1. Review: Complex numbers, Analytic functions

∘ Complex numbers, $\mathbb{C}$ form a field; addition, multiplication of complex numbers have the same properties as their counterparts in $\mathbb{R}$.

∘ There is no "good" order relation in $\mathbb{C}$. Except for that, we operate with complex numbers in the same way as we operate with real numbers.

∘ A function f of a complex variable is a function defined on some subset of $\mathbb{C}$ with complex values. Alternatively, we can view it as a pair of real valued functions of two real variables. We write $z = x + iy$ with x, y real and $i^2 = -1$ and write $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. We write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

∘ We note that $(-i)^2 = -1$ as well. There is no fundamental distinction between i and $-i$, or an intrinsic way to prefer one over the other. This entails a fundamental symmetry of the theory, symmetry with respect to complex conjugation.

∘ We can right away define a number of elementary complex functions: z, $1/z$ and more generally for $m \in \mathbb{Z}$ we easily define z^m and in fact any polynomial $\sum_{m=0}^{K} c_m(z - z_0)^m$.

∘ To be able to define and work with more interesting functions we need to define continuity, derivatives and so on. For this we need to define limits, for which purpose we need a measure of smallness (a "topology on $\mathbb{C}$"). Seen as a pair of real numbers (x, y), the modulus of z, $|z| = \sqrt{x^2 + y^2}$ gives a measure of length and thus of smallness.

Exercise 1.1. *Look up the notion of norm in any topology book and show that the modulus defines a norm on* $\mathbb{C}$.

Then z is small if its length is small. Starting here we can define limits: $z_n \to z$ if the distance between z_n and z goes to zero:

$$z_n \to z \quad \Leftrightarrow |z - z_n| \to 0 \quad \text{as} \;\; n \to \infty \tag{1.2}$$

Using (1.2), convergence in $\mathbb{C}$ is reduced to convergence in $\mathbb{R}$ which we are familiar with.

Exercise 1.2. *Show that* $z_n \to z$ *if and only if* $\mathrm{Re}(z_n) \to \mathrm{Re}(z)$ *and* $\mathrm{Im}(z_n) \to \mathrm{Im}(z)$

2. Series

A series is written as

$$\sum_{k=0}^{\infty} a_k \tag{2.2}$$

where a_k are complex, and is said to converge if, by definition, the sequence of *partial sums*

$$S_N := \sum_{k=0}^{N} a_k \tag{2.3}$$

converges as $N \to \infty$.

The series converges absolutely if the real-valued series

$$\sum_{k=0}^{\infty} |a_k| \tag{2.4}$$

converges.

Exercise 2.3. *Check that the convergence criteria that you know from real analysis: the ratio test, the n-th root test, in fact any test that does not rely on positivity (as are those using monotonicity) carry over to complex series. Check that a series which converges absolutely converges. Check that a* necessary *condition of convergence is $a_k \to 0$ as $k \to \infty$. The proofs require minor modifications, if they require any modifications at all.*

3. Power series

A power series is a series of the form

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{3.2}$$

where c_k, z, z_0 are complex.

Theorem 3.4 (Abel). *If*

$$\sum_{k=0}^{\infty} c_k (z_1 - z_0)^k \tag{3.3}$$

converges, then

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{3.4}$$

converges absolutely and uniformly in the region $|z - z_0| < |z_1 - z_0|$.

Proof. Use Exercise 2.3 to provide a short proof. □

4. Continuity, differentiability, integrals

Definition. A complex function is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$.

Exercise 4.5. *Show that polynomials are continuous functions.*

Likewise, we can now define differentiability.

Definition. A function f is differentiable at z_0 if, by definition, there is a number, call it $f'(z_0)$ such that

$$\frac{f(z) - f(z_0)}{z - z_0} \to f'(z_0) \quad \text{as} \quad z \to z_0$$

Exercise 4.6. *Show that differentiation has the properties we are familiar with from real variables: sum rule, product rule, chain rule etc. hold for complex differentiation.*

Exercise 4.7. *Write $f(z) = u(x,y) + iv(x,y)$. Take $z = z_0 + \epsilon$ with ϵ real and show that if f is differentiable, then u_x and v_x exist at (x_0, y_0). Take then $z = z_0 + i\epsilon$ and show that u_y and v_y exist at (x_0, y_0).*

5. The Cauchy-Riemann (C-R) equations

Differentiability in $\mathbb{C}$ is far more demanding than differentiability in $\mathbb{R}$. For the same reason, complex differentiable functions are much more regular and have better properties than real-differentiable ones.

Theorem 5.8 (CR)**.** *(1) Assume that f is continuously differentiable in an open region $\mathcal{D}$ in $\mathbb{C}$ (a region that contains together with any point z all sufficiently close points; intuitively, this is a domain without its boundary). Then*

$$u_x = v_y; \qquad u_y = -v_x \tag{5.2}$$

throughout $\mathcal{D}$.

(2) Conversely, if u, v are continuously differentiable in $\mathcal{D}$ ("belong to $C^1(\mathcal{D})$") and satisfy (5.2) in $\mathcal{D}$, then f is differentiable in $\mathcal{D}$.

Proof (1) Let $f'(z_0) = a + ib$. We have

$$(5.3) \quad f(z) - f(z_0) = [f'(z_0) + \epsilon(z)](z - z_0)$$
$$= (a + ib)[x - x_0 + i(y - y_0)] + \epsilon(z)(z - z_0)$$
$$= a(x - x_0) - b(y - y_0) + ia(y - y_0) + ib(x - x_0) + \epsilon(z)(z - z_0)$$

where $\epsilon(z) \to 0$ as $z \to z_0$.

On the other hand, from the *consequence* that u and v are in $C^1(\mathcal{D})$ (show this!) we have

$$(5.4) \quad f(z) - f(z_0) = u(x, y) - u(x_0, y_0) + iv(x, y) - iv(x_0, y_0)$$
$$= u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0)$$
$$+ iv_x(x_0, y_0)(x - x_0) + iv_y(x_0, y_0)(y - y_0) + \epsilon(x, y)(x - x_0) + \eta(x, y)(y - y_0)$$

where ϵ and η go to zero as $z \to z_0$.

Exercise 5.9. *Show that (5.3) and (5.4) are compatible if and only if (5.2) hold. Hint: the real and imaginary parts must be equal to each other, and $x - x_0$ and $y - y_0$ are independent quantities.*

The converse is shown by essentially reverting the steps above.

Theorem 5.10. *If the series*

$$(5.5) \qquad S(z) \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

converges in the open disk $D = \{z : |z - z_0| < R\}$ *(see Theorem 3.4), then $S(z)$ is differentiable any number of times in D. In particular,*

$$(5.6) \qquad S'(z) = \sum_{k=0}^{\infty} k c_k (z - z_0)^{k-1}$$

$$(5.7) \qquad S''(z) = \sum_{k=0}^{\infty} k(k-1) c_k (z - z_0)^{k-2}$$

$$(5.8) \qquad \ldots\ldots$$

$$(5.9) \qquad S^{(p)}(z) = \sum_{k=0}^{\infty} k(k-1)\cdots(k-p+1) c_k (z - z_0)^{k-p}$$

$$(5.10) \qquad \ldots\ldots\ldots$$

and all these series converge in D.

Proof. For the proof we only need to show the result for S' since S'' is obtained by differentiating S' and inductively $S^{(p)}$ is obtained by differentiating $S^{(p-1)}$. Furthermore, we can denote $\zeta = z - z_0$ and reduce the problem to the case when $z_0 = 0$. Let $|z| < \rho < R$ and choose h small enough so that $|z| + |h| < \rho$. Note that

$$(z+h)^n - z^n = nz^{n-1}h + \frac{n(n-1)}{2}z^{n-2}h^2 + \cdots + h^n$$

and thus

$$\frac{(z+h)^n - z^n}{h} = nz^{n-1} + \frac{n(n-1)}{2}z^{n-2}h + \cdots + h^{n-1}$$

hence

$$\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| = \left|\frac{n(n-1)}{2}z^{n-2}h + \cdots + h^{n-1}\right| \tag{5.11}$$
$$\leq \frac{n(n-1)}{2}|z|^{n-2}|h| + \cdots + |h|^{n-1} \leq \frac{n(n-1)}{2}|\rho|^{n-2}h$$

The last inequality is obtained by (1) repeating the same calculation with $|z|$ instead of z and $|h|$ instead of h, assuming h is small so that $|z| + |h| < \rho$, and applying Taylor's formula with remainder to the *real valued* function $(|z| + |h|)^n$ (check this!)

Thus,

$$\left|\frac{S(z+h) - S(z)}{h} - S'(z)\right| \leq \frac{h}{R^2}\sum_{k=0}^{\infty}\frac{k(k-1)}{2}\left(\frac{\rho}{R}\right)^{k-2} \tag{5.12}$$

The series on the rhs of (5.12) is convergent (why?), ρ is independent of h and thus the rhs of (5.12) converges to zero as $h \to 0$ $\square$.

6. Other simple functions

○ *The exponential.* We define

$$e^z = \sum_{k=0}^{\infty}\frac{z^n}{n!} \tag{6.2}$$

This series converges for any $z \in \mathbb{C}$ (check!) and thus it is differentiable for any z in $\mathbb{C}$ (why?). We have, by (5.6)

$$(e^z)' = e^z \tag{6.3}$$

Thus,

$$(e^z e^{-z})' = 0 \tag{6.4}$$

so that $e^z e^{-z}$ does not depend on z, and takes the same value everywhere, the value for $z = 0$. But we see immediately that $e^0 = 1$. Thus

$$e^z e^{-z} = 1 \quad \Leftrightarrow e^{-z} = 1/e^z \tag{6.5}$$

In the same way,

$$(e^{z+a}e^{-z})' = 0 \quad \Leftrightarrow e^{z+a}e^{-z} = e^a e^0 = e^a \quad \Leftrightarrow e^{z+a} = e^z e^a \tag{6.6}$$

which provides us with the fundamental property of the exponential. Also, we immediately check Euler's formula: for $\phi \in \mathbb{R}$ we have

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{6.7}$$

Exercise 6.11.

$$e^s = 1 \quad \Leftrightarrow s = 2N\pi i, \quad N \in \mathbb{Z} \tag{6.8}$$

◦ *The logarithm.* In the complex domain the log is a much trickier function. We will now look at a simple question, that of defining $\log(1+z)$ for $|z| < 1$. This is done via the convergent Taylor series

$$\log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots \tag{6.9}$$

By (5.6) we get

$$\log(1+z)' = 1 - z + z^2 - z^3 + \cdots = \frac{1}{1+z} \quad \text{if} \quad |z| < 1 \tag{6.10}$$

(why?).

Exercise 6.12. *Show that if s is small we have*

$$\log(e^s) = s; \qquad e^{\log(1+s)} = 1 + s$$

showing that exp *and* log*, now defined in the complex domain have the expected properties.*

We will return to the properties of the log in $\mathbb{C}$ later and study it carefully. It is one of the fundamental "branched" complex functions.

7. Operations with power series

If S and T are convergent power series, then so are $S+T, S\cdot T, S/T$ (if $T \neq 0$), $S(T)$ if $T(0) = 0$ and z is small enough etc. The formulas for these new series is obtained by working with the series as if they were polynomials. For instance,

$$ST = s_0t_0 + (s_1t_0 + s_0t_1)z + (s_2t_0 + s_1t_1 + s_0t_2)z^2 + \cdots \tag{7.2}$$

Exercise 7.13. ** Write a few terms of the series S/T. What is the radius of convergence of $S(T)$?*

8. Integrals

If $f(t) = u(t) + iv(t)$ is a complex-valued function of one real variable t then $\int_a^b f(t)dt$ is defined by

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt \tag{8.2}$$

This reduces the questions of complex integration to the familiar real integration. Likewise, a curve γ in $\mathbb{R}^2$ is usually given in terms of a pair of functions $(x(t), y(t))$, where $t \in [a, b]$. The curve is differentiable if (x, y) are differentiable functions of t.

We define using (8.2)

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Exercise 8.14. *Show that*

$$\int_\gamma f(z)dz = \int_\gamma \Big(udx - vdy\Big) + i \int_\gamma \Big(udy + vdx\Big) \tag{8.3}$$

Intuitively, a simply connected region is a region without holes. Review this notion if you need.

Theorem 8.15 (Cauchy)**.** *Assume $\mathcal{D}$ is a simply connected region and that f is continuously differentiable in $\mathcal{D}$. If γ is a simple closed curve contained in $\mathcal{D}$ then*

$$\oint_\gamma f(z)dz := \int_\gamma f(z)dz = 0 \tag{8.4}$$

Proof. By (8.3) and (5.2) this follows from Green's theorem. (Exercise: fill in all details.) □

Definition. We will call *regions* open connected sets in $\mathbb{C}$.

9. Cauchy's formula

Let $\mathcal{D}$ be a region in $\mathbb{C}$ and $z_0 \in \mathcal{D}$. The functions $f_n(z) = (z - z_0)^{-n}$, $n = 1, 2, ...$ are analytic in $\mathcal{D} \setminus \{z_0\}$. Thus, if γ_1 and γ_2 are two curves in $\mathcal{D} \setminus \{z_0\}$ *homotopic* to each-other (that is, they can be smoothly deformed into each-other without crossing the boundary of $\mathcal{D}$ or touching z_0) then

$$\int_{\gamma_1} f_n(z)dz := \int_{\gamma_2} f_n(z)dz \tag{9.2}$$

These integrals are zero if γ_i does not contain z_0 inside.

To calculate them for a simple close curve encircling z_0 it suffices, by (9.2), to make this calculation when the curve is a circle. But in this case we can easily do the calculation explicitly. Indeed, a circle centered at z_0 with radius ρ is parametrized by $z = z_0+\rho(\cos(t)+i\sin(t)) = \rho e^{it}$, $t \in [0, 2\pi]$ (where we used Euler's formula), and we get

$$\oint \frac{dz}{(z-z_0)^n} = \frac{i}{\rho^{n-1}} \int_0^{2\pi} e^{-i(n-1)t} dt = \begin{cases} 2\pi i & \text{if } \ n=1 \\ 0 & \text{otherwise} \end{cases} \tag{9.3}$$

Theorem 9.16 (Cauchy's formula)**.** *If f is analytic in the simply connected region $\mathcal{D}$ and γ is a simple closed curve in $\mathcal{D}$ around z, we have*

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(s)}{s-z} ds \tag{9.4}$$

Proof. We write
(9.5)

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(s)-f(z)}{s-z} ds + \frac{1}{2\pi i} \oint \frac{f(z)}{s-z} ds = \frac{1}{2\pi i} \oint \frac{f(s)-f(z)}{s-z} ds + f(z)$$

where we used (9.4). It remains to show that the first integral is zero. We will show that it is, in absolute value, less than any positive number, so it must be zero.

The function

$$g(s) = \frac{f(s)-f(z)}{s-z}$$

is analytic in s except at $s = z$; it is continuous at z because f is differentiable. We let M be the maximum value of g. Now, we deform the contour as shown in Fig. 1 and in the process the value of the integral is not changed. If we take the inner curve to be a circle C_ϵ of radius ϵ around z we have

$$\left| \frac{1}{2\pi i} \oint \frac{f(s)-f(z)}{s-z} ds \right| \le \frac{M}{2\pi} 2\pi\epsilon = M\epsilon$$

which shows the inequality we claimed.

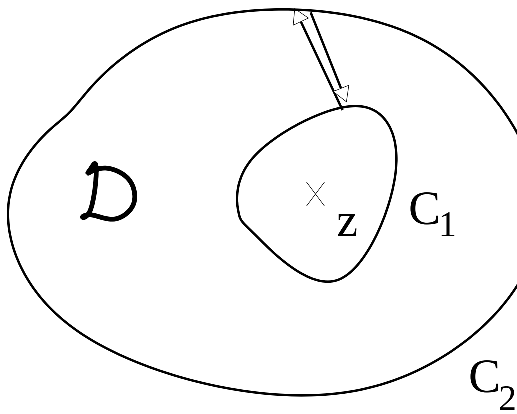


Fig. 1

10. Taylor series of analytic functions

A function which is continuously differentiable in a region $\mathcal{D}$ is called **analytic** in $\mathcal{D}$.

Assume f is analytic in $\mathcal{D}$ and let $z_0 \in \mathcal{D}$. Take ρ small enough so that the disk

$$D(z_0; \rho) := \{s : |s - z_0| < \rho\}$$

is contained in $\mathcal{D}$. Let $C(z_0, \rho) = \partial D(z_0; \rho)$ be the circle of radius ρ centered at z_0. By theorem 9.16 we have, for $z \in D(z_0; \rho)$

$$f(z) = \frac{1}{2\pi i} \oint_{C(z_0,\rho)} \frac{f(s)}{s - z} ds \tag{10.2}$$

We write

$$\frac{1}{s-z} = \frac{1}{s - z_0 - (z - z_0)} \tag{10.3}$$

$$= \frac{1}{s - z_0}\left[1 + \frac{z - z_0}{s - z_0} + \cdots + \left(\frac{z - z_0}{s - z_0}\right)^n\right] + \frac{1}{s - z}\left(\frac{z - z_0}{s - z_0}\right)^{n+1}$$

and thus

$$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{n} \oint_{C(z_0,\rho)} (z - z_0)^k \frac{f(s)}{(s - z_0)^{k+1}} ds \tag{10.4}$$

$$+ \frac{1}{2\pi i} \oint_{C(z_0,\rho)} \frac{f(s)}{(s - z)} \frac{(z - z_0)^{n+1}}{(s - z_0)^{n+1}}$$

We note that on $C(z_0, \rho)$ we have $|s - z_0| > |z - z_0|$. Thus we can choose n_0 so that for $n > n_0$ we have $|z - z_0|^n |s - z_0|^{-n} < \epsilon$. Let, as before, M be the maximum of $|f||s-z|^{-1}$ on $C(z_0, \rho)$. We thus have

$$f(z) = \sum_{k=0}^{n} c_k (z - z_0)^k + E(z, z_0, n) \tag{10.5}$$

where $E(z, z_0, n) \to 0$ as $n \to \infty$. By definition, the series $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ converges then to $f(z)$. Furthermore, we can easily obtainan estimate for $E(z, z_0, n)$ from (10.4). Namely, let $R < \rho$ and $|z - z_0| < R$. As long as $|z - z_0| < R$, $C(z_0, \rho)$ can be replaced by $C(z_0, R)$ in (10.4). Then a direct estimate of the last term in (10.5) gives

$$|E(z, z_0, n)| \le \frac{|z - z_0|^{n+1}}{R^n} \frac{MR}{R - |z - z_0|} \tag{10.6}$$

where M is the maximum of f on the circle $\partial D(z_0, R)$.

Theorem 10.17 (Taylor series; Cauchy's formula for higher derivatives)**.** *If $f(z)$ is continuously differentiable in $\mathcal{D}$ and $z_0 \in \mathcal{D}$ then there exists ρ such that, for $z \in D(z_0; \rho)$ we have*
(10.7)

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k; \quad c_k = \frac{1}{2\pi i} \oint_{C(z_0;\rho)} \frac{f(s)}{(s - z)^{k+1}} ds = \frac{f^{(k)}(z_0)}{k!}$$

We have already proved everything in this theorem except the last equality, which follows from Theorem 5.10.□

Remark 10.18. We know from Abel's theorem that the region of convergence of a series is a disk. The disk may be degenerate: in one extreme situation it is a point, $z = z_0$ in the other, the whole complex domain. So the Taylor series of an analytic function converges in a disk too. By the analyticity assumption, the disk cannot be a point. We claim that the radius of convergence of the series exactly equals the radius of the largest disk centered at z_0 where f is analytic ("the distance to the nearest singularity"). (This disk can be the whole of $\mathbb{C}$. Then "the radius of convergence is infinite".)

Indeed, in any smaller disk we can apply Theorem 10.17 above. If the radius of convergence were larger, f would be analytic in a larger domain since convergent power series are, we have seen it, analytic.

Corollary 10.19. *An analytic function is automatically infinitely often differentiable. If f is analytic, so are f', f'', etc.*

From now on we will call connected open sets in $\mathbb{C}$ *regions.*

11. More properties of analytic functions

Assume f is analytic in $D(z_0, \epsilon)$ and all derivatives of f are zero at z_0. Then f is zero in the whole of $D(z_0, \epsilon)$ (why?). More is true.

Proposition 11.20. *Assume f is analytic in a* region *$\mathcal{D}$ and all derivatives at $z_0 \in \mathcal{D}$ of f are zero. Then f is identically zero in $\mathcal{D}$.*

Proof. We use the fact from elementary topology of $\mathbb{R}^2$ that if $\mathcal{D}$ is connected then any two points z_i, z_f in $\mathcal{D}$ can be joined by a polygonal line P wholly contained in $\mathcal{D}$. Since $\partial\mathcal{D} = (\mathbb{C} \setminus \mathcal{D}) \cap \overline{\mathcal{D}}$, then $\partial\mathcal{D}$ is closed. Then $\text{dist}(P, \partial\mathcal{D}) := a > 0$ (why?). Therefore P is contained in the union of the disks $P \subset \cup_{z \in P} D(z, a) \subset \mathcal{D}$. We can then choose a finite subset $\{z_i = z_1, ..., z_n = z_f\} \in P$ (where z_i are considered as ordered successively on P) such that $P \subset \cup_{z_1,...,z_n} D(z_i, a) \subset \mathcal{D}$ (why?). The set $\{z : f(z) = 0\}$ is closed since f is continuous. Then either f is identically zero or else there is a smallest j so that $f(z_j) \neq 0$. Now, $f \equiv 0$ in $D(z_{j-1}, a)$ and $D(z_{j-1}, a) \cap D(z_j, a) = S \neq \emptyset$ for the covering of P to be possible. By elementary geometry, we can find a finite set of disks of radius $\epsilon < a$, the first one contained in S and the center of every one of them contained in the previous disk, the last one centered at z_j. By local Taylor expansions of f in these disks we get a contradiction.

Exercise 11.21. ** Justify the statement in the last sentence in the preceding proof.*

Exercise 11.22. *** Permanence of relations. Use Proposition 11.20 to show that* $\sin^2 z + \cos^2 z = 1$ *in $\mathbb{C}$. Relations between analytic functions that hold in $\mathbb{R}$ extend in $\mathbb{C}$. Formulate and prove a theorem to this effect.*

Theorem 11.23 (Morera)**.** *Let f be continuous in a simply connected region $\mathcal{D}$ and such that $\oint_\gamma f ds = 0$ for any simple closed curve γ contained in $\mathcal{D}$ (or, as it can be similarly shown, for any triangle γ in $\mathcal{D}$). Then f is analytic in $\mathcal{D}$.*

Proof. Let $z_0 \in \mathcal{D}$ and let $F(z) = \int_{z_0}^{z} f(s) ds$. Then F is well defined and differentiable in $\mathcal{D}$ (why?). Thus F is analytic and so is $F' = f$.

We have now three equivalent views of analytic functions: as differentiable functions of z, as sums of power series, and as continuous functions with zero loop integrals. All these points of view are quite valuable.

Corollary 11.24. *Complex derivatives are expressed as* **integrals***. This makes differentiation a "smooth" operation on analytic functions, unlike the situation in real analysis.*

Corollary 11.25 (Liouville's theorem). *A function which is* **entire** *(meaning analytic in $\mathbb{C}$) and bounded in $\mathbb{C}$ is constant.*

Proof. Let M be the maximum of $|f|$ in $\mathbb{C}$. We have, by Theorem 10.17

$$f'(z) = \frac{1}{2\pi i}\oint_{C(0;\rho)} \frac{f(s)}{(s-z)^2}ds \tag{11.2}$$

and thus

$$|f'(z)| \le \frac{1}{2\pi} M \frac{1}{\rho^2} 2\pi\rho = M/\rho \tag{11.3}$$

Since this is true for any ρ, no matter how large, it follows that $f'(z) \equiv 0$. Then f is a constant.

Exercise 11.26. ** Show that an entire function other than a polynomial must grow faster than any power of $|z|$ along some path as $z \to \infty$.*

12. The fundamental theorem of algebra

One classical application of this theorem is the fundamental theorem of algebra: *a polynomial $P_n(z)$ of degree n has exactly n roots in $\mathbb{C}$, counting multiplicity.* As is known from elementary algebra, it is enough to show that any nonconstant polynomial has at least one root in $\mathbb{C}$. But this is clear, since otherwise $1/P_n(z)$ would be a nonconstant bounded entire function (why?).

13. Harmonic functions

A C^2 function u which satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{13.2}$$

in some region $\mathcal{D}$ is called harmonic in $\mathcal{D}$.

Theorem 13.27. *Let $\mathcal{D}$ be a simply connected region in $\mathbb{C}$. A function is harmonic in $\mathcal{D}$ if and only if $u = Re(f)$ in $\mathcal{D}$ with f analytic in $\mathcal{D}$; f is unique up to an arbitrary imaginary constant.*

Proof If $u = \mathrm{Re}(f)$ then $u \in C^\infty$, by Corollary 10.19. Then (45.4) follows immediately from the CR equations. In the opposite direction, consider the field $\mathbf{E} = (-u_y, u_x)$. We check immediately that this is a potential field and thus $\mathbf{E} = \nabla v$ for some v (unique up to an arbitrary constant). But then, by the CR theorem, $u + iv$ is analytic in $\mathcal{D}$. $\square$

14. The maximum modulus principle

Theorem 14.28. *Assume f is analytic in the region $\mathcal{D}$. Then $|f|$ has no interior maximum strictly inside $\mathcal{D}$, unless f is a constant.*

Exercise 14.29. *Show that $|f|$ can have a minimum strictly inside $\mathcal{D}$, if and only if this minimum is zero.*

Usually the proofs use Cauchy's formula. Look up these proofs, because they extend to harmonic functions in more than two dimensions.

I will give a slightly shorter proof based on Taylor series.

Proof. Assume that z_0 is an interior point of maximum. The result is easy if $M = 0$ (why?). Taking $f \mapsto f/M$ and $z \mapsto z - z_0$, without loss of generality, we can assume that $M = 1$ and $z_0 = 0$. If f is not 1 everywhere, then there exists $k > 0$ so that the Taylor coefficient c_k of f at 0 is nonzero, in $D(\rho, 0)$ we have

$$f(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \cdots = 1 + c_k z^k (1 + d_k z + d_{k+1} z^2 + \cdots)$$

By taking z_1 small enough and such that $c_k z_1^k \in \mathbb{R}^+$ (convince yourself that this is possible) we get $|f(z_1)| > |f(z_0)|$ (why?). A direction at z_0 such that $c_k(z - z_0)^k \in \mathbb{R}^+$ is a **steepest ascent** direction of the analytic function. We see that these exist at any point. A line that follows at each point a steepest ascent direction is a **steepest ascent line**.

Exercise 14.30. * *Find the maximum and minimum value of $|\sin z|$ inside the closed unit disk.*

Theorem 14.31. *Assume u is harmonic in the region $\mathcal{D}$. Then u achieves its* maximum and minimum *on the boundary of $\mathcal{D}$.*

Proof. Let $u = \mathrm{Re}(f)$ and define $g = e^f$. We saw that g is analytic in $\mathcal{D}$. By the properties of the exponential that we have shown already, we have $e^f = e^u e^{iv}$; $|e^f| = e^u$ and then u has a maximum if and only if $|g|$ has a maximum. But this cannot happen strictly inside $\mathcal{D}$. For the minimum, note that $\min(u) = -\max(-u)$!

14.1. **Application.** The soap film picked up by a thin closed wire has the minimum possible area compatible with the constraint that it is bordered by the wire, since the potential energy is proportional to the surface area. It then follows easily that the shape function u satisfies Laplace's equation. This is shown in many books (e.g. in Fisher's book on complex analysis). It follows from Theorem 14.31 that this minimal surface is flat if the wire is flat. This is probably not a surprise. We will however be able to solve Laplace's equation with any boundary constraint, and this will provide us with a lot of insight on these minimal surfaces.

14.2. **Principal value integrals.** Suppose that f is analytic in a region containing the simple closed curve C. By Cauchy's theorem we have

$$\frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}ds = \begin{cases} f(z) \text{ if } z \text{ is inside } C \\ 0 \text{ otherwise} \end{cases} \tag{14.2}$$

What if z lies *on* C? Then of course the integral is not defined as it stands. A number of reasonable definitions can be given though, and they agree as far as they apply. In one such definition a symmetric segment of the curve centered at z of length ϵ is cut and then ϵ is taken to zero. Another definition is to take the half sum of the integral on a curve circumventing z from the outside and of the integral on a curve circumventing z from the inside. This latter procedure gives the "Cauchy principal part integral" denoted $P\oint$ (and in many other ways).

Exercise 14.32. *Show that if C is smooth, then*

$$\frac{1}{2\pi i}P\oint_C \frac{f(s)}{s-z}ds = \frac{1}{2}f(z) \tag{14.3}$$

and it coincides with the symmetric cutoff value defined above. See also S. Tanveer's notes for other cases.

15. Linear fractional transformations: a first look

Exercise 15.33. *** Let $a \in (0,1)$, $\theta \in \mathbb{R}$. Show that*

$$z \mapsto T(z) := e^{i\theta}\frac{a+z}{1+az} \tag{15.2}$$

is a one-to-one transformation of the closed unit disk onto itself.

16. Poisson's formula

Proposition 16.34. *Assume u is harmonic in the open unit disk and continuous in the closed unit disk. Then*

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})dt \tag{16.2}$$

Proof. If v is the harmonic conjugate of u then $u+iv$ is analytic in the open unit disk, and we have by Cauchy's formula for any $\rho<1$,

$$(16.3)\quad u(0)+iv(0)=f(0)=\frac{1}{2\pi i}\oint_{C(0;\rho)}\frac{f(s)}{s}ds=\frac{1}{2\pi}\int_0^{2\pi}f(\rho e^{it})dt$$
$$=\frac{1}{2\pi}\int_0^{2\pi}u(\rho e^{it})dt+i\frac{1}{2\pi}\int_0^{2\pi}v(\rho e^{it})dt$$

We get (16.3) by taking the real part of (16.3) and passing to the limit $\rho\to 1$.

Exercise 16.35. ** (i) Let u be the function defined in Proposition 16.34. Use Exercise 15.33 to show that*

$$(16.4)\qquad U(z)=u(T(z))$$

is harmonic in the open unit disk and continuous in the closed unit disk.

(ii) Show that, if $z_0=ae^{i\theta}$ we have

$$(16.5)\qquad u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}u\left(e^{i\theta}\frac{a+e^{is}}{1+ae^{is}}\right)ds$$

Proposition 16.36 (Poisson's formula)**.** *Let u be as in Proposition 16.34 and $z_0=ae^{i\theta}$ with $a<1$. We have*

$$(16.6)\qquad u(ae^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-a^2}{1-2a\cos(t-\theta)+a^2}u(e^{it})dt$$

Exercise 16.37. *** Prove this formula by making the change of variable*

$$(16.7)\qquad e^{i\theta}\frac{a+e^{is}}{1+ae^{is}}=e^{it}$$

in (16.5).

Note. Formula (16.6) is important: it gives the solution of Laplace's equation in two dimensions with Dirichlet boundary conditions, when the domain is the closed unit disk D. A simple change of variables adapts this formula to any disk. More generally, we will see that the formula can be adjusted to accommodate for the general case of the region lying in the interior of any simple, closed curve. This is a consequence of *Riemann's mapping theorem.*

Exercise 16.38. ** Formulate and solve the equation for the shape of a soap film when bounding wire is described by $t\mapsto(\sin t,\cos t,C+\sin(t))$, $t\in[0,2\pi]$.*

16.1. **The Neumann problem.** This is another important problem associated with Laplace's equation, one in which the normal derivative of the harmonic function $v \in C^1(D)$ is specified on the boundary of the unit disk.

We can address this problem similarly. Since v is harmonic in the unit disk, it has a harmonic conjugate $u \in C^1(D)$, and we have on S_1 $v_n = u_s$ where u_s is the tangential derivative (why?). Since u is well defined we see that

$$\oint u_s ds = 0 = \oint v_n ds \tag{16.8}$$

which is a necessary condition for the problem to have a solution. Given that, u_s determines u up to an additive constant on the boundary, and using Poisson's formula (16.6) we get u in D, therefore v.

17. Isolated singularities

By definition f has an isolated singularity at z_0 if f is analytic in a disk $D(z_0, \epsilon) \setminus \{z_0\}$ for some $\epsilon > 0$. Note that we allow for the possibility that z_0 is a point of analyticity of f, or, to be precise, that there exists an extension of f analytic in $D(z_0, \epsilon)$.

18. Laurent series

Proposition 18.39. *Assume that f is analytic in $D(0,1) \setminus \{0\}$. Then, in $D(0,1) \setminus \{0\}$ we have the convergent representation*

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k \tag{18.2}$$

where

$$a_k = \frac{1}{2\pi i} \oint f(s) s^{-k-1} ds, \qquad k \in \mathbb{Z}$$

Note. The number a_{-1} is called the residue of f at z_0, $\mathrm{Res}(f; z_0)$.

Proof. Take z in the annulus A between the circles in Fig. 2. below. Make a cut in the annulus as shown. The remaining region is simply connected and Cauchy's formula applies there:

$$f(z) = \frac{1}{2\pi i} \oint_{C_o} \frac{f(s)}{s - z} ds - \frac{1}{2\pi i} \oint_{C_i} \frac{f(s)}{s - z} ds \tag{18.3}$$

where the integrals are taken in an anticlockwise direction and C_o, C_i denote the outer and inner circles of the annulus respectively.

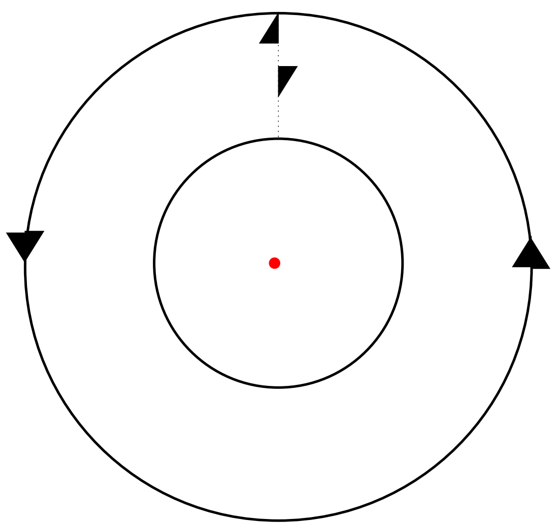


Fig 2.

Exercise 18.40. *Complete the proof of formula (18.2) by expanding the integrands in (18.3) in powers of z/s and s/z respectively, and estimating the remainders as we did for obtaining formula (10.2).*

Note. Convince your selves that (18.3) gives a decomposition of f into a part f_1 analytic in the disk D_o and a function f_2 analytic in $1/z$ in a disk of radius $1/r_i$. Ahlfors takes this decomposition for a nice proof of (18.2); look at the proof.

Definitions. A singularity z_0 of f is a pole of order M if $a_k = 0$ for all $k < -M$, it is a removable singularity if it is "a pole of order 0" (in this case, f extends to a function $\tilde{f}$ analytic in the whole disk and given by the Taylor series of f at z_0) and an essential singularity otherwise. By slight abuse of notation we often don't distinguish $\tilde{f}$ from f itself.

For example $e^{1/z}$ has an essential singularity at $z = 0$. Application of (18.2) yields (check!)

$$e^{1/z} = \sum_{k=0}^{\infty} z^{-k}/k! \tag{18.4}$$

Note. The part of the Laurent series containing the terms with negative k is called the principal part of the series.
Note. Laurent series are of important theoretical value. However, calculating effectively a function near the singularity from its Laurent series is another matter and it is usually not very practical to use Laurent series for this purpose. A Laurent series is *antiasymptotic*: its convergence gets slower as the singularity is approached.

19. Calculating Taylor series of simple functions

One easy way to calculate Taylor series is to use §7 judiciously.
Example. (1) The Taylor series of the function $z^{-1}\sin z$ is

$$\frac{\sin z}{z} = 1 - z^2/6 + z^4/120 + \cdots \tag{19.2}$$

(2) The Taylor series of the function $z/\sin z$ is

(19.3)
$$\frac{z}{\sin z} = \frac{1}{1 - (z^2/6 - z^4/120 + \cdots)} = 1 + (z^2/6 - z^4/120 + \cdots) +$$
$$(z^2/6 - z^4/120 + \cdots)^2 + \cdots = 1 + z^2/6 - z^4/120 + z^4/36 + \cdots$$
$$= 1 + z^2/6 + 7z^4/360 + \cdots$$

The first functions define is entire, the second one is not. What is the radius of convergence of the second series?

Exercise 19.41. * *Find the integral of* $1/\cos z$ *on a circle of radius* $1/2$ *centered at* $z_0 = \pi/2$.

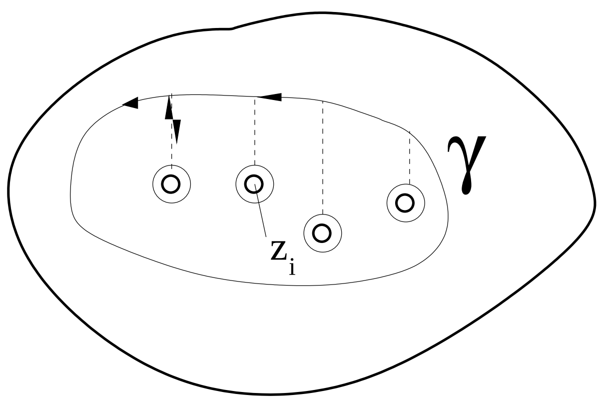


Fig. 3

20. Residues and integrals

Proposition 20.42. *Let $\mathcal{D}$ be a simply connected region. Consider a function f which is analytic in the region $\mathcal{D}\setminus\{z_1,\dots,z_n\}$ and consider a simple closed curve γ which encircles each singularity once, see Figure 3. We have*

$$\oint_\gamma f(s)ds = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f)_{z=z_i} \tag{20.2}$$

Exercise 20.43. *Complete the proof of this proposition.*

Example. Calculate

$$\oint \frac{dz}{\sin^3 z}$$

on a circle of radius $1/2$ around the origin.
Solution We have, in $D(1/2, 0)$,

(20.3)

$$\frac{1}{\sin^3 z} = \frac{1}{(z - z^3/6 + z^5/120 \cdots)^3} = \frac{1}{z^3}\frac{1}{(1 - z^2/6 + z^4/120 \cdots)^3} = \frac{1}{z^3}\left(1 + z^2/2 + 17z^4/120\right) + \cdots)$$

and thus the residue of $\sin^{-3}(z)$ at $z = 0$ is $1/2$ and the integral equals πi.

Exercise 20.44. *Show that if f has a pole of order m at $z = z_i$ then*

$$\mathrm{Res} f_{z=z_i} = \frac{[(z - z_i)^m f(z)]^{(m-1)}_{z=z_i}}{(m-1)!} \tag{20.4}$$

by applying Laurent's formula near $z = z_i$.

21. Integrals of trigonometric functions

Contour integration is very useful in calculating or estimating Fourier coefficients of periodic functions. Consider the integral

$$I = \int_0^{2\pi} \frac{e^{int}}{2 + \cos t} dt \tag{21.2}$$

Let $z = e^{it}$. Then

$$I = -i \oint_C \frac{z^{n-1}}{2 + (z + 1/z)/2} dz = -2i \oint_C \frac{z^n}{z^2 + 4z + 1} dz \tag{21.3}$$

where C is the unit circle. The roots of $z^2 + 4z + 1$ are $-2 \pm \sqrt{3}$ and only one, $z_0 = -2 + \sqrt{3}$ lies in the unit disk. Thus,

$$I = -2i \cdot 2\pi i \frac{z_0^n}{2z_0 + 4} \tag{21.4}$$

22. Counting zeros and poles

Notations and definitions (1) Assume f is analytic in a disk $D(z_0,\epsilon)$ and $f(z_0)=0$. Then, in $D(z_0,\epsilon)$ we have

$$f(z)=\sum_{k=1}^{\infty}c_k(z-z_0)^k \tag{22.2}$$

If f is not identically zero in $\mathcal{D}$ then there exists some k_0 such that $c_{k_0}\neq 0$ (see Proposition 11.20). The smallest such k_0 is called **the order or multiplicity** of the zero z_0.

(2) The function f is **meromorphic** in $\mathcal{D}$ if it only has isolated singularities in $\mathcal{D}$ none of which is an essential singularity. The order of a pole at z_0 is the multiplicity of $1/f$ at z_0 (why?).

Exercise 22.45. ****(The zeros of an analytic function are isolated)** *Assume $f\not\equiv 0$ is analytic near z_0 and $f(z_0)=0$. Use Taylor series to show that there is some disk around z_0 where $f(z)=0\Rightarrow z=z_0$.*

Assume f is meromorphic in $\mathcal{D}$; let γ be a simple closed curve contained in $\mathcal{D}$ together with its interior Γ. Note that by assumption the region of analyticity of f strictly exceeds Γ. For the purpose of the next proposition, the assumptions can be relaxed, allowing γ to be the boundary of the analyticity domain of f if we impose continuity conditions on f and f'. Check this.

Proposition 22.46. *Let N be the total number of zeros of f in Γ counting multiplicities and let P be the number of poles, each pole being counted p times if it has order p. Then*

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f'(s)}{f(s)}ds=N-P \tag{22.3}$$

Proof. The function

$$\frac{f'(s)}{f(s)}$$

is also meromorphic (why?). It has a pole of order 1 and residue n_i at a zero of order n_i of f and a pole of order 1 and residue $-p_i$ at a pole of order p_i of f (Explain!). The rest follows from (20.2).

Proposition 22.47 (Rouché's theorem)**.** *Assume f and h are analytic in the interior Γ of the simple closed curve γ, continuous in the closure $\overline{\Gamma}$ and that on γ we have $|h|<|f|$. Then the number of zeros of f and $f+h$ in Γ is the same (we can think of $f+h$ as a "small" perturbation of f).*

Proof. Note that all the assumptions hold in a small neighborhood of γ too. Note also that f can have no zeros on γ: the condition $|h| < |f|$ prevents them. We have

$$f + h = f(1 + h/f) = fQ \Rightarrow \frac{f' + h'}{f + h} = \frac{f'}{f} + \frac{Q'}{Q} \tag{22.4}$$

Since we have $|h/f| < 1$ the series $q = \sum_{k=1}^{\infty} k^{-1}(-1)^{k+1}(h/f)^k$ gives an analytic function in a region around γ and $q' = Q'/Q$. But then, evidently, $\oint q' = 0$ and the proposition follows. Check the details of this calculation.

Exercise 22.48. * *Reformulate and prove the proposition when f and h, f' and h' are continuous up to the boundary γ but not necessarily beyond.*

23. Inverse function theorem

Proposition 23.49. *Assume f is analytic at z_0 and $f'(z_0) = a \neq 0$. Then there exists a disk $D(z_0, \epsilon)$ such that f is invertible from $D(z_0, \epsilon)$ to $\mathcal{F} = \phi(D(z_0, \epsilon))$ and the inverse is analytic*

Without loss of generality, we may assume that $z_0 = 0$ and $f(z_0) = 0$ (why?). We have $f(z) = az + z^2 g(z)$ where $g(z) \to const$ as $z \to 0$. We take $M > |const|$ and take $\epsilon < 5^{-1}|a|M^{-1}$ small enough so that $|g| < M$ for $|z| < 3\epsilon$. Let $z_1 \in D(0, \epsilon)$ and $f(z_1) = w$. We show that $f(z) = w$ and $z \in D(0, \epsilon)$ implies $z = z_1$ which means that f is one-to-one from the disk $D(0, \epsilon)$ to $\mathcal{F} = f(D(0, \epsilon))$. We have $f(z) - w = az + z^2 g(z) - az_1 - z_1^2 g(z_1) = a(z - z_1) + z^2 g(z) - z_1^2 g(z_1)$. We apply Rouché's theorem in $f(D(0, 3\epsilon))$. Since $|z_1| < \epsilon$ and $|z| = 3\epsilon$ we have $|a(z - z_1)| > 2|a|\epsilon$. On the other hand, $|z^2 g(z) - z_1^2 g(z_1)| < 9\epsilon^2 M + \epsilon^2 M$. By direct calculation we see that Rouché's theorem applies, with $f = a(z - z_1)$ and $h = z^2 g(z) - z_1^2 g(z_1)$, if $10\epsilon^2 M < 2|a|\epsilon$ which holds by construction. But the equation $f = a(z - z_1) = 0$ has only one root in $D(0, 3\epsilon)$ and thus so does $f(z) - f(z_1) = 0$. (Note that z_1 is in the smaller disk $D(0, 3\epsilon)$.) Differentiability follows as in usual analysis.

24. Analytic continuation

Assume that f is analytic in $\mathcal{D}$ and f_1 is analytic in $\mathcal{D}_1$, $\mathcal{D}_1 \supset \mathcal{D}$ and $f = f_1$ in $\mathcal{D}$. Then f_1 is an *analytic extension* of f. We also say that f_1 has been obtained from f by *analytic continuation*.

The point of view favored by Weierstrass was to regard analytic functions as properly defined chains of Taylor series, each the analytic continuation of the adjacent ones. Riemann's point of view was more geometric.

If f is analytic at z_0, then there exists a disk of radius ϵ centered at z_0 such that f is the sum of this series; we take ϵ_0 to be the largest ϵ with this property. If we take a point z_1 inside this disk, f is analytic at z_1 too, and thus near z_1 it is given by a series centered at z_1. The disk of convergence of this series is, as we know, at least equal to the distance $d(z_1, \partial D(z_0, \epsilon))$, but in general it could be larger. (Convince yourselves that this is the case with the function $1/(1 + z)$ if we take a disk centered at $z = 0$ and then a disk centered at $z = 1/2$.) In the latter case, we have found a function f_1, piecewise given by the two Taylor series, which is analytic in the union $D(z_0, \epsilon) \cup D(z_1, \epsilon_1)$.
Uniqueness. If there is an analytic continuation in $D(\epsilon, z_0) \cup D(z_1, \epsilon_1)$, then it is unique (use Proposition 11.20) to show this.

In fact, we can continue this process and define chains $z_0, z_1, ...$ such that f is analytic in $D(z_i, \epsilon_i)$. These are elements of a "global analytic function". This "global analytic function" is not necessarily a function, since chains redefine the function upon every self-intersection, and the definitions may not agree.

It is useful to experiment with this procedure on $\log(1+z)$ which we have defined as an analytic function for $|z| < 1$. What happens if we take a chain around $z = -1$?

One can also find that there is a region in $\mathbb{C}$ where the function is well defined by this procedure, but no Taylor disk crosses the boundary. Then we have found a maximal region of analyticity, the boundary of which is called "natural boundary" or "singularity barrier".

Exercise 24.50. *** Consider the rational numbers $r = p/q$ and associate uniquely a positive integer, for instance $N_{pq} = 7^p 5^q$ (why is this unique?) Take the function*

$$f(z) = \sum_{N_{pq}}^{\infty} \frac{2^{-N_{pq}}}{z - p/q} \tag{24.2}$$

Show that the series converges for $z \in \mathbb{C} \setminus \mathbb{R}$ and that $\mathbb{R}$ is a singularity barrier of f.

Explain how this example can be modified to obtain an analytic function f in any region bounded by a simple closed curve γ and such that γ is a singularity barrier of f.

25. The Schwarz reflection principle

Assume f is analytic in the domains $\mathcal{D}_1, \mathcal{D}_2$ which have a common piece of boundary, a smooth piece of curve γ. Assume further that f is continuous across γ. Then, by Morera's theorem, f is analytic in $\mathcal{D}_1 \cup \mathcal{D}_2$. This allows us to do analytic continuation, in some cases.

Abbreviations We denote the upper half plane $\{z : \Im z > 0\}$ by UHP and the unit disk $\{z : |z| < 1\}$ by D.

Proposition 25.51 (The Schwarz reflection principle)**.** *Assume f is analytic in a domain $\mathcal{D}$ in UHP whose boundary contains an interval $I \subset \mathbb{R}$ and assume f is continuous on $\mathcal{D} \cup I$ and real valued on I. Then f has analytic continuation across I, in a domain $\mathcal{D} \cup \mathcal{D}^*$ where $\mathcal{D}^* = \{\overline{z} : z \in \mathcal{D}\}$.*

Proof. Consider the function $F(z)$ equal to f in $\mathcal{D} \cup I$ and equal to $\overline{f(\overline{z})}$ in $\mathcal{D}^* \cup I$. This function is clearly continuous in $\mathcal{D} \cup I \cup \mathcal{D}^*$. It is also analytic in $\mathcal{D}^*$ as it can be immediately checked using Taylor series (check it!). The proposition is proved.

Note. When we learn more about conformal mappings, we shall see that much more generally, a function admits a continuation across a curve γ if the curve is an analytic arc (we will define this precisely) and $f(\gamma)$ is an analytic arc as well.

26. Multi-valued functions

As we discussed, as a result of analytic continuation in the complex plane we may get a *global analytic function* which is not necessarily a function on $\mathbb{C}$ since the definition is path–dependent; the function is thus defined on a space of paths or curves, modulo homotopies.

As long as the domain of continuation is simply connected, we still get a function in the usual sense:

Exercise 26.52. *** Assume that f is analytic in $D(z_0, \epsilon)$ and that we have a family of homotopic curves in $\mathbb{C}$, starting at z_0 and ending at z_1 along which f can be analytically continued.*

That is, say there is a smooth map $\gamma : [0,1]^2 \mapsto \mathbb{C}$ such that $\gamma(s, 0) = z_0\ \forall s \in [0,1]$ and $\gamma(s, 1) = z_1 \forall s \in [0,1]$ and furthermore f admits analytic continuation from z_0 to z_1 along $t \mapsto \gamma(s,t), t \in [0,1]$ for any $s \in [0,1]$.

Assume that $\mathcal{D} = \gamma((0,1)^2)$ is simply connected. Show that there is an analytic function F in $\mathcal{D}$ which coincides with f in $D(z_0, \epsilon)$. As we know, this continuation is then unique.

The simplest example is perhaps the logarithm. In real analysis $\ln x = \int_1^x s^{-1} ds$. Clearly the function z^{-1} is analytic in $\mathbb{C} \setminus \{0\}$ and we can define

$$\ln^{[C]} z = \int_{1;C}^{z} s^{-1} ds \tag{26.2}$$

where integration starts at 1 and is performed along the curve $\mathbb{C}$. This integral only depends on the homotopy class of the curve C in $\mathbb{C} \setminus \{0\}$.

Let C be a smooth curve starting at 1 which does not contain 0. Drop for now the superscript: $\ln^{[C]} z = \ln z$ (this is customary, but we keep in mind that dependence on C persists). The function $e^{\ln z}$ is well defined along the curve C and analytic in a neighborhood of any point in C. We find

$$\left(\frac{e^{\ln z}}{z}\right)' = 0 \tag{26.3}$$

and thus

$$e^{\ln z} = ze^{\ln 1} = z \tag{26.4}$$

The log thus defined is the inverse of the exponential. Therefore, if we write $z = \rho e^{i\phi}$ then by Exercise 6.11

(26.5)

$$\rho e^{i\phi} = e^{\ln \rho + i\phi} = e^{\ln z} \quad \Leftrightarrow \ln z = \ln \rho + i\phi + 2N\pi i \quad (\text{for some } N \in \mathbb{Z})$$

for any choice of the curve of integration.

Thus, we have proved

Proposition 26.53. *For any curves C_1 and C_2 we have there is an $N \in \mathbb{Z}$ so that $\ln^{[C_1]} z - \ln^{[C_2]} z = 2N\pi i$.*

This is the log "function" as a *global analytic function.* It has a *branch point* at $z = 0$ and it is *multivalued.* We sometimes say it is defined in $\mathbb{C}$ up to an additive integer multiple of $2\pi i$. If we choose a value of N, then we have chosen a "branch" of the log. But what does this choice mean?

27. Branches of the log. The natural branch of the log

Alternatively, as in Exercise 26, we can take a simply connected domain in $\mathbb{C}$ and define a *function* ln, relative to that domain.

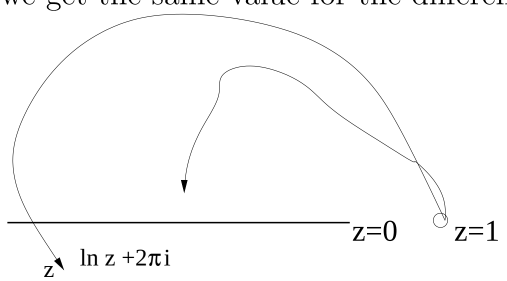
Let R be a ray in $\mathbb{C}$, that is, for some $z_0 \neq 0$, $R = \{\lambda z_0 : \lambda \geq 0\}$. Note that we can choose w.l.o.g. $|z_0| = 1$; only the argument of z_0 matters. Then the region $\mathcal{D} = \mathbb{C} \setminus R$ is simply connected and the function s^{-1} is analytic in $\mathcal{D}$. Assume that $1 \in \mathcal{D}$. Then, the function $\ln z = \int_1^z s^{-1} ds$ (we omit, as it is customary, to stress the dependence on R) **a branch of the log** is well defined and analytic in $\mathcal{D}$. Evidently, by the same argument as before, we have $e^{\ln z} = z$. The definition depends on the choice of R but by Proposition 26.53 it only has an integer parameter free.

The price that we pay is that ln is no longer defined along R even though R *is not special.* What we mean by this is that if we take any point near R we find an analytic continuation of the log *through* R. No singularity is present, except at zero. The line R is not a singular line. The problem is that the analytic continuation of log through R is different, by $\pm 2\pi i$ from the definition of the log that we already had. This multivaluedness shows you that $\mathbb{C} \setminus R$ is a maximal domain of analyticity of the chosen branch.

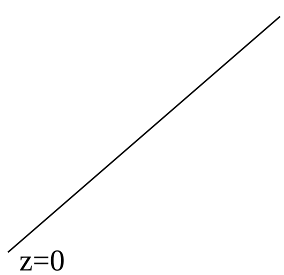
The natural branch of the log. If $R = \mathbb{R}^- \cup \{0\}$ then we have the natural branch of the log. It is natural since in real analysis ln is defined on $(0, \infty)$ and not on the negative line.

Note. *We could take any simply connected domain in $\mathbb{C}$ not containing zero and we would get another branch of the log. In particular, instead of a ray, choose any curve without self-intersections starting at zero and ending at ∞ in $\mathbb{C}$. In this sense, there are uncountably many branches of the log, as many as domains. Yet, Proposition 26.53 shows that the values at any point in $\mathbb{C}$ differ only by an integer multiple of $2\pi i$.*

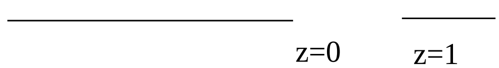
Look at the figures below, and try to understand for which values of z we get the same value for the different branches of $\ln z$ defined here.



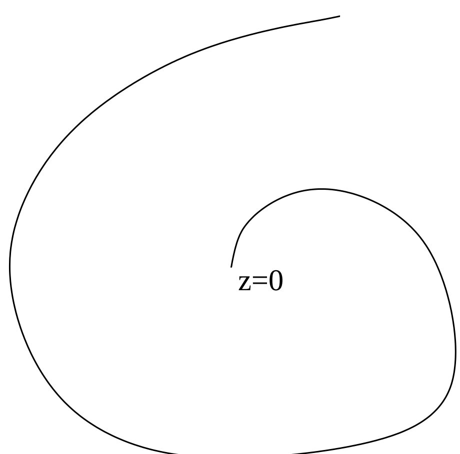
The natural branch of the log.



A branch at a slanted angle.



Another branch of the log
is defined in this cut regio



Yet another branch is in the plane except for the
line drawn.

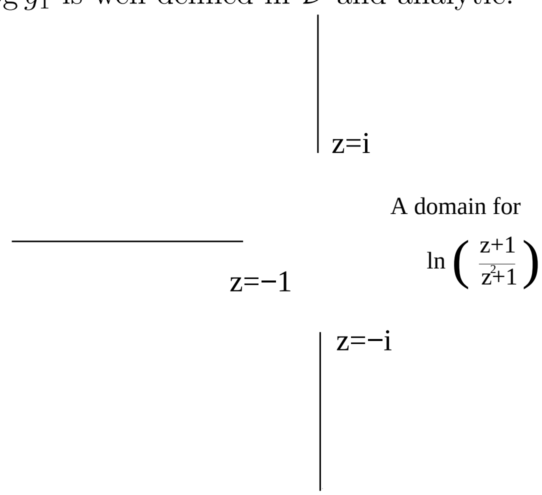
27.1. **Generalization: log of a function.** If g is a function defined in a region in $\mathbb{C}$ we can define $\ln g$ by

$$\ln g = \int_a^z \frac{g'(s)}{g(s)} ds \tag{27.2}$$

Now, depending on the properties of g, the homotopy classes will be in general more complicated.

For instance, if $g = g_1$ is a rational function, all the zeros and poles $S = \{z_i, p_j\}$ of g_1 are points where the integral, thus the log, is not defined. We are now dealing with homotopy classes in $\mathbb{C} \setminus S$.

It is convenient to define a branch of $\ln g_1$ by cutting the plane along rays originating at the points in S. Convince yourselves that this can be done such that the remaining region $\mathcal{D}$ is simply connected. Then $\log g_1$ is well defined in $\mathcal{D}$ and analytic.



27.2. **General powers of z.** Once we have defined the log, it is natural to take

$$z^\alpha = e^{\alpha \ln z} \tag{27.3}$$

Since $\ln z$ is defined along a curve, modulo homotopies in $\mathbb{C} \setminus \{0\}$, so is z^α. For a general $\alpha \in \mathbb{C}$, the multivaluedness of z^α is the same as that of the log. Note however that if $p \in \mathbb{Z}$ then the value does not depend on the homotopy class and the definition (27.3) defines a function in $\mathbb{C} \setminus \{0\}$, with a pole at zero if $p < 0$ and a removable singularity if $p \geq 0$. Convince yourselves that this definition coincides with the usual power, defined algebraically.

Another special case is that when $\alpha = p/q$, p, q relatively prime integers. Since $(z^\alpha)^q = z^p$, there are only q possible values of z^α, and these, again, are the same as those defined algebraically, by solving the equation $x^q = z$ and then taking x^p.

Note.

$$e^{\ln z_1 + \ln z_2} = e^{\ln z_1} e^{\ln z_2} = z_1 z_2 \tag{27.4}$$

However, this does not mean $\ln z_1 + \ln z_2 = \ln z_1 z_2$, but just that

$$\ln z_1 + \ln z_2 = \ln z_1 z_2 + 2N\pi i \tag{27.5}$$

For the same reason, $z^{\alpha_1} z^{\alpha_2}$ is not necessarily $z^{\alpha_1+\alpha_2}$. Beware of possible pitfalls. Note the fallacious calculation

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \quad (?!) \tag{27.6}$$

28. Evaluation of definite integrals

Contour integrals, and because of this, many definite integrals for which the endpoints are at infinity, or at special singular points of functions can be evaluated using the residue theorem. We have the following simple consequence of this theorem.

Proposition 28.54. *Let R be a rational function, continuous on $\mathbb{R}$ and such that $R(z) = O(z^{-2})$ as $z \to \infty$. Then*

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{z_i \in UHP} Res(R; z = z_i) \tag{28.2}$$

where z_i are poles of R.

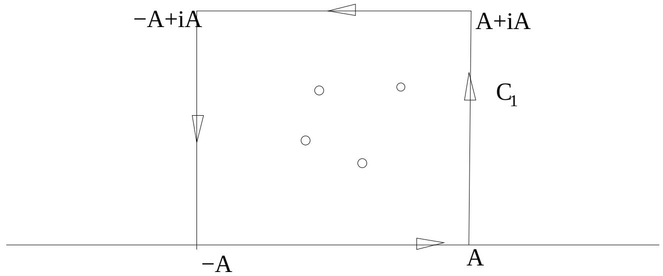
Exercise 28.55. *The upper half plane is evidently not special; formulate and prove a similar result for the LHP.*

Proof. Under the given assumptions, we take as a contour the square in the figure below and write

$$\begin{aligned}(28.3)\quad \int_{-\infty}^{\infty} R(x)dx &= \lim_{A\to\infty} \int_{-A}^{A} R(x)dx \\ &= \lim_{A\to\infty} \oint_{[-A,A]\cup C_1} R(z)dz - \lim_{A\to\infty} \int_{C_1} R(z)dz \\ = 2\pi i \sum_{z_i \in UHP} \mathrm{Res}(R; z = z_i) &- \lim_{A\to\infty} \int_{C_1} R(z)dz = 2\pi i \sum_{z_i \in UHP} \mathrm{Res}(R; z = z_i)\end{aligned}$$

since

$$\left| \int_{C_1} R(z)dz \right| \le const A^{-2}(3A) = 3A^{-1} \to 0 \text{ as } A \to \infty \tag{28.4}$$



Note It is useful to interpret the method used above as starting with the integral along the real line and pushing this contour towards $+i\infty$. Every time a pole is crossed, a residue is collected. Since there are only finitely many poles, from a certain "height" on the contour can be pushed all the way to infinity, and that integral vanishes since the integrand vanishes at a sufficient rate.

Example. Find

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$$

Solution The singularities of R in the upper half plane are at $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$ with residues $1/[(1+x^4)']_{z=z_i}$ (why?) The result is $I = \pi/\sqrt{2}$.

29. Certain integrals with rational and trigonometric functions

We focus on integrals often occurring in integral transforms, of a type which can be reduced to

$$\int_{-\infty}^{\infty} e^{iax} Q(x) dx \tag{29.2}$$

$a > 0$, where Q has appropriate decay so that the integral makes sense. We would like to push the contour, as above, towards $+i\infty$ since the exponential goes to zero in the process. We need Q to satisfy decay and analyticity assumptions too, for this process to be possible. Jordan's lemma provides such a result suitable for applications.

Lemma 29.56 (Jordan)**.** *Assume $a > 0$ and that Q is analytic in the domain $\mathcal{D} = \{z : \Im(z) \geq 0, |z| > c\}$ and that γ in UHP is the semicircle of radius $\rho > c$ centered at zero. Assume furthermore that $Q(z) \to 0$ as $|z| \to \infty$ in $\mathcal{D}$. Then,*

$$\int_{\gamma} e^{iaz} Q(z) dz \to 0 \quad \textit{as} \quad \rho \to \infty \tag{29.3}$$

Proof. Choose $\epsilon > 0$ and let ρ_0 be such that $|Q(z)| < \epsilon$ for all z with $|z| > \rho_0$. Then, for $\rho > \rho_0$ and γ as above we have

$$(29.4)\qquad \left|\int_\gamma e^{iaz}Q(z)dz\right| = \left|\int_0^\pi e^{ia\rho e^{i\phi}}Q(\rho e^{i\phi})\rho i e^{i\phi}d\phi\right|$$
$$\leq \epsilon\int_0^\pi \rho e^{-\rho a\sin\phi}d\phi = 2\epsilon\int_0^{\frac{\pi}{2}}\rho e^{-\rho a\sin\phi}d\phi$$

To calculate the last integral we bound below $\sin\theta$ by $b\theta$ for some $b > 0$. Since $t \leq \tan t$ in $[0, \pi/2]$ we see that $t^{-1}\sin t$ is decreasing on this interval. Thus $\sin\theta \geq 2\theta/\pi$ in $[0, \pi]$ and we get

$$(29.5)\qquad \left|\int_\gamma e^{iaz}Q(z)dz\right| \leq \epsilon\int_0^\pi 2\rho e^{-2\rho a\phi/\pi}d\phi \leq \frac{\epsilon\pi}{a}$$

and the result follows. ∎

Proposition 29.57. *Assume $a > 0$ and Q is a rational function continuous on $\mathbb{R}$ and vanishing as $|z| \to \infty$ (which just means that the degree of the denominator exceeds the degree of the numerator). Then*

$$(29.6)\qquad \int_{-\infty}^{\infty} Q(x)e^{iax}dx = 2\pi i\sum_{z_i\in UHP}\mathrm{Res}(Q(z)e^{iaz}; z = z_i)$$

The proof is left as an exercise: it is a simple combination of Jordan's lemma and of the arguments in Proposition 28.54.

Example Let $\tau > 0$ and find

$$(29.7)\qquad I = \int_0^\infty \frac{\cos\tau x}{x^2+1}dx$$

Solution. The function is even; thus we have

$$(29.8)\qquad 2I = \int_{-\infty}^{\infty}\frac{\cos\tau x}{x^2+1}dx = \Re\int_{-\infty}^{\infty}\frac{e^{i\tau x}}{x^2+1}dx$$

which is of the form in Proposition 29.57 and thus a little algebra shows

$$I = \frac{\pi}{2}e^{-\tau}$$

Note that we have calculated the cos Fourier transform of an even function which is *real–analytic* (this means it is analytic in a neighborhood of the real line). The result is exponentially small as $\tau \to \infty$. This is not by accident: formulate and prove a result of this type for cos transforms of even rational functions.

Example(Whittaker &Watson pp. 116) Assume $\Re z > 0$. Show that

$$(29.9)\qquad I(z) = \int_0^\infty t^{-1}(e^{-t} - e^{-tz})dt = \log z$$

Solution (for another solution look at the reference cited) Note that the integrand is continuous at zero and the integral is well defined. Furthermore, it depends analytically on z (why?) We have

$$I'(z) = \int_0^\infty e^{-tz} dt = z^{-1} \quad \Leftrightarrow I(z) = \log z + C \tag{29.10}$$

The constant C is zero (why?)

Example: A common definite integral. Show that

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} \tag{29.11}$$

Solution This brings something new, since a naive attempt to write

$$\int_{-\infty}^\infty \frac{\sin t}{t} dt = \Im \int_{-\infty}^\infty \frac{e^{it}}{t} dt \quad (??) \tag{29.12}$$

cannot work as such, since the rhs is ill–defined. But we can still apply the ideas of the residue calculations in these lectures. Here is how.

(1) Use the box argument (see figure below) to show that

$$\int_{-\infty}^\infty \frac{\sin t}{t} dt = \int_{-\infty+i}^{\infty+i} \frac{\sin t}{t} dt = \int_{-\infty}^\infty \frac{\sin(t+i)}{t+i} dt$$

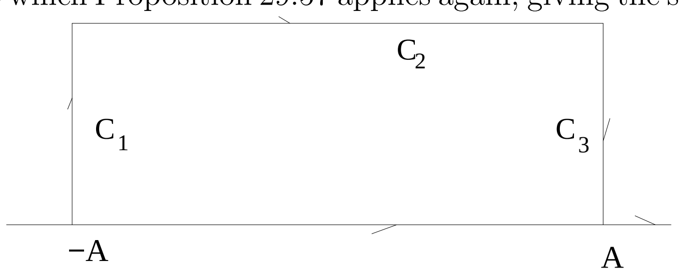
(2) Now we can write

$$\int_{-\infty}^\infty \frac{\sin(t+i)}{t+i} dt = \int_{-\infty}^\infty \frac{e^{i(t+i)} - e^{-i(t+i)}}{2i(t+i)} dt = \int_{-\infty}^\infty \frac{e^{i(t+i)}}{2i(t+i)} dt - \int_{-\infty}^\infty \frac{e^{-i(t+i)}}{2i(t+i)} dt$$

The first integral is zero, by Proposition 29.57. The last term equals

$$\overline{\int_{-\infty}^\infty \frac{-e^{i(t-i)}}{-2i(t-i)} dt}$$

to which Proposition 29.57 applies again, giving the stated result (check!)



Exercise 29.58. *** Find*

$$\int_0^\infty \frac{\sin^4 t}{t^4} dt$$

30. Integrals of branched functions

We now show that, for $\alpha \in (0,1)$ we have

$$\int_0^\infty \frac{t^{-\alpha}}{t+1}dt = \frac{\pi}{\sin \pi\alpha} \tag{30.2}$$

Note that the integrand has an integrable singularity at $t = 0$ and decays like $t^{-\alpha-1}$ for large t, thus the integral is well defined. The integral is performed along $\mathbb{R}^+$ so we know what $t^{-\alpha}$ means. We extend $t^{-\alpha}$ to a global analytic function; it has a branch point at $t = 0$ and no other singularities. Consider the region in the figure below. $t^{-\alpha}$ is analytic in $\mathbb{C} \setminus \mathbb{R}^+ \setminus \{0\}$. Thus

$$\oint_{\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3} \frac{t^{-\alpha}}{t+1}dt = 2\pi i \mathrm{Res}\left(\frac{t^{-\alpha}}{t+1}; z=-1\right) = 2\pi i e^{-\pi i\alpha} \tag{30.3}$$

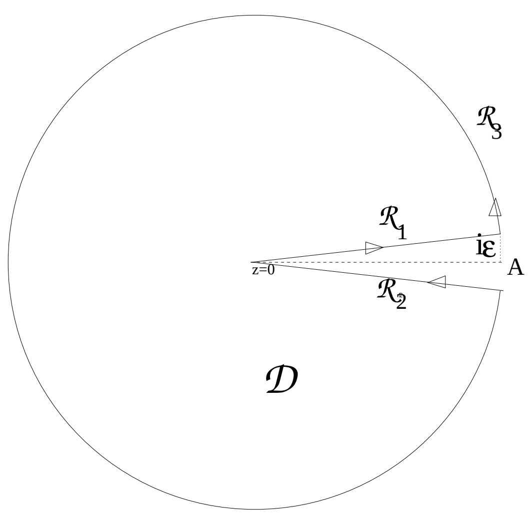
In the limit $\epsilon \to 0$ we get (check)

$$\int_0^A \frac{t^{-\alpha} - t^{-\alpha}e^{-2\pi i\alpha}}{t+1}dt + \int_{\mathcal{R}_3} \frac{t^{-\alpha}}{t+1}dt = 2\pi i e^{-\pi i\alpha}$$

In the limit $A \to \infty$, $\int_{\mathcal{R}_3}$ vanishes and $\int_0^A$ converges to $\int_0^\infty$. We get

$$\int_{\mathbb{R}^+} \frac{t^{-\alpha} - t^{-\alpha}e^{-2\pi i\alpha}}{t+1}dt = (1-e^{-2\pi i\alpha})\int_{\mathbb{R}^+} \frac{t^{-\alpha}}{t+1}dt = 2\pi i e^{-\pi i\alpha} \tag{30.4}$$

The rest is straightforward.



More generally, we have the following result.

Proposition 30.59. *Assume $\Re a \in (0,1)$ and Q is a rational function which is continuous on $\mathbb{R}^+$ and is such that $x^a Q(x) \to 0$ as $x \to 0$* and *as $x \to \infty$. Then*

$$\int_0^\infty x^{a-1} Q(x)dx = -\frac{\pi e^{-\pi i a}}{\sin a\pi} \sum Res(z^{a-1}Q(z); z_i) \tag{30.5}$$

where z_i are the poles of Q.

Exercise 30.60. *** Prove Proposition 30.59.*

Exercise 30.61. *** Let $a \in (0,1)$. Calculate*

$$P\int_0^\infty \frac{x^{a-1}}{1-x}dx$$

where P denotes the Cauchy principal part, as defined before.

Exercise 30.62.

$$\int_0^\infty \frac{x^{-1/2}\ln x}{x+1}dx$$

(There is a simple way, using the previous results.)

31. Conformal Mapping

Laplace's equation in two dimensions

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{31.2}$$

describes a number of problems in physics; it describes the flow of an incompressible fluid, the space dependence of the electric potential in a region free of charges among many others. For instance, the latter equation derives simply from $\nabla \cdot \mathbf{E} = 0$ (Maxwell's equation) and $\mathbf{E} = -\nabla V$, where $\mathbf{E}$ and V are the electric field and the electric potential respectively. Since the electric field is produced by charges, the boundary conditions are expected physically to determine the solution. A typical problem would be to solve eq. (31.2) with $u = V$ in $\mathcal{D}$ with V given on $\partial\mathcal{D}$ (Dirichlet problem). Another possible setting is (31.2) with $\mathbf{E}$ given on $\mathcal{D}$ (Neumann problem).

31.1. **Uniqueness.** The solution of the Dirichlet problem is unique. For if we had two solutions u_1, u_2 then $u = u_1 - u_2$ would satisfy (31.2) with $u = 0$ on ∂D. But a harmonic function reaches both its maximum and minimum on the boundary. Thus $u \equiv 0$. A similar argument shows that in the Neumann problem, u is determined up to an arbitrary constant.

Example 31.63. The Faraday cage. (In two dimensions) explain why a region surrounded by a conductor does not feel the electrical influence of static outside charges.

Solution. The electric potential along a conductor, at equilibrium, is zero. For otherwise, there would be a potential difference between two points, thus an electric current $i = V/\rho$ where ρ is the resistivity. This would contradict equilibrium.

Thus we deal with (31.2) with $V = C$ on $\partial\mathcal{D}$. Since $V = C$ is a solution, it is *the* solution. But then $\mathbf{E} = -\nabla V = 0$ which we wanted to prove $\Box$.

31.2. **Existence.** We have already solved Laplace's equation in a very special setting: $\mathcal{D} = D_1$, the unit disk (or any other disk). This problem then has a unique solution. What about other domains?

It is often the case in PDEs that a symmetry group exist and then it is very useful in solving the equation and/or determining its properties.

It turns out that (31.2) has a *huge* symmetry group: the equation is *conformally invariant*. This means the following.

Proposition 31.64. *If u solves (31.2) in $\mathcal{D}$ and $f = f_1 + if_2$ is analytic and such that $f : \mathcal{D}_1 \to \mathcal{D}$, then $u(f_1(s,t), f_2(s,t))$ is a solution of (31.2) in $\mathcal{D}_2 := f^{-1}(\mathcal{D})$.*

Proof. Let $D \subset \mathcal{D}$ be a disk. We know that u has a harmonic conjugate v determined up to an additive constant. Let $g = u + iv$. Then g is analytic in D. Let $D_1 = f^{-1}(D)$, which is an open set in $\mathcal{D}_1$ since in particular f is continuous. Then the composite function $g(f)$ is analytic in $\mathcal{D}_1$, and in particular $u(f_1(s,t), f_2(s,t))$ and $v(f_1(s,t), f_2(s,t))$ satisfy the CR equations in D_1. But then $u(f_1(s,t), f_2(s,t))$ is harmonic in D_1. Since this holds near any point in $\mathcal{D}_2$, the statement is proved.

We will be mostly interested in analytic *homeomorphisms* which have many nice properties. Two regions that are analytically homeomorphic to each-other are called *conformally equivalent*.

The Riemann mapping theorem, which we will prove later, states that any simply connected region other that $\mathbb{C}$ itself is conformally equivalent to the unit disk. The boundary of the region is then mapped onto the unit circle. The "orbit" of the disk under the group of conformal homeomorphisms group contains every simply connected region other that $\mathbb{C}$ itself.

The conformal group is large enough so that by its action we can solve Laplace's equation in any domain!

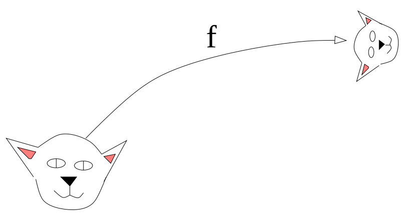
This motivates a careful study of conformal maps.

31.3. **Heuristics.** Let f be analytic at z_0, $f'(z_0) = a \neq 0$ (wlog $z_0 = 0$, $f(0) = 0$) and consider a tiny neighborhood $\mathcal{N}$ of zero. If z_β are points in $\mathcal{N}$ then

$$f(z_\beta) \approx az_\beta \tag{31.3}$$

All these points get multiplied by the *same* number a. Multiplication by a complex number rescales it by $|a|$ and rotates it by $\arg a$. If we think of z_β as describing a figure, then $f(z_\beta)$ describes the same figure, rotated and rescaled. The shape (form) of the figure is thus preserved and the transformation is *conformal*.

Since a tiny square of side ϵ becomes a square of side $|a\epsilon|$ areas are changed by a factor of $|a^2|$.



We make this rigorous in what follows.

31.4. **Preservation of angles.** Assume f is analytic in a disk D and that $f' \neq 0$. The angle between two smooth curves $\gamma(t)$ and $\Gamma(t)$ which cross at a point $z = \gamma(t_0) = \Gamma(t_1)$ (wlog we can take $t_0 = t_1 = 0$)is by definition the angle between their tangent vectors, that is $\arg \gamma'(0) - \arg \Gamma'(0)$, assuming of course that these derivatives don't vanish.

The angle between the images of these curves is given by

(31.4)
$$\arg[f(\gamma)'(0)] - \arg[f(\Gamma)'(0)] = \arg[f'(\gamma(0))\gamma'(0)] - \arg[f'(\Gamma(0))\Gamma'(0)]$$
$$= \arg f'(\gamma(0)) + \arg \gamma'(0) - \Big(\arg f'(\Gamma(0)) + \arg \Gamma'(0)\Big) = \arg \gamma'(0) - \arg \Gamma'(0)$$

That is to say the image of two curves intersecting at an angle α is a pair of curves intersecting at the same angle α.

31.5. **Rescaling of arc length.** The arc length along a curve $\gamma(t)$ is given by

$$L(\gamma) = \int_a^b |\gamma'(t)| dt =: \int_\gamma d|z| \tag{31.5}$$

If f is analytic, then
(31.6)
$$L(f(\gamma)) = \int_a^b |f(\gamma)'(t)|dt = \int_a^b |f'(\gamma(t))|\,|\gamma'(t)|dt = \int_\gamma |f'(z)|d|z|$$
and thus the arc length is stretched by the factor $|f'(z)|$.

31.6. **Transformation of areas.** The area of a set A is

$$\iint_A dxdy \tag{31.7}$$

while after the transformation $(x, y) \mapsto (u(x, y), v(x, y))$ the area becomes

$$\iint_A |J|dxdy \tag{31.8}$$

where the Jacobian J is, using the CR equations, $|f'|^2$ (check!).

Note 31.65. It is interesting to remark that it is enough that (u, v) is a smooth transformation that preserves angles for $u + iv$ to be analytic. It is also enough that it rescales any figure by the same amount for it to be analytic or anti-analytic ($\overline{f}$ is analytic). Try to prove these statements. For a reference, see Ahlfors, p 74. This gives a very nice characterization of analytic functions: they are those which are "locally Euclidian".

Note 31.66. Observe that we did not require f to be globally one-to-one. The simple fact that f is analytic with nonzero derivative makes it conformal. We need to impose bijectivity for two regions to be conformally equivalent. On the other hand, if f is an analytic homeomorphism between $\mathcal{D}_1$ and $\mathcal{D}_2$ then f is conformal (that is, $f' \neq 0$ in $\mathcal{D}$). This follows from the following proposition.

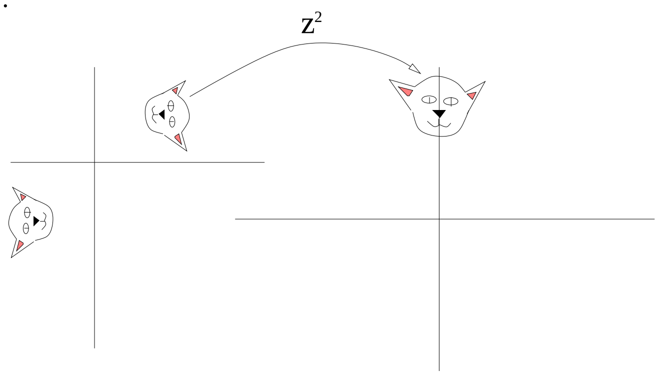
Proposition 31.67. *Assume that $f : \mathcal{D}_1 \mapsto \mathcal{D}_2$ is analytic, that for some $z_0 \in \mathcal{D}_1$ we have $f^{(j)}(z_0) = 0$ if $j = 1, ..., m-1$ and $f^{(m)}(z_0) = a \neq 0$. Wlog assume $z_0 = 0$ and $f(z_0) = 0$. Then, in some disk $D(\epsilon, 0)$, f is m-to-one, that is for any $0 \neq w \in f(D(\epsilon, z_0))$ the set $f^{-1}(w)$ consists in precisely m different points.*

Proof. In a neighborhood of 0 we have, with $a \neq 0$
$$f(z) = z^m(a + b_1 z + b_2 z^2 + \cdots) = z^m g(z)$$
and $g(z)$ is analytic, $g(0) = a$ and thus $g \neq 0$ in some disk $D = D(0, \epsilon_1)$. Then in $D(0, \epsilon_1)$ $\ln g$ is well defined (by $\int g'/g$) and analytic and so is

therefore $h = \exp(m^{-1} \ln g) = g^{1/m}$. The function zh is analytic at zero and $(zh)'(0) = h(0) \neq 0$. By the inverse function theorem, the equation $zh = y$ has exactly one solution in $D = D(0, \epsilon)$ for some ϵ. On the other hand in D we have

$$f(z) = w \Leftrightarrow (zh)^m = |y|e^{i\phi} \Leftrightarrow zh = |y|^{\frac{1}{m}} e^{im^{-1}\phi + \frac{2\pi i k}{m}},\ k = 0, 1, ..., m-1$$

$\square$.



32. Linear fractional transformations (Möbius transformations)

We assume some familiarity with these transformations, and we review their properties.

A linear fractional transformation (LFT) is a map of the form

$$S(z) = \frac{az + b}{cz + d} \tag{32.1}$$

where $ad - bc \neq 0$. If $c = 0$ we have a linear function. If $c \neq 0$ we write

$$\frac{az + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c^2(z + d/c)} \tag{32.2}$$

and we see that S is meromorphic, with only one pole at $z = -d/c$. It is also clear from (32.2) that $S(z_1) = S(z_2)$ iff $z_1 = z_2$ and in particular $S'(z) \neq 0$. They are one-to-one transformations on the Riemann sphere too (that is, the point at infinity included). Linear fractional transformation are conformal transformations.

Proposition 32.1. *Linear fractional transformations form a group.*

Proof: Exercise *

Exercise 32.2. *Show that $z \mapsto 1/z$ maps a line or a circle into a line or a circle. Hint: Show first that the equation*

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$ and $|\beta|^2 > \alpha\gamma$ is the most general equation of a line or a circle. Then apply the transformation to the equation.

As a result we have an important property of linear fractional transformations:

Proposition 32.3. *A linear fractional transformation maps a line or a circle into a line or a circle.*

Proof. To obtain a linear fractional transformation, we make the sequence of transformations

$$z \mapsto w_1 = z + d/c \mapsto w_2 = c^2 w_1 \mapsto w_3 = \frac{1}{w_2} \mapsto w_4 = -(ad - bc)w_3 \mapsto w_5 = w_4 + \frac{a}{c} \tag{32.3}$$

It is clear that the statement holds for all linear transformations. We only need to show that this is also the case for inversion, $z \mapsto 1/z$. This follows from Exercise 32.2 □

32.1. **Finding specific linear fractional transformations.** As we know from elementary geometry a line is determined by two of its points and a circle is determined by three. We now show that for any two circles/lines there is a linear fractional transformation mapping one into the other, and in fact they can be determined explicitly. Let z_1, z_2, z_3 be three points in $\mathbb{C}$. Then the transformation

$$S = \frac{z_1 - z_3}{z_1 - z_2} \frac{z - z_2}{z - z_3} \tag{32.4}$$

maps z_1, z_2, z_3 into $1, 0, \infty$ in this order. If one of z_1, z_2, z_3 is ∞, we pass the transformation to the limit. The result is

$$\frac{z - z_2}{z - z_3}, \quad \frac{z_1 - z_3}{z - z_3}, \quad \frac{z - z_2}{z_1 - z_2} \tag{32.5}$$

respectively.

Exercise 32.4. *Check that a linear fractional transformation that takes $(1, 0, \infty)$ into itself is the identity.*

To find a transformation that maps z_1, z_2, z_3 into $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3$ in this order, clearly we apply $\tilde{\tilde{S}} := \tilde{S}^{-1} S$. By Exercise 32.4 this transformation is unique.

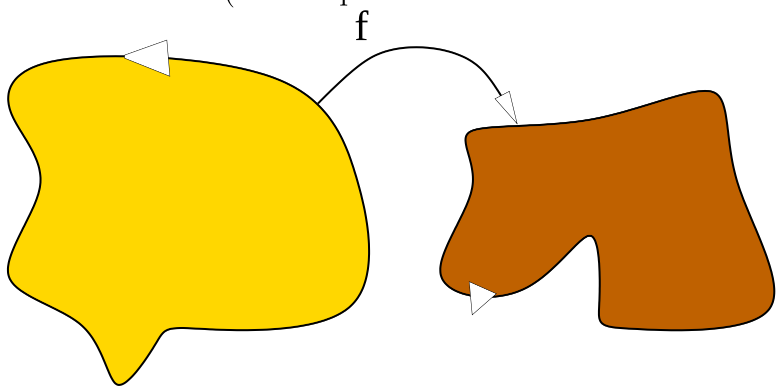
32.1.1. *Cross ratio.* If $z_i, i = 1...4$ are four distinct points and $w_i = S(z_i)$ then (check!)

$$\frac{w_1 - w_2}{w_1 - w_3}\frac{w_3 - w_4}{w_2 - w_4} = \frac{z_1 - z_2}{z_1 - z_3}\frac{z_3 - z_4}{z_2 - z_4}$$

This is often a handy way to determine the image of a fourth point when the transformation is calculated using three points.

32.2. **Mappings of regions.** We know that linear fractional transformations are *conformal and one-to-one* and transform circles/lines onto circles/lines. What about their interior? We look at this problem more generally.

By definition (a parameterization is such that) a curve is traversed in anticlockwise direction if the interior is to the left of the curve, as the curve is traversed (brush up the notions of orientation etc. if needed).



Proposition 32.5. *Assume that $f : \mathcal{D} \mapsto \mathcal{D}_1$ is analytic and γ is a simple smooth* closed *curve contained in $\mathcal{D}$ together with its interior.*

If f is one-to-one on γ then f maps one-to-one conformally Int*(γ) onto $f($*Int*$(\gamma))$ and preserves the orientation of the curve.*

Proof. Let $w_0 \in$Int$(f(\gamma))$. Then by the definition of the interior and of a simple curve we have

$$\frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w - w_0} = 1 \tag{32.6}$$

On the other hand by assumption f is one-to-one on γ and we can change variables $w = f(z)$, $z \in \gamma$, and we get

$$1 = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)dz}{f(z) - w_0} = \frac{1}{2\pi i}\int_{\gamma} \frac{(f(z) - w_0)'dz}{f(z) - w_0} \tag{32.7}$$

and by Proposition 22.46 (and since f is analytic) this shows that $f(z) - w_0$ has exactly one zero in Int(γ), or there is exactly one z_0 such that $f(z_0) = w_0$. Then f is conformal, one-to-one onto between Int(γ)

and $f(\mathrm{Int}(\gamma))$. This also shows that f preserves orientation, otherwise the integral would be -1.

Exercise 32.6. * (i) Find a linear fractional transformations that maps the unit disk onto the upper half plane.

(ii) Find a linear fractional transformations that maps the disk $(x-1)^2+(y-2)^2=4$ onto the unit circle and the center is mapped to $i/2$.

(iii) Find the most general linear fractional transformation that maps the unit disk onto itself.

32.3. As usual, we let D be the unit disk.

Theorem 32.7 (Schwarz lemma)**.** *Let $f : D \to D$ be analytic and such that $f(0) = 0$. Then*

(i)

$$|f(z)| \le |z| \tag{32.8}$$

for all $z \in D$.

(ii) If there is some $z_0 \in D$ such that for $z = z_0$ we have equality in (32.8) then $f(z) = e^{i\phi} z$ for some $\phi \in \mathbb{R}$.

(iii) $|f'(0)| \le 1$ and if equality holds then again $f(z) = e^{i\phi} z$ for some $\phi \in \mathbb{R}$.

Proof. (i) Since $f(0) = 0$, the function $f(z)/z$ extends analytically in D. By the maximum modulus principle,

$$\left|\frac{f(z)}{z}\right| \le \lim_{r\uparrow 1} \max_{|z|=r} \left|\frac{f(z)}{z}\right| = 1$$

(ii) If z_0 is such that equality in (32.8) holds, then z_0 is a point of maximum of $|f(z)/z|$, which cannot happen unless $f(z)/z = C = f(z_0)/z_0$.

(iii) The inequality follows immediately from (32.8). Assume $f'(0) = e^{i\phi}$, $\phi \in \mathbb{R}$. If $f(z) \not\equiv e^{i\phi} z$, then we can write

$$f(z)/z = e^{i\phi}(1 + z^m e^{i\psi} h(z))$$

where h is analytic and $h(0) \in \mathbb{R}^+$. If we then take $z = \epsilon \exp(-i\psi/m)$ with ϵ small enough we contradict (i). $\square$

Corollary 32.8. *If h is an* automorphism *of the unit disk and $h(0) = 0$ then $h(z) = e^{i\phi} z$ for some $\phi \in \mathbb{R}$.*

Proof. We must have, by Theorem 32.7 $|h(z)| \le |z|$. But the inverse function h^{-1} is also an automorphism of the unit disk and $h^{-1}(0) = 0$. Thus $|h^{-1}(z)| \le |z|$ for all z, in particular $|z| = |h^{-1}(h(z))| \le |h(z)|$ or $|z| \le |h(z)|$. Thus $|h(z)| = |z|$ for all z and the result follows from Theorem 32.7 (ii).$\square$

32.4. **Automorphisms of the unit disk.** . We have seen that

$$S(z) = e^{i\phi}\frac{\alpha - z}{1 - \overline{\alpha} z} \quad \text{with } \phi \in \mathbb{R} \text{ and } |\alpha| < 1 \tag{32.9}$$

maps the unit disk one-to-one onto itself.

The converse is also true:

Theorem 32.9. *Any automorphism f of D into itself is of the form (32.9) with $\alpha = f^{-1}(0)$.*

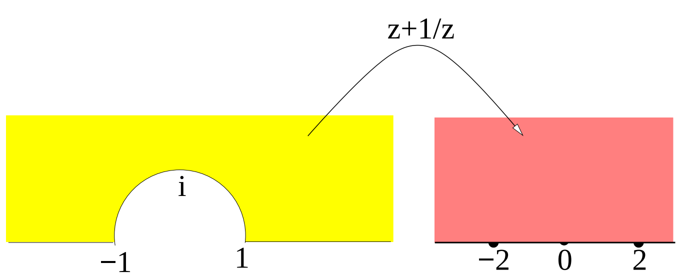
Proof. The function $h = S \circ f^{-1}$ is an automorphism of the unit disk and $h(0) = S(\alpha) = 0$. But then Corollary 32.8 applies and the result follows. □.

32.5. **Miscellaneous transformations.** We illustrate below a number of useful transformations; look in a book for more examples. A good number of interesting domains can be mapped to the unit disk using combinations of these transformations. Note that by Proposition 32.5 it suffices to examine carefully the way the boundaries are mapped to understand the action of a map on a whole domain.

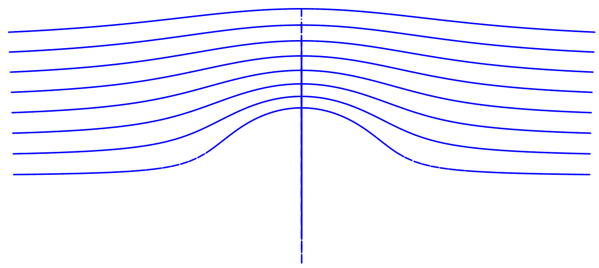
32.5.1. *The Joukovski transformation.* This is an interesting map which straightens the region in the upper half plane above the unit circle (of course, by slight modifications, you can choose other radii or centers along $\mathbb{R}^+$) to the upper half plane. It is given by

$$z \mapsto z + \frac{1}{z} \tag{32.10}$$

Exercise 32.10. *Explain the effect of this map on the region depicted.*

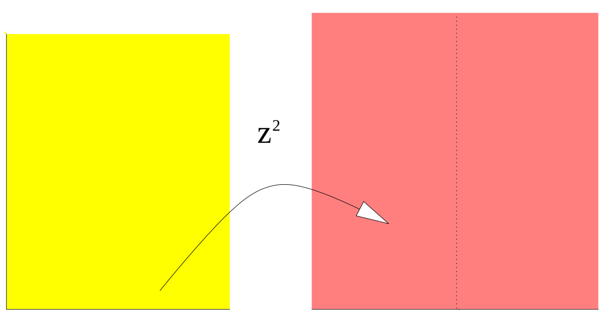


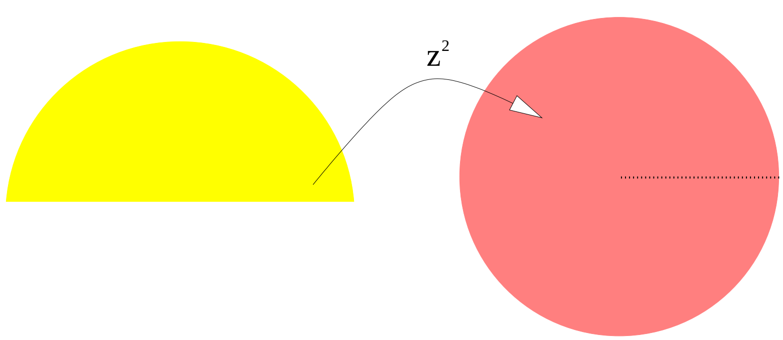
As a nice application, we can find the flow lines of a river passing above a cylindrical obstacle. Indeed, we do understand the free flow lines above a flat bottom (the problem in the upper half plane): they are just straight horizontal lines. All we have to do, remembering our discussion about the conformal invariance of Laplace's equation is to map these lines through the inverse of $z+1/z$. The result, plotted with Maple, is shown below.

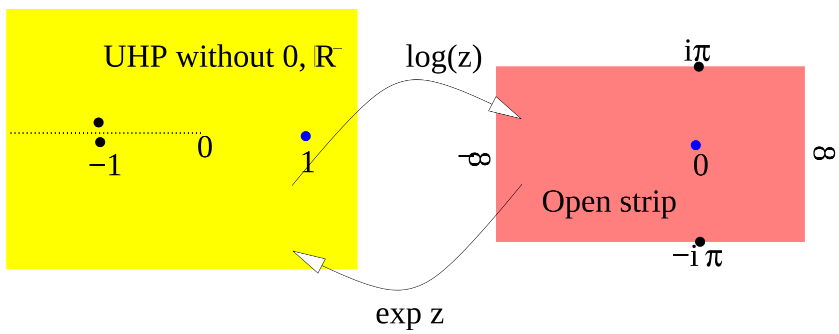


Exercise 32.11. ** Find an explicit formula for flow lines in the previous example.*

Other common mappings are depicted in the following figures.







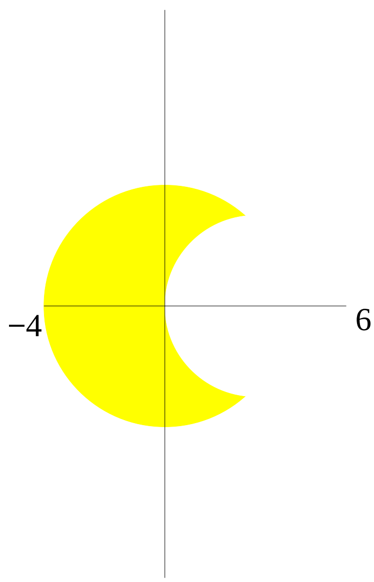
Exercise 32.12. *** (i) Draw a similar picture for the mapping* $\sin z$ *from the upper half strip bordered by the half-lines* $x = \pm\pi/2, y > 0$.

(ii) Find a conformal homeomorphism of the quarter disk $|z| < 1, \arg(z) \in (0, \pi/2)$ *onto the upper half plane.*

(iii) Find a conformal homeomorphism of the half disk $|z| < 1, \arg(z) \in (0, \pi)$ *onto the half strip* $x < 0, y \in (0, \pi)$.

(iv) Find a conformal homeomorphism of the right half plane with a cut along $[0, 1]$ *in the right half plane.*

Example 1 It is useful to remark that we can find linear fractional transformations which map a region between two circles into a half plane (or disk) using very simple transformations. Let us map the "moon crescent" M below into a half plane.



The equations of the circles are $x^2 + y^2 = 16$ and $(x - 3)^2 + y^2 = 9$. Solving the system we get $a_1 = 8/3 \pm 4i\sqrt{5}/3 = 4\exp(i \arctan \sqrt{5}/2) = 3 - 3\exp(i \arctan 4\sqrt{5}) = 3 - a_2$. Direct calculation of a_1/a_2 shows that the angle between the circles is $\phi_0 = \arctan(\sqrt{5}/2)$.

If we send the point $z_0 = 8/3 - 4i\sqrt{5}/3$ to zero, the point $z_\infty = 8/3 + 4i\sqrt{5}/3$ to infinity and finally the point 0 to 1, so that part of the boundary is $\mathbb{R}^+$, by a linear fractional transformation

$$\frac{z_\infty}{z_0}\frac{z - z_0}{z - z_\infty} \tag{32.11}$$

then both circle arcs become *rays* (since they end at ∞) starting at the origin. Which rays? If we traverse the crescent positively, z_0 is mapped to 0 and 0 is mapped to 1, in this order. Thus the small arc becomes $\mathbb{R}^+$. Because of conformality, the large arc becomes a ray to the left of $\mathbb{R}^+$ at angle ϕ_0. Now the transformation $z \mapsto z^{\pi/\phi_0}$ maps M into the upper half plane.

Example 2 (From S. Tanveer's notes) Solve $\Delta u = 0$ in the region $|z| < 1, \arg(z) \in (0, \pi/2)$ such that on the boundary we have: $u = 1$ on the arc and $u = 0$ otherwise.

Solution. Strategy: We find conformal homeomorphism of this region into the strip $\{z = x + iy : y \in (0,1)\}$ such that the arc goes into $y = 1$ and the segments into $y = 0$. The solution of the problem in this region is clear: $u = \Im z$. Then we map back this function through the transformations made.

How to find the transformation? We are dealing with circles, strips, etc so it is hopeful we can get the job done by composing elementary transformations. There is no unique way to achieve that, but the end result must be the same.

(1) The transformation $z \mapsto z^2$ opens up the quarter disk into a half disk. On the boundary we still have: $u = 1$ on the arc and $u = 0$ otherwise.

(2) We can now open the half disk into a quarter plane, by sending the point $z = 1$ to infinity, as in Example 1, by a linear fractional transformation. We need to place a pole at $z = 1$ and a zero at $z = -1$. Thus the second transformation is $z \mapsto \frac{1+z}{1-z}$. The segment starting at -1 ending at 1 is transformed in a line too, and the line is clearly $\mathbb{R}^+$ since the application is real and positive on $[0, 1)$ and 1 is a pole. What about the half circle? It must become a ray since the image starts at $z = 0$ and ends at infinity. Which line? The image of $z = i$ is $w = i$. Now we deal with the first quadrant with boundary condition $u = 1$ on $i\mathbb{R}^+$ and $u = 0$ on $\mathbb{R}^+$.

(3) We open up the quadrant onto the upper half plane by $z \mapsto z^2$.

(4) We now use a rescaled log to complete the transformation. The composite transformation is

$$\frac{2}{\pi} \ln\left(\frac{1+z^2}{1-z^2}\right)$$

Exercise** (1) Map M onto a strip as in Example 2. What is the distribution of temperature in the domain M if the temperature on the larger arc is 1 and 0 on the smaller one? (Temperature distribution also satisfies Laplace's equation). What shape do the lines of constant temperature have?

(2) What is the distribution of temperature in the domain and with the boundary conditions described in example 2? Draw an approximate picture of the lines of constant temperature.

33. Boundary behavior, The Schwarz-Christoffel formula

The Riemann Mapping theorem was formulated by Riemann but was proved later by other mathematicians.

As usual, D is the unit disk.

Theorem 33.1 (Riemann Mapping theorem)**.** *Given any simply connected region $\mathcal{D}$* other than $\mathbb{C}$ *there is an analytic homeomorphism between $\mathcal{D}$ and D.*

This map is unique if for some $z_0 \in \mathcal{D}$ it is normalized by the conditions $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}^+$.

We postpone its proof until §36. We will not use any of the results we obtain until §36 on Theorem 33.1.

33.1. **Behavior at the boundary, a weaker result.**

We derive an easy but very useful result [3]. We show that as we approach the boundary of a domain, the Riemann conformal map approaches the boundary of its image. This does not allow us to infer convergence of the images if the original points converge.

Let $\mathcal{D}$ be a region. A sequence or an arc approaches the boundary if it eventually recedes away from any point in the region. **Abbreviation:** In both cases above we will just write $z \to \partial\mathcal{D}$. More precisely,

Definition. A sequence $z_n \to \partial\mathcal{D}$ as $n \to \infty$ if for any compact set $K \subset \mathcal{D}$ there exists n_0 such that for all $n > n_0$ we have $z_n \notin K$.

The corresponding definition for an arc is very similar.

Definition. Let $z : [0, 1) \mapsto \mathbb{C}$. Then $z(t) \to \partial\mathcal{D}$ as $t \to 1$ if for any compact set $K \subset \mathcal{D}$ there exists $t_0 \in [0, 1)$ such that for all $t \in (t_0, 1)$ we have $z(t) \notin K$.

Convince yourselves that the definitions correspond to the intuitive description at the beginning of this section by taking $\mathcal{D}$ to be D, UHP or a cut disk.

Theorem 33.2. *If $f : \mathcal{D} \to \mathcal{D}'$ is an analytic homeomorphism and $z \to \partial\mathcal{D}$, then $f(z) \to \partial\mathcal{D}'$.*

Proof. We prove the statement for sequences; the one for arcs is almost identical. Let $z_n \to \partial\mathcal{D}$ and let $K' \subset \mathcal{D}'$ be any compact set. Then $K = f^{-1}(K') \subset \mathcal{D}$ is compact (f^{-1} is analytic). By definition,

$$z_n \to \partial\mathcal{D} \ \Rightarrow \exists n_0 \ s.t. \forall n > n_0 \ z_n \notin K$$

Since f is one to one, $f(z_n) \notin K'$ either.

Corollary 33.3. *If $h : \mathcal{D} \to D$ then $|h(z)| \to 1$ as $z \to \partial\mathcal{D}$.*

33.2. **Behavior at the boundary, a stronger result.** We recall that a Jordan curve in $\mathbb{C}$ is a *continuous map* γ defined (say) on $[0, 1]$ with values in $\mathbb{C}$ which is injective, that is $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$ or $t_1 = 0$ and $t_2 = 1$ where in the latter case it is a closed Jordan curve. We also recall that a closed Jordan curve divides $\mathbb{C}$ into exactly two regions, one bounded and one unbounded. The bounded region is called the interior of the curve. A Jordan region is the interior of a Jordan curve.

Theorem 33.4 (Boundary behavior)**.** *If $\mathcal{D}$ is a Jordan region then the function f extends continuously as a homeomorphism between the closures $\overline{\mathcal{D}}$ and $\overline{D}$.*

We will show this shortly for a polygon in §34.1.8. For other regions, for space limitations we do not prove this interesting result. We shall not use it in this generality either.

33.3. **Free boundary arcs.** A region $\mathcal{D}$ contains a free boundary arc if there is a line segment $s \subset \partial\mathcal{D}$ (wlog we may assume $s = (a, b)$) such that any point in (a, b) is the center of a disk D_ϵ such that $D_\epsilon \cap \partial\mathcal{D} = D_\epsilon \cap s$. (The points in s are not "junction" points of boundary pieces.) Then each of the half-disks delimited by $D_\epsilon \cap s$ is either completely inside or completely outside $\mathcal{D}$. For if this was not the case, it would intersect the boundary $\partial\mathcal{D}$ in some other point (why?). A point on s is one-sided if the corresponding the two half disks are on opposite sides of $\mathcal{D}$ and two-sided otherwise. A point on s cannot be "zero-sided" since this would contradict the fact that it is on $\partial\mathcal{D}$ (why?). All points on s are of the same kind by continuity.

(Note that if we are dealing with Jordan region *with a free boundary arc*, then this arc is free (why?).)

We have now an interesting result showing analytic continuabililty of f through s past $\mathcal{D}$.

Theorem 33.5 (Continuation past the boundary)**.** *(i) Suppose $\mathcal{D}$ is a simply connected region which contains a one-sided free boundary arc s. Then the function f in Theorem 33.1 extends analytically in a neighborhood of $\mathcal{D} \cup s$, and the image of s is an arc of D.*

(ii) Furthermore, this extension is one-to-one on s, thus in a neighborhood of $\mathcal{D} \cup s$.

Proof. Take $s_1 \subset (a,b)$ be a proper closed subarc. Let $z_0 \in \mathcal{D}$ be the unique point in $\mathcal{D}$ such that $f(z_0) = 0$ and let $0 < \delta_1 < \mathrm{dist}(z_0, s)$ (the distance between a point in an open set and the boundary is positive). From the disks in the definition of the free arc s extract a finite covering of s_1. Let the minimum of their radii be δ_2, let $\delta < \min(\delta_1, \delta_2)$ and extract again a finite subcovering of s_1 by disks of radius δ. Let $\mathcal{D}_1$ be the region defined by the half disks in $\mathcal{D}$. Wlog, say $\mathcal{D}_1 \in$ UHP. Since f is one-to-one, $f(z) \neq 0$ in $\mathcal{D}$ and a branch of $\ln z$ (see §27.1) is analytic. Furthermore, by Corollary 33.3, $|f(x+iy)| \to 1$ as $y \downarrow 0$, and so $i \ln f$ satisfies the assumptions of the Schwarz reflection principle 25.51. Thus $\ln f$ and then $f = e^{\ln f}$ is analytic in $\{x \pm iy : x \in s_1; x + iy \in \mathcal{D}_1\}$.

For the proof of (ii) we simply note that if this was not the case, then f' would be zero at some $z \in s$ which is impossible since, if for instance $f''(z) \neq 0$ f would map the open half disk H_z at z lying in $\mathcal{D}$ into a region of opening 2π at $f(z)$, which is not possible since $f(H_z) \in D$.

Exercise 33.6. ** Complete the details of the proof rigorously.*

Real analytic functions. A function $f : (a,b) \mapsto \mathbb{C}$ is real-analytic if its Taylor series converges to it at every point in (a,b). Equivalently, f extends to an analytic function in a neighborhood of (a,b) (why?)
Simple regular analytic arcs. These are defined as $t \mapsto f(t)$ where f is real analytic, injective on (a,b) with nonzero derivative. We can define free analytic arcs in the same way as we defined free arcs. By the Riemann mapping theorem, a free arc can be analytically and homeomorphically straightened to the segment (a,b). Then we have the following theorem.

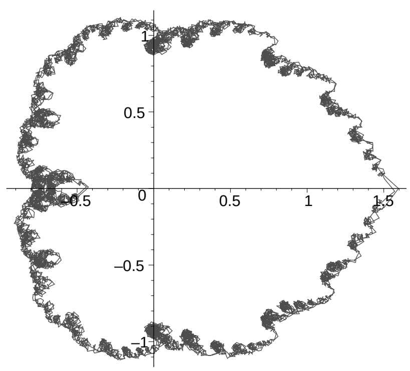
Theorem 33.7. *If $\partial\mathcal{D}$ contains a free one-sided arc γ, then the mapping f has analytic extension to $\partial\mathcal{D} \cup \gamma$ and γ is mapped onto an arc of S_1.*

Exercise 33.8. *Prove the theorem carefully.*

“Illustration of “bad behavior” at the boundary of analyticity. The image of the unit circle through the function

$$\sum_{n=1}^{\infty} n^{-2} z^{2^n}$$

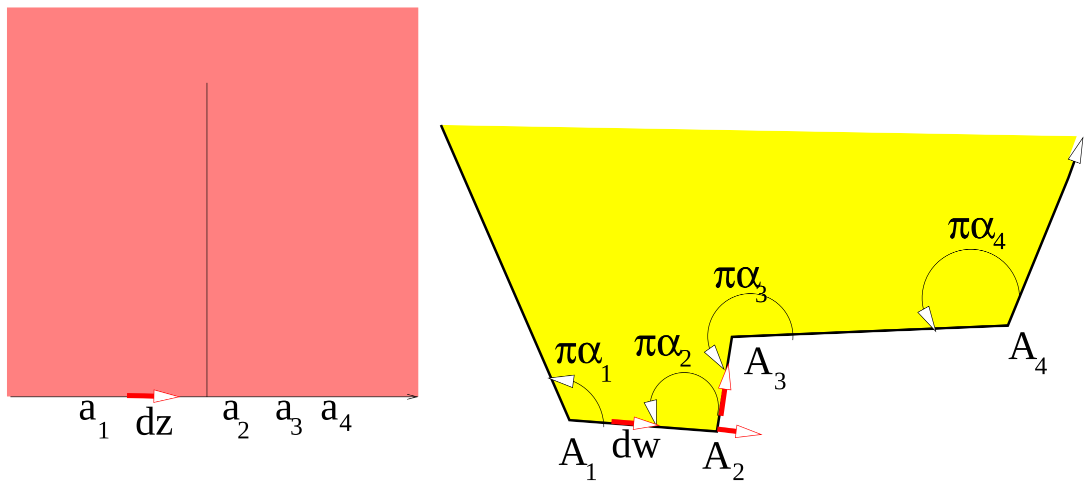
is given below. Note that the unit circle is a barrier of singularities of the function, and no piece of the image boundary can be nice.



34. Conformal mappings of polygons and the Schwarz-Christoffel formulas

For polygonal regions, the conformal map to the unit circle (or to UHP obviously) can be done by quadratures. The transformation is still usually nonelementary, but the integral representation gives us enough control to describe the transformation quite well.

34.1. **Heuristics.** For this part we roughly follow the construction in [1]. Suppose we have P is a polygon (without self-intersections; we drew it with a vertex at ∞), and let $\alpha_i\pi$ be the interior angles at the vertices, in the order in which the curve is traversed, let $A_1, ..., A_n$ be the vertices of the polygon. We assume the curve is traversed from left to right and the region lies to the left of the polygon. We wish to map this region into the upper half plane. The mapping in the unit disk follows a similar strategy. See figure.



It is convenient to work out the derivative of the mapping f from UHP to the polygonally bounded region.

Suppose we are traversing the polygon between A_1 and A_2. We have $dw = f'(z)dz$. (We think of dz and dw as infinitesimals; we are in a heuristic framework after all.) As long as we are away from the vertices, dw has fixed direction, while $\arg dz = 0$ (see again figure). Thus

$$\arg f'(z) = \arg(dw/dz) = \arg dw - \arg dz = \arg dw = const.$$

At a_2 though, $\arg dw$ changes by the amount $\pi - \pi\alpha_2$ (see figure). The function whose argument has this behavior *at* a_2 is $(z - a_2)^{\alpha_2 - 1}$. (To check the signs, I find it clearer to move backwards. Then $z - a_k$ changes argument by π while the image changes argument by the –possibly negative– amount $\pi(\alpha - 1)$.) If we take

$$f'(z) = const. \prod_{k=1}^{n} (z - a_k)^{\alpha_k - 1} \tag{34.1}$$

there is no change in the argument of f' thus of dw at points other than a_k, since the f' is nonzero. Indeed, the $\arg f' = \Im \ln f'$ and we have

$$d \ln f' = \sum_{k=1}^{n} (\alpha_k - 1) \frac{dz}{z - a_k} \tag{34.2}$$

a purely real quantity since dz, $z - a_k$ and $\alpha_k - 1$ are all real. So $f(t)$, $t \in \mathbb{R}$, evolves along straight lines except at a_k where it changes abruptly its slope by the required amount.

Theorem 34.1. *(i) The function*

$$\Phi = z \mapsto C \int_0^z \prod_{k=1}^{n-1} (s - a_k)^{\alpha_k - 1} ds + C' \tag{34.3}$$

with $\alpha_k \in (0, 2)$ and a_k on $\mathbb{R}$ and C, C' are complex constants, maps UHP into a polygon with angles $\alpha_k \pi$, $k = 1, 2, ..., n$.

(ii) The function

$$\Phi = z \mapsto C \int_0^z \prod_{k=1}^{n} (s - b_k)^{\alpha_k - 1} ds + C' \tag{34.4}$$

with $\alpha_k \in (0, 2)$ and b_k on the unit circle and C, C' are complex constants, maps the unit circle into a polygon with angles $\alpha_k \pi$, $k = 1, 2, ..., n$.

(iii) Moreover, all transformations between UHP or D and polygons are of this form.

Remark 34.2. (i) In formula (34.3) the last angle is determined by the fact that the sum of exterior angles of a *closed* polygon is 2π: only $n-1$ angles are independent. Also, the last point is $z = \infty$ as it should, since we are mapping UHP. If P is closed, thus compact, ∞ must be mapped into one of its vertices, *thus the integral is convergent in this case.*

(ii) It is important to note what freedom we have in such transformations. Suppose we want to map a triangle Δ. All triangles with same angles are similar, and a mapping between two similar triangles reduces to scaling, rotation and translation. Thus we need to understand one triangle with given angles α_i. We take say $a_1 = 0$ and $a_2 = 1$, use the α's, and see what triangle Δ_1 is obtained. (Even for a triangle, the integral may not be expressible in terms of elementary functions.) Then, we can choose C and C' so that we remap Δ_1 to Δ. Thus we are able to map any triangle to UHP, prescribing the position of the images of the vertices at will.

For $n > 3$, we can still place three points at will but the position of the fourth one etc cannot be chosen arbitrarily. This can be seen also by noting that we don't have enough many parameters in the problem. Indeed, suppose we have a polygon with n sides and given angles.

How much freedom is there in a polygon with given angles? We can place its first vertex arbitrarily; this means two real constants. We can place its second vertex arbitrarily; two real constants again. Then we have to preserve the first angle but can move arbitrarily far; 1 real constant, and so on until the last vertex, which must be chosen so as to close back the polygon. A total of $n + 1$ real constants. How many free constants are there in formula

(34.3)? $n-1+4$. Infinity is already prescribed, we can prescribe two more points.

The b_k (or a_k), $k>3$ are called accessory parameters; they usually need to be calculated numerically and symmetries, if any, help.

Proof. (i), (ii). That the function thus defined has the mapping property stated follows from the arguments given in §34.1. They need to be adapted to the circle instead of the real line (the adaptation is not difficult) and made rigorous, properly replacing "infinitesimal calculus" arguments with tangent vectors etc.

Exercise 34.3. *Make parts (i), (ii) of the proof rigorous.*

For part (iii) we need to do some more work, since it is not immediately clear why other transformations would not have similar properties. Equivalently, why can we solve the mapping problem for *any* polygon P by these formulas? It would not be clear why such transformations f exist, if we did not have Theorem 33.1; we use it at this point. We place ourselves in the context of (ii).

We next use Theorems 33.5 &33.7. We have to derive that a mapping with the same property as say, f (ii) must coincide with it (modulo the constants given).

First a **sketch of the proof**. *The sketch below is almost all that would be needed for these notes.*

We find the general properties of any conformal homeomorphism F between D and P. Namely, F^{-1} extends analytically past the sides of the polygon by Theorem 33.5 and is ramified-analytic at the vertices (since it straightens them out). Then we show that $H=F'/\Phi'$ with Φ given by (34.4) is analytic all around the boundary, since Φ' takes care of the ramifications of F'. But neither F' nor Φ' change argument except at the points corresponding to vertices so $\arg H$ *is constant on the boundary. From the max modulus principle for harmonic functions we infer that H is constant.*

Exercise 34.4. ** Work out the proof yourselves following these guidelines. The details of the proof are given below if you get stuck anywhere.*

Now we give the proof. The proof is essentially a detailed version of the one in [3].

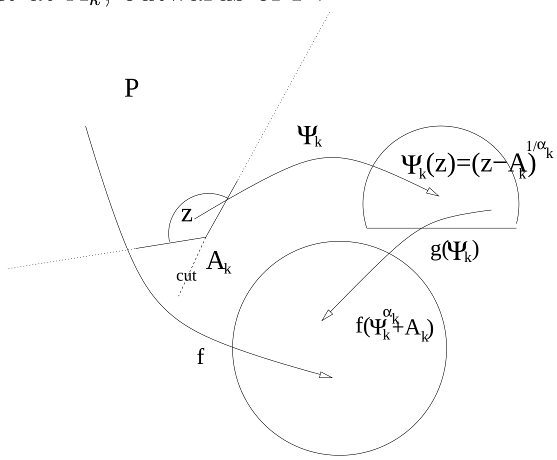
Note 34.5. We break the arguments into a number of smaller steps to increase readability.

First we take function $F:D\mapsto P$ obtained from Theorem 33.1. By Theorem 33.5 $f=F^{-1}$ can be continued past any open segment on P.

We want to check first that adjacent sides go to adjacent arcs/segments, and that latter do not overlap.

Let A_k be a vertex of P. Take a small enough disk about A_k; its intersection with $\mathrm{int}(P)$ is an open disk sector S_k.

34.1.1. There is a branch of $\Psi_k = (z - A_k)^{1/\alpha_k}$, analytic in S_k with a cut at A_k, outwards of P.



34.1.2. Ψ_k maps S_k one-to-one onto an open half disk H_k centered at 0, mapping arc to half-circle and each side to a half-diameter.

34.1.3. Thus the inverse function $\Psi_k \mapsto \Psi_k^{\alpha_k} + A_k$ maps analytically H_k to S_k. We see that the function $g(\Psi_k) = f(\Psi_k^{\alpha_k} + A_k) : H_k \mapsto g(H_k)$ is analytic and $g(H_k) \subset f(P) \subset D$. It is one-to-one analytic since it is a composition of such maps.

34.1.4. $|g| = |f(\Psi_k^{\alpha_k} + A_k)| \to 1$ as $\Psi_k \to d$ where d is the diameter of H_k. This is so by Corollary 33.3 and because in this case $\Psi_k^{\alpha_k} + A_k \to \partial P$, by construction.

34.1.5. As in the proof of Theorem 33.5 we conclude by the Schwarz reflection principle, Theorem 25.51 that g has analytic continuation to the *whole disk complementing* H_k. (It does *not* follow that f has analytic continuation though, but a form of ramified analytic continuation! See formula (34.6) below.)

34.1.6. Thus in particular

$$g(\Psi_k) = f(A_k + \Psi_k^{\alpha_k}) = b_k + \sum_{m=1}^{\infty} a_m \Psi_k^m \tag{34.5}$$

$$\left(\text{or:}\quad f(A_k + s) = b_k + \sum_{n=1}^{\infty} a_m s^{m/\alpha_k}\right) \tag{34.6}$$

where the series converges for small Ψ_k and s. Here $b_k = f(A_k)$ is on the unit circle.

34.1.7. Since $\alpha_k > 0$, Formula (34.6) shows that f is continuous at A_k. In particular, adjacent lines become adjacent arcs and they clearly cannot overlap near A_k (i.e., for small s). Also, by Theorem 33.5, f is analytic on ∂P except at the vertices.

34.1.8. We saw in §34.1.7 that f extends continuously and piecewise analytically to $\overline{\partial P}$. The function f' is L^1 (check!) and nowhere is f' zero (again by Theorem 33.5). We now calculate

$$\oint_{\partial D} \frac{dw}{w - w_0}; \quad w_0 \in D$$

in two ways as in Proposition 32.5 and conclude that f is one-to-one on the boundary (check that!).

34.1.9. We have proved Theorem 33.4 in the special case where the boundary is a polygon.

34.1.10. Note that $a_1 \neq 0$ in (34.5). Otherwise, by looking at the rhs in (34.5) –as in the proof of Theorem 14.28– $f(A_k + z^{\alpha_k})$ would overshoot the unit disk (check that!)

34.1.11. Then the series (34.5) can be analytically inverted and we find for some coefficients $\{\beta_m\}_{m\in\mathbb{N}}$ (writing $f = w$),

$$\Psi_k = \sum_{m=1}^{\infty} \beta_m (w - b_k)^m = (w - b_k)G(w) \tag{34.7}$$

Remembering that $s = \Psi^{\alpha_k}$ we raise (34.7) to power α_k, we take (34.5) and write

$$s = (w - b_k)^{\alpha_k} G_k(w) \tag{34.8}$$

with $G_k := G^{a_k}$ analytic at zero because as we noted $G(0) = \beta_1 = 1/a_1 \neq 0$. If we write as usual $w = f(A_k + s)$ we have

$$s = f^{-1}(w) - A_k := F(w) - A_k \tag{34.9}$$

and rewrite (34.6) as

$$\begin{aligned} F(w) - A_k = s = (w - b_k)^{\alpha_k} G_k(w) \\ \Rightarrow F'(w) = (w - b_k)^{\alpha_k - 1} G_k(w) + (w - b_k)^{\alpha_k} G_k'(w) \end{aligned} \tag{34.10}$$

and hence the lhs in the following equation:

$$(w - b_k)^{1-\alpha_k} F'(w) = G_k(w) + (w - b_k) G_k'(w)$$

extends analytically near b_k *and does not vanish at* b_k.

34.1.12. Let now

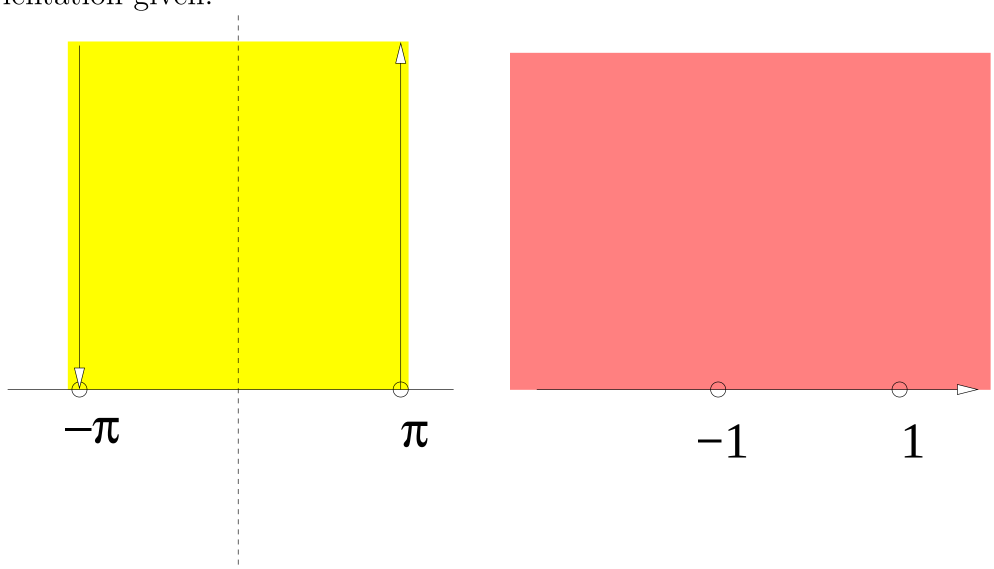
$$H := F'(w) \prod_{m=1}^{n} (w - b_m)^{1-a_m} \tag{34.11}$$

This function is analytic in D (by the conclusion of §34.1.11 and since $b_m \in S_1$).

34.1.13. We know that H is *also* analytic along S_1, by §34.1.11. If we show that $\arg H$ is a constant between any A_k and A_{k+1}, then it is piecewise constant and continuous, and by the maximum modulus principle applied to $\Im \ln H$ (recall, this is well defined in D) we are done. This is a mere calculation that we leave to the reader:

Exercise 34.6. * *Calculate* $\arg H$ *between* A_k *and* A_{k+1}*, much as in* §*34.1. Use the fact that the sum of all outer angles of a polygon is* 2π *to show that* $H = const$*. Complete the proof of the theorem.*

34.2. **Another look at the sine function.** *Problem.* Map the strip indicated into UHP preserving the points marked with circles and the orientation given.



Solution The α's at $-\pi$ and π are both $1/2$. We apply formula (34.3) with $a_1 = -1$, $a_2 = 1$ and the integrand is then $(s^2-1)^{-1/2}$. Eq. (34.3) therefore gives, for two arbitrary constants,

$$\Phi = C \arcsin z + C' \tag{34.12}$$

and therefore our map $f = \Phi^{-1}$ has the general form

$$\Phi^{-1}(w) = \sin(cw + c') \tag{34.13}$$

We have now to choose c and c' to match the prescribed points. We must have $\sin(-\pi c + c') = -1$ and $\sin(c\pi + c') = -1$; the choice $c' = 0$ and $c = 1/2$ matches these conditions. We get

$$f(w) = \sin(w/2) \tag{34.14}$$

35. Mapping of a rectangle: Elliptic functions

We map UHP in a rectangle following, roughly, [3]. All the α's in (34.3) are $1/2$, as in §34.2. We choose three a_k as simple as possible, $0, 1, \&\rho > 1$, and study the resulting rectangle. The freedom allows us to place three vertices wherever we want; we choose $C = 1$ and $C' = 0$. The integrand is now $s^{-1/2}(s-1)^{-1/2}(s-\rho)^{-1/2}$. We agree on a choice of branch of the square root (or the log for that matter). With cuts at $0, 1, \rho$ anywhere in the LHP, the argument of $z - a_k$ in UHP varies between 0 and π, and the square root is positive if the argument is positive and is in $i\mathbb{R}^+$ if the argument is negative. Throughout UHP any square root is in the first quadrant, as our branch halves the argument. We start the at $z = 0$ in (34.3) and let it increase. In $(0, 1)$, $s^{1/2} > 0$, two other square roots are purely imaginary, giving a net negative sign to the integrand. Thus in this region

$$\Phi = -\int_0^z \frac{dt}{\sqrt{t(1-t)(\rho-t)}} \tag{35.1}$$

where now all square roots are positive. This is a nonelementary integral an *elliptic integral*. As we know, for $z \in (0, 1)$we evolve along a straight line. This is obvious in this particular case: we go from zero along $\mathbb{R}^-$. The first vertex A_1 of the rectangle is at $-K$,

$$K = -\Phi(1)$$

As we know, the curve Φ is continuous, and at $s = 1$ the argument changes by $\pi/2$. We can also see this in the fact that for $z \in (1, \rho)$ the integrand is in $i\mathbb{R}^+$. Thus $\Phi - \Phi(1) \in \ i\mathbb{R}^-$. The second vertex of the rectangle is thus at $-K - iK'$ where

$$K' = \int_1^\rho \frac{dt}{\sqrt{t(1-t)(\rho-t)}} \tag{35.2}$$

Now we know all the effects of the transformation: the rectangle must close as we go from $\rho + 0$ to $+\infty$ but there is no harm in checking this directly. We could do so by changes of variables (what should they be? think of a qualitative argument!) or we can note that the integral

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{t(1-t)(\rho-t)}} = 0 \tag{35.3}$$

(The integral is well defined: convergence at infinity is ensured, the singular points are integrable.) Eq. (35.3) holds indeed because, with the given branch cuts, we can push the contour of integration towards $+\infty$ in the usual way (check this). This shows directly that the polygon closes. We can take a closer look at (35.3) and note that the imaginary part originates in the integral on the segments $(-\infty, 0)$ and $(1, \rho)$ while the other two segments give purely real contribution. The fact that the sum is zero means that the vertical sides are equal to each other and the horizontal ones too.

35.1. **Differential equation.** If we write the Schwarz-Christoffel formula for the rectangle in the form

$$\frac{d\Phi}{dz} = \frac{1}{\sqrt{z(z-1)(z-\rho)}} \quad or \quad \left(\frac{dz}{d\Phi}\right)^2 = z(z-1)(z-\rho) \tag{35.4}$$

differentiate with respect to ρ and divide by $dz/d\Phi$ we get

$$z'' = \frac{3}{2}z^2 - (\rho+1)z + \frac{\rho}{2} \tag{35.5}$$

a second order autonomous equation. A great deal of information can be extracted from this equation alone, and we will return to the subject in the second quarter.

35.2. **The symmetric version of the elliptic integral.** The double symmetry of the rectangle suggests a symmetric choice of a_i. Consider the following integral used in the literature of elliptic integrals

$$F(z) = \int_0^z \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{35.6}$$

This transformation is similar to but not quite in the form of (34.3). The square roots are combined in pairs and the signs are different. We could make the correspondence with (34.3) by carefully monitoring the products but we prefer to recalculate the effects of the transformation.

The square root branches $\sqrt{1-s^2}$ and $\sqrt{1-k^2s^2}$ are defined as follows. We make cuts in the lower half plane along $s = 1 - i\lambda$, and $k^{-1} - i\lambda$, $\lambda \geq 0$ respectively. The integrand is well defined and analytic in s, in the UHP.

With this definition, as we go from 0 to 1, F is positive and increasing. We reach a first vertex at

$$2K = \int_{-1}^1 \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{35.7}$$

When $s = 1^+$, $\arg(1-s^2) = -\pi$, thus $\sqrt{1-s^2} \in -i\mathbb{R}^+$, $(1-s^2)^{-1/2} \in i\mathbb{R}^+$ and $F(z)$ turns by an angle of $\pi/2$. After $1/k$, both square roots are imaginary and $\Re F$ decreases and $\Im F$ is constant. We can also look at the evolution of F when z goes from zero along $\mathbb{R}^-$. By the symmetry of the integrand, the integral becomes negative, then turns by $-\pi/2$ etc. The same symmetry argument shows that figure closes and it is a *rectangle* (how come it is possible to use *four* points instead of *three*?). The vertices are then, in a positive direction,

$$-K;\quad K;\quad \frac{K}{2}+iK';\quad -K+iK' \tag{35.8}$$

with

$$K' = \int_1^{1/k} \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{35.9}$$

Since the boundaries are traversed in a consistent direction, we map UHP into the *interior* of the rectangle. What happens to the point at infinity? It must be mapped on the boundary of the rectangle. It is clear where that is since

$$\int_0^\infty \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} = iK' \tag{35.10}$$

(the last equality follows by the symmetry of the function, since integrating from 0 to $-\infty$ would produce a reflected half-rectangle). Observe that for $s>1/k$ we can write

$$\sqrt{1-s^2}\sqrt{1-k^2s^2} = ks^2\sqrt{1-s^{-2}}\sqrt{1-(ks)^{-2}} \tag{35.11}$$

and that the function $\sqrt{1-\zeta}$ is *analytic* for $|\zeta|<1$. Thus the series

$$1/\sqrt{1-s^{-2}} = 1 - \frac{1}{2s^2} - \frac{1}{8s^2}\cdots \tag{35.12}$$

is convergent absolutely and uniformly and can be integrated term by term. For this reason for $s>1/k$ we can integrate term by term the series

$$\frac{1}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} = \frac{1}{ks^2} + \frac{k^{-1}+k^{-3}}{2s^4} + \cdots \tag{35.13}$$

and we get that, with $h=(1-s^2)^{-1/2}(1-k^2s^2)^{-1/2}$ and $x=1/z$,

$$F(z) = \int_0^z h(s)ds = \int_0^\infty h(s) + \int_\infty^z h(s)ds \tag{35.14}$$

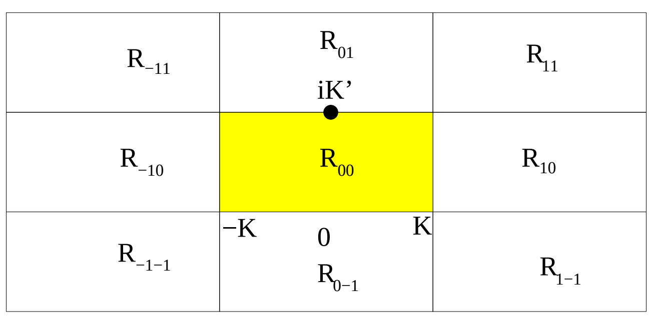
$$= iK' + \int_{1/k+\epsilon}^z \left[\frac{1}{ks^2} + \frac{k^{-1}+k^{-3}}{2s^4} + \cdots\right] ds = iK' - \frac{x}{k} - \frac{k^{-1}+k^{-3}}{6}x^3 \cdots$$

Note 35.1. The last series in (35.14) can clearly be inverted analytically and therefore x is analytic in $F - iK'$ *i.e.* $1/z$ is analytic in $F - iK'$ implying that $\mathcal{E}(F) := z(F)$ is **meromorphic** (since the Laurent series from the two sides of iK' match) with a simple pole at $F = iK'$. By the Schwarz reflection principle, at all other points on the boundary, $\mathcal{E}$ is analytic, see also §35.3. Thus $\mathcal{E}$ is meromorphic on the closed rectangle R_{00}.

Exercise 35.2. *** (i) What is the most general biholomorphic automorphism of UHP?*

(ii) How can we use this information to connect (35.1) to (35.6). Can you find changes of coordinates that would link them?

35.3. **Continuation to the whole of $\mathbb{C}$. Double periodicity.** We can analytically continue F past $(-1, 1)$. We get by Theorem 25.51 a similar rectangle where F is analytic and one-to one; therefore $\mathcal{E}$ extends analytically to $R_{00} \cup [-K, K] \cup R_{0,-1}$. We can do the same on $(1, 1/k)$ and get an extension to $R_{1,0}$. We can likewise reflect $R_{0,-1}$ down and get $R_{0,-2}$, etc. They fit, due to congruence of their sides. See figure.



We can continue reflecting and checking. What about $(1/k, \infty)$? The function F is analytic there and Theorem 25.51 applies. Thus, F extends analytically past this segment. Same for $(-\infty, -1/k)$.

By noting that two successive complex conjugations amount to the identity we see that the function thus defined in $R_{0,1}$ and $R_{0,-1}$ are the same! Same is true for $R_{-1,0}$ and $R_{1,0}$. We have obtained a *doubly periodic function*

$$\mathcal{E}(z + 2K) = \mathcal{E}(z + 2iK') = \mathcal{E}(z) \tag{35.15}$$

defined on $\mathbb{C}$. Functions with two periods in $\mathbb{C}$ are **elliptic functions**. In [3] we find that "The connection between elliptic integrals and elliptic functions was discovered by Gauss, but not published; it was rediscovered by Abel and Jacobi."

35.4. **Schwarz triangle functions.** The upper half plane is homeomorphically mapped on a triangle of angles $\alpha_1\pi, \alpha_2\pi, \pi(1-\alpha_1-\alpha_2)$ by a Schwarz-Christoffel transformation which has no auxiliary parameters, as we discussed:

$$\Phi(z) = \int_0^z s^{\alpha_1-1}(s-1)^{\alpha_2-1}ds \tag{35.16}$$

In this case too we can apply the reflection-continuation procedure of §35.3. We now imagine the reflections having a common vertex. To insure a single valued function, we must return to the starting triangle with no overlap or gap. A quick calculation shows we must have $1/\alpha_i \in \mathbb{N}$. By elementary geometry (check that!) we must have

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

with solutions $\alpha_i = 3$ (an equilateral triangle) $(2, 3, 6)$ (half of an equilateral triangle) and $(2, 4, 4)$ (isosceles right triangle). Then the reflected images cover the whole plane and the mapping functions are restrictions of meromorphic functions. These are the so-called *Schwarz triangle functions*. Each triangle function corresponds to an elliptic function. We will return to this topic in the second quarter.

36. The Riemann Mapping Theorem

The proof of this major theorem involves concepts and results that are very important and useful of their own. We will study these in detail.

36.1. **Equicontinuity.** We look at functions $f : M \mapsto M'$ where M, M' are metric spaces. We recall that if the metrics are d and d', a function is uniformly continuous if

$$\forall\delta\ \exists\epsilon\ \Big(\forall(z, z_0) \in M^2,\ d(z, z_0) < \epsilon \Rightarrow d'(f(z), f(z_0)) < \delta\Big) \tag{36.1}$$

We can assume that the metric d' is a **bounded function**, for we can always replace it by $d'' = d'/(1 + d')$ (check that d' is a metric) and convergence with respect to d' is the same as convergence with respect to d.

Definition. An equicontinuous family $\mathcal{F}$ is a collection of uniformly continuous functions with the same uniform continuity parameters:
(36.2)

$$\forall\delta\ \exists\epsilon\ \Big(\forall(z, z_0, f) \in M^2 \times \mathcal{F},\ d(z, z_0) < \epsilon \Rightarrow d'(f(z), f(z_0)) < \delta\Big)$$

36.2. **Weierstrass's theorem.**

Theorem 36.1. *Assume that f_n are analytic in the region Ω and converge uniformly on any compact set in Ω to f. Then f is analytic in Ω. Furthermore, $f_n' \to f'$ uniformly on any compact set in Ω.*

Proof. Let C be a simple closed curve contained in the compact $K \subset \Omega$. Then, by analyticity,

$$\int_C f_n(z)dz = 0 \tag{36.3}$$

Uniform convergence implies f is continuous. Furthermore, by dominated convergence we have from (36.3),

$$\int_C f(z)dz = 0 \tag{36.4}$$

Using Morera's theorem, we see that f is analytic. The properties of the derivatives are immediate, by Cauchy's formula. ∎

Definition: Normal families. Let M be a metric vector space. Then $\mathcal{F}$ is a normal family on M if any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ contains a convergent subsequence that converges uniformly on compact subsets of M.

Exhaustion by compact sets. We note that if the metric space Ω is $\mathbb{C}$, $\mathbb{R}^n$ or a subset M of these, we write

$$M = \bigcup_{n\in\mathbb{N}} K_n, \quad K_n = \{x \in M : d(x,0) + 1/d(x,\partial\Omega) \le n\}$$

Note that the sets K_n are closed and bounded, therefore compact; their union covers M. On each K_n we define the distance between two functions f and g in a manner analogous to the L^∞ distance:

$$\delta_n(f,g) = \sup_{x\in K_n} d'(f(x), g(x)) \tag{36.5}$$

and we create a distance on the whole of M which takes advantage of the compact exhaustion:

$$\rho(f,g) = \sum_{n=1}^{\infty} w_n \delta_n(f,g) \tag{36.6}$$

where $\sum_{n=0}^{\infty} w_n < \infty$ (recall, we assumed the distance is bounded by 1). Let's specifically take $w_n = 2^{-n}$.

Exercise 36.2. *Check that ρ is a metric on $\mathcal{F}$. Check that convergence with respect to ρ is equivalent with uniform convergence on compact sets. Check that $\mathcal{F}$ is a complete metric space if M' is a complete metric space.*

Theorem 36.3. *A family $\mathcal{F}$ is normal iff its closure $\overline{\mathcal{F}}$ with respect to ρ is compact.*

Proof. This follows from the fact that a space is compact iff any sequence has a convergent subsequence. □

36.2.1. *The Ascoli-Arzela Theorem.*

Theorem 36.4. *A family $\mathcal{F}$ of continuous functions in the region $\Omega \subset \mathbb{C}$ with values in a metric space M is* **normal** *in Ω iff the following conditions are both satisfied:*
(i) $\mathcal{F}$ is equicontinuous on every compact $K \subset \Omega$.
(ii) $\forall z \in \Omega\ \exists K_1$ compact in M such that $\forall f \in \mathcal{F}, f(z) \in K_1$.

Necessity (i) Suppose $\mathcal{F}$ is not equicontinuous on some compact K. Then on K
(36.7)
$$\exists(\epsilon > 0,\ \{z_n\}, \{z_n'\}, \{f_n\})\ s.t.(|z_n - z_n'| \to 0\ \&\ d(f_n(z_n), f_n(z_n') > \epsilon)$$
Since K is compact and $\mathcal{F}$ is normal from any sequence we can extract a convergent subsequence. Convince yourself that wlog we can assume $\{z_n\}, \{z_n'\}, \{f_n\}$ themselves convergent. Let $z_n \to z$, $f_n \to f$ ($z_n' \to z$ too). The limit f is continuous, thus uniformly continuous. We have
$$\lim_{n\to\infty} \sup_{x\in K} d'(f(x), f_n(x)) = 0$$
thus for n large enough,
$$d'(f_n(z_n'), f(z_n')) < \frac{\epsilon}{4},\ d'(f(z_n'), f(z)) < \frac{\epsilon}{4}, \quad (36.8)$$
$$d'(f(z), f(z_n)) < \frac{\epsilon}{4} \quad \text{and} \quad d'(f(z_n), f_n(z_n)) < \frac{\epsilon}{4}$$
implying by the triangle inequality,
$$d'(f_n(z_n'), f_n(z_n)) < \epsilon$$
a contradiction.

(ii) Fix z and take $K = \overline{\{f(z) : f \in \mathcal{F}\}}$. Take a sequence $\{w_n\} \subset K$ By the definition of K, if $w_n \in K\ \exists f_n \in \mathcal{F}$ such that $d(f_n(z), w_n) < 1/n$. By the normality of $\mathcal{F}$, there exists a subsequence of functions, wlog $\{f_n\}$ themselves, $f_n \to f$. But then $w_n \to f(z)$ □.

Sufficiency. The sufficiency of the two conditions is shown by Cantor's famous diagonal argument. Let $\{f_n\} \subset \mathcal{F}$. We take a denumerable everywhere dense set $Z = \{z_k\}$ of points in Ω, e.g., those with rational coordinates and we let $\mathcal{K}$ be any compact in Ω. Take $z_1 \in Z$. By (ii), there is a convergent subsequence $\{f_{n_{j1}}(z_1)\}_{j\in\mathbb{N}}$. Take now $z_2 \in Z$. From $\{f_{n_{j1}}(z_2)\}$we can extract a subsequence $\{f_{n_{j2}}(z_2)\}_{j\in\mathbb{N}}$

which converges as well. So $\{f_{n_{j2}}(z)\}_{j\in\mathbb{N}}$ converges both at z_1 and z_2. Inductively we find a subsequence $\{f_{n_{jm}}(z)\}_{j\in\mathbb{N}}$ such that it converges at the points $z_1, ..., z_m$. But then, the subsequence $\{g_j\} := \{f_{n_{jj}}\}$ converges at all points in Z. We aim to show that g_j *converges uniformly* in any compact set $K \in \Omega$. By equicontinuity,
(36.9)
$$\forall\epsilon > 0\, \exists\delta\, s.t.\ \forall(a, b, f) \in K^2 \times \mathcal{F}(|a - b| < \delta \Rightarrow d(f(a), f(b)) < \frac{\epsilon}{3})$$

Consider a finite covering of K by balls of radius $\delta/2$. Since Z is everywhere dense, there is a z_k in each of these balls. They are finitely many, so that for $l, m > n_0$,

$$d(g_l(z_k), g_m(z_k)) < \frac{\epsilon}{3} \tag{36.10}$$

On the other hand, any $a \in \mathcal{K}$ is, by construction, at distance at most δ from some z_k and thus by (36.9) (for any $f \in \mathcal{F}$, in particular) for g_{ni}, g_{nj} we have

$$d(g_l(a), g_l(z_k)) < \frac{\epsilon}{3} \tag{36.11}$$

$$d(g_m(a), g_m(z_k)) < \frac{\epsilon}{3} \tag{36.12}$$

We thus see by the triangle inequality that

$$d(g_l(a), g_m(a)) < \epsilon \tag{36.13}$$

Thus $g_n(a)$ converges. Convergence is uniform since the pair ϵ, δ is independent of a. $\square$

Proposition 36.5. *Let now $M \in \mathbb{R}^n$ and $\mathcal{F}$ be a normal family from Ω to M. Let $K \subset \Omega$ be compact. Then the bound on $f(z)$ can be made z independent in K:*

$$\sup_{z\in K, f\in\mathcal{F}} |f(z)| = m < \infty \tag{36.14}$$

Proof. Since $\mathcal{F}$ is a normal family we can find δ such that

$$|a - b| < \delta \Rightarrow |f(a) - f(b)| < 1 \tag{36.15}$$

Consider now a finite covering of K by balls of radius δ and let z_j be their centers. We denote $m_j = \sup\{|f(z_j)| : f \in \mathcal{F}\}$ and $m = 1 + \max_j\{m_j\}$. Then, for any $a \in \mathcal{K}$ there is a z_j such that $|a - z_j| < \delta$. Therefore, by the choice of m and by (36.15) we have

$$|f(a)| \le |f(a) - f(z_j)| + |f(z_j)| = 1 + m_i \le m \quad \square \tag{36.16}$$

Theorem 36.6. *Consider a region $\Omega \in \mathbb{C}$ and assume $\mathcal{F}$ is a family of analytic functions Ω such that for every compact $K \in \Omega$ we have $\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} < \infty$. Then the family is normal.*

Proof. It suffices to consider the case when Ω is bounded. Let $K \subset \Omega$ be compact. All we have to show is that the family is equicontinuous in K. Every $z \in \Omega$ is contained in some disk itself contained in Ω. Ω is the union of these disks:

$$\Omega = \bigcup_{z\in\Omega} D(z, \epsilon(z))$$

K is compact in the open set Ω. Let $0 < 2r = \mathrm{dist}(K, \partial\Omega)$. Consider the set $E = \{z : \mathrm{dist}(z, \partial\Omega) \geq r\}$. The distance is a continuous function, thus E is a compact set. Clearly

$$K \subset \bigcup_{z\in K} D(z, r) \subset E \subset \Omega$$

Let $m = \sup\{|f(z)| : f \in \mathcal{F}, z \in E\}$. Choose $z, z_0 \in K$, $|z - z_0| < \epsilon < r/2$ and let $S = \partial D(r, z_0)$. By Cauchy's formula we write

$$|f(z) - f(z_0)| \leq \frac{1}{2\pi} \int_S \left| \frac{1}{s-z} - \frac{1}{s-z_0} \right| |f(s)| d|s| \tag{36.17}$$
$$\leq \frac{|z-z_0|}{2\pi} \int_S \frac{|f(s)|}{|s-z||s-z_0|} d|s| \leq \frac{4m|z-z_0|}{r} \leq \frac{4m\epsilon}{r} \qquad \square$$

36.3. **Boundedness of derivatives.**

Theorem 36.7. *If the functions in $\mathcal{F}$ are analytic and uniformly bounded on any compact set, then their derivatives (of any order) are bounded too.*

The proof is left as an exercise: it relies on a simple use of the Cauchy's formula for derivatives. $\square$.

36.4. **Hurwitz's theorem.**

Theorem 36.8 (Hurwitz)**.** *If f_n are analytic and nonzero in a region $\Omega \in \mathbb{C}$ and f_n converge to $f \not\equiv 0$ uniformly on compact sets, then $f(z)$ has no zeros on Ω either.*

Proof We aim to show that f, which is analytic as a uniform limit of analytic functions, has no zeros. We take an arbitrary z_0 and show it is not a zero by Proposition 22.46. Whether z_0 is a zero or not, there is a ρ such that there is no zero of f in *some* punctured disk $D_1 = \{z : 0 < |z - z_0| < \rho\}$, because the zeros of analytic functions are isolated. Taking ρ a bit smaller if needed, we have

$$\min_{\partial D_1} |f| > 0$$

Thus $1/f$ is analytic near ∂D_1. The functions $1/f_n$ clearly do not vanish either, and converge to f on ∂D_1 (check this). Likewise $f_n' \to f'$ on ∂D_1 uniformly. But then

$$N = \frac{1}{2\pi i}\int_{\partial D_1} \frac{f'(s)}{f(s)} ds = \lim_n \frac{1}{2\pi i}\int_{\partial D_1} \frac{f_n'(s)}{f_n(s)} ds = 0 \tag{36.18}$$

□

36.5. **The Riemann Mapping Theorem. Statement and proof.**

Theorem 36.9. *Given any simply connected region $\Omega \subset \mathbb{C}$ other than $\mathbb{C}$ itself, and a point $z_0 \in \Omega$ and the normalization conditions $f(z_0) = 0, f'(z_0) \in \mathbb{R}^+$ there exists a unique biholomorphism $f(z)$ between Ω and D.*

Note 36.10. *The fact that $\mathbb{C}$ itself is an exception follows from the fact that an entire bounded function is constant.*

Proof. Uniqueness is easy; it relies on the fact that if f_1 and f_2 are two functions with the stated properties, then $S := f_1(f_2)^{-1}$ is a biholomorphism of the unit disk. Remember that Schwarz's lemma implies that these are linear transformations. The normalization given picks the identity.
Existence. **Definition** A function is univalent (*schlicht*) if $g(z_1) = g(z_2) \Rightarrow z_1 = z_2$.

The Riemann mapping function is selected from the class of all functions $\mathcal{F}$ defined on Ω with values in D (not necessarily covering the whole of D) and the normalization conditions $g(z_0) = 0, g'(z_0) > 0$. It is the $f \in \mathcal{F}$ with maximal $f'(z_0)$; this may not be surprising if we think of Schwarz's lemma.

We have to show that $\mathcal{F}$ is nonempty, that a function with maximal derivative exists, and, of course, that f has the desired properties.
(1) Nonemptiness. There is an $a \notin \Omega$. Since Ω is simply connected, we can integrate $(z-a)^{-1}$ to get a well defined branch of $\ln(z-a)$ in Ω (strange way to use that $\Omega \ne \mathbb{C}$!). Thus $\sqrt{z-a}$ is well defined in Ω and is single valued. Also, we cannot have $\sqrt{z-a} = -\sqrt{z'-a}$ for different $z, z' \in \Omega$. Indeed, we know our (well defined) square root is the inverse of the square. Squaring both sides, we get $z = z'$. Let $h(z) = \sqrt{z-a}$. If $z_0 \in \Omega$, $-h(z_0) \notin h(\Omega)$ and thus $|h(z_0) + h(z_0)| > 2\rho'' > 0$. We must have, for some ρ'

$$|h(z) + h(z_0)| \ge \rho' \ \forall z \in \Omega \tag{36.19}$$

Indeed, otherwise $h(z_k)$ would approach $-h(z_0)$ arbitrarily for some sequence $\{z_k\}$. Again, by squaring, we get that $z_k \to z_0$ in the process. But $h(z_0) \neq 0$ and we would get that $\sqrt{z_k - a}$ is arbitrarily close $-\sqrt{z_0 - a}$ with $z_k \to z_0$. This contradicts the continuity of h at z_0. We let ρ be the least of ρ', ρ''. We now construct a $g_0 \in \mathcal{F}$. It is

$$g_0 = \frac{\rho}{3} \frac{|h'(z_0)|}{|h(z_0)|^2} \frac{h(z_0)}{h'(z_0)} \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \tag{36.20}$$

Univalence is preserved by composition, linear fractional transformations are univalent, and g_0 is gotten from the univalent h by a LFT. Thus g_0 is univalent. Obviously $g_0(z_0) = 0$ and a calculation gives

$$g'(z_0) = \frac{\rho}{6} \frac{|h'(z_0)|}{|h(z_0)^2|} \in \mathbb{R}^+$$

We have on the other hand

$$\left| \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \right| = |h(z_0)| \left| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \right| \leq \frac{3|h(z_0)|}{\rho} \tag{36.21}$$

where we used (36.19) implying that $|g_0| < 1$.

Since all the functions g in $\mathcal{F}$ have the property $|g| < 1$ (they have values in D), then $\mathcal{F}$ is normal. The derivatives g' are bounded on compact sets too, by Theorem 36.7. Let $\sup |g'(z_0)| = B < \infty$. There is therefore a sequence g_n such that $g_n'(z_0) \to B$. We extract a subsequence g_{n_k} uniformly convergent on compact sets $g_{n_k}(z) \to f(z)$; f is analytic too. Since all $|g_{n_k}| \leq 1$ we must have $|f| \leq 1$. This is our f, as we shall prove.

(2) Univalence. Since $f'(z_0) = B > 0$ f cannot be a constant. Now take $z_1 \in \Omega$ and look at the functions $G = g(z) - g(z_1)$ in $\Omega_1 = \Omega \setminus \{z_1\}$. Since g are univalent, the functions G are never zero in Ω_1. Hence, by Hurwitz's theorem $f(z) - f(z_1) \neq 0$ in $\Omega \setminus \{z_1\}$. Since z_1 was arbitrary, f is univalent.

(3) f covers the whole disk. This is shown by contradiction. Assuming that $w_0 \in D$ and $w_0 \notin f(\Omega)$, we take

$$\frac{f(z) - w_0}{1 - \overline{w_0} f(z)}$$

It does not vanish in the simply connected domain Ω, thus we can define by integration its log and from it

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w_0} f(z)}} \tag{36.22}$$

We have $|F| \leq 1$ (remember LFT are automorphisms of D, and we only need to check values on ∂D). We now ensure the function vanishes at

z_0 and the derivative is positive there. This is done by another LFT, an automorphism of the disk again,

$$G(z) = \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)} \frac{|F'(z_0)|}{F'(z_0)} \tag{36.23}$$

But a simple calculation shows

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)^2|} > B \tag{36.24}$$

in contradiction with the maximality of f. (The fact that the calculation yields this contradiction comes for Schwarz's lemma since (36.22) and (36.23) allow us to define f as a function of G from D to D. Then $|H'(0)| < 1$.)

37. Asymptotic series

We have seen in the Schwarz-Christoffel section that the behavior of analytic functions near a point of nonanalyticity can be given by a series in noninteger powers of the distance to the singularity. The behavior can be more complicated, containing exponentially small corrections, logarithmic terms and so on. The series themselves may have zero radius of convergence. It is not the purpose of this part of the course to classify these behaviors, but it can be done for a fairly large class of functions. Here we look how simple behaviors can be determined for relatively simple functions.

Example 37.1. *Consider the following integral related to the so–called error function*

$$F(z) = e^{z^{-2}} \int_0^z s^{-2} e^{-s^{-2}} ds$$

It is clear that the integral converges at the origin, if the origin is approached through real values (see also the change of variable below). **Definition of** $F(z)$. *We define the integral* to $z \in \mathbb{C}$ as being taken on a curve γ with $\gamma'(0) > 0$, and define $F(0) = 0$.

Check that this is a consistent definition and the resulting function is analytic except at $z = 0$ (this is essentially the contents of Exercise 37.3 below.

What about the behavior at $z = 0$? It depends on the direction in which 0 is approached! Let's look more carefully. Replace z by $1/x$, make a corresponding change of variable in the integral and you are led to

$$E(x) = e^{x^2} \int_x^\infty e^{-s^2} ds =: \frac{\sqrt{\pi}}{2} e^{x^2} \mathrm{erfc}(x) \tag{37.1}$$

Let us take x (and thus z) real and integrate by parts m times

(37.2)

$$E(x) = \frac{1}{2x} - \frac{e^{x^2}}{2}\int_x^\infty \frac{e^{-s^2}}{s^2}ds = \frac{1}{2x} - \frac{1}{4x^3} + \frac{3e^{x^2}}{4}\int_x^\infty \frac{e^{-s^2}}{s^4}ds = ...$$
$$= \sum_{k=0}^{m-1} \frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} + \frac{(-1)^m e^{x^2}\Gamma(m+\frac{1}{2})}{\sqrt{\pi}}\int_x^\infty \frac{e^{-s^2}}{s^{2m}}ds$$

On the other hand, we have, by L'Hospital

$$\left(\int_x^\infty \frac{e^{-s^2}}{s^{2m}}ds\right)\left(\frac{e^{-x^2}}{x^{2m+1}}\right)^{-1} \to \frac{1}{2} \text{ as } x \to \infty \tag{37.3}$$

and the last term in (37.2) is $O(x^{-2m-1})$ as well. On the other hand it is also clear that the series in (37.2) is alternating and thus

$$\sum_{k=0}^{m-1} \frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} \le E(x) \le \sum_{k=0}^{m} \frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} \tag{37.4}$$

if m is even.

Remark 37.2. *Using (37.3) and Exercise 37.13 below we conclude that at zero $F(z)$ has a Taylor series,*

$$\tilde{F}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2\sqrt{\pi}}\Gamma(k+\frac{1}{2})z^{2m+1} \tag{37.5}$$

that $F(z)$ is C^∞ on $\mathbb{R}$ and analytic away from zero.

Exercise 37.3. *** Show that $z = 0$ is an isolated singularity of $F(z)$. Using Remark 37.2, show that F is unbounded as 0 is approached along some directions in the complex plane.*

Notes (1) It is *not* the Laurent series of f at 0 that we calculated! Laurent series converge. Think carefully about this distinction and why the positive index coefficients do not coincide.

(2) The rate of convergence of the Laurent series is slower as 0 is approached, quickly becoming numerically useless. By contrast, the precision gotten from (37.4) near zero is such that for $z = 0.1$ the error in calculating f is of order 10^{-45} ! However, of course (37.4) is divergent and it cannot be used to calculate *exactly* for any nontrivial value of z.

(3) We have illustrated here a simple method of evaluating the behavior of integrals, the method of integration by parts.

37.1. **More general asymptotic series.** Classical asymptotic analysis typically deals with the qualitative and quantitative description of the behavior of a function close to a point, usually a singular point of the function. This description is provided in the form of an *asymptotic expansion.* The expansion certainly depends on the point studied and, as we have noted, often on the direction along which the point is approached (in the case of several variables, it also depends on the relation between the variables as the point is approached). If the direction matters, it is often convenient to change variables to place the special point at infinity.

Asymptotic expansions are formal series[1] of simpler functions f_k,

$$\tilde{f} = \sum_{k=0}^{\infty} f_k(t) \tag{37.6}$$

in which each successive term is much smaller than its predecessors (one variable is assumed for clarity). For instance if the limiting point is t_0 approached from above along the real line this requirement is written

$$f_{k+1}(t) = o(f_k(t)) \quad \text{or} \quad f_{k+1}(t) \ll f_k(t) \quad \text{as} \quad t \downarrow t_0 \tag{37.7}$$

denoting

$$\lim_{t \to t_0^+} f_{k+1}(t)/f_k(t) = 0 \tag{37.8}$$

We will often use the variable x when the limiting point is $+\infty$ and z when the limiting point is zero. Simple examples are the Taylor series, e.g.

$$\sin z + e^{-\frac{1}{z}} \sim z - \frac{z^3}{6} + \ldots \quad (z \to 0^+)$$

and the expansion in the Stirling formula

$$\ln \Gamma(x) \sim x \ln x - x - \frac{1}{2} \ln x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}, \ x \to +\infty$$

where B_k are the Bernoulli numbers.

[1]That is, there are no convergence requirements. More precisely, they are defined as sequences $\{f_k\}_{k \in \mathbb{N} \cup \{0\}}$, the operations being defined in the same way as if they represented convergent series; see also §37.2.

(The asymptotic expansions in the examples above are the formal sums following the "$\sim$" sign, the meaning of which will be explained shortly.)

Examples of expansions that are *not* asymptotic expansions are

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (x \to +\infty)$$

which converges to $\exp(x)$, but it is not an asymptotic series for large x since it fails (37.7); another example is the series

$$\sum_{k=0}^{\infty} \frac{x^{-k}}{k!} + e^{-x} \quad (x \to +\infty) \tag{37.9}$$

(because of the exponential terms, this is not an ordered *simple series* satisfying (37.7)). Note however expansion (37.9), *does* satisfies all requirements in the *left* half plane, if we write e^{-x} in the first position.

We also note that in this particular case the first series is convergent, and if we replace (37.9) by

$$e^{1/x} + e^{-x} \tag{37.10}$$

then (37.10) *is* a valid asymptotic expansion, of a very simple kind, with two nonzero terms. Since convergence is relative to a topology, this elementary remark will play a crucial role when we will speak of Borel summation.

Functions asymptotic to a series, in the sense of Poincaré. The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits:

Definition 37.4. *A function f is asymptotic to the formal series $\tilde{f}$ as $t \to t_0^+$ if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) =: f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{37.11}$$

We note that condition (37.11) can then be also written as

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{37.12}$$

where $g(t) = O(h(t))$ means $\limsup_{t \to t_0^+} |g(t)/h(t)| < \infty$. Indeed, this follows from (37.11) and the fact that $f(t) - \sum_{k=0}^{N+1} \tilde{f}_k(t) = o(\tilde{f}_{N+1}(t))$.

37.2. **Asymptotic power series.** In many instances the functions f_k are exponentials, powers and logarithms. This is not simply a matter of choice or an accident, but reflects some important fact about the relation between asymptotic expansions and functions which will be clarified later.

A special role is played by power series which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \quad z \to 0^+ \tag{37.13}$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$) the series can be centered at t_0 (or $+\infty$, respectively).

Remark. If a c_k is zero then Definition 37.4 fails trivially in which case (37.13) is not an asymptotic series. This motivates the following definition.

Definition 37.5 (Asymptotic power series)**.** *A function possesses an asymptotic* power *series if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \tag{37.14}$$

We use the boldface notation $\boldsymbol{\sim}$ for the stronger asymptoticity condition in (37.11) when confusion is possible.

Example Check that the Taylor series of an analytic function at zero is its asymptotic series there.

In the sense of (37.14), the asymptotic power series at zero of e^{-1/x^2} is the zero series. It is however surely not the case that e^{-1/x^2} behaves like zero as $x \to 0$ on $\mathbb{R}$. Rather, in this case, the asymptotic *behavior* of e^{-1/x^2} is e^{-1/x^2} itself (only exponentials and powers involved).

Asymptotic power series form an algebra; addition of asymptotic power series is defined in the usual way:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} c'_k z^k = \sum_{k=0}^{\infty}(Ac_k + Bc'_k) z^k$$

while multiplication is defined as in the convergent case

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} c'_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j c'_{k-j}\right) z^k$$

Remark 37.6. *If the series $\tilde{f}$ is convergent and f is its sum (note the ambiguity of the "sum" notation) $f = \sum_{k=0}^{\infty} c_k z^k$ then $f \sim \tilde{f}$.*

The proof of this remark follows directly from the definition of convergence.

Lemma 37.7. *(Uniqueness of the asymptotic series to a function) If $f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$ as $z \to 0$ then the $\tilde{f}_k$ are unique.*

Proof. Assume that we also have $f(z) \sim \tilde{F} = \sum_{k=0}^{\infty} \tilde{F}_k z^k$. We then have (cf. (37.11))

$$\tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = o(z^N)$$

which is impossible unless $g_N(z) = \tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = 0$, since g_N is a polynomial of degree N in z. ∎

Corollary 37.8. *The asymptotic series at the origin of an analytic function is its Taylor series at zero. More generally, if F has a Taylor series at 0 then that series is its asymptotic series as well.*

The proof of the following lemma is immediate:

Lemma 37.9. *(Algebraic properties of asymptoticity to a power series) If $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ and $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ then*
(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$
(ii) $fg \sim \tilde{f}\tilde{g}$

Sometimes it is convenient to check a formally weaker condition of asymptoticity:

Lemma 37.10. *Let $\tilde{f} = \sum_{n=0}^{\infty} a_n z^n$. If f is such that there exists a sequence $p_n \to \infty$ such that*

$$\Big(\forall n \exists p_n\Big) \ s.t. \ \ f(z) - \tilde{f}^{[p_n]}(z) = o(z^n) \quad as \quad z \to 0$$

then $f \sim \tilde{f}$.

Proof. We let m be arbitrary and choose $n > m$ such that $p_n > m$. We have

$$f(z) - \tilde{f}^{[m]} = (f(z) - \tilde{f}^{[p_n]}) + (\tilde{f}^{[p_n]} - \tilde{f}^{[m]}) = o(z^m) \ (z \to 0)$$

by assumption and since $\tilde{f}^{[p_n]} - \tilde{f}^{[m]}$ is a polynomial for which the smallest power is z^{m+1} (we are dealing with truncates of the same series). ∎

37.3. **Integration and differentiation of asymptotic power series.** While asymptotic power series can be safely integrated term by term as the next proposition shows, differentiation is more delicate. In suitable spaces of functions and expansions, we will see the asymmetry largely disappears if we are dealing with analytic functions in suitable regions.

Anyway, for the moment note that the function $e^{-1/z}\sin(e^{1/z^2})$ is asymptotic to the zero power series as $z \to 0^+$ although the derivative is unbounded and thus not asymptotic to the zero series.

Proposition 37.11. *Assume f is integrable near $z = 0$ and that*

$$f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} \tilde{f}_k z^k$$

Then

$$\int_0^z f(s)ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{k+1} z^{k+1}$$

Proof. This follows from the fact that $\int_0^z o(s^n)ds = o(z^{n+1})$ as can be seen by immediate estimates. ▮

Asymptotic power series of analytic function, if they are valid in wide enough regions can be differentiated.

Asymptotics in a strip. Assume $f(x)$ is analytic in the strip $S_a = \{x : |x| > R, |\Im(x)| < a\}$. Let $\alpha < a$ and and $S_\alpha = \{x : |x| > R, |\Im(x)| < \alpha\}$ and assume that

$$f(x) \sim \tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k} \quad (|x| \to \infty, x \in S_\alpha) \tag{37.15}$$

It is assumed that that the limits implied in (37.15) hold uniformly in the given strip.

Proposition 37.12. *If (37.15) holds, then, for $\alpha' < \alpha$ we have*

$$f'(x) \sim \tilde{f}'(x) := \sum_{k=0}^{\infty} -\frac{kc_k}{x^{k+1}} \quad (|x| \to \infty, x \in S_{\alpha'})$$

Proof. We have $f(x) = \tilde{f}^{[N]}(x) + g_N(x)$ where clearly g is analytic in S_a and $|g_N(x)| \le Const.|x|^{-N-1}$ in S_α. But then, for $x \in S_{\alpha'}$ and $\delta = \frac{1}{2}(\alpha - \alpha')$ we get

$$|g_N'(x)| = \frac{1}{2\pi}\left|\oint_{|x-s|=\delta}\frac{g_N(s)ds}{(s-x)^2}\right| \le \frac{1}{\delta}\frac{Const.}{(|x|-|\delta|)^{N+1}}$$
$$= O(x^{-N-1}) \quad (|x|\to\infty,\ x\in S_{\alpha'})$$

By Lemma 37.10, the proof follows. ▮

Exercise 37.13. *** Show that if $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ and $f'(x)\to L$ as $x\downarrow 0$, then f is differentiable to the right at zero and this derivative equals L. Use this fact, Proposition 37.12 and induction to show that the Taylor series at the origin of $F(z)$ is indeed given by (37.5).*

37.4. **Watson's Lemma.** In many instances integral representations of functions are amenable to Laplace transforms

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-xp}F(p)dp \tag{37.16}$$

The behavior of $\mathcal{L}F$ for large x relates to the behavior for small p of F.

It is shown in the later parts of this book that solutions of generic analytic differential equations, under mild assumptions can be conveniently expressed in terms of Laplace transforms.

For the error function note that

$$\int_N^\infty e^{-s^2}ds = N\int_1^\infty e^{-N^2u^2}du = \frac{\sqrt{x}e^{-x}}{2}\int_0^\infty \frac{e^{-xp}}{\sqrt{p+1}}dp;\quad x=N^2$$

For the Gamma function, writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ in

$$n! = \int_0^\infty e^{-t}t^n dt = n^{n+1}\int_0^\infty e^{n(-s+\ln s)}ds \tag{37.17}$$

we can make the substitution $t-\ln t = p$ in each integral and obtain

$$n! = n^{n+1}e^{-n}\int_0^\infty e^{-np}G(p)dp$$

Watson's Lemma

This important tool states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 37.14. *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^{\infty} c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants β_i with $\Re(\beta_i) > 0$, $i = 1, 2$. Then*

$$\mathcal{L}F \sim \sum_{k=0}^{\infty} c_k \Gamma(k\beta_1 + \beta_2) x^{-k\beta_1-\beta_2}$$

along any ray ρ in the open right half plane H.

Proof. Induction, using the simpler version, Lemma 37.15, proved below. □

Lemma 37.15. *Let $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ and assume*

$$F(p) \sim p^{\beta} \quad \text{as } p \to 0^+$$

with $\Re(\beta) > -1$. Then

$$\int_0^{\infty} F(p) e^{-px} dp \sim \Gamma(\beta+1) x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $U(p) = p^{-\beta} F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let $\mathcal{X}_A$ be the characteristic function of the set A and $\phi = \arg(x)$. We choose C and a positive so that $|F(p)| < C|p^{\beta}|$ on $[0, a]$. Since

$$\left| \int_a^{\infty} F(p) e^{-px} \mathrm{d}p \right| \le e^{-|x|a\cos\phi} \|F\|_1 \tag{37.18}$$

we have by dominated convergence, and after the change of variable $s = p|x|$,

(37.19)
$$x^{\beta+1} \int_0^{\infty} F(p) e^{-px} \mathrm{d}p = e^{i\phi(\beta+1)} \int_0^{\infty} s^{\beta} U(s/|x|) \mathcal{X}_{[0,a]}(s/|x|) e^{-se^{i\phi}} \mathrm{d}s$$
$$+ O(|x|^{\beta+1} e^{-|x|a\cos\phi}) \to \Gamma(\beta+1) \quad (|x| \to \infty)$$

37.5. **Example: the Gamma function.** We start from the representation

$$n! = \int_0^{\infty} t^n e^{-t} dt = n^{n+1} \int_0^{\infty} e^{-n(s-\ln s)} ds$$
$$= n^{n+1} \int_0^1 e^{-n(s-\ln s)} ds + n^{n+1} \int_1^{\infty} e^{-n(s-\ln s)} ds \tag{37.20}$$

On $(0,1)$ and $(1,\infty)$ separately, the function $s-\ln(s)$ is monotonic and we may write, after inverting $s-\ln(s)=t$ on the two intervals to get $s_{1,2}=s_{1,2}(t)$,

$$n! = n^{n+1}\int_1^\infty e^{-nt}(s_2'(t)-s_1'(t))dt = n^{n+1}e^{-n}\int_0^\infty e^{-np}G(p)dp \tag{37.21}$$

where $G(p)=s_2'(1+p)-s_1'(1+p)$. In order to determine the asymptotic behavior of $n!$ we need to determine the small p behavior of the function $G'(p)$

Remark 37.16. *The function $G(p)$ is an analytic function in $\sqrt{p}$ and thus $G'(p)$ has a convergent Puiseux series*

$$\sum_{k=-1}^{\infty} c_k p^{k/2} = \sqrt{2}p^{-1/2} + \frac{\sqrt{2}}{6}p^{1/2} + \frac{\sqrt{2}}{216}p^{3/2} - \frac{139\sqrt{2}}{97200}p^{5/2} + \ldots$$

Thus, by Watson's Lemma, for large n we have

$$n! \sim \sqrt{2\pi n}\,n^n e^{-n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}+\ldots\right) \tag{37.22}$$

Proof. We write $s=1+S$ and $t=1+p$ and the equation $s-\ln(s)=t$ becomes $S-\ln(1+S)=p$. Note that $S-\ln(1+S)=S^2U(S)/2$ where $U(0)=1$ and $U(S)$ is analytic for small S; with the natural branch of the square root, $\sqrt{U(S)}=H(S)$ is also analytic. We rewrite $S-\ln(1+S)=p$ as $SH(S)=\pm\sqrt{2}\sigma$ where $\sigma^2=p$. Since $(SH(S))'_{|S=0}=1$ the implicit function theorem ensures the existence of two functions $S_{1,2}(\sigma)$ (corresponding to the two choices of sign) which are analytic in σ. The concrete expansion may be gotten by implicit differentiation in $SH(S)=\pm\sqrt{2}\sigma$, for instance. ∎

38. Riemann-Hilbert problems: an introduction

An impressive number of problems can be reduced to so-called Riemann Hilbert problems, and for many of them the only known way of solution is via the associated Riemann-Hilbert problem.

A core RH problem to which many of the others can be brought to is: given a simple smooth contour C and $f(t)$ a suitably regular function on C, find an analytic function whose jump across C is f:

$$\Phi^+(t)-\Phi^-(t)=f(t) \tag{38.1}$$

Problems which can be solved with RH techniques include

(1) Finding the inverse Radon transform, a transform which is measured in computerized tomography,

(2) solving integral equations of the type

$$f(t) + \int_0^\infty \alpha(t - t')f(t')dt' = \beta(t) \tag{38.2}$$

(under suitable integrability conditions)

(3) solving singular integral equations of the type

$$f(t) + P\int_a^b \frac{\alpha(t')}{t' - t} f(t')dt' = \beta(t) \tag{38.3}$$

where $P\int$ is the principal value integral we introduced in §14.2.

(4) inverse scattering problems: find, from the scattering data the potential $q(x)$ in the time–independent Schrödinger equation

$$\psi_{xx} + (k^2 + q(x))\psi = 0 \tag{38.4}$$

(5) Solving the nonlinear initial value problem for the KdV (Korteweg–deVries) equation

(38.5)

$$u_t + u_{xxx} + uu_x = 0;\ \ u(x,0) = u_0(x),\ u \to 0 \text{ as } |x| \to \infty\ \ (x \in \mathbb{R}, t \in \mathbb{R}^+)$$

(6) solving transcendental Painlevé equations e.g. $y'' = 6y^2 + x$ (P_I). and many others.

38.1. **Generalization: $\overline{\partial}$ (DBAR) problems.** A particular case of a RH problem is to find an analytic function with a given jump across the real line:

$$\Phi^+(x) - \Phi^-(x) = f(x) \tag{38.6}$$

with $\Phi^\pm$ analytic in the UHP (LHP) respectively.

If we let Φ be defined by Φ^+ in the UHP and by Φ^- in the LHP, then we have

$$\frac{\partial \Phi}{\partial y} = \frac{1}{2} f(x)\delta(y) \tag{38.7}$$

A general $\overline{\partial}$ problem would be, given g, to solve

$$\frac{\partial \Phi}{\partial \overline{z}} = g(x, y) \tag{38.8}$$

in some region $\mathcal{D} \subset \mathbb{C}$.

39. Cauchy type integrals

We recall that a function is Hölder continuous of order λ on a smooth curve C if $f(x) - f(y) = O((x-y)^\lambda)$ as $x \to y$ on C. The condition is compatible with continuity if $\lambda > 0$ and nontrivial if $\lambda \le 1$ (if $\lambda > 1$ then $df/ds = 0$).

Let C for now be a closed contour or a compact curve and ϕ be Hölder continuous on C. Then the function

$$\Phi(z) = \frac{1}{2\pi i}\int_C \frac{\phi(s)}{s-z}ds \tag{39.1}$$

is manifestly analytic for $z \notin C$ (you can check this by Morera's theorem using Fubini).

39.1. **Asymptotic behavior.** It is a simple exercise to show that we have

$$\Phi(z) \sim -\frac{\int_C \phi(s)ds}{2\pi i}\,\frac{1}{z} \quad \text{as} \quad z \to \infty \tag{39.2}$$

as $z \to \infty$.

Let C be a simple smooth contour and let t be an interior point of C. By this we mean that $C = \{\gamma(s) : s \in [0,1]\}$ and $t = \gamma(s_1)$ with $s_1 \in (0,1)$. We can then draw a small circle centered at t which intersects L in two points (a_1 and a_2). One arc of circle together with the curve segment between a_1 and a_2 form a closed Jordan curve, and so does the other arc circle and the curve segment between a_1 and a_2. A sequence approaches C from the left side if it eventually belongs to the closed Jordan curve whose interior is to the left of C as the curve is traversed positively (a similar definition applies to right limits).

Theorem 39.1 (Plemelj's formulas)**.** *Assume ϕ is Hölder continuous on the simple smooth contour C and let t be an interior point of C and z_n approach $t \in \mathrm{Int}\ C$ from the left (right). Then,*

$$\lim_{n\to\infty} \Phi(z_n) = \Phi^{\pm}(t) \tag{39.3}$$

where

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i}P\int \frac{\phi(s)}{s-t}ds \tag{39.4}$$

and

$$P\int \frac{\phi(s)}{s-t}ds = \lim_{\epsilon\to 0}\int_{C-C_\epsilon} \frac{\phi(s)}{s-t}ds \tag{39.5}$$

and C_ϵ is a curve segment of length 2ϵ centered at t.

Note 39.2. *A similar statement can be obviously made when t is approached along a curve, since all limits along subsequences coincide.*

It is clearly enough to show the formula as the contour is approached from the left. It is also easy, and left as an exercise to extend the proof from the case when C is a piece of $\mathbb{R}$, say $[-1, 1]$ to a more general curve (open or not): parametrize the curve and do similar estimates).

We reduce to the case when ϕ is a constant in the following way. Let $z_n = t + \epsilon_n + iy_n$. Then $\epsilon_n \to 0$ and $y_n \downarrow 0$. Consider the auxilliary function ($t_n = t + \epsilon_n$)

$$\Psi_n := \frac{\phi(s) - \phi(t_n)}{s - t_n - iy_n} \tag{39.6}$$

We will show the following result

Proposition 39.3. *(i) Let $z_n = t + \epsilon_n + iy_n$, $\epsilon_n \to 0$ and $y_n \downarrow 0$. We have, as $z_n \to t$ in this way,*

$$\lim_{z_n \to t} \int_{-1}^{1} \Psi_n(s) ds = \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds \tag{39.7}$$

(ii)

$$\lim_{z_n \to t} \int_{-1}^{1} \frac{1}{s - z_n} ds = \pi i + P \int_{-1}^{1} \frac{1}{s - t} ds \tag{39.8}$$

(That the last integral exists can be checked directly, and is shown at the end of the proof of Lemma 39.1.)

(iii)

$$\int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds = P \int_{-1}^{1} \frac{\phi(s)}{s - t} ds - \phi(t) P \int_{-1}^{1} \frac{1}{s - t} ds \tag{39.9}$$

39.2. **Proof of Plemelj's formulas *assuming* Proposition 39.3.**

Proof. We have

$$
\begin{aligned}
(39.10)\quad & \lim_{z_n \to t} \int_{-1}^{1} \frac{\phi(s)}{s - z_n} ds = \\
& \lim_{z_n \to t} \int_{-1}^{1} \frac{\phi(s) - \phi(t_n)}{s - z_n} ds + \lim_{z_n \to t} \phi(t_n) \int_{-1}^{1} \frac{1}{s - z_n} ds \\
& = \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds + \pi i \phi(t) + \phi(t) P \int_{-1}^{1} \frac{1}{s - t} ds \\
& = P \int_{-1}^{1} \frac{\phi(s)}{s - t} ds - \phi(t) P \int_{-1}^{1} \frac{1}{s - t} ds + \pi i \phi(t) + \phi(t) P \int_{-1}^{1} \frac{1}{s - t} ds \\
& = P \int_{-1}^{1} \frac{\phi(s)}{s - t} ds + \pi i \phi(t)
\end{aligned}
$$

where we used Proposition 39.3 and the continuity of ϕ to show that the limits on the second line of (39.10) exist and Proposition 39.3 (iii) to go from the third to the fourth line of (39.10). ∎

39.3. **Proof of Proposition 39.3.**

Proof. We first show that (ii) implies (iii). Indeed note that

$$
(39.11)\qquad \lim_{\epsilon \to 0} \int_{[-1, t-\epsilon] \cup [t+\epsilon, 1]} \frac{\phi(s) - \phi(t)}{s - t} ds = \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds
$$

since the integrand is in L^1. On the other hand, by Proposition 39.3 (ii) we have

$$
(39.12)\qquad \lim_{\epsilon \to 0} \int_{[-1, t-\epsilon] \cup [t+\epsilon, 1]} \frac{1}{s - t} ds = P \int_{-1}^{1} \frac{1}{s - t} ds
$$

exists. On the other hand, we have

$$
\begin{aligned}
(39.13)\quad & \int_{[-1, t-\epsilon] \cup [t+\epsilon, 1]} \frac{\phi(s) - \phi(t)}{s - t} ds \\
& = \int_{[-1, t-\epsilon] \cup [t+\epsilon, 1]} \frac{\phi(s)}{s - t} ds - \phi(t) \int_{[-1, t-\epsilon] \cup [t+\epsilon, 1]} \frac{1}{s - t} ds
\end{aligned}
$$

Since the left hand side has a limit as $\epsilon \to 0$ and by (39.12) so does the first term on the right hand side. By definition, that limit equals $P \int \phi(s)(s-t)^{-1} ds$. In conclusion, indeed,

$$
(39.14)\qquad \int_{-1}^{1} \frac{\phi(s) - \phi(t)}{s - t} ds = P \int_{-1}^{1} \frac{\phi(s)}{s - t} dt - \phi(t) P \int_{-1}^{1} \frac{1}{s - t} dt
$$

Proof of Proposition 39.3 (i) To avoid having variable limits of integration we write

$$\int_{-1}^{1} \Psi_n(s)ds = \int_{-2}^{2} \Psi_n(s)\chi(s)ds \tag{39.15}$$

where χ is the characteristic function of the interval $[-1,1]$. We write

$$\begin{aligned}\int_{-2}^{2} \Psi_n(s)\chi(s)ds &= \int_{-2}^{2} \frac{\phi(s)-\phi(t+\epsilon_n)}{s-t-\epsilon_n-iy_n}\chi(s)ds \\ &= \int_{-2}^{2} \frac{\phi(\sigma+\epsilon_n)-\phi(t+\epsilon_n)}{\sigma-t-iy_n}\chi(\sigma+\epsilon_n)d\sigma\end{aligned} \tag{39.16}$$

By the Hölder condition we have

$$|\phi(\sigma+\epsilon_n)-\phi(t+\epsilon_n)| \le C|\sigma-t|^{\alpha} \tag{39.17}$$

and thus

$$\left|\frac{\phi(\sigma+\epsilon_n)-\phi(t+\epsilon_n)}{\sigma-t-iy_n}\right| \le \frac{C|\sigma-t|^{\alpha}}{\sqrt{(\sigma-t)^2+y_n^2}} \le C|\sigma-t|^{\alpha-1} \tag{39.18}$$

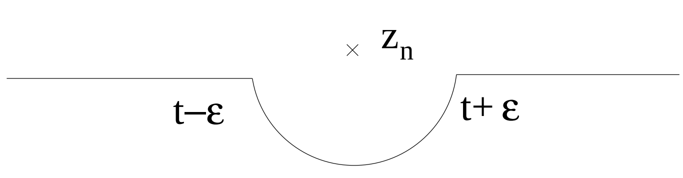
Since $|\chi| \le 1$ we can apply dominated convergence and get

$$\int_{-1}^{1} \Psi_n(s)ds \to \int_{-2}^{2} \frac{\phi(\sigma)-\phi(t)}{\sigma-t}\chi(\sigma)d\sigma = \int_{-1}^{1} \frac{\phi(s)-\phi(t)}{s-t}ds \tag{39.19}$$

Proof of Proposition 39.3 (ii). We have, by analyticity and homotopic deformation for z to the left of the curve,

$$\int_{-1}^{1} \frac{1}{s-z}ds = \int_{C_\epsilon} \frac{1}{s-z}ds \tag{39.20}$$

where C_ϵ is the contour depicted below, where a line segment of length 2ϵ centered at zero is replaced by an semicircle of radius ϵ in the LHP.



The integral around the half circle is easily calculated by parametrization and (again using dominated convergence) we get

$$\int_{-\pi}^{0} i\frac{\epsilon e^{i\phi}}{\epsilon e^{i\phi}-\epsilon_n-iy_n}d\phi \to \pi i \quad (n\to\infty) \tag{39.21}$$

by dominated convergence which also shows that

$$P\int_{-1}^{1}\frac{1}{s-t}ds \tag{39.22}$$

exists since the integrals along symetrically cut intervals do not even depend on ϵ. ▮

Note 39.4. (i) The function defined by the Cauchy type integral (39.1) is called **sectionally analytic**. With the convention about the sides of the curve mentioned before, functions that are boundary values of Cauchy type integrals are sometimes denoted $\oplus$ and $\ominus$ functions. respectively.

(ii) Plemelj's formulas allow us immediately to solve problems of the following kind. Assume Φ is analytic except on a curve C where it has limiting values, and, with the sign convention agreed,

$$\Phi^+(t) - \Phi^-(t) = \phi(t) \tag{39.23}$$

The solution is given by the expression (39.1).

39.4. **Examples.**

39.4.1. *A very simple example.* As usual S_1 is the unit circle. Find a function Φ analytic in $\mathbb{C}\setminus S_1$ such that along S_1 we have

$$\Phi^+(t) - \Phi^-(t) = 1 \tag{39.24}$$

Note that the set of analyticity is disconnected, and it is not a region in our sense. There is no reason to think of Φ as one analytic function, unless we find that the two pieces are analytic continuations of each other across S_1.

For the problem in this section, Plemelj's formula reads

$$\Phi(z) = \frac{1}{2\pi i}\int_{S_1}\frac{1}{s-z}ds \tag{39.25}$$

Clearly, if z is inside the unit disk, which, according to our convention, is to the left of S_1 oriented positively, we have

$$\Phi_{\rm in}(z) = \frac{1}{2\pi i}\int_{S_1}\frac{1}{s-z}ds = 1 \tag{39.26}$$

Likewise, if z is outside the unit disk we have

$$\Phi_{\rm out}(z) = \frac{1}{2\pi i}\int_{S_1}\frac{1}{s-z}ds = 0 \tag{39.27}$$

Both $\Phi_{\rm in}$ and $\Phi_{\rm out}$ are analytic, but not analytic continuations of each-other, so in this case our sectionally analytic function is really a pair of distinct analytic functions. We leave the question of uniqueness to the next subsection when the contour is open and which leads to a more interesting discussion.

39.4.2. *A simple example.* Find a function Φ analytic in $\mathbb{C} \setminus [-1,1]$ such that along $[-1,1]$ we have

$$\Phi^+(t) - \Phi^-(t) = 1 \tag{39.28}$$

39.4.3. *A solution.* According to Plemelj's formulas this function is given by

$$\Phi(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{s-z} ds \tag{39.29}$$

(It is clear Φ is well defined and analytic in $\mathbb{C} \setminus [-1,1]$, which is now a region.)

39.4.4. *Formula for the solution in §39.4.3.* Can we say more about this function? For $z \in (1,\infty)$, for example, we can calculate the integral explicitly and it gives

$$\Phi(z) = \frac{1}{2\pi i} \ln\left(\frac{z-1}{z+1}\right) \tag{39.30}$$

with the ususal branch of the log, which makes $i\Phi$ negative for $z > 2$. Since $(z-1)(z+1)^{-1} = 1 - 2/(z+1)$, Φ admits a uniformly convergent series representation for $|z+1| > 2$ and thus it is analytic in $\{z : |z+1| > 2$. It coincides by construction with our Φ on $(1,\infty)$. They are therefore identical to each other on $\mathbb{C} \setminus [-1,1]$.

39.4.5. *A simple verification.* As an exercise of working with branched functions, let's check that indeed the function in (39.30) solves our problem. Let

$$w = \frac{z-1}{z+1}$$

As z approaches $[-1,1]$ from above, w approaches $(-\infty,0)$ from above as well, and thus $\arg w \to \pi$.

$$\ln\left(\frac{z-1}{z+1}\right) \to \ln\left(\frac{1-t}{1+t}\right) + i\pi \tag{39.31}$$

and similarly, as z approaches $[-1,1]$ from below we get

$$\ln\left(\frac{z-1}{z+1}\right) \to \ln\left(\frac{1-t}{1+t}\right) - i\pi \tag{39.32}$$

and indeed $\Phi^+ - \Phi^- = 1$ along $[-1,1]$. Note that Φ is analytic on the universal covering on $\mathbb{C} \setminus \{-1,1\}$ and in this picture, only the points $\{-1,1\}$ are special, not the whole segment. In any event, we can think of Φ as one analytic function, either in $\mathbb{C} \setminus [-1, 1]$ or on the universal covering of $\mathbb{C} \setminus \{-1, 1\}$.

39.4.6. *Calculating principal value integrals.* Plemelj's formulas help us calculate principal values integrals as well, sometimes in a simpler way. For instance, in our case, by definition the integral

$$P \int_{-1}^{1} \frac{ds}{s - t} \tag{39.33}$$

must be real. We can obtain it by taking the real part of the upper analytic continuation of the log, as calculated in §39.4.4:

$$P \int_{-1}^{1} \frac{ds}{s - t} = \ln \left(\frac{1 - t}{1 + t} \right) \tag{39.34}$$

Extension. Let now C be a simple smooth closed curve and assume that $f(z)$ is analytic in $\mathrm{Int}(C)$ and Hölder continuous in the closure of $\mathrm{Int}(C)$. We can immediately derive from Plemelj's formulas that

$$\frac{1}{2\pi i} P \int_{C} \frac{f(s)}{s - t} ds = \frac{1}{2} f(t) \tag{39.35}$$

a "limiting case" of a Cauchy formula.

39.4.7. *Uniqueness issues.* Is the solution of our problem unique? Certainly not. We can add to Φ any analytic function with isolated singularities at ± 1. Can we achieve uniqueness in such a problem? Yes, if we rule out this frreedom by providing conditions at infinity and near the endpoints of the curve. Indeed, assume that we take (39.28) and insist that $\Phi \to 0$ as $z \to \infty$ and does not grow faster than logarithmically at ± 1. Our Φ satisfies this condition, as it is easy to verify: note that we have the convergent representation for large z

$$\frac{z - 1}{z + 1} = 1 - \frac{2}{z} + \frac{2}{z^2} - \frac{2}{z^3} \cdots \tag{39.36}$$

Assume Φ_1 is another solution with the same properties. Then $f = \Phi - \Phi_1$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and continuous on $(-1, 1)$, thus analytic in $\mathbb{C} \setminus \{-1, 1\}$. The points ± 1 are isolated singularities for f, where it is polynomially bounded. These points can at most be poles, but this is ruled out by the logarithmic bounds. Thus f is entire. But since it goes to zero at infinity, then it equals zero.

40. Extensions

We mention the following extension.

Lemma 40.1. *Assume that L is the real line, that there exists L such that $\phi(s) \to L$ as $|t| \to \infty$ that the Hölder condition is uniform on $\mathbb{R}$ and at infinity, the latter condition being expressed as*

$$|\phi(s) - L| = O(s^{-\mu}), \quad |s| \to \infty; \quad \mu > 0. \tag{40.1}$$

Then the conclusion of Theorem 39.1 holds.

Exercise 40.2. ** The proof of this lemma is very similar to the one in the case the curve is compact. Work out the details.*

40.1. **Boundary behavior.** The boundary behavior of Φ at the edges of the curve is essential for uniqueness, as we saw in our simple example. Even the classification of possible behaviors would take too much space, so for some results in this direction we refer the interested reader to [1] and references therein.

40.2. **Scalar homogeneous RH problems.** This is a problem of the type

$$\Phi^+ = g\Phi^- \quad \text{on} \quad C \tag{40.2}$$

where C is a smooth simple closed contour, g nonzero on C and satisfying a Hölder condition on C. We are looking for solutions of *finite order* and assume that the *index of g w.r.t. C* is k. We now explain these last two notions.

40.2.1. *Index of a function with respect to a curve.* We first need to define the index of a function ϕ with respect to a smooth closed curve $C = \{\gamma(t) : t \in [0, 1]\}$. Assume ϕ does not vanish along C and is Hölder continuous of exponent α and constant A. If ϕ is in fact differentiable, the definition of the index is simply

$$\mathrm{ind}_C\phi := \frac{1}{2\pi i}\int_C \frac{\phi'(s)}{\phi(s)} ds \tag{40.3}$$

(If ϕ is meromorphic inside of C, then clearly $\mathrm{ind}_C\phi = N - P$, the number of zeros minus the number of poles inside C.) If ϕ is not differentiable, we can still define the index by noting that in the differentiable case $\phi'/\phi = (\ln\phi)'$ and then

$$\mathrm{ind}_C\phi := \frac{1}{2\pi i}[\log\phi]_C \tag{40.4}$$

the total variation of the argument of ϕ when C is traversed once. Because of the nonvanishing of ϕ, a branch of the log can be consistently chosen and followed along C. Indeed, since $\phi \neq 0$ on C, then $\min_C |\phi| = a > 0$. Let $\Gamma = \max_{t\in[-1,1]} |\gamma'(t)$. We choose ϵ such that $A\Gamma\epsilon^\alpha < a/2$ and then we have $\phi(\gamma(t+\epsilon)) = \phi(\gamma(t)) + \delta$ where $|\delta| < a/2$. If we partition $[0,1]$ in intervals of size ϵ and choose a branch of $\log(\phi(\gamma(0)))$ we can calculate inductively the log in any interval of size ϵ by taking $0 < \epsilon' < \epsilon$ and writing $\phi(\gamma(k\epsilon + \epsilon'))) = \phi(\gamma(k\epsilon)) + \delta'$, noting that $|\delta'| < |\phi(\gamma(k\epsilon))|$ and thus

$$\log\Big(\phi(\gamma(k\epsilon+\epsilon'))\Big) = \log\Big(\phi(\gamma(k\epsilon))\Big) + \log\Big(1+\delta'/\phi(\gamma(k\epsilon))\Big) \tag{40.5}$$

can be calculated by Taylor expanding the last log. Since the log and ϕ are well defined, and the condition $\exp(\log z) = z$ is preserved in the process, the index of ϕ must be an integer.

40.2.2. *Degree of a function at infinity.* By definition Φ has degree k at infinity if for some $C \neq 0$ we have

$$\Phi(z) = Cz^k + O(z^{k-1}) \quad \text{as} \quad z \to \infty \tag{40.6}$$

The function Φ has finite degree at infinity if $\Phi = o(z^m)$ for some m.

40.2.3. *Solution to the homogeneous RH problem.* First we note that if Φ is a solution and P is a polynomial of order m, then by homogeneity ΦP is also a solution.

Let us assume C is a simple smooth closed curve. Without loss of generality we assume $0 \in\mathrm{Int}(C)$. We can rewrite the problem as

$$\Phi^+(t) = (t^{-k}g(t))(t^k\Phi^-) \quad \text{on} \quad C \tag{40.7}$$

or

$$\ln \Phi^+(t) = \ln(t^{-k}g(t)) + \ln(t^k\Phi^-) \quad \text{on} \quad C \tag{40.8}$$

or finally, with obvious notation,

$$\Gamma^+(t) = f(t) + \Gamma^-(t) \quad \text{on} \quad C \tag{40.9}$$

The reason we formed the combination $t^{-k}g(t)$ was to ensure Hölder continuity of the function. Otherwise, since $\arg\phi$ changes by $2k\pi$ upon traversing C, the function $\log g$ is not continuous (unless $k = 0$). But we already know a solution to (40.9), given by Plemelj's formulas

$$\Gamma(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds \tag{40.10}$$

We recall that $\Gamma^- = O(z^{-1})$ for large z, and thus $\log(z^k\Phi^-) = O(z^{-1})$ too. This means that $z^k\Phi^- \to 1$ as $z \to \infty$, or, which is the same,

$\Phi^- = z^{-k} + o(z^{-k})$ for large z. We finally get the solution of degree m at infinity,

$$\Phi(z) = X(z)P_{m+k}(z) \tag{40.11}$$

where

$$X = \begin{cases} e^{\Gamma(z)}, & z \text{ inside } C \\ z^{-k}e^{\Gamma(z)}, & z \text{ outside } C \end{cases} \tag{40.12}$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_C \frac{\log(s^{-k}g(s))}{s-z}ds \tag{40.13}$$

The polynomial P is appended to ensure, if possible, the desired begavior at infinity. We will not, for reasons of space, discuss uniqueness issues here.

40.2.4. *Ingomogeneous RH problems.* These are equations of the form

$$\Phi^+ = g\Phi^- + f \tag{40.14}$$

again under suitable assumptions on g and f. These can be brought to Plemelj's formulas in the following way. We first solve the homogeneous problem

$$\Psi^+ = g\Psi^- \tag{40.15}$$

and look for a solution of (40.14) in the form $\Phi = U\Psi$. We get (40.16)

$$U^+\Psi^+ = gU^-\Psi^- + f \Rightarrow U^+g\Psi^- = U^-g\Psi^- + f \Rightarrow U^+ - U^- = \frac{f}{g\Psi^-}$$

which is of the form we already solved.

Exercise 40.3. * *Check that f/Ψ^+ is Hölder continuous.*

40.3. **Applications.**

40.3.1. *Ingomogeneous singular integral equations.* These are equations of the form

$$a(t)\phi(t) + b(t)P\int_C \frac{\phi(s)}{s-z}ds = c(t) \tag{40.17}$$

with a, b, c Hölder continuous and the further condition $i\pi a(t) \pm b(t) \neq 0$. We attempt to write, guided by Plemelj's formulas

$$\phi(t) = \Phi^+(t) - \Phi^-(t) \tag{40.18}$$

and

$$\Phi(z) = \frac{1}{2\pi i}\int \frac{\phi(s)}{s-z}ds \tag{40.19}$$

and then

$$P\int_C \frac{\phi(s)}{s-t}ds = i\pi\Big[\Phi^+(t)+\Phi^-(t)\Big] \tag{40.20}$$

where, of course ϕ is still unknown. The equation becomes

$$a(t)\Big[\Phi^+(t)-\Phi^-(t)\Big]+b(t)i\pi\Big[\Phi^+(t)+\Phi^-(t)\Big]=c(t) \tag{40.21}$$

or

$$\Phi^+(t)(a(t)+b(t)i\pi)+\Phi^-(t)(b(t)i\pi-a(t))=c(t) \tag{40.22}$$

or, finally,

$$\Phi^+(t)=\frac{a(t)-b(t)i\pi}{a(t)+b(t)i\pi}\Phi^-(t)+\frac{c(t)}{a(t)+b(t)i\pi} \tag{40.23}$$

which is of the form (40.14) which we addressed already. Care must be taken that the chosen solution Φ is such that $\Phi^+ + \Phi^-$ has the behavior (39.2) at infinity. Then the substitution is a posteriory justified. We did not discuss whether there are other solutions of the integral equation. A complete discussion of this and related equations can be found in [6]. Many interesting examples are given in [1].

40.3.2. *The heat equation.* The heat equation with prescribed boundary data too can be reduced to a RH problem; since however we have already solved this problem for the unit disk and we can use conformal mapping for other domains, we refer the interested reader to [1].

41. Entire and Meromorphic functions

Analytic and meromorphic functions share with polynomials and rational functions a number of very useful properties, such as decomposition by partial fractions and factorization. These notions have to be carefully analyzed though, since questions of convergence arise.

41.1. **Partial fraction decompositions.** First let $R = P_0/Q = P_1 + P/Q$ be a rational function. where P_i and Q are polynomials and $\deg(P) < \deg(Q)$. We aim at a partial fraction decomposition of R; if $\deg(Q) = 0$ there is nothing further to do. Otherwise let $z_1, ..., z_n$, $n \geq 1$, be the zeros of Q, where we don't count the multiplicities, and let n_j be the multiplicities of these roots. Let's look at the singular part of the Laurent expansion of P/Q at z_j:

$$\frac{P}{Q}=\sum_{k=1}^{n_j}\frac{c_{jk}}{(z-z_j)^k}+\text{analytic at } z_j \tag{41.1}$$

We claim that

$$\frac{P}{Q} = \sum_{j=1}^{n} \sum_{k=1}^{n_j} \frac{c_{jk}}{(z - z_j)^k} \tag{41.2}$$

Indeed,

$$E(z) := \frac{P}{Q} - \sum_{j=1}^{n} \sum_{k=1}^{n_j} \frac{c_{jk}}{(z - z_j)^k} \tag{41.3}$$

is an entire function. By assumption, $P/Q \to 0$ as $z \to \infty$ and the rhs of (41.2) also, clearly, goes to zero as $z \to \infty$. Thus $E(z) \to 0$ as $z \to \infty$ and therefore $E \equiv 0$. Let's try a less trivial example. The function

$$\frac{\pi^2}{\sin^2 \pi z} \tag{41.4}$$

has zeros for $z_i = N$, $N \in \mathbb{Z}$, and the singular part of the Laurent series at $z = N$ is, as it can be quickly checked

$$\frac{1}{(z - N)^2} \tag{41.5}$$

We claim that in fact

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{k \in \mathbb{Z}} \frac{1}{(z - k)^2} \tag{41.6}$$

and in fact the proof is similar to that for rational functions. We first note that the series on the rhs of (41.6) converges in $\mathbb{C} \setminus \mathbb{Z}$, uniformly on compact sets; it thus defines an analytic f function in $\mathbb{C} \setminus \mathbb{Z}$. Let $E(z)$ be the difference between the lhs and rhs of (41.6).

(i) If $z = x + iy$, $x \in [0, 1]$ and y large, the terms of the series are bounded by

$$\frac{1}{1 + (k - x)^2}$$

and in particular, by dominated convergence, for fixed x, $f \to 0$ as $y \to \infty$.

(ii) With $z = x + iy$ we have

$$|\sin(x + iy)^2| = \frac{1}{2}\Big(\cosh(2y) - \cos(2x)\Big) \geq \frac{1}{2}\left(\cosh(2y) - 1\right)$$

(iii) Clearly $E(z)$ is periodic with period 1 and it is, by construction and the form of f, an entire function.

(iv) By analyticity, $E(z)$ is bounded in the rectangle $\{(x, y) : |x| \leq 1, |y| \leq 2\}$ and by (i) and (ii) it is also bounded in the strip $\{(x, y) : |x| \leq 1, |y| \geq 2$ and since it is periodic, it is bounded in $\mathbb{C}$. Thus E is a constant. It can only be zero, since by (i) and (ii) $E \to 0$ as x is fixed and $y \to \infty$.

41.2. **The Mittag-Leffler theorem.** How generally is it possible to decompose meromorphic functions by partial fractions? Completely general, as we'll see in a moment, provided we are careful with the questions of convergence of series. Had we attempted instead to write in the same spirit

$$\frac{\pi}{\sin \pi z} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{z-n} \quad (?) \tag{41.7}$$

we would have run into the problem that the series in (41.7) is not convergent. But provided we add and subtract terms so as to ensure convergence, the partial fraction decomposition is general.

Theorem 41.1 (Mittag-Leffler)**.** *(i) Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers such that $b_n \to \infty$ as $n \to \infty$ and let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials without constant term. Then there are meromorphic functions in $\mathbb{C}$ such that the only poles are at $z = b_n$ and the singular part of the Laurent expansion at b_n is $P_n((z-b_n)^{-1})$.*

(ii) Conversely let f be meromorphic in $\mathbb{C}$ with poles at $z = b_n$ and the singular part of the Laurent expansion at b_n is $P_n((z-b_n)^{-1})$. Then there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ and an entire function g such that

$$f = \sum_{n\in\mathbb{N}} \left[P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right] + g(z) := S(z) + g(z) \tag{41.8}$$

where the series converges uniformly on compact sets in $\mathbb{C} \setminus \{b_n\}_{n\in\mathbb{N}}$.

Proof. We start by proving (ii). We can assume without loss of generality that $b_n \neq 0$ for if say $b_0 = 0$ then we can prove the theorem for $f' = f - P_0(1/z)$.

The function $P_n((z-b_n)^{-1})$ is analytic in the open disk at zero of radius $R = |b_n|$. We take a smaller disk, of radius say $R = |b_n|/4$ and denote by p_n the Taylor polynomial of order n_m at zero of $P_n((z-b_n)^{-1})$. In the disk of radius R we have (cf. (10.6)

$$\left|P_n((z-b_n)^{-1}) - p_n(z)\right| \le 2M \frac{(2|z|)^{n_m+1}}{|b_n|^{n_m+1}} \tag{41.9}$$

We now choose n_m so that, for $|z| < b_n/4$ we have

$$\left|P_n((z-b_n)^{-1}) - p_n(z)\right| \le 2^{-n} \tag{41.10}$$

We make the same construction for all n. Note that (41.10) is a fortiori true for all $|b_m| > |b_n|$, since in the estimate (41.9, b_n would get replaced by (the larger) $|b_m|$.

Thus, if we exclude the terms with $|b_n| < 4R$ from the series S (a finite number of terms), the remaining series R_1 converges uniformly in the closed disk of radius R, and thus S_1 is analytic in $D(0, R)$. By adding back the finitely many terms we subtracted out, we obviously get a function analytic in the disk $D(0, R) \setminus \{b_n\}_{n\in\mathbb{N}}$. Since R is arbitrary, S is analytic in $\mathbb{C} \setminus \{b_n\}_{n\in\mathbb{N}}$. But, by construction, $f - S(z)$ is entire.

(i) We note that, by the same arguments, the function

$$h(z) = \sum_{n\in\mathbb{N}} P_n((z - b_n)^{-1}) - p_n(z) \tag{41.11}$$

constructed in (ii) is analytic in $\mathbb{C} \setminus \{b_n\}_{n\in\mathbb{N}}$ and has the required singular Laurent part. ∎

41.3. **Further examples.** We have

$$\pi(\cot \pi z)' = -\frac{\pi^2}{\sin^2 \pi z} \tag{41.12}$$

On the other hand, the series

$$S(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z - n} + \frac{1}{n} \right) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n(z - n)} \tag{41.13}$$

converges uniformly in $\mathbb{C} \setminus \mathbb{Z}$ and, by Weierstrass's theorem can be differentiated termwise.

We get

$$-\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(z - n)^2} = S'(z) \tag{41.14}$$

and thus

$$S'(z) = \pi(\cot \pi z)' + \frac{1}{z^2} \tag{41.15}$$

and thus

$$\pi \cot \pi z = C + \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z - n} + \frac{1}{n} \right) \tag{41.16}$$

On the other hand, combining pairwise the term with n with the term with $-n$ we get

$$\pi \cot \pi z = C + \frac{1}{z} + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{2z}{z^2 - n^2} \tag{41.17}$$

since the left side is odd, we must have $C = 0$ and thus

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{N} \setminus \{0\}} \frac{2z}{z^2 - n^2} \tag{41.18}$$

We can now use this identity to calculate easily some familiar sums. Note that the lhs of (41.18) has the Laurent expansion at $z = 0$

$$\pi \cot \pi z = \frac{1}{z} - \frac{\pi^2 z}{3} - \frac{\pi^4 z^3}{45} - \frac{2\pi^6 z^5}{945} - \cdots \tag{41.19}$$

Since on the other hand the series on the rhs of (41.18) converges uniformly near $z = 0$, by Weierstrass's theorem it converges together with all derivatives. On the other hand we have

$$\frac{2z}{z^2 - n^2} = -2\left(\frac{z}{n^2} + \frac{z^3}{n^4} + \frac{z^5}{n^6} + \cdots\right) \tag{41.20}$$

and we get immediately,

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n \geq 1} \frac{1}{n^6} = \frac{\pi^6}{945} \cdots \tag{41.21}$$

Exercise 41.2. ** The definition of the Bernoulli numbers B_k is*

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1} \tag{41.22}$$

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k} \tag{41.23}$$

Also using (41.18) it is not difficult to show that

$$\frac{\pi}{\sin \pi z} = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z - n} = \frac{1}{z} + \sum_{m=1}^{\infty} (-1)^m \frac{2z}{z^2 - m^2} \tag{41.24}$$

giving a precise meaning to (34.14).

42. Infinite products

An infinite product is the limit

$$\prod_{n=1}^{\infty} p_n := \lim_{k \to \infty} \prod_{n=1}^{k} p_n = \lim_{k \to \infty} \Pi_k \tag{42.1}$$

We adopt here the convention of existence of a nontrivial limit used by Ahlfors. Evidently, if one of the factors is zero, the infinite product would be zero regardless of the behavior of the other terms. On the other hand, we will be able to express analytic functions as infinite products, and we should allow them to vanish. Then (42.1) is said to converge *if only finitely many terms p_n are zero, and the rest of the product has a finite nonzero limit*. Omitting the zero factors and writing $p_n = P_n/P_{n-1}$, $P_0 = 1$ we see that the limit of Π_k is the same as the limit of the P_k, and thus $p_n \to 1$ is a necessary condition of convergence of the infinite product. We should then better write the products as

$$\prod_{n=1}^{\infty}(1 + a_n) \tag{42.2}$$

and then a necessary condition of convergence is $a_n \to 0$.

Theorem 42.1. *The infinite product (42.2) converges iff*

$$\sum_{n=1}^{\infty} \ln(1 + a_n) \tag{42.3}$$

converges. We use the principal branch of the log, extended by continuity when $\arg(z) \uparrow \pi$ *and omit, as usual, the terms with* $a_n = -1$.

Before proving the theorem, a word of caution. We know that in the complex domain, $\log ab$ is not always $\log a + \log b$. The limit of the sum will not, in general, be the log of the infinite product. So the reasonning is not that obvious.

If the sum (42.3) converges, then P_n converges, since the exponential of a finite sum is a finite product.

In the opposite direction, assume the product converges. It is clear that the definition of convergence is independent of a finite initial set of terms that we can modify at will. We can thus assume that the limit is $P = 1$.

We successively calculate $\ln P_n$ by adding the real and imaginary parts of the logs we thus defined. We can write, absorbing the small imaginary contribution in ϵ_n, $\ln P_n = \epsilon_n + 2i\pi h_n$ where $\epsilon_n \to 0$ and $h_n \in \mathbb{Z}$.

We claim that for large enough n, h_n is constant. Since $a_n \to 0$ we have $\ln(1 + a_n) = a_n + o(a_n)$. Thus, $\ln P_{n+1} = \epsilon_n + 2i\pi h_n + o(1) =: \epsilon_{n+1} + 2i\pi h_{n+1}$. Since h_n is an integer and cannot jump by one upon addition of a term $o(1)$, we must have $h_n = const$ for large n and then $\ln P_n = \epsilon_n + 2i\pi\, const \to 2i\pi\, const$ and hence the sum of the logs converges $\square$.

Absolute convergence is easier to control in terms of series. An infinite product is absolutely convergence, by definition, iff

$$(42.4) \qquad \sum_{n=1}^{\infty} |\ln(1+a_n)|$$

is convergent.

Theorem 42.2. *The sum (42.4) is absolutely convergent iff $\sum a_k$ is absolutely convergent.*

Proof. Assume $\sum a_k$ converges absolutely. Then in particular $a_n \to 0$. Also, if $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges absolutely then $\ln(1+a_n) \to 0$ and $a_n \to 0$. But then for large enough n we can use the fact that $|a_n| < 1/4$ and elementary inequality

$$\frac{1}{2}|a_n| < |\ln(1+a_n)| < \frac{3}{2}|a_n|$$

to show that absolute convergence occurs simultaneously for the two series. ∎

Note 42.3. *Conditional (not absolute) convergence of $\sum a_n$ and of $\prod(1+a_n)$ are unrelated notions. (Consider, e.g., the product $\prod(1-(-1)^n n^{-1/2})$. Is the associated series $\sum(-1)^n n^{-1/2}$ convergent? Is the product convergent?)*

42.1. **Uniform convergence of products.**

Exercise 42.4. *** Assume that $p_n(z)$ are analytic in the region Ω and $f(z) = \prod_{n\ge 1} p_n(z)$ converges absolutely and uniformly on every compact set in the region Ω. Show that f is analytic in Ω. Show that*

$$(42.5) \qquad f'(z) = \sum_{k=1}^{\infty} \prod_{n=1}^{\infty} \frac{p_k'}{p_k} p_n$$

where the sum is also uniformly convergent. Hint: Use Weierstrass's theorem.

42.2. **Example: the** sin **function.** We note that the zeros of $\sin \pi z$ are at the integers and we would like to write sin in terms of the products of these roots. We can start with (41.18) and note that

$$(42.6) \qquad \ln(C\pi \sin \pi z)' = \pi \cot \pi z; \quad (\ln C_n(z^2-n^2))' = \frac{2z}{z^2-n^2}$$

so we end up with the *formal* identity

$$(42.7) \qquad C\pi \sin \pi z \stackrel{?}{=} z \prod_{n>0} C_n(z^2-n^2))$$

(it is formal because, in principle, we are not allowed to combine the logs the way we did) where the constants C_n need to be chosen so that the product is convergent. Other than that, the constants C_n would contribute to an overall immaterial constant, since we already have one on the left side. A good choice is $C_n = -n^{-2}$ which gives us the tentative identity

$$C\pi \sin \pi z \stackrel{?}{=} z \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{42.8}$$

The constant C can only be $1/\pi^2$ if we look at the behavior near $z = 0$. Thus,

$$\frac{\sin \pi z}{\pi} \stackrel{?}{=} z \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{42.9}$$

But this equality is only plausible; it needs to be proved. This is not too difficult, and it can be done in the same way as we proved equalities stemming from partial fractions decompositions.

First we note that the product on the rhs of (42.9) is absolutely and uniformly convergent on any compact z set; this can be easily checked. It thus defines an entire function $g(z)$. Motivated by the way we obtained this possible identity, let us look at the expression $f'/f - g'/g$ where $\pi f(z) = \sin \pi z$. We get, using Exercise 42.4,

$$f'/f - g'/g = \pi \cot \pi z - \frac{1}{z} + \sum_{n \in \mathbb{N} \setminus \{0\}} \frac{2z}{z^2 - n^2} = 0 \tag{42.10}$$

This means that

$$\frac{f'g - fg'}{fg} = 0 \tag{42.11}$$

in $\mathbb{C} \setminus \mathbb{Z}$ or, equivalently,

$$\frac{f'g - fg'}{g^2} = 0 = \left(\frac{f}{g}\right)' \tag{42.12}$$

or $f/g = const$; we already calculated the constant based on the behavior at zero, it is one. Thus indeed,

$$\frac{\sin \pi z}{\pi z} = \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{42.13}$$

How general is this decomposition possible? Again, if we are careful about convergence issues it is perfectly general. This is what we are going to study in the next subsection.

42.3. **Canonical products.** The simplest possible case is that in which we have a function with no zeros.

Theorem 42.5. *Assume f is entire and $f \neq 0$ in $\mathbb{C}$ Then f is of the form*

$$f = e^g \tag{42.14}$$

where g is also entire.

Proof. Note that f'/f is entire. Since $\mathbb{C}$ is simply connected, $h(z) = \int_0^z f'(s)/f(s)ds$ is well defined and also entire. Now we note that $(fe^{-g})' = 0$ in $\mathbb{C}$ and thus $f = \exp(h + C)$ proving the result. ∎

Assume now that f has finitely many zeros, a zero of order $m \geq 0$ at the origin, and the nonzero ones, possibly repeated are $a_1, ...a_n$.

Then

$$f = z^m \prod_{k=1}^{n} \left(1 - \frac{z}{a_n}\right) e^{g(z)}$$

where g is entire.

This is clear, since if we divide f by the prefactor of e^g we get an entire function with no zeros.

We cannot expect, in general, such a simple formula to hold if there are infinitely many zeros. Again we have to take care of convergence problems. This is done in a manner similar to that used in the Mittag-Leffler construction.

Theorem 42.6 (Weierstrass)**.** *(i) If $\{a_n\}$ is a finite set or a sequence with the property $a_n \to \infty$ as $n \to \infty$ then there exists an entire function with zeros at a_n and no other zeros.*

(ii) Assume f is an entire function with zeros at a_n. Then there exist integers m, m_n and an entire function $g(z)$ such that

$$f(z) = e^{g(z)} z^m \prod \left[\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \right] \tag{42.15}$$

Proof. As in the proof of Mittag-Leffler's theorem, we prove part (ii) and (i) is an easy byproduct of this proof. We shall see that we can take $m_n = n$. We some estimates on f we shall be able to make a much sharper choice (Hadamard's theorem).

We show that the product is absolutely and uniformly convergent. Let $R > 0$ be arbitrary and let us examine the product for $x \in D(0, R)$. There are finitely many roots thus terms with $|a_n| \leq R$ and we need not worry about them.

Consider now the a_n with $|a_n| > R$. We have, for $|z| < R$,

$$\ln(1 - z/a_n) = -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{a_n}\right)^k \tag{42.16}$$

We can write

$$\ln(1 - z/a_n) = -\sum_{k=1}^{m_n} \frac{1}{k} \left(\frac{z}{a_n}\right)^k + \mathcal{E}_{m_n} := -p_{m_n} + \mathcal{E}_{m_n} \tag{42.17}$$

where, by (10.6) we have

$$|\mathcal{E}_{m_n}| \le \frac{1}{m_n + 1} \left(\frac{R}{|a_n|}\right)^{m_n+1} \left(1 - \frac{R}{|a_n|}\right)^{-1} \tag{42.18}$$

We can choose, say, $m_n = n$. Then the series $\sum |\mathcal{E}_{m_n}|$ is majorized by $\sum \lambda^n$ for some $\lambda < 1$ and thus converges absolutely and uniformly for $|z| > R$.

Since $\sum_k |\ln(1 - z/a_k)e^{p_k}| = \sum_k |\mathcal{E}_{m_k}|$ converges, then the product

$$\prod_k \left(1 - \frac{z}{a_k}\right) e^{p_k}(z) \tag{42.19}$$

converges absolutely and uniformly (see Exercise 42.4) to an analytic function in the disk $|z| < R$. On the other hand,

$$f z^{-m} \left[\prod_k \left(1 - \frac{z}{a_k}\right) e^{p_k}(z)\right]^{-1}$$

is entire and has no zeros, and thus it is of the form e^g with g entire. ∎

Corollary 42.7. *Any meromorphic function is a ratio of entire functions.*

Proof. Let F be meromorphic with poles at b_n of order m_n. Let G be any entire function with zeros at b_n of order m_n. Then FG has only removable singularities. ∎

42.4. **Counting zeros of analytic functions. Jensen's formula.** The rate of growth of an analytic function is closely related to the density of zeros. We have a quite effective counting theorem, due to Jensen.

Theorem 42.8 (Jensen)**.** *Assume $f \not\equiv 0$ is analytic in the closed disk $\overline{D(0,r)}$ and $f(z) = cz^m g(z)$ with $m \ge 0$ and $g(0) = 1$. Let a_i be the* *nonzero roots of f in $D(0,r)$, repeated according to their multiplicity. Then*

$$\ln|c| = -\sum_{i=1}^{n} \ln\left(\frac{r}{|a_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta - m\ln r \tag{42.20}$$

Proof. The proof essentially boils down to the case where $f(0) \neq 0$ and it has no zeros inside the disk of radius r. In this simple case, a consistent branch of $\ln f$ can be defined inside and $\Re \ln f = \ln|f|$ is harmonic in $D(0,r)$. For $r' < r$ we have be the mean value theorem for harmonic functions we have,

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r'e^{i\theta})|d\theta \tag{42.21}$$

Since f is analytic in the closed disk and $\ln|x|$ is in $L^1(-a,a)$, it is easy to see by dominated convergence (check) that (42.21) holds in the limit $r = r'$ too, even if there are zeros on the circle of radius r:

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \tag{42.22}$$

Assume now f has zeros, with the convention in the statement of the theorem. We then build a function which has no zeros inside $D(0,r)$ and has the same absolute value for $|z| = r$. Such a function is

$$h(z) = \frac{r^m}{z^m} f(z) \prod_{i=1}^{n} \frac{r^2 - \overline{a_i}z}{r(z - a_i)} \tag{42.23}$$

Then

$$\ln|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|h(re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \tag{42.24}$$

The formula now follows by expanding out $\ln|h(0)|$. ∎

Corollary 42.9. *Assume f is analytic in the closed disk of radius R and $f(0) \neq 0$. Let $\nu(r)$ denote the number of zeros of f in the disk of radius $r \leq R$. Then*

$$\int_0^R \frac{\nu(x)}{x} dx \leq \ln \max_{|z|=R} |f(z)| - \ln|f(0)| \tag{42.25}$$

Of course, $\nu(x)$ is an increasing discontinuous function of x.

Proof. Note that

$$\ln(R/|a_i|) = \int_{|a_i|}^R \frac{dx}{x} = \int_0^R \chi_{[|a_i|,R]}(x)\frac{dx}{x}$$

Thus

$$\sum_{i=1}^{n} \ln\left(\frac{r}{|a_i|}\right) = \int_0^R \sum_{i=1}^{n} \chi_{[|a_i|,R]}(x)\frac{dx}{x} = \int_0^R \frac{\nu(x)}{x}dx$$

The rest follows immediately from (42.20). ∎

Example 42.10. *Assume $f(z)$ is entire, and for large $|z|$ there are positive constants C, c and ρ such that $|f(z)| \le Ce^{c|z|^\rho}$. Then, for large r there is a $c_2 > 0$ such that we have*

$$\nu(r) \le c_2 r^\rho$$

Indeed

$$c|R|^\rho - \ln|f(0)| \ge \int_0^R \frac{\nu(x)}{x}dx \ge \int_{R/2}^R \frac{\nu(x)}{x}dx \ge \frac{\nu(R/2)}{R}\frac{R}{2} \tag{42.26}$$

and the rest is immediate. We can optimize the inequality by including the constants of Jensen's inequality, choosing R/a instead of $R/2$ and optimizing over a.

42.5. **Estimating analytic functions by their real part.**

Theorem 42.11 (Borel-Carathéodory)**.** *Let $f = u + iv$ be analytic in a closed disk of radius R. Let $A_R = \max_{|z|=R} u(z)$. Then for $r < R$ we have*

$$\max_{|z|\le r} |f(z)| \le \frac{2rA_R}{R-r} + \frac{R+r}{R-r}|f(0)| \tag{42.27}$$

Note that if, say, as $z \to \infty$ we have $|f| \to \infty$ then, since $|u| \le |f|$, the theorem above shows that u and f are roughly of the same order of magnitude.

Proof. Assume first that $f(0) = 0$. Then $u(0) = 0$ and by the mean value theorem $A_R \ge 0$. If $A = 0$ then by the same argument $u \equiv 0$ on $\partial D(0,R)$ and by Poisson's formula $u \equiv 0$ in $D(0,R)$. Then $v \equiv$ *const* $= 0$ since $f(0) = 0$, thus $f \equiv 0$ and the formula holds trivially.

We now take $A_R > 0$. Since the maximum of a harmonic function is reached on the boundary, we have $2A_R - u \ge u$ in $D(0,R)$ and the inequality is strict in the interior. The function

$$g(z) = \frac{1}{2A - f(z)}\frac{f(z)}{z} \tag{42.28}$$

is holomorphic in $D(0,R)$ and on the disk of radius R we have

$$|2A - f| = \sqrt{(2A-u)^2 + v^2} \ge \sqrt{u^2 + v^2} = |f| \tag{42.29}$$

and thus in $D(0,R)$ we have

$$|g(z)| = \left|\frac{1}{2A - f(z)}\frac{f(z)}{z}\right| \le \frac{1}{R} \tag{42.30}$$

hence

$$\left|\frac{f(z)}{z}\right| \le \frac{1}{R}\,|2A - f(z)| \le \frac{1}{R}(2A + |f(z)|) \tag{42.31}$$

Solving for $|f(z)|$ we get

$$|f(z)| \le \frac{2|z|A_R}{R - |z|} \tag{42.32}$$

as claimed. The general case is obtained by applying this inequality to $f(z) - f(0)$ (exercise). ∎

Corollary 42.12. *Assume $\rho \ge 0$, $f = u + iv$ is entire and as $|z| \to \infty$ we have*

$$|u(z)| \le C|z|^\rho \tag{42.33}$$

Then f is a polynomial of degree at most ρ.

Proof. Let $R = 2|z|$. We have, from Theorem 42.11 for large $r = |z|$,

$$|f(z)| \le \frac{2Crr^\rho}{r} + 3|f(0| \le C' r^\rho \tag{42.34}$$

The rest is standard. ∎

42.6. **Entire functions of finite order.** Let f be an entire function. We denote by $\|f\|_R$ the maximum value of $|f(z)|$ for $|z| \le R$, or which is the same, $\|f\|_R$ is the maximum value of $|f(z)|$ for $|z| = R$. A function is of **order** $\le \rho$ if for any $\epsilon > 0$ there is some $c > 0$ such that for all R large enough we have

$$\|f\|_R \le e^{cR^{\rho+\epsilon}} \tag{42.35}$$

or equivalently

$$\ln \|f\|_R = O(R^{\rho+\epsilon}) \tag{42.36}$$

Note 42.13. *We can always check the condition for $R \in \mathbb{N}$ large enough since $(N+1)^\rho = O(N^\rho)$.*

The function f has **strict order** $\le \rho$ if there is some $c > 0$ such that for all R large enough we have

$$\|f\|_R \le e^{cR^\rho} \tag{42.37}$$

A function has **order equal** ρ if it has order $\le \rho'$ iff $\rho' \ge \rho$. A function has **strict order equal** ρ if it has order ρ' iff $\rho' \ge \rho$.

42.7. **Counting zeros of entire functions of finite order.**

Theorem 42.14. *Assume f is an entire function of strict order ρ and let $\nu_f(R)$ be the number of zeros of f in the disk of radius R centered at the origin $D(0,R)$. Then for large R we have*

$$\nu_f(R) = O(R^\rho) \tag{42.38}$$

Proof. We can obtain this result from Example 42.10 which gives us a more accurate estimate. However the result is important enough and a direct proof is simple so it's worth deriving again the result. We can always assume that $z=0$ is not a zero of f since for some finite $m \geq 0$ f/z^m is a function of same order which does not vanish at zero. Let $z_1, ..., z_n$ be the zeros of f in $D(0,R)$. The function

$$g(z) = \frac{f(z) z_1 \cdots z_n}{(z-z_1)\cdots(z-z_n)} \tag{42.39}$$

is also entire and $|g(0)| = |f(0)| \neq 0$. But (with some constants $C_i > 0$)

$$|f(0)| = |g(0)| \leq \max_{|z|=3R} |g(z)| \leq \max_{|z|=3R} \frac{|f(z)|R^n}{(2R)^n} \leq C_1 \frac{e^{(3R)^\rho}}{e^{n \ln 2}} \tag{42.40}$$

i.e., solving for n we get for some $C_2 > 0$,

$$n \leq C_2 R^\rho \tag{42.41}$$

∎

43. Hadamard's theorem

Let $\rho > 0$ and let k_ρ be the smallest integer *strictly* greater than ρ, $k_\rho = \lfloor \rho \rfloor + 1$. We consider again the truncates of the series of $-\ln(1-z)$, namely, with $k = k_\rho$,

$$P_k(z) = z + \frac{z^2}{2} + \cdots + \frac{z^{k-1}}{k-1} \tag{43.1}$$

Theorem 43.1 (Hadamard)**.** *Let f be entire of order ρ, let z_n be its nonzero zeros and let $k = k_\rho$. Then, with $m \geq 0$ the order of the zero of f at zero, there is a polynomial h of degree $\leq \rho$ such that*

$$f(z) = e^{h(z)} z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{P_k(z/z_n)} = e^{h(z)} E(z) \tag{43.2}$$

The proof of this important theorem requires a number of intermediate results, notably the *minimum modulus principle* proved in the following section, a very useful result in its own right.

Lemma 43.2. *Let ϵ be such that $\lambda := \rho + \epsilon < k_\rho := k$. There is a $c > 0$ such that*

$$|(1-\zeta)\exp P_k(\zeta)| \le \exp(c|\zeta|^\lambda) \tag{43.3}$$

Proof. For $|\zeta| \le 1/2$ we have

$$\ln(1-\zeta) + P_k(\zeta) = \sum_{n=k}^{\infty} \frac{\zeta^n}{n} = \zeta^k C_k;\ |C_k| \le \sum_{n=k}^{\infty} 2^{-n} \le 2 \tag{43.4}$$

$$\Rightarrow (1-\zeta)e^{P_k(\zeta)} \le e^{2|\zeta|^k} \le e^{2|\zeta|^\lambda} \tag{43.5}$$

For $|\zeta| \in [1/2, 1]$ we have

$$\begin{aligned} |(1-\zeta)\exp P_k(\zeta)| &\le \frac{1}{2}\exp\left[|\zeta|^k\left(\frac{1}{|\zeta|^{k-1}} + \cdots + \frac{1}{|\zeta|(k-1)}\right)\right] \\ &\le \frac{1}{2}\exp(2^k|\zeta|^k) \le \frac{1}{2}\exp(2^k|\zeta|^\lambda) \end{aligned} \tag{43.6}$$

For $|\zeta| > 1$ we have

(43.7)

$$|(1-\zeta)\exp P_k(\zeta)| \le |(1-\zeta)|\exp\left[|\zeta|^{k-1}\left(\frac{1}{k-1} + \cdots + \frac{1}{|\zeta|^{k-2}}\right)\right]$$
$$\le \exp\left(k|\zeta|^{k-1} + \ln|1+|\zeta||\right) \le \exp\left(k|\zeta|^\lambda + \ln|1+|\zeta||\right) \le \exp\left(C_2|\zeta|^\lambda\right)$$

for some C_2 independent of $\zeta, |\zeta| > 1$. This is because $t^{-\lambda}\ln(1+t)$ is continuous on $[1,\infty)$ and goes to zero at infinity (fill in the details). ∎

43.1. **Canonical products.** Take any sequence $\{z_n\}_n$ where the terms are ordered by absolute value, with the property that for some $\rho > 0$ and any $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\rho+\epsilon}} < \infty \tag{43.8}$$

Definition 43.3. *The canonical product determined by the sequence $\{z_n\}$, denoted by $E^{(k)}(z, \{z_n\})$ or simply $E(z)$ is defined by*

$$E(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)\exp[P_k(z/z_n)] \tag{43.9}$$

Theorem 43.4. *$E(z)$ is an entire function of order $\le \rho$.*

Proof. Take again any ϵ be such that $\lambda := \rho + \epsilon < k_\rho$. Then, by Lemma 43.2 we have
(43.10)
$$|E(z)| \le \prod_{n=1}^{\infty} \exp(c|z/z_n|^\lambda) = \exp\left(c|z|^\lambda \sum_{k=1}^{\infty} |z_n|^{-\lambda}\right) \le \exp\left(c_1|z|^\lambda\right)$$

proving, in the process, uniform convergence of the product. ∎

Theorem 43.5. *Let f be entire of strict order $\le \rho$ and let $\{z_n\}$ be its nonzero zeros, repeated according to their multiplicity and ordered increasingly by their absolute value. Then for any $\epsilon > 0$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\rho+\epsilon}} \tag{43.11}$$

is convergent.

Proof. We can obviously discard the roots with $|z_i| \le 1$ which are in finite number. Without loss of generality we assume there are none. We have, with $N \in \mathbb{N}$ and estimating the sum by annuli,

$$\sum_{|z_n|\le N} \frac{1}{|z|^{\rho+\epsilon}} \le \sum_{k=1}^{N} \frac{\nu(k+1)-\nu(k)}{k^{\rho+\epsilon}} \tag{43.12}$$

we can now use the method of Abel summation by parts. We write

(43.13)
$$\frac{\nu(k+1)-\nu(k)}{k^{\rho+\epsilon}} = \nu(k+1)\left(\frac{1}{k^{\rho+\epsilon}} - \frac{1}{(k+1)^{\rho+\epsilon}}\right) + \left(\frac{\nu(k+1)}{(k+1)^{\rho+\epsilon}} - \frac{\nu(k)}{k^{\rho+\epsilon}}\right)$$

and note that by summation, the terms in the last parenthesis cancel out to

$$\frac{\nu(N+1)}{(N+1)^{\rho+\epsilon}} - \nu(1)$$

Note that by usual calculus we have for some $\gamma = \gamma(k)$

$$\frac{\nu(k+1)}{k^{\rho+\epsilon}} - \frac{\nu(k+1)}{(k+1)^{\rho+\epsilon}} = \frac{(\rho+\epsilon)\nu(k+1)}{(k+\gamma)^{\rho+\epsilon+1}} \le \frac{Ck^\rho(\rho+\epsilon)}{k^{\rho+\epsilon+1}} \tag{43.14}$$

and the sum converges. ∎

44. The minimum modulus principle; end of proof of Theorem 43.1

This important theorem tells us, roughly, that if a function does not grow too fast it cannot decrease too quickly either, aside from zeros. More precisely we have

Theorem 44.1 (Minimum modulus principle)**.** *Let f be an entire function of order $\le \rho$. As before, let $\{z_n\}$ be its zeros with $|z_i| > 1$, repeated according to their multiplicity and let $\epsilon > 0$. At every root, take out a disk $D(z_n, r_n)$ with $r_n = |z_n|^{-\rho-\epsilon}$ and consider the complement U in $\mathbb{C}$ of these disks. Then in U, for large r there is a constant c such that*

$$|f(z)| \ge \exp(-c|z|^{\rho+\epsilon}) \quad \textit{or} \quad \frac{1}{|f(z)|} = O(\exp(|z|^{\rho+\epsilon})) \tag{44.1}$$

Proof. We start with the case when the entire function is a canonical product. We take $|z| = r$ and write

$$E(z) = \prod_{|z_n|<2r} E_k(z, z_n) \prod_{|z_n|\ge 2r} E_k(z, z_n) \tag{44.2}$$

and estimate the two terms separately. We note that in the second product, all ratios $\zeta =: \zeta_n = z/z_n$ have the property $|\zeta| \le 1/2$. Taking one term of the product, we have to estimate *below*

$$E(\zeta) = (1-\zeta)e^{P(\zeta)} \tag{44.3}$$

Since $|\zeta| \le 1/2$, $\ln(1-\zeta)$ exists; we take the principal branch and write

$$\begin{aligned} \left|(1-\zeta)e^{P(\zeta)}\right| &= \left|e^{\ln(1-\zeta)+P(\zeta)}\right| = \left|\exp\left(-\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right)\right| \\ &= \exp\left(-\Re\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right) \ge \exp\left(-\left|\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right|\right) \ge \exp\left(-\sum_{n=k}^{\infty}\frac{|\zeta|^n}{n}\right) \\ &\ge \exp\left(-|\zeta|^k\sum_{n=k}^{\infty}\frac{(1/2)^n}{n}\right) \ge e^{-2|\zeta|^k} \ge e^{-2|\zeta|^{\rho+\epsilon}} \end{aligned} \tag{44.4}$$

Thus for the second product in (44.2) we have

$$\prod_{|z_n|\ge 2r} E_k(z, z_n) \ge \exp\left(-2|z|^{\rho+\epsilon}\sum_{|z_n|\ge 2r}\frac{1}{|z_n|^{\rho+\epsilon}}\right) \ge e^{-c|z|^{\rho+\epsilon}} \tag{44.5}$$

since the infinite sum converges by Theorem 43.5. We now examine the convergence improving factors, for $|z_n| < 2r$.

$$\left|\sum_{|z_n|<2r} P_k(z/z_n)\right| \le \left|\sum_{r<|z_n|<2r} P_k(z/z_n)\right| + \left|\sum_{|z_n|\le r} P_k(z/z_n)\right| \tag{44.6}$$

For the first term on the right we note that when $|z/z_n| = |\zeta_n| =: |\zeta| < 1$ and we have

$$\left|\sum_{n=1}^{k-1} \frac{\zeta^n}{n}\right| \le \sum_{n=1}^{k-1} \frac{1}{n} =: c_1 \tag{44.7}$$

and thus

$$\sum_{r<|z_n|<2r} |P_k(z/z_n)| \le \nu(2r)c_1 \le c_2 r^{\rho+\epsilon} \tag{44.8}$$

For the second term on the right of (44.6) we note that $|z/z_n| \ge 1$ and thus, with $\zeta = z/z_n$ we have

$$\left|\sum_{n=1}^{k-1} \frac{\zeta^n}{n}\right| \le |\zeta|^{k-1} \sum_{n=1}^{k-1} \frac{1}{n} =: c_1 |\zeta|^{k-1} = c_1 r^{k-1} |z_n|^{-k+1} \tag{44.9}$$

We use Abel summation by parts (we are careful that r is not necessarily an integer)

$$\begin{aligned} \sum_{|z_n|\le r} |z_n|^{-k+1} &\le \sum_{m\le r} \frac{\nu(m+1)-\nu(m)}{m^{k-1}} \\ &= \sum_{m\le r} \nu(m+1)\left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}}\right) + \frac{\nu(r+1)}{r^{k-1}} - \nu(1) \\ &\le \sum_{m\le r} \nu(m+1)\left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}}\right) + \frac{\nu(r+1)}{r^{k-1}} \\ &\le \sum_{m\le r} \frac{kCm^{k-\delta}}{m^k} + c_3 r^{\rho+\epsilon-k+1} \\ &\le C_1 \sum_{m\le r} \frac{1}{m^\delta} + c_3 r^{\rho+\epsilon-k+1} \le C_2 r^{1-\delta} + c_3 r^{\rho+\epsilon-k+1} \end{aligned} \tag{44.10}$$

for δ such that $k-\delta = \rho+\epsilon$ and some constant $C = C(\delta)$ and we majorized the sum by an integral in the usual way. Multiplying by $c_1 r^{k-1}$ we get that the second term on the right of (44.6) is bounded by

$$c_4 r^{k-\delta} + c_5 r^{\rho+\epsilon} = (c_4+c_5) r^{\rho+\epsilon} \tag{44.11}$$

since $\rho + \epsilon > k - 1$. Thus (44.6) is bounded by

$$(c_2 + c_5) r^{\rho+\epsilon} \tag{44.12}$$

Finally, we look at the products $(1 - z/z_n)$ for $|z_n| < 2r$. Since by the construction of U we have $|z - z_n| \geq |z_n|^{-\rho-\epsilon}$ we get

$$|1 - z/z_n| = \frac{|z - z_n|}{|z_n|} \geq |z_n|^{-\rho-\epsilon-1} \geq (2r)^{-\rho-\epsilon-1} \tag{44.13}$$

and thus

(44.14)
$$\prod_{|z_n|<2r} |1 - z/z_n| \geq [(2r)^{-\rho-\epsilon-1}]^{\nu(2r)} = e^{-\nu(2r)(\rho+\epsilon+1)\ln(2r)} \geq e^{-c_6 r^{\rho+\epsilon'}}$$

for any $\epsilon' > \epsilon$ if r is large enough ($r^{\epsilon-\epsilon'} \ln r \to 0$ as $r \to \infty$).

We now finish the proof of Theorem 43.1.

Proof. We take $\epsilon > 0$ and $s = \rho+\epsilon$. We order the roots nondecreasingly by $|z_n|$. For each root z_n we consider the annulus $A_n = \{z : |z| \in [|z_n|-2|z_n|^{-s}, |z_n|+2|z_n|^{-s}]$. Consider $J^c := \mathbb{R}^+ \setminus J$ where J is the union of all intersections of the A_n with $\mathbb{R}^+$. Since the Lebesgue measure of J does not exceed $4\sum_n |z_n|^{-s} < \infty$, there exist arbitrarily large numbers in the complement J^c. Take r be any number in J^c and consider the circle $\partial D(0, r)$. Consider the function $g = f/E$. g is clearly an entire function with no zeros. Then, by Theorem 42.5, $g = e^h$ with h entire. Since $\Re h \leq (C_f + C_E) r^{\rho+\epsilon}$ for some $C_f + C_E$ independent of r in $D(0, r)$ for arbitrarily large r (check), we have by Corollary 42.12 that h is a polynomial of degree at most $\rho + \epsilon$. Since ϵ is arbitrary, h is a polynomial of degree at most ρ. ∎

To finish the proof of the minimum modulus principle, we use Hadamard's theorem and the fact that e^{-h} satisfies the required bounds. (Exercise: fill in the details.) ∎

Example 44.2. *Let us show that $f(z) = e^z - z$ has infinitely many roots in $\mathbb{C}$. Indeed, first note that $f(z)$ has order 1 since $|z| \leq e^{|z|}$ for all z. Suppose f had finitely many zeros. Then*

$$e^z - z = P(z) e^{h(z)} \tag{44.15}$$

where $P(z)$ is a polynomial and $h(z)$ is a polynomial of degree one, and without loss of generality we can take $h(z) = cz$, $c = \alpha + i\beta$. As $z = t \to +\infty$ we have

$$P(t) e^{(c-1)t} = 1 - te^{-t} \to 1 \tag{44.16}$$

In particular $|P(t)|e^{(\alpha-1)t} \to 1$ *which is only possible if* $\alpha = 1$. *But then* $|P(t)| \to 1$ *which is only possible if* $P(t) = const = e^{i\phi}$. *We are then left with*

$$e^{i(\phi+t\beta)} \to 1 \quad as \quad t \to +\infty \tag{44.17}$$

Then $\beta = 0$, *otherwise, say if* $\beta > 0$ *then for* $t = \beta^{-1}((2k+1)\pi - \alpha), k \in \mathbb{N}$ *the lhs of (44.17) is* -1. *Then* $e^{i\phi} = 1$. *We are left with the identity*

$$e^z - z \equiv e^z \tag{44.18}$$

which is obviously false.

Exercise 44.3. ** Let* $P \not\equiv 0$ *be a polynomial. Show that the equation* $e^z = P(z)$ *has infinitely many roots in* $\mathbb{C}$.

Exercise 44.4. *** (i) Rederive formula (42.13) using Hadamard's theorem.*

(ii) Write down a product formula of the function

$$f(z) = \sin z + 3\sin(3z) + 5\sin(5z) + 7\sin(7z)$$

The final formula should be explicit except for arcsins of roots of a cubic polynomial.

44.1. **Some applications.**

Corollary 44.5 (Borel)**.** *Assume that* ρ *is not an integer and* f *has order strictly* ρ. *Then* f *takes every value in* $\mathbb{C}$ *infinitely many times.*

Proof. It suffices to show that such a function has infinitely many zeros, since f and $f - z_0$ have the same strict order. Assume to get a contradiction f had finitely many zeros. Then $g(z) = f(z)\prod_{i=1}^{n}(z - z_i)^{-1}$ would be entire, with no zeros, and as it is easy to check, of order strictly ρ. Then $g = e^h$ with h a polynomial whose degree can only be an integer. ∎

Let $\exp^{(n)}$ be the exponential composed with itself n times.

Corollary 44.6 (A weak form of Picard's theorem)**.** *A nonconstant entire function which is bounded by* $\exp^{(n)}(C|z|)$ *for some* n *and large* z *takes every value with at most one exception.*

Proof. We prove this by induction on n. We first show that a nonconstant entire function of finite order takes every value with at most one exception. Assume a is an exceptional (*lacunary*) value. Then $f(z) - a$ is entire with no zeros, thus of the form e^h with h a polynomial, $f = e^h - a$. If the degree of h is zero, then f is a constant. Otherwise, we must show that $e^h - a$ takes all values with at most one exception ($-a$ of course), or, which is the same, e^h takes all values with at most one exception. The equation $e^h = b$, $b \neq 0$ is solved if $h - \ln b$ has roots, which is true by the fundamental theorem of algebra.

Assume now the property holds for $n \leq k-1$ and we wish to prove it for $n = k$. Let f be an entire function bounded by $\exp^{(n)}(C|z|)$ which avoids the value a. Then $f - a$ is entire with no zeros, $f - a = e^h$ with h entire. It is easy to show that h is bounded by $\exp^{(n-1)}(C|z|)$. Thus it avoids at most one value, by the induction hypothesis. The equation $e^h - a = b$, for $b \neq -a$ always has a solution. Indeed, if $\ln(b-a)$ is not an avoided value of h this is obvious. On the other hand, if $\ln(b-a)$ is avoided by h, then again by the induction hypothesis $\ln(b-a) + 2\pi i$ is not avoided. ∎

Exercise 44.7. ** Show that the equation

$$\cos(z) = z^4 + 5z^2 + 13 \tag{44.19}$$

has infinitely many roots in $\mathbb{C}$.

Exercise 44.8. ** *(Bonus)* Show that the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{44.20}$$

takes every complex value infinitely many times. (Hint: using L'Hospital show that $\operatorname{erf}(is)/(e^{s^2}/s) \to const.$ as $s \to +\infty$.)

We will return to the error function later and use asymptotic methods to locate, for large x, these special points.

45. The Phragmén-Lindelöf Theorem

Theorem 45.1 (Phragmén-Lindelöf). *Let U be the open sector between two rays from the origin, forming an angle π/β, $\beta > \frac{1}{2}$. Assume f is analytic in U, and continuous on its closure, and for some $C_1, C_2, M > 0$ and $\alpha \in (0, \beta)$ it satisfies the estimates*

$$|f(z)| \leq C_1 e^{C_2 |z|^\alpha}; \quad z \in U; \quad |f(z)| \leq M; \quad z \in \partial U \tag{45.1}$$

Then

$$|f(z)| \leq M; \quad z \in U \tag{45.2}$$

Proof. By a rotation we can make $U = \{z : 2|\arg z| < \pi/\beta\}$. Making a cut in the complement of U we can define an analytic branch of the log in U and, with it, an analytic branch of z^β. By taking $g = f(z^{1/\beta})$, we can assume without loss of generality that $\beta = 1$ and $\alpha \in (0, 1)$ and then $U = \{z : |\arg z| < \pi/2\}$. Let $\alpha' \in (\alpha, 1)$ and consider the analytic function

$$e^{-C_2 z^{\alpha'}} f(z) \tag{45.3}$$

Since $|e^{-C_2 z^{\alpha'}}| < 1$ in U (check) and $|e^{-C_2 z^{\alpha'} + C_2 z^{\alpha}}| \to 0$ as $|z| \to \infty$ on the half circle $|z| = R, \Re z \geq 0$ (check), the usual maximum modulus principle completes the proof. ∎

45.1. **An application to Laplace transforms.** We will study Laplace and inverse Laplace transforms in more detail later. For now let $F \in L^1(\mathbb{R})$. Then it by Fubini and dominated convergence, the Laplace transform

$$\mathcal{L}F := \int_0^\infty e^{-px} F(p) dp \tag{45.4}$$

is well defined and continuous in x in the closed right half plane and analytic in the open RHP (the open right half plane). (Obviously, we could allow $Fe^{-|\alpha|p} \in L^1$ and then $\mathcal{L}F$ would be defined for $\Re x > |\alpha|$.) F is uniquely defined by its Laplace transform, as seen below.

Lemma 45.2 (Uniqueness)**.** *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of x with an accumulation point. Then $F = 0$ a.e.*

We will from now on write $F = 0$ on a set to mean $F = 0$ *a.e.* on that set.

Proof. By analyticity, $\mathcal{L}F = 0$ in the open RHP and by continuity, for $s \in \mathbb{R}$, $\mathcal{L}F(is) = 0 = \hat{\mathcal{F}}F$ where $\hat{\mathcal{F}}F$ is the Fourier transform of F (extended by zero for negative values of p). Since $F \in L^1$ and $0 = \hat{\mathcal{F}}F \in L^1$, by the known Fourier inversion formula [8], $F = 0$. ∎

More however can be said. We can draw interesting conclusions about F just from the rate of decay of $\mathcal{L}F$.

Proposition 45.3 (Lower bound on decay rates of Laplace transforms)**.** *Assume $F \in L^1(\mathbb{R}^+)$ and for some $\epsilon > 0$ we have*

$$\mathcal{L}F(x) = O(e^{-\epsilon x}) \quad as \quad x \to +\infty \tag{45.5}$$

Then $F = 0$ on $[0, \epsilon]$.

Proof. We write

$$\int_0^\infty e^{-px} F(p) dp = \int_0^\epsilon e^{-px} F(p) dp + \int_\epsilon^\infty e^{-px} F(p) dp \tag{45.6}$$

we note that

$$(45.7)\quad \Big|\int_\epsilon^\infty e^{-px}F(p)dp\Big| \le e^{-\epsilon x}\int_\epsilon^\infty |F(p)|dp \le e^{-p\epsilon}\|F\|_1 = O(e^{-\epsilon x})$$

Therefore

$$(45.8)\qquad g(x) = \int_0^\epsilon e^{-px}F(p)dp = O(e^{-\epsilon x}) \quad \text{as} \quad x \to +\infty$$

The function g is entire (prove this!) Let $h(x) = e^{\epsilon x}g(x)$. Then by assumption h is entire and uniformly bounded for $x \in \mathbb{R}$ (since by assumption, for some x_0 and all $x > x_0$ we have $h \le C$ and by continuity $\max|h| < \infty$ on $[0, x_0]$). The function is also manifestly bounded for $x \in i\mathbb{R}$ (by $\|F\|_1$). By Phragmén-Lindelöf (first applied in the first quadrant and then in the fourth quadrant, with $\beta = 2, \alpha = 1$) h is bounded in the closed RHP. Now, for $x = -s < 0$ we have

$$(45.9)\qquad e^{-s\epsilon}\int_0^\epsilon e^{sp}F(p)dp \le \int_0^\epsilon |F(p)| \le \|F\|_1$$

Again by Phragmén-Lindelöf (again applied twice) h is bounded in the closed LHP thus bounded in C, and it is therefore a constant. But, by the Riemann-Lebesgue lemma, $h \to 0$ for $x = is$ when $s \to +\infty$. Thus $h \equiv 0$. But then, with χ_A the characteristic function of A,

$$(45.10)\qquad \int_0^\epsilon F(p)e^{-isp}dp = \hat{\mathcal{F}}(\chi_{[0,\epsilon]}F) = 0$$

for all $s \in \mathbb{R}$ entailing the conclusion. ∎

Corollary 45.4. *Assume $F \in L^1$ and $\mathcal{L}F = O(e^{-AX})$ as $x \to +\infty$ for all $A > 0$. Then $F = 0$.*

Proof. This is straightforward. ∎

As we see, uniqueness of the Laplace transform can be reduced to estimates. Also, no two different $L^1(\mathbb{R}^+)$ functions, real–analytic on $(0, \infty)$, can have Laplace transforms within exponentially small corrections of each–other. This will play an important role later on.

45.2. **A Laplace inversion formula.**

Theorem 45.5. *Assume $c \ge 0$, $f(z)$ is analytic in the closed half plane $H_c := \{z : \Re z \ge c\}$. Assume further that $\sup_{c' \ge c}|f(c' + it)| \le g(t)$ with $g(t) \in L^1(\mathbb{R})$. Let*

$$(45.11)\qquad F(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}f(x)dx =: (\mathcal{L}^{-1}F)(p)$$

Then for any $x \in \{z : \Re z > c\}$ we have

$$\mathcal{L}F = \int_0^\infty e^{-px} F(p) dp = f(x) \tag{45.12}$$

Proof. Note that for any $x' = x_1' + iy_1' \in \{z : \Re z > c\}$

$$\int_0^\infty dp \int_{c-i\infty}^{c+i\infty} \left| e^{p(s-x')} f(s) \right| d|s| \;\le\; \int_0^\infty dp e^{p(c-x_1')} \|g\|_1 \;\le\; \frac{\|g\|_1}{x_1' - c} \tag{45.13}$$

and thus, by Fubini we can interchange the orders of integration:

$$\begin{aligned} U(x') &= \int_0^\infty e^{-px'} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} f(x) dx \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dx f(x) \int_0^\infty dp e^{-px'+px} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(x)}{x'-x} dx \end{aligned} \tag{45.14}$$

Since $g \in L^1$ there must exist subsequences $\tau_n, -\tau_n'$ tending to ∞ such that $|g(\tau_n)| \to 0$. Let $x' > \Re x = x_1$ and consider the box $B_n = \{z : \Re z \in [x_1, x'], \Im z \in [-\tau_n', \tau_n]\}$ with positive orientation. We have

$$\int_{B_n} \frac{f(s)}{x'-s} ds = -f(x') \tag{45.15}$$

while, by construction,

$$\lim_{n\to\infty} \int_{B_n} \frac{f(s)}{x'-s} ds = \int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s} ds - \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{x'-s} dx \tag{45.16}$$

On the other hand, by dominated convergence, we have

$$\int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s} ds \to 0 \quad \text{as} \quad x' \to \infty \tag{45.17}$$

∎

45.3. **Abstract Stokes phenomena.** This theorem shows that if an analytic function decays rapidly along some direction, then it increases "correspondingly" rapidly along a complementary direction. The following is reminiscent of a theorem by Carlson [4].

Theorem 45.6. *Assume $f \not\equiv 0$ is analytic in the closed right half plane and that for all $a > 0$ we have $f(t) = O(e^{-at})$ for $t \in \mathbb{R}^+, t \to \infty$. Then, for all $b > 0$ the function*

$$e^{-bz} f(z) \tag{45.18}$$

is unbounded in the closed right half plane.

Proof. Assume that for some $b > 0$ we had $|e^{-bz}f(z)| < M$ in the closed RHP. Then, the function

$$\psi(z) = \frac{e^{-bz}f(z)}{(z+1)^2} \tag{45.19}$$

satisfies the assumptions of Theorem 45.5. But then $\psi(z) = \mathcal{L}\mathcal{L}^{-1}\psi(z)$ satisfies the assumptions of Corollary (45.4) and $\psi \equiv 0$. ∎

Let $\alpha > 2$.

Corollary 45.7. *Assume $f \not\equiv 0$ is analytic in the closed sector $S = \{z : 2|\arg z| \le \pi/\alpha\}$, $\alpha > \frac{1}{2}$ and that $f(t) \le Ce^{-t^\beta}$ with $\beta > \alpha$ for $t \in \mathbb{R}^+$. Then for any $\beta' < \beta$ there exists a subsequence $z_n \in S$ such that*

$$\left| f(z_n)e^{-z_n^{\beta'}} \right| \to \infty \quad as \quad n \to \infty \tag{45.20}$$

Proof. This follows from Theorem 45.6 by simple changes of variables. ∎

Exercise 45.8. ** Carry out the details of the preceding proof.*

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