^{m_1}) \Leftrightarrow \mathbf{k} = 0. \tag{6.4}$$

The theory of these equations is remarkably similar to that of differential equations. We consider the solutions of (6.1) which are small as n becomes large.

d .1 Transseries for difference equations

Braaksma [9] showed that the recurrences (6.1) posess l-parameter transseries solutions of the form

$$\tilde{\mathbf{x}}(t) := \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu}t} t^{\mathbf{k}\cdot\mathbf{a}} \tilde{\mathbf{x}}_{\mathbf{k}}(t) \tag{6.5}$$

(6.5) with $t = n$ where $\tilde{\mathbf{x}}_{\mathbf{k}}(n)$ are formal power series in powers of n^{-1} and $l \le m$ is chosen such that, after reordering the indices, we have $\Re(\mu_j) > 0$ for $1 \le j \le l$.

It is shown in [9] that these transseries are generalized Borel summable in any direction and Borel summable in all except m of them and that

$$\mathbf{x}(n) = \sum_{\mathbf{k}\in\mathbb{N}^l} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} n^{\mathbf{k}\cdot\mathbf{a}} \mathbf{x}_{\mathbf{k}}(n) \tag{6.6}$$

is a solution of (6.1), if $n > y_0$, t_0 large enough.

There is a freedom of composition with periodic functions. For example, the general solution of $x_{n+1} = x_n$ is an arbitrary 1-periodic function. This freedom permeates both the formal and analytic theory. It can be ruled out by disallowing purely oscillatory terms in the transseries.

Chapter 7

The principle of transasymptotic matching

A Expansion regeneration

Much as in the case of analytic continuation, when transseries break down on the boundary can be matched in many cases with expansions valid in other regions. Often when an expansion approaches the edge of validity, the blow-up structure suggests a new expansion which is valid beyond the breakdown of the original one. Usually there is then a common region of validity, allowing for matching the two. We revisit example 3.155 , but now at a purely formal level to observe this interesting phenomenon. We assumed that $U \in C^\infty$ and U has finitely many zeros. Suppose $U(0) = 0$ and $U'(0) = a > 0$. If we look at the expansion (3.164) in a neighborhood of $x = 0$ and approximate U by its Taylor series at zero $U(x) = ax + bx^2 + \ldots$

$$h_2 = \frac{\sqrt{ax}}{\epsilon}\left(1 + \frac{bx}{2a}\right) + -\frac{1}{4x}\left(1 + \frac{bx}{a} + \cdots\right) - \frac{5z}{32\sqrt{ax^5}}\left(1 - \frac{bx}{10a} + \cdots\right) \tag{7.1}$$

and in general we would get

$$h_n = \frac{\sqrt{x}}{\epsilon}\left(y_0 + \xi y_1 + \xi^2 y_2 + \cdots\right); \quad \xi = \frac{\epsilon}{x^{3/2}} \tag{7.2}$$

where

$$y_j = a_{j0} + a_{j1}x + a_{j2}x^2 + \cdots = a_{j0} + a_{j1}\frac{\epsilon^{2/3}}{\xi^{2/3}} + a_{j2}\frac{\epsilon^{4/3}}{\xi^{4/3}} + \cdots \tag{7.3}$$

We note that now the expansion has *two* small parameters, ϵ and x; these cannot be chosen small *independently*: the condition if $\xi \ll 1$ has to be satisfied to make asymptotic sense of (7.2). This would carry us down to values of x such that, say, $\xi \ll 1/\ln|\epsilon|$. In such

B Introduction

We take first the relatively simple Abel's equation (4.226). Eq. (4.226) is known to be non-integrable from Kruskal's poly-Painlevé analysis [37]. Its normal form is (4.233). We write the decaying formal asymptotic series solution as

$$y \sim \sum_{j=2}^{\infty} \frac{a_{j,0}}{x^j} \equiv \tilde{y}_0(x) \tag{7.4}$$

where $a_{j,0}$ can be determined algorithmically. and their value is immaterial for now. If y_0 is a particular solution to (4.232) with asymptotic series $\tilde{y}_0$ then, y_0 and $y_0 + \delta$ will have the same asymptotic series if $\delta = o(x^{-n})$ for any n, i.e, if δ is a term beyond all orders for the asymptotic series $\tilde{y}_0$. Furthermore, δ satisfies

$$\delta' = -\delta + \frac{1}{5x}\delta \tag{7.5}$$

which has the solution $\delta \sim Cx^{1/5}e^{-x}$, where C is an arbitrary constant. The full trannseries solution is obtained as usual by substiting

$$y = y_0 + \sum_{k=1}^{\infty} C^k x^{k/5} e^{-kx} y_k \tag{7.6}$$

in (4.232) and equating coefficients of e^{-kx} to determine a set of differential equations for y_k, in which we look for solutions which are not exponentially growing in the right half plane; the only such solutions are of the form

$$y_k(x) \sim \sum_{j=0}^{\infty} \frac{a_{j,k}}{x^j} \equiv \tilde{y}_k(x) \tag{7.7}$$

Arbitrariness only appears in the choice of $a_{0,1}$; all other coefficients are determined recursively. Since C an arbitrary constant multiplying $\tilde{y}_1$, there is no loss of generality in setting $a_{0,1} = 1$. We write (7.6) in the form The transseries of y is defined to be the formal expansion

$$\tilde{y}_0(x) + \sum_{k=1}^{\infty} \xi^k \tilde{y}_k(x) \tag{7.8}$$

with $\xi = x^{1/5}e^{-x}$. We know from Theorem 5.11 that the transseries is Borel summable and

$$y = y_0(x) + \sum_{k=1}^{\infty} \xi^k y_k(x) \quad , \quad \text{where } \xi = C x^{1/5} e^{-x} \tag{7.9}$$

where

$$y_k(x) \;=\; \mathcal{L}Y_k \;=\; \int_C e^{-px}Y_k(p)dp \;=\; \mathcal{L}\mathcal{B}\tilde{y}_k \tag{7.10}$$

and

$$Y_k(p) = \mathcal{B}[\tilde{y}_k] \tag{7.11}$$

By the definition of Borel summation , the contour in the Laplace transforms in (7.10) are taken such that $-px$ is real and positive. Thus, analytic continuation in x in the upper half plane entails simultaneous analytic continuation in p in the lower half plane. We note, again using Theorem 5.11 that Y_k are analytic $\mathbb{C} \setminus \mathbb{R}^+$. Then, y_k are analytic in x (and bounded by some c^k) in a sector with angles $(-\pi/2, 5\pi/2)$. Convergence of the series (7.8) depends in an essential way on the size of effective variable $\xi \;=\; C\; x^{1/5}\; e^{-x}$. The solution $y(x)$ is analytic in a sector in the RHP of any angle $< \pi$. But ξ becomes large in the left half plane. The series is not expected to converge there.

The key to understanding the behavior of $y(x)$ for x beyond its analyticity region is to look carefully at the borderline region, where (7.9) converges, barely and see what expansion is adequate there and beyond. Convergence is marginal along curves such that ξ is small enough, but as $|x| \;\to\; \infty$, is nevertheless larger than all *negative* powers of x. In this case, any term in the transseries of the form $\xi^k a_{0,k}$ is larger than any other term of the form $\xi^l\; a_{j,l}x^{-j}$, if $k, l \;\geq\; 0$ and $j \;>\; 0$. Then though the transseries is still valid, and its summation converges, thye terms are disordered: smaller terms are followed by both smaller and larger terms.

The natural thing to do is to properly reorder the terms. This will give the expansion in a form that is suited for this marginal region, and as it turns out, beyond it as well.

In the aforementioned domain, the largest terms are those containing no inverse power of x, namely

$$y(x) \sim \sum_{k\geq 0} \xi^k a_{0,k} \equiv F_0(\xi) \tag{7.12}$$

Next in line, insofar as orders of magnitudes are concerned, are the terms containing only the first power of x^{-1} and any power of ξ, followed by the group of terms containing x^{-2} and any power of ξ and so on. The result is

$$y(x) \sim \sum_{j=0}^{\infty} x^{-j} \sum_{k=0}^{\infty} \xi^k a_{j,k} \equiv \sum_{j=0}^{\infty} \frac{F_j(\xi)}{x^j} \tag{7.13}$$

This is a new type of expansion.

It is intuitively clear that the region of validity of (7.13), while overlapping as expected with the transseries region, goes beyond it. This is because unless ξ approaches some singular value of F_j, F_j is much smaller than x. By the same token, we can read, with high accuracy, the location of the singularities of y from this expansion. All this will be shown rigorously in §C.

This is a simple but relevant example of transasymptotic matching. The new expansion (7.13) might break down further on, in which case we do exactly the same, namely push the expansion close to its boundary of validity, rearrange the terms there and obtaining a new expansion. This works until true singularities of y are reached.

It has a two-scale structure, with scales ξ and x, with the ξ-series of each F_j *analytic* in ξ for small ξ. This may seem paradoxical, as it suggests that we have started with a series with zero radius of convergence and ended up, by mere rearrangement, with a convergent one. This is not the case. The new series still diverges factorially, because the F_k *as a function of* k grow factorially.

Above we have obtained (7.13) by rearranging the series by hand. This procedure which is quite cumbersome and fortunately there is a better way to obtain it.

Namely, now that we know how the expansion should look like, we can plug in (7.13) in the original differntial equation and identify the terms order by order in $1/x$, thinking of ξ as an independent variable. In view of the simple changes of coordinates involved, we can make this substitution in (4.231), which is simpler.

We obtain

$$9\xi F_0' = (3F_0)^3 - 1; \qquad F_0'(0) = 1;\ F_0(0) = 1/3 \tag{7.14}$$

while for $k \geq 1$ we have

$$-\xi F_k' + 9F_0^2 F_k = \left(k - 1 - \frac{\xi}{5}\right) F_{k-1}' + \sum_{\substack{j_1+j_2+j_2+j_3=k \\ j_i \neq 0}} F_{j_1} F_{j_2} F_{j_3} \tag{7.15}$$

The condition $F_0'(0) = 1$ comes from the fact that the coefficient of $\xi = Ce^{-x}x^{1/5}$ in the transseries is one, while $F_0(0) = h(\infty)$. Of course, the equation for F_0 can be solved in closed form. First we treat it abstractly. If we take $F_0 = 1/3 + xG$, then it can be written in integral form as

$$G = 1 + 3\xi \int_0^{\xi} (G^2(s) + G^3(s))ds \tag{7.16}$$

which is contractive in the ball of radius say 2 in the sup norm of functions analytic in ξ for $|\xi| < \epsilon$, for small enough ϵ. Thus F_0 is analytic in ξ small, that is, the series (7.12) converges.

We see that the equations for F_k are linear.

Exercise 7.17 *Show that for* $k = 1$ *we have a one parameter family of solutions which are analytic at* $\xi = 0$*, of the form* $-1/15 + c\xi + \cdots$*. There is a* *choice of c_1 so that the equation for F_2 has a one-parameter family of solutions analytic at $\xi = 0$, parametrized by c_2, and by induction there is a choice of c_k so that the equation for F_{k+1} has a one-parameter family of solutions parametrized by c_{k+1} and so on.*

Remark 7.18 *With this choice of constants, clearly, F_j is singular only if F_0 is singular.*

Remark 7.19 *Of course, the fact that F_j are analytic at zero and thus have convergent power series in ξ does not mean that (7.13) is a convergent (double) series. Rearrangements of factorially divergent series are still factorially divergent. The growth of F_k* as a function of k *is still factorial.*

Exercise 7.20 *Let $X > 0$ be large and $\epsilon > 0$ be small. The expansion (7.12) is asymptotic along any curve of the form in Fig 1, if with the properties*

- *$|x| > X$ along the curve, the length of the curve is $O(X^m)$ and no singularity of F_0 is approached at a distance less than ϵ.*

For example, a contractive mapping integral equation can be written for the remainder

$$y(x) - \sum_{j=0}^{N} \frac{F_j(\xi)}{x^j} \tag{7.21}$$

for N conveniently large.

Ba Equation (4.226)

Let $f = F_0 - 1/3$. The equation for $f(\xi)$ is, cf. (7.48),

$$\xi f' = f(1 + 3f + 3f^2); \qquad f'(0) = 1 \tag{7.22}$$

so that

$$\xi = \xi_0 f(\xi)(f(\xi) + \omega_0)^{-\theta}(f(\xi) + \overline{\omega_0})^{-\overline{\theta}} \tag{7.23}$$

with $\xi_0 = 3^{-1/2}\exp(-\frac{1}{6}\pi\sqrt{3})$, $\omega_0 = \frac{1}{2} + \frac{i\sqrt{3}}{6}$ and $\theta = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. and, cf. (7.49),

$$\xi F_k' = (3f+1)^2 F_k + R_k(f, ..., F_{k-1})$$
$$(\text{ for } k \geq 1 \text{ and where } R_1 = \frac{3}{5}f^3) \tag{7.24}$$

The functions F_k, $k \geq 1$ can also be obtained in closed form, order by order.

By Theorem 7.52 below, the relation $y \sim \tilde{y}$ holds in the sector

$$S_{\delta_1} = \{x \in \mathbb{C} : \arg(x) \geq -\frac{\pi}{2} + \delta,\ |Cx^{1/5}e^{-x}| < \delta_1\}$$

for some $\delta_1 > 0$ and any small $\delta > 0$. Theorem 7.52 insures that $y \sim \tilde{y}$ holds in fact on a larger region, surrounding singularities of F_0 (and thus of y). To apply this result we need the surface of analyticity of F_0 and an estimate for the location of its singularities.

Lemma 7.25 *(i) The function F_0 is analytic on the universal covering $\mathcal{R}_\Xi$ of $\mathbb{C} \setminus \Xi$ where*

$$\Xi = \{\xi_p = (-1)^{p_1}\xi_0 \exp(p_2\pi\sqrt{3}) : p_{1,2} \in \mathbb{Z}\} \tag{7.26}$$

and its singularities are algebraic of order $-1/2$, located at points lying above Ξ.

(ii) (The first Riemann sheet) The function F_0 is analytic in $\mathbb{C}\setminus\Big((-\infty, \xi_0]\cup [\xi_1, \infty)\Big)$.

(iii) The Riemann surface associated to F_0 is represented in Fig. 2.

Proof

Singularities of F_0. The RHS of (7.14) is analytic except at $F_0 = \infty$, thus F_0 is analytic except at points where $F_0 \to \infty$. From (7.23) it follows that $\lim_{F_0\to\infty} \xi \in \Xi$ and (i) follows straightforwardly; in particular, as $\xi \to \xi_p \in \Xi$ we have $(\xi - \xi_p)^{1/2}F_0(\xi) \to \sqrt{-\xi_p/6}$.

(ii) We now examine on which sheets in $\mathcal{R}_\Xi$ these singularities are located, and start with a study of the first Riemann sheet (where $F_0(\xi) = \xi + O(\xi^2)$ for small ξ). Finding which of the points ξ_p are singularities of F_0 on the first sheet can be rephrased in the following way. On which constant phase (equivalently, steepest ascent/descent) paths of $\xi(F_0)$, which extend to $|F_0| = \infty$ in the plane F_0, is $\xi(F_0)$ uniformly bounded?

Constant phase paths are governed by the equation $\Im(\mathrm{d}\ln\xi) = 0$. Thus, denoting $F_0 = X + iY$, since $\xi'/\xi = \big(F_0 + 3F_0^2 + 3F_0^3\big)^{-1}$ one is led to the *real* differential equation $\Im(\xi'/\xi)\mathrm{d}X + \Re(\xi'/\xi)\mathrm{d}Y = 0$, or

$$\begin{aligned} &Y(1 + 6X + 9X^2 - 3Y^2)\mathrm{d}X \\ &\qquad - (X + 3X^2 - 3Y^2 + 3X^3 - 9XY^2)\mathrm{d}Y = 0 \end{aligned} \tag{7.27}$$

We are interested in the field lines of (7.27) which extend to infinity. Noting that the singularities of the field are $(0, 0)$ (unstable node, in a natural parameterization) and $P_\pm = (-1/2, \pm\sqrt{3}/6)$ (stable foci, corresponding to $-\overline{\omega_0}$ and $-\omega_0$), the phase portrait is easy to draw (see Fig. 2) and there are only two curves starting at $(0, 0)$ so that $|F_0| \to \infty$, ξ bounded, namely $\pm\mathbb{R}^+$, along which $\xi \to \xi_0$ and $\xi \to \xi_1$, respectively.

(iii) Thus Fig. 2 encodes the structure of singularities of F_0 on $\mathcal{R}_\Xi$ in the following way. A given class $\gamma \in \mathcal{R}_\Xi$ can be represented by a curve composed of rays and arcs of circle. In Fig. 2, in the F_0-plane, this corresponds to a curve

γ' composed of constant phase (dark gray) lines or constant modulus (light gray) lines. Curves in $\mathcal{R}_\Xi$ terminating at singularities of F_0 correspond in Fig 2. to curves so that $|F_0| \to \infty$ (the four dark gray separatrices $S_1, ..., S_4$). Thus to calculate where, on a particular Riemann sheet of $\mathcal{R}_\Xi$, is F_0 singular, one needs to find the limit of ξ in (7.23), as $F_0 \to \infty$ along along γ' followed by S_i. This is straightforward, since the branch of the complex powers $\theta, \overline{\theta}$, is calculated easily from the index of γ' with respect to $P_\pm$. $\square$

Theorem 7.52 can now be applied on relatively compact subdomains of $\mathcal{R}_\Xi$ and used to determine a uniform asymptotic representation $y \sim \tilde{y}$ in domains surrounding singularities of $y(x)$, and to obtain their asymptotic location. Going back to the original variables, similar information on $u(z)$ follows. For example, using Theorem 7.52 for the first Riemann sheet.

$$\mathcal{D} = \{|\xi| < K \mid \xi \notin (-\infty, \xi_1) \cup (\xi_0, +\infty)\ ,\ |\xi - \xi_0| > \epsilon, |\xi - \xi_1| > \epsilon, \}$$

(for any small $\epsilon > 0$ and large positive K) the corresponding domain in the z-plane is shown in Fig. 3.

In general, we fix $\epsilon > 0$ small, and some $K > 0$ and define $\mathcal{A}_K = \{z : \arg z \in \left(\frac{3}{10}\pi - 0, \frac{9}{10}\pi + 0\right),\ |\xi(z)| < K\}$ and let $\mathcal{R}_{K,\Xi}$ be the universal covering of $\Xi \cap \mathcal{A}_K$ and $\mathcal{R}_{z;K,\epsilon}$ the corresponding Riemann surface in the z plane, with $\epsilon-$ neighborhoods of the points projecting on $z(x(\Xi))$ deleted.

Proposition 7.28 *(i) The solutions $u = u(z; C)$ described in the beginning of §D have the asymptotic expansion*

$$u(z) \sim z^{1/3}\left(1 + \frac{1}{9}z^{-5/3} + \sum_{k=0}^{\infty} \frac{F_k\left(C\xi(z)\right)}{z^{5k/3}}\right) \qquad (\textit{as } z \to \infty; \quad z \in \mathcal{R}_{z;K,\epsilon}) \quad (7.29)$$

where

$$\xi(z) = x(z)^{1/5} \mathrm{e}^{-x(z)}, \ \textit{and } x(z) = -\frac{9}{5} z^{5/3} \quad (7.30)$$

(ii) In the "steep ascent" strips $\arg(\xi) \in (a_1, a_2)$, $|a_2 - a_1| < \pi$ starting in $\mathcal{A}_K$ and crossing the boundary of $\mathcal{A}_K$, the function u has at most one singularity, when $\xi(z) = \xi_0$ or ξ_1, and $u(z) = z^{1/3}\mathrm{e}^{\pm 2\pi i/3}(1 + o(1))$ as $z \to \infty$ (the sign is determined by $\arg(\xi)$).

(iii) The singularities of $u(z; C)$, for $C \neq 0$, are located within $O(\epsilon)$ of the punctures of $\mathcal{R}_{z;K,0}$.

Applying Theorem 7.52 to (4.232) it follows that for $n \to \infty$, a given solution y is singular at points $\tilde{x}_{p,n}$ such that $\xi(\tilde{x}_{p,n})/\xi_p = 1 + o(1)$ ($|\tilde{x}_{p,n}|$ large).

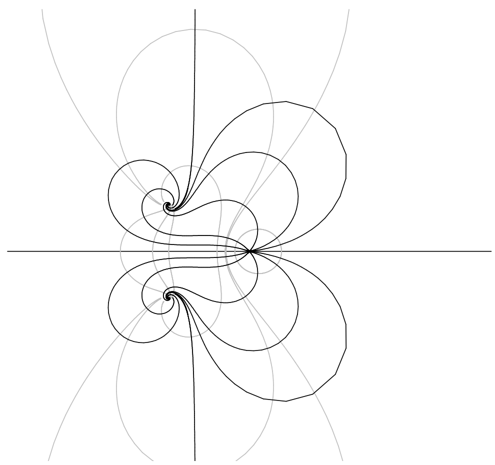


FIGURE 7.1: The dark lines represent the phase portrait of (7.27), as well as the lines of steepest variation of $|\xi(u)|$. The light gray lines correspond to the orthogonal field, and to the lines $|\xi(u)| = const$.

Now, y can only be singular if $|y| \to \infty$ (otherwise the r.h.s. of (4.232) is analytic). If $\tilde{x}_{p,n}$ is a point where y is unbounded, with $\delta = x - \tilde{x}_{p,n}$ and $v = 1/y$ we have

$$\frac{\mathrm{d}\delta}{\mathrm{d}v} = vF_s(v,\delta) \tag{7.31}$$

where F_s is analytic near $(0,0)$. It is easy to see that this differential equation has a unique solution with $\delta(0) = 0$ and that $\delta'(0) = 0$ as well.

The result is then that the singularities of u are also algebraic of order $-1/2$.

Proposition 7.32 *If z_0 is a singularity of $u(z;C)$ then in a neighborhood of z_0 we have*

$$u = \pm\sqrt{-1/2}(z-z_0)^{-1/2}A_0((z-z_0)^{1/2}) \tag{7.33}$$

where A_0 is analytic at zero and $A_0(0) = 1$.

Notes. 1. The local behavior near a singularity could have been guessed by local Painlevé analysis and the method of dominant balance, with the standard ansatz near a singularity, $u \sim Const.(z - z_0)^p$. The results however are **global**: Proposition 7.28 gives the behavior of *a fixed* solution at infinitely many singularities, and gives the **position** of these singularities as soon as C_1 (or the position of only one of these singularities) is known (and in addition show that the power behavior ansatz is correct in this case).

2. By the substitution $y = v/(1 + v)$ in (4.232) we get

$$v' = -v - 27\,\frac{v^3}{1+v} - 10\,v^2 + \frac{1}{5t}v + g^{[1]}(t^{-1}, v) \tag{7.34}$$

The singularities of v are at the points where $v(t) = -1$.

3. **It is not always the case** that the singularities of y must be of the same *type* as the singularities of F_0. The position, as we argued is asymptotically the same, but near singularities the expansion (7.13) becomes invalid and it must either be re-matched to an expansion valid near singularities or, again, we can rely on the differential equation to see what these singularities are.

Further examples and discussions follow, in §Da and §Db .

C Rigorous results for generic nonlinear differential systems

We describe the results in [42] but omit proofs which follow the lines sketched in §B, but are rather lengthy. The region where the formal or summed transseries is valid is

$$S_{trans} = \left\{ x \in \mathbb{C}\,;\ \text{if } C_j \neq 0 \text{ then } x^{a_j} e^{-\lambda_j x} = o(1), j = 1, ..., n \right\} \tag{7.35}$$

This sector might be the whole $\mathbb{C}$ if all $C_j = 0$; otherwise it lies between two antistokes lines, and has opening at most π.

If we have normalized the equation in such a way that $\lambda_1 = 1$, and λ_m is the eigenvalue in the fourth quadrant (if there is such an eigenvalue) with the most *negative* angle, then in the upper half plane, S_{trans} will be controlled, roughly, by the condition $\Re(\lambda_m x) > 0$. If there is no such eigenvalue, then the rigion in the first quadrant will be determined by $\lambda_1 = 1$, namely $x^{a_1} e^{-x} = o(1)$. If we examine the first quadrant, it is now convenient to rotate again the independent variable so that $\lambda_m = 1$, since this eigenvalue is the determining one. Since originally no exponentials associated with λ_j belonging to the second or third quadrant were allowed, then after this new rotation there will be no eigenvalue in the fourth quadrant, and the region of validity *in the first quadrant* would be, roughly, up to the imaginary line.

$$\mathbf{y}(x) = \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x)$$
$$= \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathcal{L}\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}(x) \equiv \mathcal{L}\mathcal{B}\tilde{\mathbf{y}}(x) \quad (7.36)$$

for some constants $\mathbf{C} \in \mathbb{C}^n$, where $M_j = \lfloor \Re\alpha_j \rfloor + 1$ ($\lfloor\cdot\rfloor$ is the integer part), and

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^{-\mathbf{k}\boldsymbol{\alpha}'+r}} \qquad (\boldsymbol{\alpha}' = \boldsymbol{\alpha} - \mathbf{M}) \quad (7.37)$$

(for technical reasons the Borel summation procedure is applied to the series

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) \equiv x^{\mathbf{k}\boldsymbol{\alpha}'} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \quad (7.38)$$

rather than to $\tilde{\mathbf{s}}_{\mathbf{k}}(x)$.

The key to understanding the behavior of $\mathbf{y}(x)$ for x beyond S_{an} is to look carefully at the borderline region where (7.36) converges but barely so. Because of nonresonance, for $\arg(x) = \pi/2$ we have $\Re(\lambda_j x) > 0, j = 2, ..., n_1$. All terms in (7.36) with $\mathbf{k}$ not a multiple of $\mathbf{e}_1 = (1, 0, ..., 0)$ are subdominant (small). Thus, for x near $i\mathbb{R}^+$ we only need to look at

$$\mathbf{y}^{[1]}(x) = \sum_{k\geq 0} C_1^k \mathrm{e}^{-kx} x^{kM_1} \mathbf{y}_{k\mathbf{e}_1}(x) \quad (7.39)$$

The region of convergence of (7.39) (thus of (4.225)) is then determined by the effective variable $\xi = C_1 \mathrm{e}^{-x} x^{\alpha_1}$ (since $\mathbf{y}_{k\mathbf{e}_1} \sim \tilde{\mathbf{y}}_{k\mathbf{e}_1} = \mathbf{e}_1 + o(1)$). Convergence is marginal along curves such that ξ is small enough but, as $|x| \to \infty$, is nevertheless larger than all *negative* powers of x. In this case, any term of the form $C_1^k e^{-kx} x^{kM_1} \mathbf{y}_{k\mathbf{e}_1;0}$ is much larger than the terms $C_1^l e^{-lx} x^{l\alpha_1} \mathbf{y}_{l\mathbf{e}_1}$ if $k, l \geq 0$ and $r > 0$. Hence the leading behavior of $\mathbf{y}^{[1]}$ is expected to be

$$\mathbf{y}^{[1]}(x) \sim \sum_{k\geq 0} (C_1 \mathrm{e}^{-x} x^{\alpha_1})^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;0} \equiv F_0(\xi) \quad (7.40)$$

moreover, taking into account all terms in $\tilde{\mathbf{y}}_{k\mathbf{e}_1}$ we get

$$\mathbf{y}^{[1]}(x) \sim \sum_{r=0}^{\infty} x^{-r} \sum_{k=0}^{\infty} \xi^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;r} \equiv \sum_{j=0}^{\infty} \frac{\mathbf{F}_j(\xi)}{x^j} \quad (7.41)$$

Expansion (7.41) has a two-scale structure, with the scales ξ and x.

Ca Notation

Let d be a direction in the x-plane which is not not an antistokes line. Consider a solution $\mathbf{y}(x)$ of (4.220) satisfying the assumptions in §5.0a . We define

$$S_{an} = S_{an}(\mathbf{y}(x);\epsilon) = S_\epsilon^+ \cup S_\epsilon^- \tag{7.42}$$

where

$$S_\epsilon^\pm = \Big\{x\,;\,|x|>R\ ,\ \arg(x)\in[-\frac{\pi}{2}\mp\epsilon,\frac{\pi}{2}\mp\epsilon] \text{ and } \\ \big|C_j^- e^{-\lambda_j x}x^{-\beta_j}\big| < \delta^{-1} \text{ for } j=1,...,n\Big\} \tag{7.43}$$

We use be the representation of $\mathbf{y}$ as summation of its transseries $\tilde{\mathbf{y}}(x)$ (4.225) in the direction d. Let

$$p_{j;\mathbf{k}} = \lambda_j - \mathbf{k}\cdot\boldsymbol{\lambda}\ ,\quad j=1,...,n_1\ ,\quad \mathbf{k}\in\mathbb{Z}_+^{n_1} \tag{7.44}$$

For simplicity we *assume*, what is generically the case, that no $\overline{p_{j;\mathbf{k}}}$ lies on the antistokes lines bounding S_{trans}.

We *assume* that not all parameters C_j are zero, say $C_1\neq 0$. Then S_{trans} is bounded by two antistokes lines and its opening is at most π.

We arrange that

(a) $\arg(\lambda_1)<\arg(\lambda_2)<...<\arg(\lambda_{n_1})$

and, by construction,

(b) $\Im\lambda_k\ge 0$.

The solution $\mathbf{y}(x)$ is then analytic in a region S_{an}.

The locations of singularities of $\mathbf{y}(x)$ depend on the constant C_1 (constant which may change when we cross the Stokes line $\mathbb{R}^+$). We need its value in the sector between $\mathbb{R}^+$ and $i\mathbb{R}_+$, the next Stokes line.

Fix some small, positive δ and c. Denote

$$\xi = \xi(x) = C_1 \mathrm{e}^{-x}x^{\alpha_1} \tag{7.45}$$

and

$$\mathcal{E} = \Big\{x\,;\,\arg(x)\in\Big[-\frac{\pi}{2}+\delta,\frac{\pi}{2}+\delta\Big] \text{ and } \\ \Re(\lambda_j x/|x|)>c \text{ for all } j \text{ with } 2\le j\le n_1\} \tag{7.46}$$

Also let

$$\mathcal{S}_{\delta_1} = \{x\in\mathcal{E}\,;\,|\xi(x)|<\delta_1\} \tag{7.47}$$

The sector $\mathcal{E}$ contains S_{trans}, except for a thin sector at the lower edge of S_{trans} (excluded by the conditions $\Re(\lambda_j x/|x|)>c$ for $2\le j\le n_1$, or, if $n_1=1$, by the condition $\arg(x)\ge -\frac{\pi}{2}+\delta$), and may extend beyond $i\mathbb{R}_+$ since there is no condition on $\Re(\lambda_1 x)$—hence $\Re(\lambda_1 x)=\Re(x)$ may change sign in $\mathcal{E}$ and $\mathcal{S}_{\delta_1}$.

Figure 1 is drawn for $n_1=1$; $\mathcal{E}$ contains the gray regions and extends beyond the curved boundary.

Cb The recursive system for $\mathbf{F}_m$

The functions are $\mathbf{F}_m$ recursively, from their differential equation. Formally the calculation is the following.

The series $\tilde{\mathbf{F}} = \sum_{m\geq 0} x^{-m}\mathbf{F}_m(\xi)$ is a formal solution of (4.220); substitution in the equation and identification of coefficients of x^{-m} yields the recursive system

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_0 = \xi^{-1}\left(\hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0,\mathbf{F}_0)\right) \tag{7.48}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_m + \hat{N}\mathbf{F}_m = \alpha_1\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_{m-1} + \mathbf{R}_{m-1} \qquad \text{for } m\geq 1 \tag{7.49}$$

where $\hat{N}$ is the matrix

$$\xi^{-1}(\partial_{\mathbf{y}}\mathbf{g}(0,\mathbf{F}_0) - \hat{\Lambda}) \tag{7.50}$$

and the function $\mathbf{R}_{m-1}(\xi)$ depends only on the $\mathbf{F}_k$ with $k<m$:

$$\xi\mathbf{R}_{m-1} = -\left[(m-1)I + \hat{A}\right]\mathbf{F}_{m-1} - \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbf{g}\left(z;\sum_{j=0}^{m-1} z^j\mathbf{F}_j\right)\Bigg|_{z=0} \tag{7.51}$$

For more detail see [42] Section 4.3.

To leading order we have $\mathbf{y}\sim\mathbf{F}_0$ (see also (7.40)) where $\mathbf{F}_0$ satisfies the autonomous (after a substitution $\xi = e^{\zeta}$) equation

$$\mathbf{F}_0' = \hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0,\mathbf{F}_0)$$

which can be solved in closed form for first order equations ($n=1$) (the equation for F_0 is separable, and for $k\geq 1$ the equations are linear), as well as in other interesting cases (see e.g. §Db).

(7.48), (7.49). To determine the $\mathbf{F}_m$'s associated to $\mathbf{y}$ we first note that these functions are analytic at $\xi=0$ (cf. Theorem 7.52). Denoting by $F_{m,j}$, $j = 1,..,n$ the components of $\mathbf{F}_m$, a simple calculation shows that (7.48) has a unique analytic solution satisfying $F_{0,1}(\xi) = \xi + O(\xi^2)$ and $F_{0,j}(\xi) = O(\xi^2)$ for $j=2,...,n$. For $m=1$, there is a one parameter family of solutions of (7.49) having a Taylor series at $\xi=0$, and they have the form $F_{1,1}(\xi) = c_1\xi + O(\xi^2)$ and $F_{1,j}(\xi) = O(\xi^2)$ for $j=2,...,n$. The parameter c_1 is determined from the condition that (7.49) has an analytic solution for $m=2$. For this value of c_1 there is a one parameter family of solutions $\mathbf{F}_2$ analytic at $\xi=0$ and this new parameter is determined by analyzing the equation of $\mathbf{F}_3$. The procedure can be continued to any order in m, in the same way; in particular, the constant c_m is only determined at step $m+1$ from the condition of analyticity of $\mathbf{F}_{m+1}$.

Theorem 7.52 *(i) The functions* $\mathbf{F}_m(\xi)$; $m \geq 1$, *are analytic in* $\mathcal{D}$ *(note that by construction* $\mathbf{F}_0$ *is analytic in* $\mathcal{D}$*) and for some positive* B, K *we have*

$$|F_m(\xi)| \leq K m! B^m, \quad \xi \in \mathcal{D} \tag{7.53}$$

(ii) For large enough R, *the solution* $\mathbf{y}(x)$ *is analytic in* $\mathcal{D}_x$ *and has the asymptotic representation*

$$\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m} \mathbf{F}_m(\xi(x)) \quad (x \in \mathcal{D}_x,\ |x| \to \infty) \tag{7.54}$$

In fact, the following Gevrey-like estimates hold

$$\left| \mathbf{y}(x) - \sum_{j=0}^{m-1} x^{-j} \mathbf{F}_j(\xi(x)) \right| \leq K_2 m! B_2^m |x|^{-m} \quad (m \in \mathbb{N}^+,\ x \in \mathcal{D}_x) \tag{7.55}$$

(iii) Assume $\mathbf{F}_0$ *has an isolated singularity at* $\xi_s \in \Xi$ *and that the projection of* $\mathcal{D}$ *on* $\mathbb{C}$ *contains a punctured neighborhood of (or an annulus of inner radius* r *around)* ξ_s.

Then, if $C_1 \neq 0$, $\mathbf{y}(x)$ *is singular at a distance at most* $o(1)$ $(r + o(1)$, *respectively) of* $x_n \in \xi^{-1}(\{\xi_s\}) \cap \mathcal{D}_x$, *as* $x_n \to \infty$.
The collection $\{x_n\}_{n \in \mathbb{N}}$ *forms a nearly periodic array*

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1) \tag{7.56}$$

as $n \to \infty$.

Some of the conclusions of the theorem hold with $\mathcal{D}$ noncompact, under some natural restrictions, see Proposition 7.57.

Remarks. 1. The singularities x_n satisfy $C_1 e^{-x_n} x_n^{\alpha_1} = \xi_s(1 + o(1))$ (for $n \to \infty$). Therefore, the singularity array lies slightly to the left of the antistokes line $i\mathbb{R}_+$ if $\Re(\alpha_1) < 0$ (this case is depicted in Figure 1) and slightly to the right of $i\mathbb{R}_+$ if $\Re(\alpha_1) > 0$.

2. In practice it is useful to normalize the system (4.220) so that α_1 is as small as possible.

3. By (7.55) a truncation of the two-scale series (7.54) at an m dependent on x ($m \sim |x|/B$) is seen to produce exponential accuracy $o(\mathrm{e}^{-|x/B|})$, see e.g. [10].

4. Theorem 7.52 can also be used to determine precisely the nature of the singularities of $\mathbf{y}(x)$. In effect, for any n, the representation (7.54) provides $o(\mathrm{e}^{-K|x_n|})$ estimates on $\mathbf{y}$ down to an $o(\mathrm{e}^{-K|x_n|})$ distance of an actual singularity x_n. In most instances this is more than sufficient to match to a suitable

local integral equation, contractive in a tiny neighborhood of x_n, providing rigorous control of the singularity. See also §D.

General comments. 1. The expansion scales, x and $x^{-1/2}e^{-x}$ are crucial. Only for this choice one obtains an expansion which is valid both in S_{trans} and near poles of (4.88). For instance, the more general second scale $x^a e^{-x}$ introduces logarithmic singularities in H_j, except when $a \in -\frac{1}{2} + \mathbb{Z}$. With these logarithmic terms, the two scale expansion would only be valid in an $O(1)$ region in x, what is sometimes called a "patch at infinity", instead of more than a sector. Also, $a \in -\frac{1}{2} - \mathbb{N}$ introduces obligatory singularities at $\xi = 0$ precluding the validity of the expansion in S_{trans}. The case $a \in -\frac{1}{2} + \mathbb{N}$ produces instead an expansion valid in S_{trans} but not near poles. Indeed, the so that

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\alpha}{A}$$

implying

$$\xi^2 F_0'' + \xi F_0' - F_0 = 3BF_0^2 - F_0^3$$

and, with the same initial condition as above, we now have

$$F_0 = \frac{2\xi(1 + B\xi)}{\xi^2 + 2}$$

The first normalization applies for the manifold of solutions such that $y \sim -\frac{\alpha}{x}$ (for $\alpha = 0$ y is exponentially small and behaves like an Airy function) while the second one corresponds to $y \sim -B - \frac{\alpha}{2}x^{-3/2}$.

The following is an extension, in some respects, of Theorem 7.52 (ii).

Proposition 7.57 *Assume $\mathcal{D}$ is not necessarily compact, Γ is a curve of possibly infinite length in $\mathcal{D}$ with the following properties:*
(a) For some $\epsilon > 0$, $\mathbf{T}_{1,2}(z, \boldsymbol{\delta})$ and $\hat{N}(z)$ are analytic for z in an ϵ neighborhood of Γ and for $|\boldsymbol{\delta}| < \epsilon$ and in addition $\mathbf{T}_{1,2}(z, \boldsymbol{\delta}) = O(z\boldsymbol{\delta}, \boldsymbol{\delta}^2)$
(b) $\hat{M}(\xi, \xi_{1,0})$ is bounded in an ϵ neighborhood of Γ and for some K and all $\xi \in \Gamma$ we have $\int_{\xi_{1,0}}^{\xi} \left|\hat{M}(\xi, \xi_{1,0})\right| d|s| < K$ (where $|\hat{M}|$ is some Euclidian norm of the matrix $\hat{M}(\xi, \xi_{1,0})$).

Then the conclusions of Theorem 7.52 (ii) hold in the x domain $\mathcal{D}_x$ corresponding to $\mathcal{D}$.

Cc Proof of Theorem 7.52 (iii)

To show Theorem 7.52 (iii), assume ξ_s is an isolated singularity of $\mathbf{F}_0$ (thus $\xi_s \neq 0$) and $X = \{x : \xi(x) = \xi_s\}$. By lemma 8.4 there is a circle $\mathcal{C}$ around ξ_s and a function $g(\xi)$ analytic in $B_r(\xi - \xi_s)$ such that $\oint_{\mathcal{C}} \mathbf{F}_0(\xi)g(\xi)d\xi = 1$. In a neighborhood of $x_n \in X$ the function $f(x) = e^{-x}x^{\alpha_1}$ is conformal and for large x_n

$$\oint_{f^{-1}(\mathcal{C})} \mathbf{y}(x) \frac{g(f(x))}{f(x)} dx$$
$$= -\oint_{\mathcal{C}} (1 + O(x_n^{-1}))(\mathbf{F}_0(\xi) + O(x_n^{-1}))g(\xi)d\xi = 1 + O(x_n^{-1}) \neq 0 \quad (7.58)$$

It follows from lemma 8.4 that for large enough x_n $\mathbf{y}(x)$ is not analytic inside $\mathcal{C}$ either. Since the radius of $\mathcal{C}$ can be taken $o(1)$ Theorem 7.52 (iii) follows.

Note. In many cases the singularity of $\mathbf{y}$ is of the *same type* as the singularity of $\mathbf{F}_0$. See §D for further comments.

In the following we will make rigorous these intuitive arguments and then proceed to explore further properties and consequences.

D Further examples

Da $\mathrm{P_I}$.

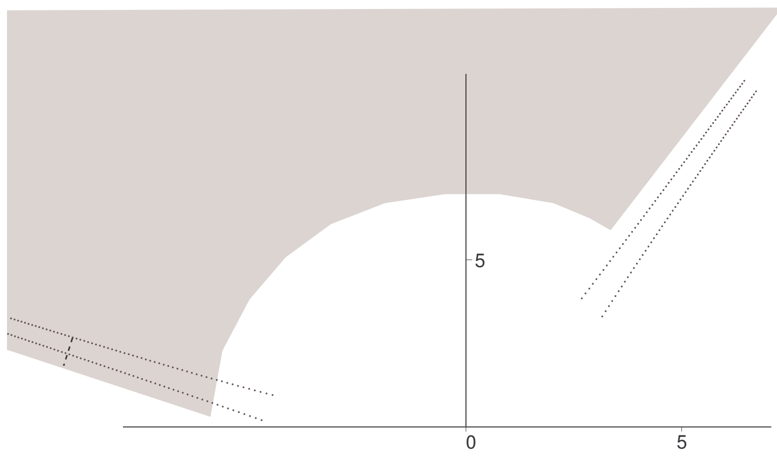
Proposition 7.59 below shows, in (i), how the constant C beyond all orders is associated to a truncated solution $y(z)$ of $\mathrm{P_I}$ for $\arg(z) = \pi$ (formula (7.60)) and gives the position of one array of poles z_n of the solution associated to C (formula (7.61)), and in (ii) provides uniform asymptotic expansion to all orders of this solution in a sector centered on $\arg(z) = \pi$ and one array of poles (except for small neighborhoods of these poles) in formula (7.63).

Proposition 7.59 *(i) Let y be a solution of (4.82) such that $y(z) \sim \sqrt{-z/6}$ for large z with $\arg(z) = \pi$. For any $\phi \in (\pi, \pi + \frac{2}{5}\pi)$ the following limit determines the constant C (which does not depend on ϕ in this range) in the transseries $\tilde{y}$ of y:*

$$\lim_{\substack{|z|\to\infty \\ \arg(z)=\phi}} \xi(z)^{-1} \left(\sqrt{\frac{6}{-z}} y(z) - \sum_{k \le |x(z)|} \frac{\tilde{y}_{0;k}}{z^{5k/2}} \right) = C \qquad (7.60)$$

(Note that the constants $\tilde{y}_{0;k}$ do not *depend on C). With this definition, if $C \neq 0$, the function y has poles near the antistokes line $\arg(z) = \pi + \frac{2}{5}\pi$ at all points z_n, where, for large n*

$$z_n = -\frac{(60\pi i)^{4/5}}{24} \left(n^{\frac{4}{5}} + iL_n n^{-\frac{1}{5}} + \left(\frac{L_n^2}{8} - \frac{L_n}{4\pi} + \frac{109}{600\pi^2} \right) n^{-\frac{6}{5}} \right) + O\left(\frac{(\ln n)^3}{n^{\frac{11}{5}}} \right) \qquad (7.61)$$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with $L_n = \frac{1}{5\pi}\ln\left(\frac{\pi i C^2}{72} n\right)$*, or, more compactly,*

$$\xi(z_n) = 12 + \frac{327}{(-24z_n)^{5/4}} + O(z_n^{-5/2}) \quad (z_n \to \infty) \tag{7.62}$$

(ii) Let $\epsilon \in \mathbb{R}^+$ *and define*

$$\mathcal{Z} = \{z : \arg(z) > \frac{3}{5}\pi; |\xi(z)| < 1/\epsilon;\ |\xi(z) - 12| > \epsilon\}$$

(the region starts at the antistokes line $\arg(z) = \frac{3}{5}\pi$ *and extends slightly beyond the next antistokes line,* $\arg(z) = \frac{7}{5}\pi$*). If* $y \sim \sqrt{-z/6}$ *as* $|z| \to \infty,\ \arg(z) = \pi$*, then for* $z \in \mathcal{Z}$ *we have*

$$y \sim \sqrt{\frac{-z}{6}}\left(1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} + \sum_{k=0}^{\infty} \frac{30^k H_k(\xi)}{(-24z)^{5k/4}}\right)$$
$$(|z| \to \infty,\ z \in \mathcal{Z}) \tag{7.63}$$

The functions H_k *are rational, and* $H_0(\xi) = \xi(\xi/12 - 1)^{-2}$*. The expansion (7.63) holds uniformly in the sector* $\pi^{-1}\arg(z) \in (3/5, 7/5)$ *and also on one of its sides, where* H_0 *becomes dominant, down to an* $o(1)$ *distance of the actual poles of* y *if* z *is large.*

Proof. We prove the corresponding statements for the normal form (4.88). To go back to the variables of (4.82) mere substitutions are needed, which we omit.

Most of Proposition 7.59 is a direct consequence of Theorems 1 and 2. For the one-parameter family of solutions which are small in the right half plane we then have

$$h \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi(x)) \tag{7.64}$$

where $\xi(x) = x^{-1/2}e^{-x}$.
As in the first example we find H_k by substituting (7.64) in (4.88).

The equation of H_0 is

$$\xi^2 H_0'' + \xi H_0' = H_0 + \frac{1}{2}H_0^2$$

The general solution of this equation are the Weierstrass elliptic functions of $\ln\xi$, as expected from the general knowledge of the asymptotic behavior of the Painlevé solutions (see [13]). For our special initial condition, H_0 analytic at zero and $H_0(\xi) = \xi(1 + o(1))$, the solution is a degenerate elliptic function, namely,

$$H_0(\xi) = \frac{\xi}{(\xi/12-1)^2}$$

Important remark. *One of the two free constants in the general solution H_1 is determined by the condition of analyticity at zero of H_1 (this constant multiplies terms in $\ln\xi$). It is interesting to note that the remaining constant is only determined in the* next *step, when solving the equation for H_2! This pattern is typical (see §Cb).*

Continuing this procedure we obtain successively:

$$H_1 = \left(216\,\xi + 210\,\xi^2 + 3\,\xi^3 - \frac{1}{60}\,\xi^4\right)(\xi-12)^{-3} \tag{7.65}$$

$$H_2 = \left(1458\xi + 5238\xi^2 - \frac{99}{8}\xi^3 - \frac{211}{30}\xi^4 + \frac{13}{288}\xi^5 + \frac{\xi^6}{21600}\right)(\xi-12)^{-4} \tag{7.66}$$

We omit the straightforward but quite lengthy inductive proof that all H_k are rational functions of ξ. The reason the calculation is tedious is that this property holds for (4.88) but *not* for its generic perturbations, and the last potential obstruction to rationality, successfully overcome by (4.88), is at $k=6$. On the positive side, these calculations are algorithmic and are very easy to carry out with the aid of a symbolic language program.

In the same way as in Example 1 one can show that the corresponding singularities of h are double poles: all the terms of the corresponding asymptotic expansion of $1/h$ are *analytic* near the singularity of h! All this is again straightforward, and lengthy because of the potential obstruction at $k=6$.

Let ξ_s correspond to a zero of $1/h$. To leading order, $\xi_s = 12$, by Theorem 7.52 (iii). To find the next order in the expansion of ξ_s one substitutes $\xi_s = 12 + A/x + O(x^{-2})$, to obtain

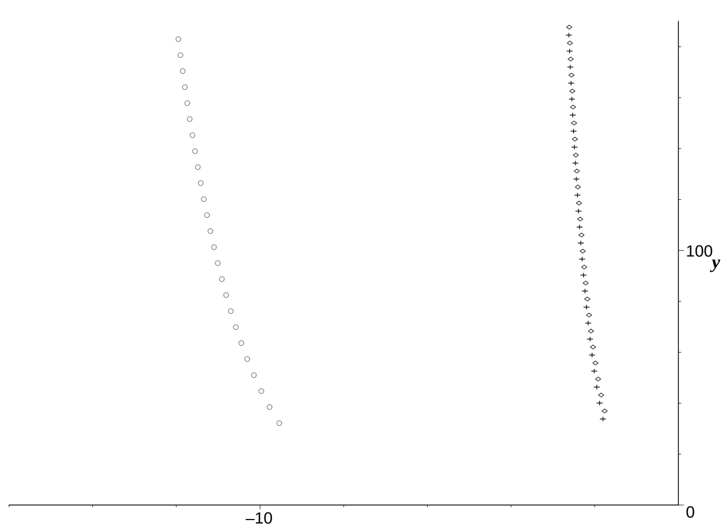
$$1/h(\xi_s) = \frac{(A-109/10)^2}{12^3x^2} + O(1/x^3)$$

whence $A = 109/10$ (because $1/h$ is analytic at ξ_s) and we have

$$\xi_s = 12 + \frac{109}{10x} + O(x^{-2}) \tag{7.67}$$

Given a solution h, its constant C in ξ for which (7.64) holds can be calculated from asymptotic information in any direction above the real line by near least term truncation, namely

$$C = \lim_{\substack{x\to\infty\\ \arg(x)=\phi}} \exp(x)x^{1/2}\left(h(x) - \sum_{k\le|x|}\frac{\tilde{h}_{0,k}}{x^k}\right) \tag{7.68}$$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(this is a particular case of much more general formulas [24]) where $\sum_{k>0} \tilde{h}_{0,k} x^{-k}$ is the common asymptotic series of all solutions of (4.88) which are small in the right half plane.

□

General comments. 1. The expansion scales, x and $x^{-1/2}\mathrm{e}^{-x}$ are crucial. Only for this choice one obtains an expansion which is valid both in S_{trans} and near poles of (4.88). For instance, the more general second scale $x^a\mathrm{e}^{-x}$ introduces logarithmic singularities in H_j, except when $a \in -\frac{1}{2} + \mathbb{Z}$. With these logarithmic terms, the two scale expansion would only be valid in an $O(1)$ region in x, what is sometimes called a "patch at infinity", instead of more than a sector. Also, $a \in -\frac{1}{2} - \mathbb{N}$ introduces obligatory singularities at $\xi = 0$ precluding the validity of the expansion in S_{trans}. The case $a \in -\frac{1}{2} + \mathbb{N}$ produces instead an expansion valid in S_{trans} but not near poles. Indeed, the substitution $h(x) = g(x)/x^n,\ n \in \mathbb{N}$ has the effect of changing α to $\alpha + n$ in the normal form. This in turn amounts to restricting the analysis to a region far away from the poles, and then all H_j will be entire. In general we need thus to make (by substitutions in (4.220)) $a = \alpha$ minimal compatible with the assumptions (a1) and (a2), as this ensures the widest region of analysis.

Db The Painlevé equation P2

This equation reads:

$$y'' = 2y^3 + xy + \alpha \tag{7.69}$$

(Incidentally, this example also shows that for a given equation distinct solution manifolds associated to distinct asymptotic behaviors may lead to different normalizations.) After the change of variables

$$x = (3t/2)^{2/3}; \quad y(x) = x^{-1}(t\,h(t) - \alpha)$$

one obtains the normal form equation

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2 + 1}{9t^2}\right) h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3 - \alpha)}{9t^3} = 0 \tag{7.70}$$

and

$$\lambda_1 = 1, \quad \alpha_1 = -1/2; \ \xi = \frac{e^{-t}}{\sqrt{t}}; \quad \xi^2 F_0'' + \xi F_0' = F_0 + \frac{8}{9}F_0^3$$

The initial condition is (always): F_0 analytic at 0 and $F_0'(0) = 1$. This implies

$$F_0(\xi) = \frac{\xi}{1 - \xi^2/9}$$

Distinct normalizations (and sets of solutions) are provided by

$$x = (At)^{2/3}; \quad y(x) = (At)^{1/3}\left(w(t) - B + \frac{\alpha}{2At}\right)$$

if $A^2 = -9/8, B^2 = -1/2$. In this case,

$$w'' + \frac{w'}{t} + w\left(1 + \frac{3B\alpha}{tA} - \frac{1-6\alpha^2}{9t^2}\right)w$$
$$- \left(3B - \frac{3\alpha}{2tA}\right)w^2 + w^3 + \frac{1}{9t^2}\left(B(1+6\alpha^2) - t^{-1}\alpha(\alpha^2-4)\right) \quad (7.71)$$

so that

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\alpha}{A}$$

implying

$$\xi^2 F_0'' + \xi F_0' - F_0 = 3BF_0^2 - F_0^3$$

and, with the same initial condition as above, we now have

$$F_0 = \frac{2\xi(1+B\xi)}{\xi^2+2}$$

The first normalization applies for the manifold of solutions such that $y \sim -\frac{\alpha}{x}$ (for $\alpha = 0$ y is exponentially small and behaves like an Airy function) while the second one corresponds to $y \sim -B - \frac{\alpha}{2}x^{-3/2}$.

Chapter 8

Appendix

A Rigorous construction of transseries

Aa Abstracting from §4.2b

1. Let $(\mathcal{G}, \cdot, \ll)$ be a finitely generated, totally ordered (any two elements are comparable) abelian group, with generators $\mu_1, \mu_2, ...\mu_n$, such that $\ll$ is compatible with the group operations, that is, $g_1 \ll g_2$ and $g_3 \ll g_4$ implies $g_1 g_3 \ll g_2 g_4$, and such that $1 \gg \mu_1 \gg \cdots \gg \mu_n$. This is the case when μ_i are transmonomials of level zero.

2. We write $\mu_{\mathbf{k}} = \mu^{\mathbf{k}} := \mu_1^{k_1} \cdots \mu_n^{k_n}$.

Lemma 8.1 *Consider the partial order relation that we introduced before on $\mathbb{Z}^n$, $\mathbf{k} > \mathbf{m}$ iff $k_i \geq m_i$ for all $i = 1, 2, ..., n$ and at least for some j we have $k_j > m_j$. If $B \subset A = \{\mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{m}\}$, then there is no infinite nonascending chain in B. That, is there is no infinite sequence in $B, b_n \neq b_m$ for $n \neq m$, and $b_{n+1} \not> b_n$ for all n.*

Proof. Assume there is an infinite nonascending sequence, $\{\mathbf{k}(m)\}_{m\in\mathbb{N}}$. Then at least for some $i \in \{1, 2, ..., n\}$ the sequence $\{k_i(m)\}_{m\in\mathbb{N}}$ must have infinitely many distinct elements. Since the $k_i(m)$ are bounded below, then the set $\{k_i(m)\}_{m\in\mathbb{N}}$ is unbounded above, and we can extract a strictly increasing subsequence $\{k_i(m_l)\}_{l\in\mathbb{N}}$. We now take the sequence $\{\mathbf{k}(m_l)\}_{l\in\mathbb{N}}$. At least for some $j \neq i$ the set $k_j(m_l)$ needs to have infinitely many elements too. Indeed if the sets $\{k_j(m_l); j \neq i\}$ are finite, we can split $\{\mathbf{k}(m_l)\}_{l\in\mathbb{N}}$ into a finite set of subsequences, in each of which all $k_j(m_l)$, $j \neq i$, are constant while k_i is strictly increasing. But every such subsequence would be strictly decreasing, which is impossible. By finite induction we can extract a subsequence $\{\mathbf{k}(m_t)\}_{t\in\mathbb{N}}$ of $\mathbf{k}(m)\}_{m\in\mathbb{N}}$ in which all $k_l(m_t)$ are increasing, a contradiction.

Remark. This is a particular, much easier result of Kruskal's tree theorem. which we briefly mention here. A relation is well-founded if and only if it contains no countable infinite descending sequence $\{x_j\}_{\in\mathbb{N}}$ of elements of X such that $x_{n+1} R x_n$ for every $n \in \mathbb{N}$. The relation R is a quasiorder if it is reflexive and transitive. Well-quasi-ordering is a well-founded quasi-ordering such that that there is no sequence $\{x_j\}_{\in\mathbb{N}}$ with $x_i \not\leq x_j \forall i < j$. A tree is a collection of vertices in which any two vertices are connected by exactly one line. *J. Kruskal's tree theorem states that the set of finite trees over a well-quasi-ordered set is well-quasi-ordered.*

3. *Exercises. (1) Show that the equation* $\mathbf{k}_1+\mathbf{k}_2=\mathbf{l}$ *has only finitely many solutions in the set* $\{\mathbf{k} : \mathbf{k} \geq \mathbf{m}\}$.

   (2) Show that for any $\mathbf{l} \in \mathbb{R}^n$ *there can only be finitely many* $p \in \mathbb{N}$ *and* $\mathbf{k}_j \in \mathbb{R}^n, j = 1, ..., p$ *such that* $\mathbf{k}_1 + \mathbf{k}_2 + \cdots \mathbf{k}_p = \mathbf{l}$.

Corollary 8.2 *For any set* $B \subset A = \{\mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{m}\}$ *there is a set* $B_1 = \operatorname{mag}(B)$ *with* **finitely many elements**, *such that* $\forall b \in B \setminus B_1$ *there exists* $b_1 \in B_1$ *such that* $b_1 < b$.

Consider the set of all elements which not greater than other elements of B, $B_1 = \{b_1 \in B | b \neq b_1 \Rightarrow b \not> b_1\}$. In particular, no two elements of B_1 can be compared with each-other. But then, by Lemma 8.1 this set cannot be infinite since it would contain an infinite non-ascending chain.

Now, if $b \in B \setminus B_1$, then by definition there is a $b' > b$ in B. If $b' \in B_1$ there is nothing to prove. Otherwise there is a $b'' > b'$ in B. Eventually some $b^{(k)}$ must belong to B_1, finishing the proof, otherwise $b < b' < ...$ would form an infinite nonascending chain.

Corollary 8.3 *For any set* $B \subset A = \{\mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{m}\}$ *there is a set* $\operatorname{Mag}(B)$ *with* finitely many elements*, such that* $\forall b \in B \setminus B_1$ *there exists* $b_1 \in B_1$ *such that* $b_1 < b$.

4. For any $\mathbf{m} \in \mathbb{Z}^n$ and any set $B \subset \{\mathbf{k} | \mathbf{k} \geq \mathbf{m}\}$, the set $A = \{\mu_{\mathbf{k}} | \mathbf{k} \in B\}$ has a largest element with respect to $>$. Indeed, if such was not the case, then we would be able to construct an infinitely ascending sequence.

Lemma 8.4 *No set of elements of* $\mu_{\mathbf{k}} \in \mathcal{G}$ *such that* $\mathbf{k} \geq \mathbf{m}$ *can contain an infinitely* ascending *chain, that is a sequence of the form*

$$g_1 \ll g_2 \ll \cdots$$

Proof. For such a sequence, the corresponding $\mathbf{k}$ would be strictly nonascending, in contradiction with Lemma 8.1.

5. It follows that for any $\mathbf{m}$ every $B \subset A_{\mathbf{m}} = \{g \in \mathcal{G} | g = \mu_{\mathbf{k}}; \mathbf{k} \geq \mathbf{m}\}$ is well ordered (every subset has a largest element) and thus B can be indexed by ordinals. By this we mean that there exists a set of ordinals Ω (or, which is the same, an ordinal) which is in one-to-one correspondence with B and $g_\beta \ll g_{\beta'}$ if $\beta > \beta'$.

6. If A is as in 4, and *if* $g \in \mathcal{G}$ has a *successor* in A, that is, there is a $\tilde{g} \in A$, $g \gg \tilde{g}$ then it has an *immediate successor*, the largest element in the subset of A consisting of all elements less than g. There may not be an immediate *predecessor* though, as is the case of e^{-x} in $A_1 = \{x^{-n}, n \in \mathbb{N}\} \cup \{e^{-x}\}$. Note also that, although e^{-x} has infinitely many predecessors, there is no infinite ascending chain in A_1.

Lemma 8.5 *For any $g \in \mathcal{G}$, and $\mathbf{m} \in \mathbb{Z}^n$, there exist finitely many (distinct) $\mathbf{k} \geq \mathbf{m}$ such that $\mu_{\mathbf{k}} = g$.*

Proof. Assume the contrary. Then for at least one i, say $i = 1$ there are infinitely many k_i in the set of $(\mathbf{k})_i$ such that $\mu_{\mathbf{k}} = g$. As in Lemma 8.12, we can extract a strictly increasing subsequence. But then, along it, $\mu_1^{k_2} \cdots \mu_n^{k_n}$ would form an infinite strictly ascending sequence, a contradiction.

Proof: Exercise.

7. For any coefficients $c_{\mathbf{k}} \in \mathbb{R}$, consider the formal multiseries, which we shall call *transseries* over $\mathcal{G}$,

$$T = \sum_{\mathbf{k} \in \mathbb{Z}^n ; \mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \mu_{\mathbf{k}} \tag{8.6}$$

Transseries actually needed in analysis are constructed in the sequel, with a particular inductive definition of generators μ_k.

8. *More generally a* **transseries over** G *is a sum which can be written in the form (8.6) for some (fixed) $n \in \mathbb{N}$ and for some* **some choice of generators** $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^n$.

9. The fact that a transseries s is small does *not* mean that the corresponding $\mu_{\mathbf{k}}$ have positive $\mathbf{k}$; s could contain terms such as xe^{-x} of $x^{\sqrt{2}}x^{-2}$ etc.). But positiveness can be *arranged* by a suitable choice of generators as follows from the next result.

10. **Note** It is important that a transseries is defined over a set of the form $A_{\mathbf{m}}$. For instance, in the group $\mathcal{G}$ with two generators x^{-1} and $x^{-\sqrt{2}}$ an expression of the form

$$\sum_{\{(m,n) \in \mathbb{Z}^2 | m\sqrt{2}+n>0\}} x^{-m\sqrt{2}-n} \tag{8.7}$$

is not acceptable. The behavior of a function whose "asymptotic expansion" is given by (8.7) is not at all manifest.

Exercise 8.8 *Consider the numbers the form $m\sqrt{2}+n$, where $m,n\in\mathbb{Z}$. It can be shown, for instance using continued fractions, that one can choose a subsequence from this set such that $s_n\uparrow 1$. Show that $\sum_n x^{-s_n}$ is not a transseries over any group of monomials of order zero.*

Expressions similar to the one in the exercise do appear in some problems in discrete dynamics. The very fact that transseries are closed under many operations, including solutions of ODEs, shows that such functions are "highly transcendental".

11. Given $\mathbf{m}\in\mathbb{Z}^n$ and $g\in\mathcal{G}$, the set $S_g=\{\mathbf{k}|\mu_{\mathbf{k}}=g\}$ contains, by Lemma 8.5 finitely many elements (possibly none). Thus the constant $d(g)=\sum_{\mathbf{k}\in S_g}c_{\mathbf{k}}$ is well defined. By 4 there is a largest $g=g_1$ in the set $\{\mu_{\mathbf{k}}|d(g)\neq 0\}$, unless all coefficients are zero. We call this g_1 the magnitude of T, $g_1=\operatorname{mag}(T)$, and we write $\operatorname{dom}(T)=d(g_1)g_1=d_1g_1$.

12. By 5, the set $\{g=\mu_{\mathbf{k}}|\mathbf{k}\geq\mathbf{m}\}$ can be indexed by ordinals, and we write

$$T=\sum_{\beta\in\Omega}d_\beta g_\beta \tag{8.9}$$

where $g_\beta\ll g_{\beta'}$ if $\beta>\beta'$. By convention, the first element in (8.9), $d_1g_1\neq 0$.

Convention. To simplify the notation and terminology, we will say, with some abuse of language, that a group element g_β appearing in (8.9) belongs to T.

Whenever convenient, we can also select the elements of $d_\beta g_\beta$ in T with nonzero coefficients. As a subset of a well ordered set, it is well ordered too, by a set of ordinals $\tilde{\Omega}\subset\Omega$ and write

$$T=\sum_{\beta\in\tilde{\Omega}}d_\beta g_\beta \tag{8.10}$$

where all d_β are nonzero.

13. **Notation** To simplify the exposition we will denote by $A_{\mathbf{m}}$ the set $\{\mu_{\mathbf{k}}|\mathbf{k}\geq\mathbf{m}\}$, $\mathbf{K}_{\mathbf{m}}=\{\mathbf{k}|\mathbf{k}\geq\mathbf{m}\}$ and $\mathcal{T}_{A_{\mathbf{m}}}$ the set of transseries over $A_{\mathbf{m}}$.

14. Any transseries can be written in the form

$$T=L+c+s=\sum_{\beta\in\Omega;g_\beta\gg 1}d_\beta g_\beta+c+\sum_{\beta\in\Omega;g_\beta\ll 1}d_\beta g_\beta \tag{8.11}$$

where L is called a purely large transseries, c is a constant and s is called a small transseries.

Note that L,c and s are transseries since, for instance, the set $\{\beta\in\Omega;g_\beta\ll 1\}$ is a subset of ordinals, thus an ordinal itself.

Lemma 8.12 *If $\mathcal{G}$ is finitely generated, if $A_{\mathbf{m}} \subset \mathcal{G}$ and s is a small transseries over $A_{\mathbf{m}}$ we can always assume, for an $n \ge n'$ that the generators $\nu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^{n'}$ are such that for all $\nu_{\mathbf{k}'} \in s$ we have $\mathbf{k}' > 0$.*

$$s = \sum_{\mathbf{k} \ge \mathbf{m}} \mu_{\mathbf{k}} c_{\mathbf{k}} = \sum_{\beta \in \tilde{\Omega}} d_\beta g_\beta = \sum_{\mathbf{k}' > 0} \nu_{\mathbf{k}'} c'_{\mathbf{k}'} \tag{8.13}$$

Proof. In the first sum on the left side we can retain only the set of indices I such that $\mathbf{k} \in I \Rightarrow \mu_{\mathbf{k}} = g_\beta$ has nonzero coefficient d_β. In particular, since all $g_\beta \ll 1$, we have $\mu_{\mathbf{k}} \ll 1$ $\forall \mathbf{k} \in I$. Let $I_1 = \mathrm{Mag}(I)$. We adjoin to the generators of $\mathcal{G}$ all the $\nu_{\mathbf{k}'} = \mu_{\mathbf{k}}$ with $\mathbf{k}' \in I_1$. The new set of generators is still finite and for all $\mathbf{k} \in I$ there is a $\mathbf{k}' \in \mathrm{Mag}(I)$ such that $\mathbf{k} \ge \mathbf{k}'$ and $\mu_{\mathbf{k}}$ can be written in the form $\nu^1_{\mathbf{k}'} \mu_{\mathbf{l}}$ where all $\mathbf{l} \ge 0$.

Remark. After the construction, generally, there will be nontrivial relations between the generators. But nowhere do we assume that generators are relation-free, so this creates no difficulty.

15. An algebra over $\mathcal{G}$ can be defined as follows. Let A and $\tilde{A}$ be well ordered sets in Ω. The set of pairs $(\beta, \tilde{\beta}) \in A \times \tilde{A}$ is well ordered (check!). For every g, the equation $g_\beta \cdot g_{\tilde{\beta}} = g$ has finitely many solutions. Indeed, otherwise there would be an infinite sequence of g_β which cannot be ascending, thus there is a subsequence of them which is strictly descending. But then, along that sequence, $g_{\tilde{\beta}}$ would be strictly ascending; then the set of corresponding ordinals $\tilde{\beta}$ would form an infinite strictly descending chain, which is impossible. Thus, in

$$T \cdot \tilde{T} := \sum_{\gamma \in A \times \tilde{A}} g_\gamma \sum_{g_\beta \cdot g_{\tilde{\beta}} = g_\gamma} d_\beta d_{\tilde{\beta}} \tag{8.14}$$

the inner sum contains finitely many terms.

16. We denote be $\mathcal{T}_{\mathcal{G}}$ the algebra of transseries over $\mathcal{G}$. $\mathcal{T}_{\mathcal{G}}$ is a commutative algebra with respect to $(+, \cdot)$. We will see in the sequel that $\mathcal{T}_{\mathcal{G}}$ is in fact a field. We make it an ordered algebra by writing

$$T_1 \ll T_2 \Leftrightarrow \mathrm{mag}(T_1) \ll \mathrm{mag}(T_2) \tag{8.15}$$

and writing

$$T > 0 \Leftrightarrow \mathrm{dom}(T) > 0 \tag{8.16}$$

17. **Product form**. With the convention $\mathrm{dom}(0) = 0$, any transseries can be written in the form

$$T = \mathrm{dom}(T)(1 + s) \tag{8.17}$$

where s is small (check).

18. Embeddings. If $\mathcal{G}_1 \subset \mathcal{G}$, we write that $\mathcal{T}_{\mathcal{G}_1} \subset \mathcal{T}_{\mathcal{G}}$ in the natural way.

19. **Topology** on $\mathcal{T}_{\mathcal{G}}$. We consider a sequence of transseries over a *common* set $A_{\mathbf{m}}$ of elements of $\mathcal{G}$, indexed by the ordinal Ω.

$$\{T^{[j]}\}_{j\in\mathbb{N}};\ T^{[j]} = \sum_{\beta\in\beta} d_\beta^{[j]} g_\beta^{[j]}$$

Definition. We say that $T^{[j]} \to 0$ as $j \to \infty$ if for any $\beta \in \Omega$ there is a $j(\beta)$ such that the coefficient $d_\beta^{[j]} = 0$ for all if $j > j(\beta)$.

Thus the transseries $T^{[j]}$ must be *eventually depleted of all coefficients*. This aspect is very important. The mere fact that $\mathrm{dom}(S) \to 0$ does not suffice. Indeed the sequence $\sum_{k>j} x^{-k} + je^{-x}$, though "rapidly decreasing" is not convergent according to the definition, and probably should not be considered convergent in any reasonable topology.

20. Equivalently, the sequence $T^{[j]} \to 0$ is convergent if there is a representation such that

$$T^{[j]} = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}}^{[j]} \mu_{\mathbf{k}} \tag{8.18}$$

and in the sum $\mu_{\mathbf{k}} = g$ has only one solution (we know that such a choice is possible), and $\min\{|k_1| + \cdots + |k_n| : c_{\mathbf{k}}^{[j]} \neq 0\} \to 0$ as $j \to \infty$.

21. Let $\mu_1, ..., \mu_n$ be any generators for $\mathcal{G}$, $\mathbf{m} \in \mathbb{Z}^d$, as in 5 and $T_j \in \mathcal{T}_{A_{\mathbf{m}}}$ a sequence of transseries. Let $N_j := \min\{k_1 + ... + k_n | \mu_1^{p_1} \cdots \mu_n^{p_n} \in T_j\}$. Note that we can write min since, by Lemma 8.1, the minimum value is attained (check this!). If $N_j \to \infty$ then $T_j \to 0$. Indeed, if this was not the case, then there would exist a g_β such that $g_\beta \in T_j$ with $d_\beta \neq 0$ for infinitely many j. Since $N_j \to \infty$ there is a sequence $\mu_{\mathbf{k}} \in A_{\mathbf{m}}$ such that $k_1 + ... + k_n \to \infty$ and $\mu_{\mathbf{k}} = g_\beta$. This would yield an infinite set of solutions of $\mu_{\mathbf{k}} = g_\beta$ in $A_{\mathbf{m}}$, which is not possible. The function $\max\{e^{-|k_1|+\cdots+|k_n|} : \sum_{\mu_k=g} c_{\mathbf{k}} \neq 0\}$ is a semimetric (it satisfies all properties of the metric except the triangle inequality) which induces the same topology.

More generally, transseries are a subset of functions f defined on $\mathcal{G}$ with real values and for which there exists a $\mathbf{k}_0(f) = \mathbf{k}_0$ such that $f(g_{\mathbf{k}}) = 0$ for all $\mathbf{k} < \mathbf{k}_0$. On these functions we can define a topology by writing $f^{[j]} \to 0$ if there exists $\mathbf{k}_0(f^{[j]})$ does not depend on j and for any g_β there is an N we have $f^{[n]}(g_\beta) = 0$ for all $n > N$ and such . The first restriction is imposed to disallow, say, the convergence of x^n to zero, which would not be compatible with a good structure of transseries.

22. This topology is metrizable. For example we can proceed as follows. Let $A_{\mathbf{m}}$ be the common set over which the transseries are defined. The elements of $\mathcal{G}$ are countable. We choose any counting on $A_{\mathbf{m}}$. We then identify transseries over $A_{\mathbf{m}}$ with the space $\mathcal{F}$ of real-valued functions defined on the natural numbers. We define $d(f,g) = 1/n$ where n is the least integer such that $f(n) \neq g(n)$ and $d(f,f) = 0$. The only property that needs to be checked is the triangle inequality. Let $h \in \mathcal{F}$. If $d(g,h) \geq 1/n$, then clearly $d(f,g) \leq d(f,h) + d(h,g)$. If $d(g,h) < 1/n$ then $d(f,h) = 1/n$ and the inequality holds too.

23. The topology cannot come from a norm, since in general $a_n\mu \not\to 0$ as $a_n \to 0$.

24. We also note that the topology is *not* compatible with the order relation. For example $s_n = x^{-n} + e^{-x} \to e^{-x}$ as $n \to \infty$, $s_n \gg e^{-\sqrt{x}}$ for all n while $e^{-x} \not\gg e^{-\sqrt{x}}$. The same argument shows that there is no distance compatible with the order relation.

25. In some sense, there is no "good" topology compatible with the order relation $\ll$. Indeed, if there was one, then the sequences $s_n = x^{-n}$ and $t_n = x^{-n} + e^{-x}$ which are interlaced in the order relation should have the same limit, but then addition would be discontinuous[1].

26. Giving up compatibility with asymptotic order allows us to ensure continuity of most operations of interest.

    Exercise. Show that a Cauchy sequence in $\mathcal{T}_{A_{\mathbf{m}}}$, is convergent, and $\mathcal{T}_{A_{\mathbf{m}}}$ is a topological algebra.

27. If $\mathcal{G}$ is finitely generated, then for any small transseries

$$s = \sum_{\beta\in\Omega: g_\beta \ll 1} d_\beta g_\beta \tag{8.19}$$

we have $s^j \to 0$ as $j \to \infty$.

Proof. Indeed, by Lemma 8.12 we may assume that the generators of $\mathcal{G}$, $\mu_1, ..., \mu_n$, are chosen such that all $\mathbf{k} > 0$ in s. Let $g \in \mathcal{G}$. The terms occurring in the formal sum of s^j are of the form $const.\mu_1^{l_1^1+...+l_1^j} \cdots \mu_n^{l_n^1+\cdots l_n^j}$ where $l_m^s \geq 0$ and at least one $l_j^s > 0$. Therefore $l_1^1 + ... + l_1^j \to \infty$ and $\sum_{l=1..M} s^l \to 0$ by 21 for any $j, M \to \infty$.

As a side remark, finite generation is not needed at this point. More generally, let $A \subset \mathcal{G}$ be well ordered. It follows from J. Kruskal's theorem that the set $\tilde{\mathcal{A}} \supset A$ of all products of elements of A is also well quasi-ordered.

[1]This example was pointed out by G. Edgar.

Note 8.20 The sum $\sum_{k=0}^{\infty} c_k s^k$ might belong to a space of transseries defined over a larger, but still finite, number of generators. For instance, if

$$\frac{1}{xe^x+1} = \frac{1}{xe^x(1+xe^{-x})} = \frac{e^{-x}}{x}\sum_{j=0}^{\infty}(-1)^j x^j e^{-jx} \tag{8.21}$$

then the generators of (8.21) can be taken to be x^{-1}, e^{-x}, xe^{-x} but certainly cannot stay e^{-x}, x^{-1} since then the power of x^{-1} would be unbounded below.

28. In particular if $f(\mu) := \sum_{k=0}^{\infty} c_k \mu^k$ is a formal series and s is a small transseries, then

$$f(s) := \sum_{k=0}^{\infty} c_k s^k \tag{8.22}$$

is well defined.

Exercise 8.23 *Show that f is continuous, in the sense that $s^{[n]} \to 0$ implies $f(s) \to c_0$.*

29. If $T_1 \gg T_2$, $T_3 \ll T_1$ and $T_4 \ll T_2$ then $T_1 + T_3 \gg T_2 + T_4$. Indeed, mag(T_1+T_3)=mag(T_1) and mag(T_2+T_4)=mag(T_2).

30. It is easily checked that $(1+s)\cdot 1/(1+s) = 1$, where

$$\frac{1}{1+s} := \sum_{j\ge 0}(-1)^j s^j \tag{8.24}$$

More generally we define

$$(1+s)^a = 1 + a\,s + \frac{a(a-1)}{2} + \cdots$$

31. Writing $S = \mathrm{dom}(S)(1+s)$ we define $S^{-1} = \mathrm{dom}(S)^{-1}(1+s)^{-1}$.

32. if μ^r is defined for a real r (this will be the case for the power-exponential transseries), then we then adjoin μ^r to $\mathcal{G}$ and define

$$T^r := d_1^r g_1^r (1+s)^r$$

33. If $\mu_j \mapsto \mu_j'$ is a "derivation" defined from the generators μ_j into $\mathcal{T}_{\mathcal{G}}$, where we assume that derivation is compatible with the relations between the generators, we can extend it by $(g_1g_2)' = g_1'g_2 + g_1g_2'$, $1' = 0$ to the whole of $\mathcal{G}$ and by linearity to $\mathcal{T}_{\mathcal{G}}$,

$$\left(\sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}\mu_{\mathbf{k}}\right)' = \sum_{j=1}^{n}\mu_j' \sum_{\mathbf{k}\in\mathbb{Z}^n} k_j c_{\mathbf{k}}\mu_1^{k_1}\cdots\mu_j^{k_j-1} \tag{8.25}$$

and the latter sum is a well defined finite sum of transseries.

Exercise. Show that with these operations, $\mathcal{T}_{\mathcal{G}}$ is a differential field.

34. If s is a small series, we define

$$e^s = \sum_{k\geq 0} \frac{s^k}{k!} \tag{8.26}$$

Exercise. Show that e^s has the usual properties with respect to multiplication and differentiation.

35. **Transseries are limits of finite sums**. We let $\mathbf{m} \in \mathbb{Z}^n$ and $\mathbf{M}_p = (p, p, ..., p) \in \mathbb{N}^n$. Note that

$$T_p := \sum_{g_\beta=\mu_{\mathbf{k}};\mathbf{m}\leq\mathbf{k}\leq\mathbf{M}_p;\beta\in\Omega} d_\beta g_\beta \underset{p\to\infty}{\to} \sum_{\beta\in\Omega} d_\beta g_\beta$$

Indeed, it can be checked that $d(T_p, T) \to 0$ as $p \to \infty$.

36. More generally, let $\mathcal{G}$ be finitely generated and $\mathbf{k}_0 \in \mathbb{Z}$. Assume $s_{\mathbf{k}} \to 0$ as $\mathbf{k} \to \infty$. Then, for any sequence of real numbers $c_{\mathbf{k}}$, the sequence

$$\sum_{\mathbf{k}_0\leq\mathbf{k}\leq\mathbf{M}_p} c_{\mathbf{k}} s_{\mathbf{k}} \tag{8.27}$$

where $\mathbf{M}_p = (p, ..., p), p \in \mathbb{N}$ is Cauchy and the limit

$$\lim_{p\to\infty} \sum_{\mathbf{k}_0\leq\mathbf{k}\leq\mathbf{M}_p} c_{\mathbf{k}} s_{\mathbf{k}} \tag{8.28}$$

is well defined. In particular, for a given transseries

$$T \triangleleft \mathbf{s} = \sum d_{\mathbf{k}} s_{\mathbf{k}} \tag{8.29}$$

we define the **transcomposition**

$$T \triangleleft \mathbf{s} = \sum_{\mathbf{k}\geq\mathbf{k}_0} d_{\mathbf{k}} s_{\mathbf{k}} \tag{8.30}$$

37. As an example of transcomposition, we see that transseries are closed under right pseudo-composition with *large* (not necessarily purely large) transseries $\mathbf{T} = T_i; i = 1, 2, ..., n$ by

$$T_1(1/\mathbf{T}) = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}} \mathbf{T}^{-\mathbf{k}} \tag{8.31}$$

if

$$T_1 = \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} \mu^{\mathbf{k}}$$

(cf. 27) We should mention that at this level of abstractness pseudo-composition may not behave as a composition, for instance it may not be compatible with chain rule in differentiation.

38. **Contractive operators** Contractivity is usually defined in relation to a metric, but given a topology, contractivity depends on the metric while convergence does not. There is apparently no natural metric on transseries.

Definition 8.32 *Let first J be a linear operator from $\mathcal{T}_{A_{\mathbf{m}}}$ or from one of its subspaces, to $A_{\mathbf{k}}$,*

$$JT = J \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} \mu_{\mathbf{k}} = \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} J \mu_{\mathbf{k}} \tag{8.33}$$

Then J is called asymptotically contractive on $\tilde{\mathcal{A}}_{\mathbf{m}}$ if

$$J\mu_{\mathbf{j}} = \sum_{\mathbf{p} > 0} c_{\mathbf{p}} \mu_{\mathbf{j}+\mathbf{p}} \tag{8.34}$$

Remark 8.35 *Contractivity depends on the set of generators.*

Remark 8.36 *It can be checked that contractivity holds if*

$$J\mu_{\mathbf{j}} = \sum_{\mathbf{p} > 0} c_{\mathbf{p}} \mu_{\mathbf{j}+\mathbf{p}} (1 + s_{\mathbf{j}}) \tag{8.37}$$

where $s_{\mathbf{j}}$ are small transseries.

Exercise 8.38 *Check that for any $\mu_{\mathbf{j}}$ we have*

$$\sup_{p>0} \sum_{k=n}^{n+p} J^k \mu_{\mathbf{j}} \to 0$$

as $n \to \infty$.

We then have

$$JT = \sum_{\mathbf{k} \ge \mathbf{m}} J \mu_{\mathbf{k}} \tag{8.39}$$

Definition 8.40 *The linear or nonlinear operator J is (asymptotically) contractive in the set $A \subset A_{\mathbf{m}}$ if $J : A \mapsto A$ and the following condition holds. Let T_1 and T_2 in A be arbitrary and let*

$$T_1 - T_2 = \sum_{\mathbf{k} \geq \mathbf{m}} c_{\mathbf{k}} \mu_{\mathbf{k}} \tag{8.41}$$

Then

$$J(T_1) - J(T_2) = \sum_{\mathbf{k} \geq \mathbf{m}} c'_{\mathbf{k}} \mu_{\mathbf{k}+\mathbf{p}_{\mathbf{k}}} (1 + s_{\mathbf{k}}) \tag{8.42}$$

where $\mathbf{p}_{\mathbf{k}} > 0$ and $s_{\mathbf{k}}$ are small.

Remark 8.43 *The sum of asymptotically contractive operators is contractive; the composition of contractive operators, whenever defined, is contractive.*

Theorem 8.44 *(i) If J is linear and contractive on $\mathcal{T}_{A_{\mathbf{m}}}$ then for any $T_0 \in \mathcal{T}_{A_{\mathbf{m}}}$ the fixed point equation $T = JT + T_0$ has a unique solution $T \in \mathcal{T}_{A_{\mathbf{m}}}$.*

(ii) In general, if $A \subset A_{\mathbf{m}}$ is closed and $J : A \mapsto A$ is a (linear or nonlinear) contractive operator on A, then $T = J(T)$ has a unique solution is A.

PROOF For (ii) we define the sequence $T_{n+1} = J(T_n)$ is convergent since for some coefficients $c_{j,\mathbf{k}}$ we have

$$J^q(T) - J(T) = \sum_{\mathbf{k} \geq m} c_{j,\mathbf{k}} \mu_{\mathbf{k}+q\mathbf{p}_{\mathbf{k}}} \to 0$$

as $q \to \infty$. Uniqueness is immediate. ☐

39. When working with transseries we often encounter this fixed point problem in the form $X = Y + \mathcal{N}(X)$, where Y is given, X is the unknown Y is given, and $\mathcal{N}$ is "small".

    Exercise. Show the existence of a unique inverse of $(1 + s)$ where s is a small transseries, by showing that the equation $T = 1 - sT$ is contractive.

40. For example ∂ is contractive on transseries of level zero. This is clear since in every monomial the power of x decreases by one. But note that ∂ is not contractive anymore if we add "terms beyond all orders", e.g., $(e^{-x^2})' = -2xe^{-x^2} \gg e^{-x^2}$.

We cannot expect any contractivity of ∂ in general, since if y_1 is the level zero solution of $T = 1/x - T'$ then $T + Ce^{-x}$ is a solution for any C so uniqueness fails.

This is one reason the WKB method works near irregular singularities, where exponential behavior is likely, and naive approximations don't.

41. We take the union

$$\mathcal{T} = \bigcup_{\mathcal{G}} \mathcal{T}_{\mathcal{G}}$$

with the natural embeddings. It can be easily checked that $\mathcal{T}$ is a differential field too. The topology is that of inductive limit, namely a sequence of transseries converges if they all belong to some $\mathcal{T}_{\mathcal{G}}$ and they converge there.

42. One can check that algebraic operations, exponentiation, composition with functions for which composition is defined, are continuous wherever the functions are "C^∞".

Exercise 8.45 *Let $T \in A_{\mathbf{m}}$. Show that the set $\{T_1 \in A_{\mathbf{m}} | T_1 \ll T\}$ is closed.*

Ab General logarithmic-free transseries

Ac Assumption on the inductive step

1. We have already constructed transseries of level zero. Transseries of any level are constructed inductively, level by level.

   Since we have already studied the properties of abstract multiseries, the construction is relatively simple, all we have to do is essentially watch for consistency of the definitions at each level.

2. Assume finitely generated transseries of level at most n have already been constructed. We assume a number of properties, and then build level $n+1$ transseries and show that these properties are conserved.

   (a) Transmonomials μ_j of order at most N are totally ordered, with respect to two order relations, $\ll$ and $<$. Multiplication is defined on the transmonomials, it is commutative and compatible with the order relations.

   (b) For a set of n small transmonomials, a transseries of level at most N is defined as expression of the form (8.6).

   It follows that the set $\{g = \mu_{\mathbf{k}} | \mathbf{k} \geq \mathbf{m}\}$ can be indexed by ordinals, and we can write the transseries in the form (8.9). The decomposition 8.11 then applies.

It also follows that two transseries are equal iff their corresponding d_β coincide.

The ordering relation on transseries of level N is defined as before, $T \gg 1$ if, by definition $g_1 \gg 1$ and $T > 0$ iff $d_1 > 0$.

Transseries of level at most N are defined as the union of all $\mathcal{T}_{\mathcal{A}_{\mathbf{m}}}$ where $A_{\mathbf{m}}$ is as before.

(c) A transmonomial or order at most N is of the form $x^a e^L$ where L is a purely large or null transseries of level $N-1$, and e^L is defined recursively. There are no transseries of level -1, so for $N = 1$ we take $L = 0$.

Exercise. Show that any transmonomial is of the form $x^a e^{L_1} e^{L_2} \cdots e^{L_j}$ *where* L_j *are of order exactly* j *meaning that they are of order* j *but not of lower order.*

(d) For any transmonomial, $(x^a e^L)^r$ is defined as $x^{ar} e^{rL}$ where the ingredients have already been defined. It may be a adjoined to the generators of $\mathcal{G}$ and then, as in the previous section, T^r is well defined.

(e) By definition, $x^a e^L = e^L x^a$ and $x^{a_1} e^{L_1} x^{a_2} e^{L_2} = x^{a_1+a_2} e^{L_1+L_2}$. Furthermore $e^{L_1} \gg x^a e^{L_2}$ for any a if $L_1 > 0$ is a purely large transseries of level strictly higher than the level of L_2.

(f) There is a differentiation with the usual properties on the generators, compatible with the group structure and equivalences. We have $(x^a e^L)' = a x^{a-1} x^L + x^a L' e^L$ where L' is a (finitely generated) transseries of level at most $N - 1$.

We define

$$T' = \sum_{\mathbf{k} \in \mathbb{Z}^n; \mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \left[(x^{-\mathbf{k}\cdot\boldsymbol{\alpha}})' e^{-\mathbf{L}\cdot\boldsymbol{\beta}} + x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} (e^{-\mathbf{L}\cdot\boldsymbol{\beta}})' \right] \tag{8.46}$$

where, according to the definition of differentiation, (8.46) is a finite sum of products of transseries of level at most N.

We have $T' = 0$ iff $T = \mathit{const}$. If $\mathrm{dom}(T_{1,2}) \neq \mathit{const.}, \mathit{then}\ T_1 \ll T_2$ implies $T_1' \ll T_2'$.

3. It can be checked by induction that $T > 0, T \gg 1$ implies $T' > 0$. In this sense, differentiation is compatible with the order relations.

4. It can then be checked that differentiation has the usual properties.

5. if c is a constant, then e^c is a constant, the usual exponential of c, and if $L + c + s$ is the decomposition of a transseries of level $N - 1$ we write $e^{L+c+s} = e^L e^c e^s$ where e^s is *reexpanded* according to formula (8.26) and the result is a *transseries* of level N.

We convene to write e^T, for any T transseries of level at most N *only* in this reexpanded form.

Then it is always the case that $e^T = T_1 e^{L_2}$ where T_1 and L_2 are transseries of level $N-1$ and L_2 is purely large or zero. The transseries e^T is finitely generated, with generators e^{-L_1} , if $L_1 > 0$ or e^{L_1} otherwise, together with all the generators of L_1.

Sometimes it is convenient to adjoin to the generators of T all the generators in the exponents of the transmonomials in T, and then the generators in exponents in the exponents of the transmonomials in T etc. Of course, this process is finite, and we end up with a finite number of generators, which we will call *the complete set of generators* of T.

6. This **defines** the exponential of any transseries of level at most $N-1$ if $L \neq 0$ and the exponential of any transseries of level at most N if $L = 0$. We can check that $e^{T_1} = e^{T_2}$ iff $T_1 = T_2$.

7. If all transseries of level N are written in the canonical form (8.9) then $T_1 = T_2$ iff all g_β at all levels have exactly the same coefficients. Transseries, in this way, have a unique representation in a strong sense.

8. The space of transseries of level N, $\mathcal{T}^{[N]}$, is defined as the union of all spaces of transseries over finitely generated groups of transmonomials of level N.
$$\mathcal{T}^{[N]} = \bigcup_{\mathcal{G}_N} \mathcal{T}_{\mathcal{G}_N}$$
with the inductive limit topology.

9. The abstract theory of transseries we have developed in the previous section applies. In particular the definition $1/(1-s) = \sum_j s^j$ $1/T = 1/\operatorname{dom}(T)(1+s)^{-1}$ and transseries of level N form a differential field closed under the contractive mappings.

10. Note that transseries of order N are closed under the contractive mapping principle.

Ad Passing from step N to step $N+1$

1. We now proceed in defining transseries of level at most $N+1$. We have to check that the construction preserves the properties in §Ac .

2. For any purely large transseries of level N we define $x^a e^L$ to equal the already defined transmonomial of order N. If L is a (finitely generated) purely large transseries of level exactly N we define a new primitive object, $x^a e^L$, a transmonomial of order $N+1$, with the properties

   (a) $e^0 = 1$.

(b) $x^a e^L = e^L x^a$.

(c) $x^{a_1} e^{L_1} x^{a_2} e^{L_2} = x^{a_1+a_2} e^{L_1+L_2}$.

(d) If $L > 0$ is a purely large transseries of level exactly N then we have $e^L \gg x^a$ for any a.

Exercise. Show that if L_1 and L_1 are purely large transseries and the level of L_1 strictly exceeds the level of L_2, then $e^{L_1} \gg x^a e^{L_2}$ for any a.

Note that $L_1 \pm L_2$ may be of lower level but it is either purely large or else zero; $L_1 L_2$ is purely large.

Note 8.47 At this stage, no meaning is given to e^L, or even to e^x; they are treated as primitives. There are possibly many models of this construction. We will interpret many of them later by finding an extended isomorphism between a family of transseries and a set of functions. Then e^x would correspond to the usual exponential, convergent multiseries will correspond to their sums etc. Finite generation would play a role throughout that process, and "good" transseries come as solutions of well defined classes of problems, with "coefficients" which are themselves "good" transseries. We will have $(1-1/x)^{-1} = \sum_j x^{-j}$ but also selected divergent series will have a meaning, e.g. $e^x \sum_{k=0}^{\infty} n!/x^{n+1} = PV \int_{-\infty}^{x} t^{-1} e^t dt$ The latter transseries, and its associated sum solve $f' + f = 1/x$. But it is not to be expected to have a summation process that applies to all series.

3. If $\alpha > 0$ and L is a *positive* transseries of level N we define a generator of order N to be $\mu = x^{-\alpha} e^{-L}$. We choose a number of generators $\mu_1, ..., \mu_n$, and define the abelian multiplicative group generated by them, with the multiplication rule just defined. We can check that $\mathcal{G}$ is a totally ordered, of course finitely generated, abelian group, and that the order relation is compatible with the group structure.

4. We can now define transseries over $\mathcal{G} = \mathcal{G}^{[N+1]}$ as in §A.

5. We define transseries of order $N+1$ to be the union over all $\mathcal{T}_{\mathcal{G}^{[N+1]}}$, with the natural embeddings. We denote these transseries by $\mathcal{T}^{[N+1]}$.

6. *Compatibility of differentiation with the order relation.* We have already assumed that this is the case for transseries of level at most N. (i) We first show that it holds for transmonomials of level $N+1$. If $L_1 - L_2$ is a positive transseries, then $(x^a e^{L_1})' \gg (x^b e^{L_2})'$ follows directly from the formula of differentiation, the fact that $e^{L_1 - L_2}$ is large and the induction hypothesis. If $L_1 = L_2$ then $a > b$ and the property follows from the fact that L_1 is either zero, or else $L \gg x^\beta$ for some $\beta > 0$ for some positive β (check!).

(ii) For the general case we note that

$$\left(\sum_{\beta} d_\beta \mu_\beta\right)' = \sum_{\beta} d_\beta \mu'_\beta$$

and $\mu'_{\beta_1} \ll \mu'_{\beta_2}$ if $\beta_1 > \beta_2$. Then $\operatorname{dom}(T)' = (\operatorname{dom}(T))'$ and the property follows.

7. Differentiation is continuous. Indeed, if $T^{[m]} \to 0$,

$$T^{[m]} = \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}}^{[m]} x^{\mathbf{k}\cdot\mathbf{a}} e^{-\mathbf{k}\cdot\mathbf{L}} \to 0 \quad \text{as} \quad m \to \infty$$

where the transseries $L_1, ..., L_n$ are purely large, then

$$(T^{[m]})' = \frac{1}{x} \sum_{\mathbf{k} \ge \mathbf{m}} (\mathbf{k} \cdot \mathbf{a}\, c_{\mathbf{k}}^{[m]}) x^{\mathbf{k}\cdot\mathbf{a}} e^{-\mathbf{k}\cdot\mathbf{L}} - \mathbf{L}' \cdot \sum_{\mathbf{k} \ge \mathbf{m}} (\mathbf{k} c_{\mathbf{k}}^{[m]}) x^{\mathbf{k}\cdot\mathbf{a}} e^{-\mathbf{k}\cdot\mathbf{L}}$$

and the rest follows from continuity of multiplication and the definition of convergence.

8. Therefore, if a property of differentiation holds for finite sums of transmonomials, then it holds for transseries.

9. By direct calculation, if μ_1, μ_2 are transmonomials of order $N+1$ then $(\mu_1\mu_2)' = \mu_1'\mu_2 + \mu_1\mu_2'$. Then, one can check by usual induction, the product rule holds for finite sums of transmonomials. Using 8 the product rule follows for general transseries.

Ad .1 Composition

10. Composition *to the right* with a *large* (not necessarily purely large) transseries T of level m is defined as follows.

    The power of a transseries $T = x^a e^L(1+s)$ is defined by $T^p = x^{ap} e^{pL}(1+s)^p$, where the last expression is well defined and $(T^p)' = pT'T^{p-1}$ (check).

    The exponential of a transseries is defined, inductively, in the following way.

$$T = L + c + s \Rightarrow e^T = e^L e^c e^s = S e^L e^c \tag{8.48}$$

where S is given in (8.26).

A general exponential-free transseries of level zero has the form

$$T_0 = \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} \tag{8.49}$$

where $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^{+n}$ for some n.

Then we take $\mathbf{T} = (T^{\alpha_1}, ..., T^{\alpha_n})$ and define $T_0(1/T)$ by (8.31); $T_0(1/T)$ has level m. If the sum (8.49) contains finitely many terms, it is clear that $[T_0(1/T)]' = T_0'(1/T)T'$. By continuity, this is true for a general T_0 of level zero.

11. Assume that composition with T has been defined for all transseries of level N. It is assumed that this composition is a transseries of level $N+m$. Then $L(T) = L_1 + c_1 + s_1$ (it is easily seen that $L(T)$ is not necessarily purely large). Then

$$(x^a e^L) \circ (T) := T^a e^{L(T)} = x^b(1 + s_1(T))e^{L_1(T)} \tag{8.50}$$

where $L_1(T)$ is purely large. Since L_1 has level $N+m$, then $(x^a e^L) \circ (T)$ has level $N+m+1$. We have $(e^{L_1})' = L_1' e^{L_1}$ and the chain rule follows by induction and from the sum and product rules.

Exercise 8.51 *If $T^{[n]}$ is a sequence of transseries, then $e^{T^{[n]}}$ is a not necessarily a valid sequence of transseries. But if it is, then there is an L_0 such that $L^{[n]} = L_0$ for all large n. If $e^{T^{[n]}}$ is a sequence of transseries and $T^{[n]} \to 0$, then $e^{T^{[n]}} \to 1$.*

12. The exponential is continuous. This follows from the Exercise 8.51 and Exercise 8.23.

13. Take now a general transseries of level $N+1$ and write $T = x^a e^L (1+s)$

$$t = \sum_{\mathbf{k} \geq \mathbf{m}} x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} e^{-\mathbf{k}\cdot\mathbf{l}} \tag{8.52}$$

Then $t(T)$ is well defined as the limit of the following finite sum with generators $x^{-|a\alpha_j|}, x^{-\alpha_j} e^{-l_j(T)}, e^{-l_j(T)};\ , j = 1, ..., n$:

$$t(T) = \sum_{\mathbf{M}_p \geq \mathbf{k} \geq \mathbf{m}} x^{-a(\mathbf{k}\cdot\boldsymbol{\alpha})} e^{-\mathbf{k}\cdot\mathbf{l_1}(T)} (1 + \mathbf{s}(T)) \tag{8.53}$$

14. The chain rule holds by continuity.

15. The general theory we developed in §A applies and guarantees that the properties listed in §Ac hold (check!).

Ad .2 Small transseries as infinitesimals; expansions beyond all orders

16. Let T be a transseries of level N over $\mathcal{G}$ and dx a small transseries with dominance e^{-L} where L is a positive large transseries of level $N+p$,

$p > 0$. Then $(T(x + dx) - T(x))/dx = T'(x) + s(T)$ where $s(T)$ is a small transseries of level $N + p$.

The proof is by induction on the level. By linearity and continuity it is enough to prove the statement for transmonomials. We have

$$(x + dx)^a e^{-L_1(x+dx)} = x^a(1 + dx/x)^a e^{L_1(x)+L_1'(x)dx+s(L)}$$

where $L_1' dx$ is a small transseries (since $L_1 e^{-L}$ is small) and $s(L_1)$ is of level $N + p$. The claim follows after reexpansion of the two terms in the product. Note that dx must be far less than all terms in T; $dx \ll 1$ is not enough.

Exercise 8.54 *Show that, under the same assumptions that*

$$T(x + dx) = \sum_{j=0}^{\infty} T^{(n)}(x) \frac{dx^n}{n!} \tag{8.55}$$

In this sense, transseries behave like analytic functions.

Ad .3 An inequality helpful in WKB analysis.

Proposition 8.56 *If $L \gg 1$ then $L'' \ll (L')^2$ (or, which is the same, $L' \ll L^2$).*

PROOF If $L = x^a e^{L_1}$ where $L_1 \neq 0$ then L_1 is purely large, then the dominance of L' is of the form $x^b e^{L_1}$, whereas the dominance of L is of the form $x^a e^{2L_1}$ and the property is obvious. If $L_1 = 0$ the property is obvious as well. ∎

In WKB analysis this result is mostly used in the form (8.58 below.

Exercise 8.57 *Show that if $T \gg 1$, T positive or negative, we have*

$$\mathrm{dom}[(e^T)^{(n)}] = \mathrm{dom}[(T')^n e^T] \tag{8.58}$$

Ae General logarithmic-free transseries

These are simply defined as

$$\mathcal{T}_e = \bigcup_{N \in \mathbb{N}} \mathcal{T}^{[N]} \tag{8.59}$$

with the natural embeddings.

The general theory we developed in §A applies to $\mathcal{T}_e$ as well. Since any transseries belongs to some level, any finite number of them share some level. There are no operations defined which involve infinitely many levels, because they would involve infinitely many generators. Then, the properties listed in §Ac hold in $\mathcal{T}_e$ (check!).

Af Écalle's notation

- $\sqcup$ —small transmonomial.
- $\sqcap$ —large transmonomial.
- $\square$ —any transmonomial, large or small.
- $\sqcup\!\sqcup$ —small transseries.
- $\sqcap\!\sqcap$ —large transseries.
- $\square\!\square$ —any transseries, small or large.

Af .1 Further properties of transseries

Definition. The level $l(T)$ of T is n if $T \in \mathcal{T}^{[n]}$ and $T \notin \mathcal{T}^{[n-1]}$.

Af .2 Further properties of differentiation

We denote $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}x}$

Corollary 8.60 *We have* $\mathcal{D}T = 0 \iff T = \mathrm{Const}$.

PROOF We have to show that if $T = L + s \neq 0$ then $T' \neq 0$. If $L \neq 0$ then for some $\beta > 0$ we have $L + s \gg x^{\beta} + s$ and then $L' + s' \gg x^{\beta-1} \neq 0$. If instead $L = 0$ then $(1/T) = L_1 + s_1 + c$ and we see that $(L_1 + s_1)' = 0$ which, by the above, implies $L_1 = 0$ which gives $1/s = s_1$, a contradiction. ∎

Proposition 8.61 *Assume $T = L$ or $T = s$. Then:*
(i) If $l(\mathrm{mag}(T)) \geq 1$ *then* $l(\mathrm{mag}(T^{-1}T')) < l(\mathrm{mag}(T))$.
(ii) $\mathrm{dom}(T') = \mathrm{dom}(T)'(1 + s)$.

PROOF Straightforward induction. ∎

Af .3 Transseries with complex coefficients

Complex transseries $\mathcal{T}_{\mathbb{C}}$ are constructed in a similar way as real transseries, replacing everywhere $L_1 > L_2$ by $\Re L_1 > \Re L_2$. Thus there is only one order relation in $\mathcal{T}_{\mathbb{C}}$, $\gg$. Difficulties arise when exponentiating transseries whose dominant term is imaginary. Operations with complex transseries are then limited. We will only use complex transseries in contexts that will prevent these difficulties.

Af .4 Differential systems in $\mathcal{T}_e$

The theory of differential equations in $\mathcal{T}_e$ is similar in many ways to the corresponding theory for functions.

Example. The general solution of the differential equation

$$f' + f = 1/x \tag{8.62}$$

in $\mathcal{T}_e$ (for $x \to +\infty$) is $T(x;C) = \sum_{k=0}^{\infty} k!x^{-k} + Ce^{-x} = T(x;0) + Ce^{-x}$.

The particular solution $T(x;0)$ is the unique solution of the equation $f = 1/x - \mathcal{D}f$ which is manifestly contractive in the space of level zero transseries.

Indeed, the fact that $T(x;C)$ is a solution follows immediately from the definition of the operations in $\mathcal{T}_e$ and the fact that e^{-x} is a solution of the homogeneous equation.

To show uniqueness, assume T_1 satisfies (8.62). Then $T_2 = T_1 - T(x;0)$ is a solution of $\mathcal{D}T + T = 0$. Then $T_2 = e^x T$ satisfies $\mathcal{D}T_2 = 0$ i.e., $T_2 =$ Const.

Ag The space $\mathcal{T}$ of general transseries

We define

$$\log_n(x) = \underbrace{\log\log ... \log(x)}_{n\ times} \tag{8.63}$$

$$\exp_n(x) = \underbrace{\exp\exp ... \exp(x)}_{n\ times} \tag{8.64}$$

$$\tag{8.65}$$

with the convention $\exp_0(x) = \log_0(x) = x$.

We write $\exp(\log x) = x$ and then any log-free transseries can be written as $T(x) = T \circ \exp_n(\log_n(x))$. This defines right composition with $\log_n$ in this trivial case, as $T_1 \circ \log_n(x)) = (T \circ \exp_n) \circ \log_n(x) := T(x)$.

More generally, we define $\mathcal{T}$, the space of general transseries , as a set of formal compositions

$$\mathcal{T} = \{T \circ \log_n : T \in \mathcal{T}_e\}$$

with the algebraic operations and inequalities (symbolized below by $\odot$) inherited from $\tilde{\tilde{\mathcal{T}}}$ by

$$(T_1 \circ \log_n) \odot (T_2 \circ \log_{n+k}) = [(T_1 \circ \exp_k) \odot T_2] \circ \log_{n+k} \tag{8.66}$$

and using (8.66), differentiation is defined by

$$\mathcal{D}(T \circ \log_n) = x^{-1} \left[(\prod_{k=1}^{n-1} \log_k)^{-1} \right] (\mathcal{D}T) \circ \log_n$$

Proposition 8.67 $\mathcal{T}$ *is an ordered differential field, closed under restricted composition.*

PROOF Exercise. ☐

The logarithm of a transseries. This is defined by first considering the case when $T \in \mathcal{T}_e$ and then taking right composition with iterated logs.

If $T = c\,\mathrm{mag}(T)(1+s) = cx^a e^L(1+s)$ then we define

$$\log(T) = \log(\,\mathrm{mag}(T)) + \log c + \log(1+s) = a\log x + L + \log c + \log(1+s) \quad (8.68)$$

where $\log c$ is the usual log, $\log(1+s)$ is defined by expansion which we know is well defined on small transseries.

1. If $L \gg 1$ is large, then $\log L \gg 1$ and if $s \ll 1$, then $\log s \gg 1$.

Ag .1 Restricted composition

Proposition 8.69 $\mathcal{T}$ *is closed under integration.*

PROOF The idea behind the construction of $\mathcal{D}^{-1}$ is the following: we first find an invertible operator J which is *to leading order* $\mathcal{D}^{-1}$; then the equation for the correction will be contractive. Let $T = \sum_{\mathbf{k}\ge\mathbf{k}_0} \mu^{\mathbf{k}} \circ \log_n$. To unify the treatment, it is convenient to use the identity

$$\int_x T(s)ds = \int_{\log_{n+2}(x)} \left(T \circ \exp_{n+2}\right)(t) \prod_{j\le n+1} \exp_j(t)\mathrm{d}t = \int_{\log_{n+2}(x)} T_1(t)\mathrm{d}t$$

where the last integrand, $T_1(t)$ is a log-free transseries and moreover

$$T_1(t) = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} \mu_1^{k_1} \cdots \mu_M^{k_M} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} e^{-k_1 L_1 - \ldots - k_M L_M}$$

The case $\mathbf{k} = 0$ is trivial and it thus suffices to find $\partial^{-1}e^{\pm L}$, where $n = l(L) \ge 1$ where $L > 0$. We analyse the case $\partial^{-1}e^{\pm L}$, the other one being similar. Then $L \gg x^m$ for any m and thus also $\partial L \gg x^m$ for all m. Therefore, since $\partial e^{-L} = -(\partial L)e^{-L}$ we expect that $\mathrm{dom}(\partial^{-1}e^{-L}) = -(\partial L)^{-1}e^{-L}$ and we look for a Δ so that

$$\partial^{-1}e^{-L} = -\frac{e^{-L}}{\partial L}(1+\Delta) \quad (8.70)$$

Then Δ should satisfy the equation

$$\Delta = -\frac{\partial^2 L}{(\partial L)^2} - \frac{\partial^2 L}{(\partial L)^2}\Delta + (\partial L)^{-1}\partial\Delta \quad (8.71)$$

Since $s_1 = 1/L'$ and $s_2 = L''/(L')^2$ are small, by Lemma 8.12, there is a set of generators in which all the magnitudes of $s_{1,2}$ are of the form $\boldsymbol{\mu}^{\mathbf{k}}$ with $\mathbf{k} > 0$.

By Proposition 8.56 and Exercise 8.45, (8.71) is contractive and has a unique solution in the space of transseries with the complete set of generators of L and x^{-1} and $\Delta \ll L$ and the generators constructed above. For the last term, note that if $\Delta = \sum c_\omega e^{-L_\omega}$ and $L = e^{L_1}$, then $\Delta'/L' = \sum c_\omega L'_\omega e^{-L_\omega} e^{-L_1}$ and $L'_\omega e^{-L} = \mu_\omega \ll 1$.

□

1. Since the equation is contractive, it follows that $\text{mag}(\Delta) = \text{mag}(L''/L'^2)$.

In the following we also use the notation $\partial T = T'$ and we write $\mathcal{P}$ for the antiderivative ∂^{-1} constructed above.

Proposition 8.72 *$\mathcal{P}$ is an antiderivative without constant terms, i.e,*

$$\mathcal{P}T = L + s$$

PROOF This follows from the fact that $\mathcal{P}e^{-L} \ll 1$ while $P(e^L)$ is purely large, since all small terms are of lower level. Check! □

Proposition 8.73 *We have*

$$\begin{aligned}
\mathcal{P}(T_1 + T_2) &= \mathcal{P}T_1 + \mathcal{P}T_2 \\
(\mathcal{P}T)' = T;\ &\mathcal{P}T' = T_{\overline{0}} \\
\mathcal{P}(T_1 T_2') &= (T_1 T_2)_{\overline{0}} - \mathcal{P}(T_1' T_2) \\
T_1 \gg T_2 &\implies \mathcal{P}T_1 \gg \mathcal{P}T_2 \\
T > 0 \text{ and } T \gg 1 &\implies \mathcal{P}T > 0
\end{aligned} \tag{8.74}$$

where

$$T = \sum_{\mathbf{k} \ge \mathbf{k}_0} c_{\mathbf{k}} \mu^{\mathbf{k}} \implies T_{\overline{0}} = \sum_{\mathbf{k} \ge \mathbf{k}_0; \mathbf{k} \ne 0} c_{\mathbf{k}} \mu^{\mathbf{k}}$$

PROOF Exercise. □

There exists only one $\mathcal{P}$ with the properties (8.74), for any two would differ by a constant.

Remark 8.75 *Let $s_0 \in \mathcal{T}$. The operators defined by*

$$J_1(T) = \mathcal{P}(e^{-x}(\text{Const.} + s_0)T(x)) \tag{8.76}$$

$$J_2(T) = e^{\pm x} x^{\sigma} \mathcal{P}(x^{-2} x^{-\sigma} e^{\mp x}(\text{Const.} + s_0)T(x)) \tag{8.77}$$

are contractive on $\mathcal{T}$.

PROOF For (8.76) it is enough to show contractivity of $\mathcal{P}(e^{-x}\cdot)$. If we assume the contrary, that $T' \not\ll Te^{-x}$ it follows that $\log T \not\gg 1$. We know that if $\log T$ is small then $\mathrm{mag}(T) = c$, c constant. But if $\mathrm{mag}(T) = c$ then the property is immediate. The proof of (8.76) is very similar.

∎

B The C^*–algebra $\mathcal{D}'_{m,\nu}$

Let $\mathcal{D}$ be the space of test functions (compactly supported C^∞ functions on $(0,\infty)$) and $\mathcal{D}(0,x)$ be the test functions on $(0,x)$.

We say that $f \in \mathcal{D}'$ is a staircase distribution if for any $k = 0,1,2,...$ there is an L^1 function on $[0,k+1]$ so that $f = F_k^{(km)}$ (in the sense of distributions) when restricted to $\mathcal{D}(0,k+1)$ or

$$F_k := \mathcal{P}^{mk} f \in L_1(0,k+1) \tag{8.78}$$

(since $f \in L^1_{loc}[0,1-\epsilon]$ and $\mathcal{P}f$ is well defined, [20]). With this choice we have

$$F_{k+1} = \mathcal{P}^m F_k \text{ on } [0,k] \text{ and } F_k^{(j)}(0) = 0 \text{ for } j \le mk-1 \tag{8.79}$$

We denote these distributions by $\mathcal{D}'_m$ ($\mathcal{D}'_m(0,k)$ respectively, when restricted to $\mathcal{D}(0,k)$) and observe that $\bigcup_{m>0} \mathcal{D}'_m \supset S'$, the distributions of slow growth. The inclusion is strict since any element of S' is of finite order.

Let $f \in L^1$. Taking $F = \mathcal{P}^j f \in C^j$ we have, by integration by parts and noting that the boundary terms vanish,

$$(F*F)(p) = \int_0^p F(s)F(p-s)\mathrm{d}s = \int_0^p F^{(j)}(s)\mathcal{P}^j F(p-s) \tag{8.80}$$

so that $F*F \in C^{2j}$ and

$$(F*F)^{(2j)} = f*f \tag{8.81}$$

This motivates the following definition: for $f, \tilde{f} \in \mathcal{D}'_m$ let

$$f * \tilde{f} := (F_k * \tilde{F}_k)^{(2km)} \text{ in } \mathcal{D}'(0,k+1) \tag{8.82}$$

We first check that the definition is consistent in the sense that

$$(F_{k+1} * F_{k+1})^{(2m(k+1))} = (F_k * F_k)^{(2mk)}$$

on $\mathcal{D}(0, k+1)$. For $p < k+1$ integrating by parts and using (8.79) we obtain

$$\frac{\mathrm{d}^{2m(k+1)}}{\mathrm{d}p^{2m(k+1)}} \int_0^p F_k(s)\mathcal{P}^{2m}\tilde{F}_k(p-s)\mathrm{d}s = \frac{\mathrm{d}^{2mk}}{\mathrm{d}p^{2mk}} \int_0^p F_k(s)\tilde{F}_k(p-s)\mathrm{d}s \tag{8.83}$$

(8.84)

The same argument shows that the definition is compatible with the embedding of $\mathcal{D}'_m$ in $\mathcal{D}'_{m'}$ with $m' > m$. Convolution is commutative and associative: with $f, g, h \in \mathcal{D}'_m$ and identifying $(f * g)$ and h by the natural inclusion with elements in $\mathcal{D}'_{2m}$ we obtain $(f * g) * h = ((F * G) * H)^{(4mk)} = f * (g * h)$.

The following staircase decomposition exists in $\mathcal{D}'_m$.

Lemma 8.85 . *For each $f \in \mathcal{D}'_m$ there is a* unique *sequence $\{\Delta_i\}_{i=0,1,..}$ such that $\Delta_i \in L^1(\mathbb{R}^+)$, $\Delta_i = \Delta_i \chi_{[i,i+1]}$ and*

$$f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \tag{8.86}$$

Also (cf. (8.79)),

$$F_i = \sum_{j \le i} \mathcal{P}^{m(i-j)} \Delta_i \ \ on\ [0, i+1) \tag{8.87}$$

Note that the infinite sum is $\mathcal{D}'$−convergent since for a given test function only a finite number of distributions are nonzero.

Proof

We start by showing (8.87). For $i = 0$ we take $\Delta_0 = F_0\chi[0,1]$ (where $F_0\chi[0,1]$ $:= \phi \mapsto \int_0^1 F_0(s)\phi(s)\mathrm{d}s$). Assuming (8.87) holds for $i < n$ we simply note that

$$\begin{aligned}\Delta_n &:= \chi_{[0,n+1]} \left(F_n - \sum_{j \le n-1} \mathcal{P}^{m(n-j)} \Delta_j \right) \\ &= \chi_{[0,n+1]} \Big(F_n - \mathcal{P}^m (F_{n-1}\chi_{[0,n]}) \Big) = \chi_{[n,n+1]} \Big(F_n - \mathcal{P}^m (F_{n-1}\chi_{[0,n]}) \Big)\end{aligned} \tag{8.88}$$

(with $\chi_{[n,\infty]}F_n$ defined in the same way as $F_0\chi[0,1]$ above) has, by the induction hypothesis and (8.79) the required properties. Relation (8.86) is immediate. It remains to show uniqueness. Assuming (8.86) holds for the sequences $\Delta_i, \tilde{\Delta}_i$ and restricting f to $\mathcal{D}(0,1)$ we see that $\Delta_0 = \tilde{\Delta}_0$. Assuming $\Delta_i = \tilde{\Delta}_i$ for $i < n$ we then have $\Delta_n^{(mn)} = \tilde{\Delta}_n^{(mn)}$ on $\mathcal{D}(0, n+1)$. It follows ([20]) that

$\Delta_n(x) = \tilde{\Delta}_n(x) + P(x)$ on $[0, n+1)$ where P is a polynomial (of degree $< mn$). Since by definition $\Delta_n(x) = \tilde{\Delta}_n(x) = 0$ for $x < n$ we have $\Delta_n = \tilde{\Delta}_n(x)$. ☐

The expression (8.82) hints to decrease in regularity, but this is not the case. In fact, we check that the regularity of convolution is not worse than that of its arguments.

Remark 8.89

$$(\cdot * \cdot) : \mathcal{D}_n \mapsto \mathcal{D}_n \tag{8.90}$$

Since

$$\chi_{[a,b]} * \chi_{[a',b']} = \left(\chi_{[a,b]} * \chi_{[a',b']}\right) \chi_{[a+a',b+b']} \tag{8.91}$$

we have

$$F * \tilde{F} = \sum_{j+k \le \lfloor p \rfloor} \mathcal{P}^{m(i-j)} \Delta_j * \mathcal{P}^{m(i-k)} \tilde{\Delta}_k = \sum_{j+k \le \lfloor p \rfloor} \Delta_j * \mathcal{P}^{m(2i-j-k)} \tilde{\Delta}_k \tag{8.92}$$

which is manifestly in $C^{2mi-m(j+k)}[0,p) \subset C^{2mi-m\lfloor p \rfloor}[0,p)$.

☐

B .2 Norms on $\mathcal{D}'_m$

For $f \in \mathcal{D}'_m$ define

$$\|f\|_{\nu,m} := c_m \sum_{i=0}^{\infty} \nu^{im} \|\Delta_i\|_{L^1_\nu} \tag{8.93}$$

(the constant c_m, immaterial for the moment, is defined in (8.106). When no confusion is possible we will simply write $\|f\|_\nu$ for $\|f\|_{\nu,m}$ and $\|\Delta\|_\nu$ for $\|\Delta_i\|_{L^1_\nu}$ (no other norm is used for the Δ's). Let $\mathcal{D}'_{m,\nu}$ be the distributions in $\mathcal{D}'_m$ such that $\|f\|_\nu < \infty$.

Remark 8.94 *$\|\cdot\|_\nu$ is a norm on $\mathcal{D}'_{m,\nu}$.*

If $\|f\|_\nu = 0$ for all i, then $\Delta_i = 0$ whence $f = 0$. In view of Lemma 8.85 we have $\|0\|_\nu = 0$. All the other properties are immediate.

Remark 8.95 *$\mathcal{D}'_{m,\nu}$ is a Banach space. The topology given by $\|\cdot\|_\nu$ on $\mathcal{D}'_{m,\nu}$ is stronger than the topology inherited from $\mathcal{D}'$.*

Proof. If we let $\mathcal{D}'_{m,\nu}(k,k+1)$ be the subset of $\mathcal{D}'{}_{m,\nu}$ where all $\Delta_i = 0$ except for $i = k$, with the norm (8.93), we have

$$\mathcal{D}'{}_{m,\nu} = \bigoplus_{k=0}^{\infty} \mathcal{D}'_{m,\nu}(k,k+1) \tag{8.96}$$

and we only need to check completeness of each $\mathcal{D}'_{m,\nu}(k,k+1)$ which is immediate: on $L^1[k,k+1]$, $\|\cdot\|_\nu$ is equivalent to the usual L^1 norm and thus if $f_n \in \mathcal{D}'_{m,\nu}(k,k+1)$ is a Cauchy sequence then $\Delta_{k,n} \stackrel{L_\nu}{\to} \Delta_k$ (whence weak convergence) and $f_n \stackrel{\mathcal{D}'_{m,\nu}(k,k+1)}{\to} f$ where $f = \Delta_k^{(mk)}$. ☐

Lemma 8.97 *The space $\mathcal{D}'{}_{m,\nu}$ is a C^* algebra with respect to convolution.*

Proof. Let $f, \tilde{f} \in \mathcal{D}'_{m,\nu}$ with

$$f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \quad , \quad \tilde{f} = \sum_{i=0}^{\infty} \tilde{\Delta}_i^{(mi)}$$

Then

$$f * \tilde{f} = \sum_{i,j=0}^{\infty} \Delta_i^{(mi)} * \tilde{\Delta}_j^{(mj)} = \sum_{i,j=0}^{\infty} \left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)} \tag{8.98}$$

and the support of $\Delta_i * \tilde{\Delta}_j$ is in $[i+j, i+j+2]$ i.e. $\Delta_i * \tilde{\Delta}_j = \chi_{[i+j,i+j+2]} \Delta_i * \tilde{\Delta}_j$.

We first evaluate the norm in $\mathcal{D}'_{m,\nu}$ of the terms $\left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)}$.

I. Decomposition formula. Let $f = F^{(mk)} \in \mathcal{D}'(\mathbb{R}_+)$, where $F \in L^1(\mathbb{R}_+)$, and F is supported in $[k, k+2]$ i.e., $F = \chi_{[k,k+2]} F$ $(k \geq 0)$. Then $f \in \mathcal{D}'_m$ and the decomposition of f (cf. (8.86)) has the terms:

$$\Delta_0 = \Delta_1 = \ldots = \Delta_{k-1} = 0 \quad , \quad \Delta_k = \chi_{[k,k+1]} F \tag{8.99}$$

and

$$\Delta_{k+n} = \chi_{[k+n,k+n+1]} G_n, \quad \text{where } G_n = \mathcal{P}^m \left(\chi_{[k+n,\infty)} G_{n-1}\right), \quad G_0 = F \tag{8.100}$$

Proof of Decomposition Formula. We use first line of (2.98) of the paper

$$\Delta_j = \chi_{[j,j+1]} \left(F_j - \sum_{i=0}^{j-1} \mathcal{P}^{m(j-i)} \Delta_i \right) \tag{8.101}$$

where, in our case, $F_k = F$, $F_{k+1} = \mathcal{P}^m F$, ..., $F_{k+n} = \mathcal{P}^{mn} F$, ...

The relations (8.99) follow directly from (8.101). Formula (8.100) is shown by induction on n. For $n = 1$ we have

$$\Delta_{k+1} = \chi_{[k+1,k+2]} \left(\mathcal{P}^m F - \mathcal{P}^m \Delta_k\right)$$

$$= \chi_{[k+1,k+2]}\mathcal{P}^m \left(\chi_{[k,\infty)}F - \chi_{[k,k+1]}F\right) = \chi_{[k+1,k+2]}\mathcal{P}^m \left(\chi_{[k+1,\infty)}F\right)$$

Assume (8.100) holds for Δ_{k+j}, $j \leq n-1$. Using (8.101), with $\chi = \chi_{[k+n,k+n+1]}$ we have

$$\Delta_{k+n} = \chi \left(\mathcal{P}^{mn}F - \sum_{i=k}^{n-1} \mathcal{P}^{m(n-i)}\Delta_i\right) = \chi\mathcal{P}^m \left(G_{n-1} - \Delta_{n-1}\right)$$

$$= \chi\mathcal{P}^m \left(\chi_{[k+n-1,\infty)}G_{n-1} - \chi_{[k+n-1,k+n]}G_{n-1}\right) = \chi\mathcal{P}^m \left(\chi_{[k+n,\infty)}G_{n-1}\right) \quad \Box$$

II. Estimating Δ_{k+n}. For f as in **I**, we have

$$||\Delta_{k+1}||_\nu \leq \nu^{-m}||F||_\nu \quad , \quad ||\Delta_{k+2}||_\nu \leq \nu^{-2m}||F||_\nu \tag{8.102}$$

and, for $n \geq 3$

$$||\Delta_{k+n}||_\nu \leq e^{2\nu-n\nu}(n-1)^{nm-1}\frac{1}{(nm-1)!}||F||_\nu \tag{8.103}$$

Proof of estimates of Δ_{k+n}.
(A) Case $n = 1$.

$$\begin{aligned}
||\Delta_{k+1}||_\nu &\leq \int_{k+1}^{k+2} dt\, e^{-\nu t}\mathcal{P}^m \left(\chi_{[k+1,\infty)}|F|\right)(t) \\
&= \int_{k+1}^{k+2} dt\, e^{-\nu t} \int_{k+1}^{t} ds_1 \int_{k+1}^{s_1} ds_2 \,... \int_{k+1}^{s_{m-1}} ds_m |F(s_m)| \\
&\leq \int_{k+1}^{k+2} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \,... \int_{s_2}^{\infty} ds_1 \int_{s_1}^{\infty} dt\, e^{-\nu t} \\
&= \int_{k+1}^{k+2} ds_m |F(s_m)| e^{-\nu s_m}\nu^{-m} \leq \nu^{-m}||F||_\nu
\end{aligned} \tag{8.104}$$

(B) Case $n = 2$:

$$||\Delta_{k+1}||_\nu \leq \int_{k+2}^{k+3} dt\, e^{-\nu t}\mathcal{P}^m \left(\chi_{[k+2,\infty)}\mathcal{P}^m \left(\chi_{[k+1,\infty)}|F|\right)\right)$$

$$= \int_{k+2}^{k+3} dt\, e^{-\nu t} \int_{k+2}^{t} dt_1 \int_{k+2}^{t_1} dt_2 \ldots \int_{k+2}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-1}} ds_m |F(s_m)|$$

$$\leq \int_{k+2}^{k+3} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 \int_{\max\{s_1,k+2\}}^{\infty} dt_m \int_{t_m}^{\infty} dt_{m-1} \ldots \int_{t_1}^{\infty} dt\, e^{-\nu t}$$

$$= \int_{k+2}^{k+3} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 e^{-\nu \max\{s_1,k+2\}} \nu^{-m-1}$$

$$\leq \int_{k+2}^{k+3} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_3}^{\infty} ds_2 e^{-\nu s_2} \nu^{-m-2} = \int_{k+2}^{k+3} ds_m |F(s_m)| e^{-\nu s_m} \nu^{-2m}$$

(C) Case $n \geq 3$. We first estimate $G_2, \ldots, G_n$:

$$|G_2(t)| \leq \mathcal{P}^m \left(\chi_{[k+2,\infty)} \mathcal{P}^m \left(\chi_{[k+1,\infty)} |F| \right) \right)(t)$$

$$= \int_{k+2}^{t} dt_1 \int_{k+2}^{t_1} dt_2 \ldots \int_{k+2}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-1}} ds_m |F(s_m)|$$

and using the inequality

$$|F(s_m)| = |F(s_m)| \chi_{[k,k+2]}(s_m) \leq |F(s_m)| e^{-\nu s_m} e^{\nu(k+2)}$$

we get

$$|G_2(t)| \leq e^{\nu(k+2)} ||F||_\nu \int_{k+1}^{t} dt_1 \int_{k+1}^{t_1} dt_2 \ldots \int_{k+1}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-2}} ds_{m-1}$$

$$= e^{\nu(k+2)} ||F||_\nu (t-k-1)^{2m-1} \frac{1}{(2m-1)!}$$

The estimate of G_2 is used for bounding G_3:

$$|G_3(t)| \leq \mathcal{P}^m \left(\chi_{[k+3,\infty)} |G_2| \right) \leq \mathcal{P}^m \left(\chi_{[k+1,\infty)} |G_2| \right)$$

$$\leq e^{\nu(k+2)} ||F||_\nu (t-k-1)^{3m-1} \frac{1}{(3m-1)!}$$

and similarly (by induction)

$$|G_n(t)| \leq e^{\nu(k+2)} ||F||_\nu (t-k-1)^{nm-1} \frac{1}{(nm-1)!}$$

Then

$$||\Delta_{k+n}||_\nu \le e^{\nu(k+2)}||F||_\nu \frac{1}{(nm-1)!}\int_{k+n}^{k+n+1} dt\, e^{-\nu t}(t-k-1)^{nm-1}$$

and, for $\nu \ge m$ the integrand is decreasing, and the inequality (8.103) follows.
III. Final Estimate. Let $\nu_0 > m$ be fixed. For f as in **I**, we have for any $\nu > \nu_0$,

$$||f|| \le c_m \nu^{km}||F||_\nu \tag{8.105}$$

for some c_m, if $\nu > \nu_0 > m$.
Proof of Final Estimate

$$||f|| = \sum_{n\ge 0}\nu^{km+kn}||\Delta_{k+n}||_\nu \le \nu^{km}||F||_\nu\left[3+\sum_{n\ge 3}\nu^{nm}e^{2\nu-n\nu}\frac{(n-1)^{nm-1}}{(nm-1)!}\right]$$

and, using $n-1 \le (mn-1)/m$ and a crude Stirling estimate we obtain

$$||f|| \le \nu^{km}||F||_\nu\left[3+me^{2\nu-1}\sum_{n\ge 3}\left(e^{m-\nu}\nu^m/m^m\right)^n\right] \le c_m\nu^{km}||F||_\nu \tag{8.106}$$

Thus (8.105) is proven for $\nu > \nu_0 > m$.
End of the proof. From (8.98) and (8.105) we get

$$||f*\tilde{f}|| \le \sum_{i,j=0}^{\infty}||\left(\Delta_i*\tilde{\Delta}_j\right)^{m(i+j)}||$$

$$\le \sum_{i,j=0}^{\infty}c_m^2\nu^{m(i+j)}||\Delta_i*\tilde{\Delta}_j||_\nu \le c_m^2\sum_{i,j=0}^{\infty}\nu^{m(i+j)}||\Delta_i||_\nu\,||\tilde{\Delta}_j||_\nu = c_m^2||f||\,||\tilde{f}||$$

□

Remark 8.107 *Let $f \in \mathcal{D}'_{m,\nu}$ for some $\nu > \nu_0$ where $\nu_0^m = \mathrm{e}^{\nu_0}$. Then $f \in \mathcal{D}'_{m,\nu'}$ for all $\nu' > \nu$ and furthermore,*

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{8.108}$$

Proof. We have

$$\nu^{mk}\int_k^{k+1}|\Delta_k(s)|\mathrm{e}^{-\nu s}\mathrm{d}s = (\nu^m\mathrm{e}^{-\nu})^k\int_0^1|\Delta_k(s+k)|\mathrm{e}^{-\nu s}\mathrm{d}s \tag{8.109}$$

which is decreasing in ν. The rest follows from the monotone convergence theorem. □

B .3 Embedding of L^1_ν in $\mathcal{D}'_m$

Lemma 8.110 *i) Let $f \in L^1_{\nu_0}$ (cf. Remark 8.107). Then $f \in \mathcal{D}'_{m,\nu}$ for all $\nu > \nu_0$.*

ii) $\mathcal{D}(\mathbb{R}^+\backslash\mathbb{N}) \cap L^1_\nu(\mathbb{R}^+)$ is dense in $\mathcal{D}_{m,\nu}$ with respect to the norm $\|\|_\nu$.

Proof.

Note that if for some ν_0 we have $f \in L^1_{\nu_0}(\mathbb{R}^+)$ then

$$\int_0^p |f(s)|\mathrm{d}s \le \mathrm{e}^{\nu_0 p}\int_0^p |f(s)|\mathrm{e}^{-\nu_0 s}\mathrm{d}s \le \mathrm{e}^{\nu_0 p}\|f\|_{\nu_0} \tag{8.111}$$

to which, application of $\mathcal{P}^{k-1}$ yields

$$\mathcal{P}^k|f| \le \nu_0^{-k+1}\mathrm{e}^{\nu_0 p}\|f\|_{\nu_0} \tag{8.112}$$

Also, $\mathcal{P}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \le {\nu_0}^{-1}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p}$ so that

$$\mathcal{P}^m\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \le {\nu_0}^{-m}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \tag{8.113}$$

so that, by (8.88) (where now F_n and $\chi_{[n,\infty]}F_n$ are in $L^1_{loc}(0, n+1)$) we have for $n > 1$,

$$|\Delta_n| \le \|f\|_{\nu_0}\mathrm{e}^{\nu_0 p}{\nu_0}^{1-mn}\chi_{[n,n+1]} \tag{8.114}$$

Let now ν be large enough. We have

$$\sum_{n=2}^{\infty}\nu^{mn}\int_0^\infty |\Delta_n|\mathrm{e}^{-\nu p}\mathrm{d}p \le \nu_0\|f\|_{\nu_0}\sum_{n=2}^{\infty}\int_n^{n+1}\mathrm{e}^{-(\nu-\nu_0)p}\left(\frac{\nu}{\nu_0}\right)^p\mathrm{d}p$$
$$= \frac{\mathrm{e}^{-2(\nu-\nu_0-\ln(\nu/\nu_0))}}{\nu-\nu_0-\ln(\nu/\nu_0)}\nu_0\|f\|_{\nu_0} \tag{8.115}$$

For $n = 0$ we simply have $\|\Delta_0\| \le \|f\|$, while for $n = 1$ we write

$$\|\Delta_1\|_\nu \le \|1^{*(m-1)} * |f|\|_\nu \le \nu^{m-1}\|f\|_\nu \tag{8.116}$$

Combining the estimates above, the proof of (i) is complete. To show (ii), let $f \in \mathcal{D}'_{m,\nu}$ and let k_ϵ be such that $c_m\sum_{i=k_\epsilon}^\infty \nu^{im}\|\Delta_i\|_\nu < \epsilon$. For each $i \le k_\epsilon$ we take a function δ_i in $\mathcal{D}(i, i+1)$ such that $\|\delta_i - \Delta_i\|_\nu < \epsilon 2^{-i}$. Then $\|f - \sum_{i=0}^{k_\epsilon}\delta_i^{(mi)}\|_{m,\nu} < 2\epsilon$. $\square$

Proof of continuity of $f(p) \mapsto pf(p)$. If $f(p) = \sum_{k=0}^\infty \Delta_k^{(mk)}$ then $pf = \sum_{k=0}^\infty (p\Delta_k)^{(mk)} - \sum_{k=0}^\infty mk\mathcal{P}(\Delta_k^{(mk)}) = \sum_{k=0}^\infty (p\Delta_k^{(mk)}) - 1 * \sum_{k=0}^\infty (mk\Delta_k)^{(mk)}$. The rest is obvious from continuity of convolution, the embedding shown above and the definition of the norms.

C Useful formulas

A straightforward computation shows that

$$\mathcal{B}\left(\frac{1}{x^n}\right) = \frac{p^{n-1}}{\Gamma(n)} \text{ or } \mathcal{L}(p^n) = \frac{\Gamma(n+1)}{x^{n+1}} \tag{8.117}$$

$$p^q * p^r = \frac{\Gamma(q+1)\Gamma(r+1)}{\Gamma(q+r+2)} p^{q+r+1} \tag{8.118}$$

Also, with $f_{1,2}(p) := p \mapsto \mathcal{H}(p - k_{1,2})g_{1,2}(p - k_{1,2})$ we have

$$\Big(f_1 * f_2\Big)(p) = \mathcal{H}(p - k_1 - k_2)\Big(g_1 * g_2\Big)(p - k_1 - k_2) \tag{8.119}$$

D Appendix

Maple 11 iteration to find the first few terms in the asymptotic series solution of $y' + y = x^{-1} + y^3 + xy^5$ as $x \to \infty$; "%" is Maple shortcut for "the previous expression". The input line following Eq. (4) is copied and pasted without change. In practice one would instead return to the line after Eq. (4) and re-run it as many times as needed. Of course, a do loop can be easily implemented, but there is no point in that unless a very high number of terms is needed.

```
> d1:=y(x) = -(diff(y(x), x))+1/x+y(x)^3+x*y(x)^5;
```

$$d1 := y(x) = -\left(\frac{d}{dx}y(x)\right) + \frac{1}{x} + y(x)^3 + x\,y(x)^5 \quad (1)$$

```
> rs:=rhs(d1);
```

$$rs := -\left(\frac{d}{dx}y(x)\right) + \frac{1}{x} + y(x)^3 + x\,y(x)^5 \quad (2)$$

```
> subs(y(x)=0,rs):asympt(%,x,8);
```

$$\frac{1}{x} \quad (3)$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} \quad (4)$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{7}{x^4} + \frac{15}{x^5} + \frac{25}{x^6} + O\left(\frac{1}{x^7}\right) \quad (5)$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{45}{x^5} + \frac{140}{x^6} + O\left(\frac{1}{x^7}\right) \quad (6)$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{69}{x^5} + \frac{308}{x^6} + O\left(\frac{1}{x^7}\right) \quad (7)$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{69}{x^5} + \frac{428}{x^6} + O\left(\frac{1}{x^7}\right) \quad (8)$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{69}{x^5} + \frac{428}{x^6} + O\left(\frac{1}{x^7}\right) \quad (9)$$

```
>
```

FIGURE 8.1: Maple 11 output in solving $y' + y = x^{-1} + y^3 + xy^5$ by asymptotic power series.

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